Limits at Infinity

We write $\lim_{x\to\infty} f(x) = L$ if,

for all very large values of x, $f(x)$ is arbitrarily close to L.

<u>Def</u>: We write $\lim_{x\to\infty} f(x) = L$ if, for all $\varepsilon > 0$, there is an N such that $f(x)$ is within ε of L for all $x > N$.

<u>Ex</u>: Compute $\lim_{x\to\infty} \frac{1}{x}$.

$\lim_{x\to\infty} \frac{1}{x} = 0$ because if you plug in values of x that are very large, $\frac{1}{x}$ is very close to 0.

More precisely, for any small positive number ε (for example $\varepsilon = .001$), there is a number N (for example, $N = 1000$)

with the property that, if $x > N$, then $\cancel{0<}\, \frac{1}{x} < \varepsilon$.

Ex: $$\lim_{x\to\infty} \frac{2x^2-3x+5}{x^2-7x+2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{\frac{2x^2}{x^2} - \frac{3x}{x^2} + \frac{5}{x^2}}{\frac{x^2}{x^2} - \frac{7x}{x^2} + \frac{2}{x^2}}$$

$$= \lim_{x\to\infty} \frac{2 - \frac{3}{x} + \frac{5}{x^2}}{1 - \frac{7}{x} + \frac{2}{x^2}}$$

$$= \frac{\lim_{x\to\infty}\left[2 - \frac{3}{x} + \frac{5}{x^2}\right]}{\lim_{x\to\infty}\left[1 - \frac{7}{x} + \frac{2}{x^2}\right]}$$

$$= \frac{\overset{2}{\lim_{x\to\infty} 2} - 3\overset{0}{\lim_{x\to\infty} \frac{1}{x}} + 5\left[\overset{0}{\lim_{x\to\infty} \frac{1}{x}}\right]^2}{\overset{1}{\lim_{x\to\infty} 1} - 7\overset{0}{\lim_{x\to\infty} \frac{1}{x}} + 2\left[\overset{0}{\lim_{x\to\infty} \frac{1}{x}}\right]^2}$$

$$= \frac{2 - 3\cdot 0 + 5\cdot 0^2}{1 - 7\cdot 0 + 2\cdot 0^2} = \frac{2}{1} = 2$$

Ex: $\displaystyle\lim_{x\to\infty} \frac{x-7}{x^2+3} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$

$$= \lim_{x\to\infty} \frac{\frac{1}{x} - \frac{7}{x^2}}{1 + \frac{3}{x^2}} = \frac{\overbrace{\lim_{x\to\infty}\frac{1}{x}}^{0} - 7\left[\overbrace{\lim_{x\to\infty}\frac{1}{x}}^{0}\right]^2}{\underbrace{\lim_{x\to\infty} 1}_{1} + 3\left[\underbrace{\lim_{x\to\infty}\frac{1}{x}}_{0}\right]^2}$$

$$= \frac{0 - 7\cdot 0^2}{1 + 3\cdot 0} = \frac{0}{1} = 0$$

If $\frac{p(x)}{q(x)}$ is a rational function,

$$\lim_{x\to\infty} \frac{p(x)}{q(x)} = \begin{cases} 0 & \text{if } \deg p(x) < \deg q(x) \\ \dfrac{\text{leading coeff. of } p}{\text{leading coeff. of } q} & \text{if } \deg p(x) = \deg q(x) \\ \text{does not exist} & \text{if } \deg p(x) > \deg q(x) \end{cases}$$

You can also take the limit of a function as x goes to $-\infty$.

We write $\lim_{x \to -\infty} f(x) = L$ if,

for every $\varepsilon > 0$ there is an N with the property that $f(x)$ is within ε of L for all $x < N$.

Ex: $\lim_{x \to -\infty} e^x$

plug in $x = -1$: $e^{-1} = \frac{1}{e^1}$

$x = -100$: $e^{-100} = \frac{1}{e^{100}}$ very small positive number

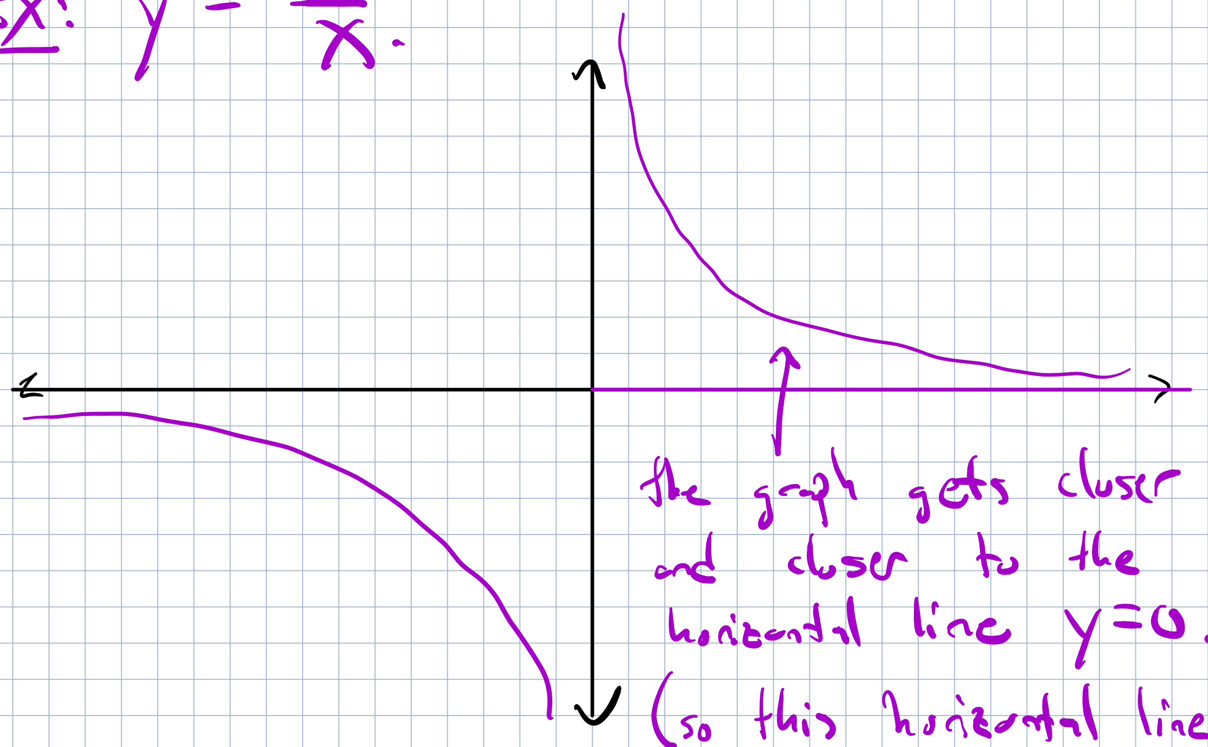
$x = -1000$ $e^{-1000} = \frac{1}{e^{1000}}$ even smaller positive number

These numbers are getting closer and closer to 0, so $\lim_{x \to -\infty} e^x = 0$.

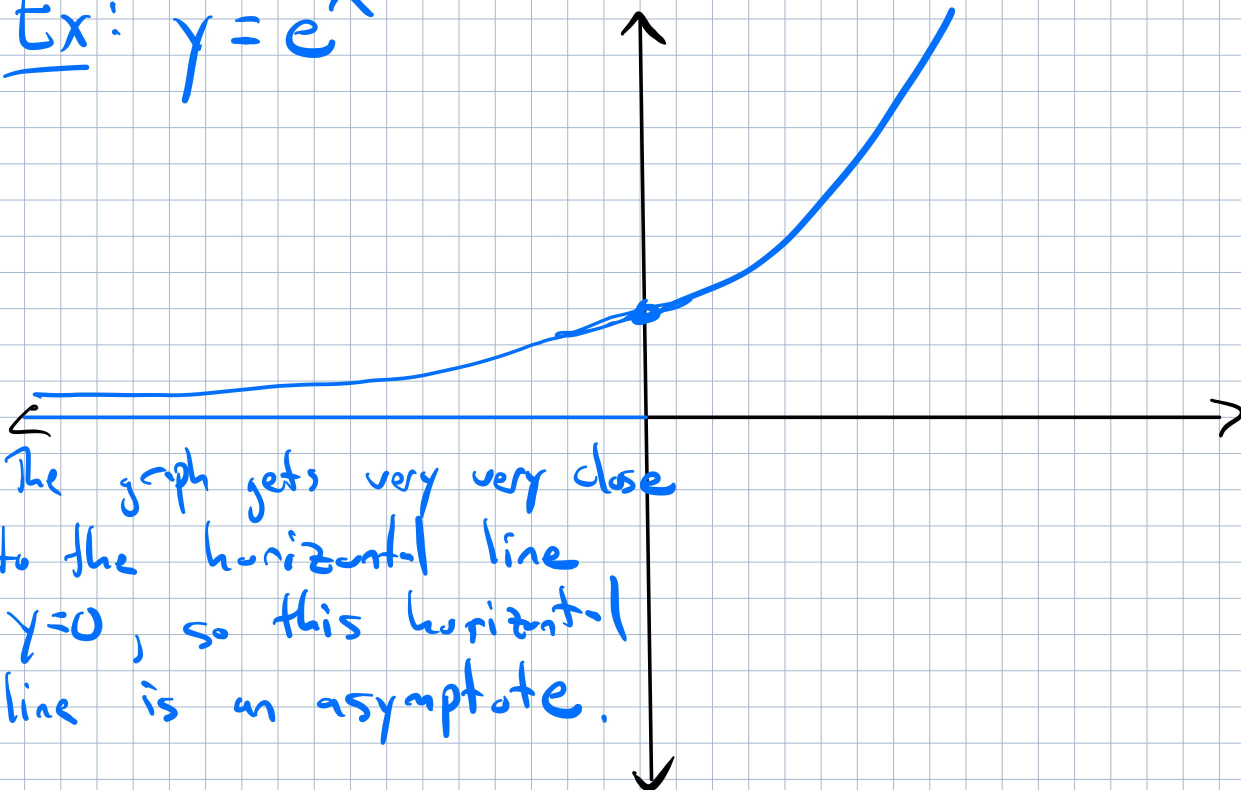
Graphical Interpretation of Limits at Infinity

Given a function $f(x)$, an __asymptote__ is a line that the graph of $f(x)$ gets closer and closer to.

__Ex:__ $y = \frac{1}{x}$.



Ex: $y = e^x$



The graph gets very very close to the horizontal line $y=0$, so this horizontal line is an asymptote.

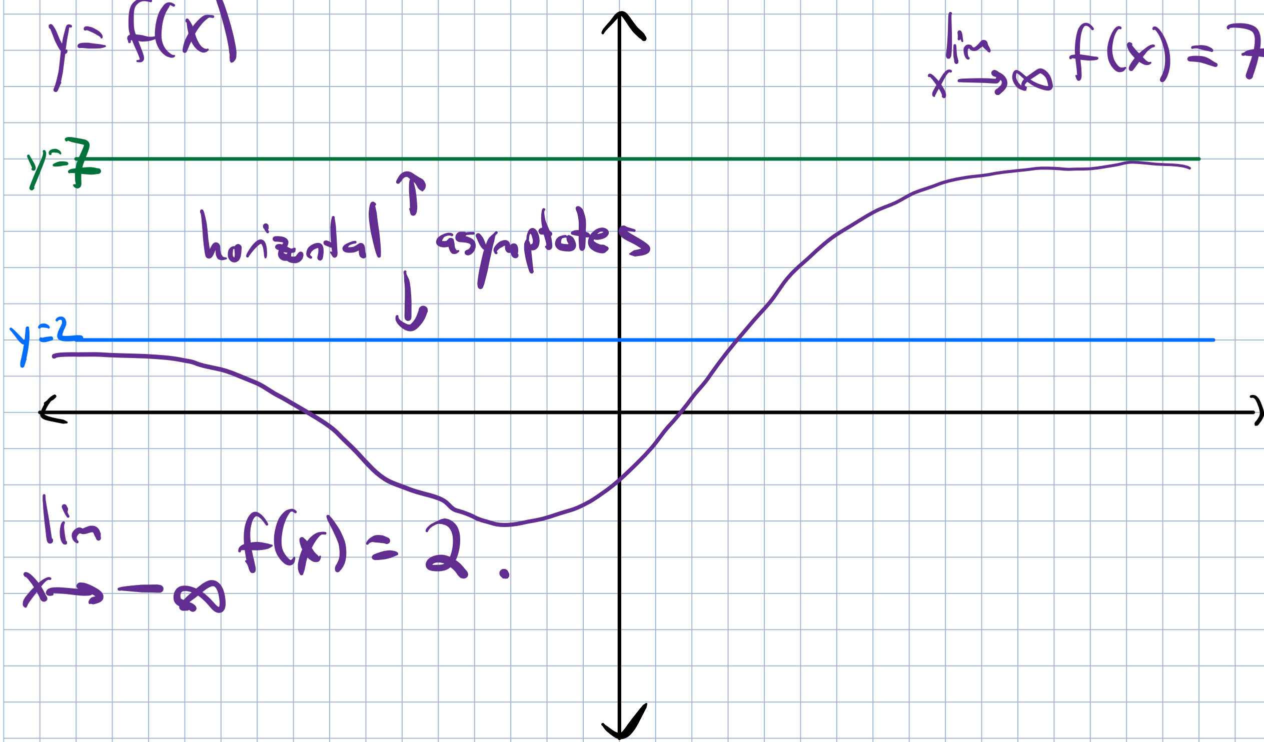
The graph of a function $y=f(x)$ has a horizontal asymptote at $y=L$ if and only if either:

1) $\lim\limits_{x \to \infty} f(x) = L$ or

2) $\lim\limits_{x \to -\infty} f(x) = L$.

$y = f(x)$

$$\lim_{x \to \infty} f(x) = 7$$



$$\lim_{x \to -\infty} f(x) = 2.$$