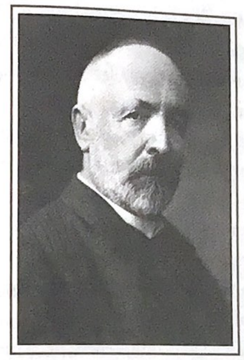
171. Theorem. … ation of a finite system Σ of n ele-
ments, then is the number of elements of the transform $\psi(\Sigma)$ less than n.

Proof. If we select from all those elements of Σ that possess one and the same transform, always one and only one at pleasure, then is the system T of all these selected elements obviously a proper part of Σ, because ψ is a dissimilar transformation of Σ (26). At the same time it is clear that the transformation by (21) contained in ψ of this part T is a similar transformation, and that $\psi(T) = \psi(\Sigma)$; hence the system $\psi(\Sigma)$ is similar to the proper part T of Σ, and consequently our theorem follows by (162), (165).

172. Final remark. Although it has just been shown that the number m of the elements of $\psi(\Sigma)$ is less than the number n of the elements of Σ, yet in many cases we like to say that the number of elements of $\psi(\Sigma) = n$. The word number is then, of course, used in a different sense from that used hitherto (161); for if α is an element of Σ and a the number of all those elements of Σ, that possess one and the same transform $\psi(\alpha)$ then is the latter as element of $\psi(\Sigma)$ frequently regarded still as representative of a elements, which at least from their derivation may be considered as different from one another, and accordingly counted as a-fold element of $\psi(\Sigma)$. In this way we reach the notion, very useful in many cases, of systems in which every element is endowed with a certain frequency-number which indicates how often it is to be reckoned as element of the system. In the foregoing case, e.g., we would say that n is the number of the elements of $\psi(\Sigma)$ counted in this sense, while the number m of the actually different elements of this system coincides with the number of the elements of T. Similar deviations from the original meaning of a technical term which are simply extensions of the original notion occur very frequently in mathematics; but it does not lie in the line of this memoir to go further into their discussion.



Georg Cantor

(1845–1918)

HIS LIFE AND WORK

Georg Cantor scaled the peaks of infinity and then plunged into the deepest abysses of the mind: mental depression.

Georg Ferdinand Ludwig Phillipp Cantor was the firstborn son and namesake of a Protestant father and a Catholic mother. His father, Georg Waldemar Cantor, was a German-born Protestant who moved to St Petersburg, the capital of Tsarist Russia, to become a stockbroker. Cantor ultimately became famous because of the mathematical talent on his father's side. However, he first achieved notoriety for his fine violin playing, no doubt a talent derived from his mother, Marie Böhm, a native Russian who came from a musical family renowned for its violin virtuosi.

Any records of Cantor's early schooling in St Petersburg must have been lost when the family moved back to Germany and settled in Frankfurt so that his father would no longer have to endure the harsh Russian winters. Cantor had a distinguished record at Gymnasia in Frankfurt and nearby Wiesbaden. Cantor's father thought that young Georg's love of mathematics could enable him to be "a shining star in the engineer firmament." Cantor acquiesced to his father's firm suggestion that the Polytechnic school in Zurich would be a good school for studying engineering. After a semester, young Georg finally summoned up the courage to ask his father's permission to transfer to the University of Berlin, where he could study pure mathematics.

Much to Cantor's surprise, his father agreed to the change. In June 1863, as he was finishing his year in Zurich, Cantor learned of his father's sudden death. The elder Cantor would never know of even his son's first accomplishments as a shining star in the mathematical firmament! Cantor sped through the mathematics curriculum at Berlin where he was a pupil of the great Weierstrass, newly resident in the chair of mathematics. Within four years he had both his undergraduate and doctoral degrees.

After teaching at a local girls school for two years, Cantor received his first university teaching appointment at the University in Halle, the birthplace of the composer George Friedrich Handel, about 100 miles south of Berlin. Ten years later, he received a full professorship. He would remain there for the rest of his career.

When Cantor arrived at Halle, his new colleague, the mathematician Heinrich Heine, challenged him to prove the uniqueness of a functions representation by a trigonometric series, a generalization of a Fourier series. This was the research that led Cantor to his study of the infinite in the 1870s.

Cantor was not the first mathematician to formalize the concept of the infinite. Prior to Cantor, Richard Dedekind (1831–1916) made the first giant step by deciding how to ***recognize*** the infinite, rather than ***construct*** it, thereby avoiding objections such as the following one made by the great Gauss:

> *I protest against the use of infinite magnitude as something completed, which in mathematics is never permissible. Infinity is merely a* façon de parler, *the real meaning being a limit which certain ratios approach indefinitely near, while others are permitted to increase without restriction.*

Dedekind took the natural numbers, 0, 1, 2, 3, 4, …, as the paradigm example of an infinite set and defined a set as infinite if the natural numbers could be put into a one-to-one correspondence with that set, or a subset of it. Thus, the natural numbers are infinite by definition and so are the integers, the rational numbers, and the real numbers because every natural number is also an integer, a rational number, and a real number.

With Dedekind having accomplished that, Cantor asked two interesting questions: First, can infinity be recognized without making reference to the natural numbers? Second, are there different degrees of infinity?

Cantor answered his first question by defining a set as being infinite if it could be put into a one-to-one correspondence with Cantor answered his first question by defining a set as being infinite if it could be put into a one to one correspondence with the set of natural numbers ({0,1,2,…}). The set of natural numbers trivially satisfy this condition. The set of natural numbers can be put into a one-to-one mapping with a subset of itself (consider, for example, the mapping that takes 0→1, 1→2, 2→3, 3→4,…). Therefore, any set that satisfies Cantor's definition of the infinite automatically satisfies Dedekind's definition. To show the opposite requires a piece of 20th century mathematics called "the Axiom of Choice".

The bulk of Cantor's groundbreaking work concerned his second question, and it built on his answer to the first question. By making the infinite more general than the natural numbers, Cantor opened the possibility of there being different degrees of the infinite. Cantor defined two sets as being *equinumerous* if they could be put into a one-to-one correspondence with each other. Thus, the positive integers are equinumerous with the negative integers by the mapping $n \leftrightarrow -n$ (for a positive integer n).

Similarly, the positive real numbers are equinumerous with the negative real numbers by a similar mapping. (Try proving to yourself that the positive integers are equinumerous with all of the integers.)

As he made these proofs, Cantor introduced the distinction between a set's *cardinality*, that is, how many members it has, and its *order-type*. He noted that although the positive and the negative integers have the same cardinality, they have a different order-type. There is a first positive integer but no last one using the standard greater-than/less-than ordering. In contrast, there is a last negative integer but no first one using the standard greater-than/less-than ordering. Cantor used Hebrew letters to denote cardinal numbers. $\aleph_0$ (The Hebrew letter Aleph with a subscript of 0 denotes the first infinite cardinal.) Greek letters denote order-types. The Greek letter ω denotes the order-type of a set like the positive integers with a first element but no last element and ω^* (ω with an asterisk as a superscript) denotes a set like the negative integers with a last element but no first element. (The order-type $\omega + 1$ describes a set like the positive integers plus a single element greater than all of the positive integers. The integers as a whole have order-type $\omega^* + \omega$. Can you guess what kind of set the order-type $\omega + \omega^*$ describes? It is not hard. It is just not one you're used to dealing with!)

Then Cantor demonstrated the power of his definition based on equinumerosity. First, he demonstrated that the positive rational numbers are equinumerous with the positive integers. Cantor realized that the positive rational numbers couldn't be put into greater-than/less-than, so he rearranged!

1/1	2/1	3/1	4/1	5/1	6/1	7/1	8/1	9/1	10/1
1/2	2/2	3/2	4/2	5/2	6/2	7/2	8/2	9/2	10/2
1/3	2/3	3/3	4/3	5/3	6/3	7/3	8/3	9/3	10/3
1/4	2/4	3/4	4/4	5/4	6/4	7/4	8/4	9/4	10/4
1/5	2/5	3/5	4/5	5/5	6/5	7/5	8/5	9/5	10/5
1/6	2/6	3/6	4/6	5/6	6/6	7/6	8/6	9/6	10/6
1/7	2/7	3/7	4/7	5/7	6/7	7/7	8/7	9/7	10/7
1/8	2/8	3/8	4/8	5/8	6/8	7/8	8/8	9/8	10/8

1/9	2/9	3/9	4/9	5/9	6/9	7/9	8/9	9/9	10/9
1/10	2/10	3/10	4/10	5/10	6/10	7/10	8/10	9/10	10/10

He rearranged them by the sum of their numerator and denominator and then started enumerating them as follows: 1/1, 2/1, 1/2, 1/3, 2/2, 3/1, 4/1, 3/2, ... thereby demonstrating that the positive rational numbers are equinumerous with the positive integers.

Cantor then tackled the real numbers and proved that they are not equinumerous with the integers. There are strictly more real numbers than integers. Here is the gist of Cantor's proof (for real numbers greater than zero and less than one): If the positive real numbers greater than zero and less than one are equinumerous with the positive integers, then they can all be listed out in a sequence such as

1st	0.**2**092119644443...
2nd	0.3**1**08131969619...
3rd	0.24**2**5129315441...
4th	0.348**0**075650872...
5th	0.0415**8**10010525...
6th	0.47027**4**2494171...
7th	0.659837**1**022485...
8th	0.4153943**6**69555...
9th	0.88325973**6**2598...
10th	0.247564657**6**200...
11th	0.7400378254**5**61...
12th	0.65230954343**7**1...
13th	0.351396247085**1**...

Now comes Cantor's great insight. He considers the highlighted digits on the diagonal and he changes all of their values adding 1 if the value is 0 through 8 and changing a 9 to a 0. This constructs the real number **0.3231952777682** ... that cannot be in the table because it differs from every real number listed in the table! Cantor used a particular form of a *reduction ad absurdum* proof called a *diagonalization argument* to demonstrate that there are strictly more real numbers than there are integers. We will see diagonalization arguments reappear with gusto in the works of Kurt Gödel and Alan Turing.

Because the word *infinity* had a long history with much baggage attached, Cantor introduced the term *transfinite numbers* to denote all of his infinite numbers, both the cardinal numbers and the ordinal numbers (order-types).

Then Cantor asked a question that remains open to this day: Given that there are strictly more real numbers than integers, how many different types of infinite

subsets of the real numbers are there? Are there only two types of infinite subsets of the real numbers, those equinumerous to the real numbers and those equinumerous to the integers? Or are there more types of subsets in between these two types? Cantor believed that there are only two types of infinite subsets of the real numbers, but he could not prove it. The resolution of this conjecture, now known as the Continuum Hypothesis, would have surprised Cantor. In 1940, Kurt Gödel showed that the Continuum Hypothesis cannot be *disproved* using the standard axioms of set theory; then in 1963, Paul Cohen proved that it cannot be *proved* using those axioms either!

Cantor's early years in Halle must have been filled with joy. In 1874, he married Vally Guttmann, a friend of his sister. They honeymooned in Switzerland and returned to a house Cantor had built with the inheritance from his father. It would be the birthplace of their five children over the next twelve years.

Cantor's growing family must have made financial demands on a professor paid at relatively meager provincial standards. Hoping to alleviate these problems, Cantor sought a professorship at his alma mater in Berlin. Cantor's work on transfinite numbers had drawn wide praise in the world of mathematics, praise that he hoped would win him an appointment at a prestigious university such as Berlin. Weierstrass, his old mentor, had been particularly full of praise for Cantor's work. But there were pockets of those opposed to any talk of actual infinities. Among them was Leopold Kronecker, a high-ranking professor in Berlin.

Mathematics, according to Kronecker, dealt with constructions, precisely what Cantor and Dedekind had avoided in their treatment of the infinite. In spite of Weierstrass's efforts, Kronecker was able to block all attempts to get Cantor a mathematics professorship at Berlin.

Just before turning forty, the combination of personal and professional stresses became too much for Cantor to bear. He had his first bout of deep mental depression, spending a few weeks in a sanitarium. After his release, Cantor wrote to a fellow mathematician:

> *I don't know when I shall return to the continuation of my scientific work. At the moment I can do absolutely nothing with it, and limit myself to the most necessary duty of my lectures; how much happier I would be to be scientifically active, if only I had the necessary mental freshness.*

Indeed, Cantor's best years as a mathematician had ended. Perhaps recognizing this, Cantor devoted significant energy to building an association of mathematicians across the newly unified Germany. He served as its first president from its founding in 1890 until 1893 when he had his next round of depression.

Cantor would be in and out of mental hospitals for the last twenty-five years of his life, fighting depression. In 1894, he published an exceptionally strange paper for a mathematician of his renown. While in the mental hospital, he had become fascinated by Goldbach's famous conjecture that every even number could be expressed as the sum of two primes. In 1894, he published a paper demonstrating all of the ways the even numbers up to 1,000 could be written as the sum of two primes, forty years after an obscure mathematician had done the same for all of the even numbers up to 10,000.

As the years went on, Cantor's mental state got worse and worse. In these later years, he devoted himself to the study of Shakespeare. He even attempted to prove that the Bard and the philosopher Francis Bacon were one and the same person!

The German mathematical community had planned to have a major celebration in 1915 in honor of Cantor's seventieth birthday. However, the privations of World War I made that impossible. Cantor entered a mental hospital for the last time in June 1917. On January 6, 1918, he died, unaware that Imperial Germany would also perish by the end of the same year.

Selections from Contributions to the Founding of the Theory of Transfinite Numbers

(First Article)

"Hypotheses non fingo."

"Neque enim leges intellectui aut rebus damus ad arbitrium nostrum, sed tanquam scribæ fideles ab ipsius naturæ voce latas et prolatas excipimus et describimus."

"Veniet tempus, quo ista quæ nunc latent, in lucem dies extrahat et longioris ævi diligentia."

§ 1

THE CONCEPTION OF POWER OR CARDINAL NUMBER

By an "aggregate" (*Menge*) we are to understand any collection into a whole (*Zusammenfassung zu einem Ganzen*) M of definite and separate objects *m* of our intuition or our thought. These objects are called the "elements" of M.

In signs we express this thus:

$$M = \{m\}. \tag{1}$$

We denote the uniting of many aggregates M, N, P, ..., which have no common elements, into a single aggregate by

$$(M, N, P, \ldots). \tag{2}$$

The elements of this aggregate are, therefore, the elements of M, of N, of P, ..., taken together.

We will call by the name "part" or "partial aggregate" of an aggregate M any other aggregate M_1 whose elements are also elements of M.

If M_2 is a part of M_1 and M_1 is a part of M, then M_2 is a part of M.

Every aggregate M has a definite "power," which we will also call its "cardinal number."

We will call by the name "power" or "cardinal number" of M the general concept which, by means of our active faculty of thought, arises from the aggregate M when we make abstraction of the nature of its various elements *m* and of the order in which they are given.