CHIP FIRING

2. The Degree of a Divisor

In this lecture, we consider a fundamental invariant of divisors on graphs.

**Definition 2.1.** The *degree* of a divisor $D = \sum_{v \in V(G)} D(v)v$ is the integer

$$\deg(D) = \sum_{v \in V(G)} D(v).$$

Note that the degree is invariant under chip-firing. This allows us to make the following definition.

**Definition 2.2.** The *Jacobian* $Jac(G)$ of a graph $G$ is the group of linear equivalence classes of divisors of degree 0 on $G$. (The Jacobian is also known as the sandpile group, or critical group, and probably many other things besides.)

The degree is a group homomorphism $\text{Pic}(G) \xrightarrow{\deg} \mathbb{Z}$. It is easy to see that this map is surjective, and the kernel is the group of divisors of degree 0. In other words, we have the short exact sequence

$$0 \rightarrow \text{Pic}^0(G) \rightarrow \text{Pic}(G) \xrightarrow{\deg} \mathbb{Z} \rightarrow 0.$$  

Note that, since $\mathbb{Z}$ is free, the exact sequence above splits, so

$$\text{Pic}(G) \cong \mathbb{Z} \oplus \text{Jac}(G).$$

It follows that the degree $d$ part of the Picard group, $\text{Pic}^d(G)$, is a $\text{Jac}(G)$-torsor. That is, the action of $\text{Jac}(G)$ on $\text{Pic}^d(G)$ by addition is free and transitive.

In the previous lecture, we saw that the Picard group of a graph can be computed using the graph Laplacian $\Delta$. Note that $\det(\Delta) = 0$, because the sum of the columns of $\Delta$ is zero. From this we see that $\text{Pic}(G)$ is infinite. Of course, this also follows from the fact that the degree homomorphism maps $\text{Pic}(G)$ surjectively onto the integers. The order of the Jacobian is the absolute value of the determinant of the reduced Laplacian, which is the matrix $\widetilde{\Delta}$ obtained by removing any row from $\Delta$ and the corresponding column. More precisely, $\text{Jac}(G) \cong \mathbb{Z}^{V(G)-1}/\text{Im}(\Delta)$. In this way, we see that Jacobians of graphs are easily computable. Indeed, if one puts the reduced Laplacian in Smith normal form, one obtains a decomposition of $\text{Jac}(G)$ as a direct sum of cyclic groups.

**Example 2.3.** In the previous lecture, we computed the Picard group of the graph $G$ depicted in Figure 1. We found that $\text{Pic}(G) \cong \mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. The reduced Laplacian

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obtained by removing the third row and column is

\[ \tilde{\Delta} = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \]

The determinant of \( \tilde{\Delta} \) is \(-3\), so \( \text{Jac}(G) \cong \mathbb{Z}/3\mathbb{Z} \).

Recall that a spanning tree in a graph \( G \) is a subgraph that contains every vertex and is a tree. Perhaps the most well-known result concerning reduced graph Laplacians is Kirchoff’s Matrix Tree Theorem.

**Matrix Tree Theorem.** The absolute value of the determinant of the reduced graph Laplacian of a graph \( G \) is equal to the number of spanning trees in \( G \).

To prove the Matrix Tree Theorem, we choose an orientation of the graph \( G \), and let \( E \) be the matrix with columns indexed by the edges of \( G \) and rows indexed by the vertices of \( G \), given by

\[
E_{ve} = \begin{cases} 
0 & \text{if } e \text{ is not incident to } v \\
1 & \text{if } v \text{ is the head of } e \\
-1 & \text{if } v \text{ is the tail of } e.
\end{cases}
\]

**Lemma 2.4.** Let \( G \) be a graph. Then \( \Delta(G) = -EE^T \).

**Proof.** If \( i \neq j \), then the \((i,j)\)th entry of \( EE^T \) is given by multiplying the row of \( E \) corresponding to vertex \( i \) by the column of \( E^T \) corresponding to vertex \( j \). We see that the nonzero entries of each vector correspond to edges incident to the given vertex, and the two vectors share a nonzero entry when there is an edge incident to both vertex \( i \) and vertex \( j \). For each such edge, one of the vectors contains a 1 and the other contains a \(-1\). Thus, the \((i,j)\)th entry of \( EE^T \) is the negative of the number of edges incident to both vertex \( i \) and vertex \( j \).

The \((i,i)\)th entry of \( EE^T \) is given by multiplying the row of \( E \) corresponding to vertex \( i \) by its own transpose. Again, the nonzero entries of this vector correspond to edges incident to vertex \( i \). Thus, the \((i,i)\)th entry is the number of edges incident to vertex \( i \).

**Proof of the Matrix Tree Theorem.** Let \( \tilde{\Delta} \) denote the reduced graph Laplacian obtained by removing the row and column corresponding to some vertex \( v \) from the
graph Laplacian $\Delta$. Let $\tilde{E}$ denote the minor of $E$ obtained by removing the row of $E$ corresponding to the same vertex $v$. By the Cauchy-Binet formula for the determinant, we have

$$
\det \Delta = \sum_S \det \tilde{E}_S \det \tilde{E}_S^T = \sum_S \det \tilde{E}_S^2,
$$

where the sum is over all subsets of the edges of size $|V(G)| - 1$. Now, if no edge of $S \subset E(G)$ is incident to the vertex $w \neq v$, then $\tilde{E}_S$ contains a row of all zeros, and therefore has determinant zero. If no edge of $S$ is incident to the vertex $v$, then the sum of the rows of $\tilde{E}_S$ is the zero vector, hence $\tilde{E}_S$ has determinant zero. We therefore see that, if $S$ is not a spanning tree, then $\det \tilde{E}_S = 0$.

On the other hand, if $S$ is a spanning tree, we will show that $\det \tilde{E}_S = \pm 1$. For every vertex $w \in V(G)$, consider the unique path in $S$ from $w$ to $v$. Adding the columns of $\tilde{E}_S$ corresponding to the edges in this path, we obtain a vector with $\pm 1$ in the row corresponding to $w$, and a 0 in every other entry. By performing these elementary column operations to every column of $\tilde{E}_S$, we obtain a matrix in which every row and column has precisely one nonzero entry, and this nonzero entry is $\pm 1$. The determinant of such a matrix is $\pm 1$. \qed

**Corollary 2.5.** For any graph $G$, the order of $\text{Jac}(G)$ is equal to the number of spanning trees in $G$. In particular, $\text{Jac}(G)$ is a finite abelian group.