Chapter Four

The Beginnings of Written Mathematics: Mesopotamia

Fleshing Out the History

Studying ancient Mesopotamian history is rather like going on a long and unfamiliar journey: we are not sure whether we are on the right road until we reach our destination. The abridged chronology given in table 4.1 will be of some help in plotting our course across this difficult terrain; the places mentioned in the table are shown in the accompanying map (figure 4.1). The earliest protocuneiform written records are from around the last few centuries of the fourth millennium BC, and the last cuneiform records are from around the end of the first millennium BC. With the Persian conquest in 539 BC, Mesopotamia ceased to exist as an independent entity. The subsequent history of this region cannot be separated from the histories of other countries such as Persia, Greece, Arabia, and, more recently, Turkey.

Along the fertile crescent between the Tigris and Euphrates rivers emerged the first cities occupied by the people who had originally migrated from the present-day Armenian region of the Black and Caspian seas. By 3500 BC, the population pressure was such that the naturally irrigated floodplains could no longer sustain the basic needs of the inhabitants, especially since, unlike the case of Egypt, the flooding occurred somewhat erratically. The rivers were not navigable, making the city-states culturally and economically isolated from one another. The whole region was wide open and flat, lacking in natural defenses, making it vulnerable to external invasions. Thus the physical environment of Mesopotamia influenced both the economy and the habitat of its inhabitants.

It has been suggested by certain historians, notably Wittfogel (1957) that, just as in the case of Egypt, a society that had mastered the principles of hydraulics (irrigation) was well equipped to initiate the beginnings of

Table 4.1: Chronology of Ancient Mesopotamia from 4000 BC to 64 BC

Dates	*Historical/socioeconomic background*	*Mathematical developments*
4000–3500	Early urbanization in the south.	Early accounting practices based on tokens. Development of separate systems of notations for (1) counting numbers on base 60 (sexagesimal), (2) area numbers, (3) weight numbers, (4) grain capacity numbers. The earliest school texts from Uruk.
3500–2500	Early Bronze Age. Emergence of city-states of Sumeria with centers of power at Ur, Nippur, Eridu, and Lagash.	Discovery of earliest school texts from Fara (Shuruppak). Development of sexagesimal numerals and phonetic writing, more advanced accounting practices.
2500–2000	Establishment of the empires of Sumer and Akkad (centers of power: Ur, Agade). Notable rulers: Sargon I (c. 2350) and Shulgi (2100).	Old Akkadian school texts. About 2000: tables of reciprocals and use of sexagesimal place-value notation.
2000–1500	Conflicts and wars; rule by city-state; establishment of the Old Babylonian empire (center of power: Babylon). Notable ruler: Hammurabi (1792–1752).	Widespread evidence of early concrete algebra and geometry, quantity surveying, often found as adjuncts to scribal training. Sophisticated Babylonian mathematical texts.
1500–1000	Late Bronze Age. International contacts.	Spread of sexagesimal numeracy. Development of astronomy.
1000–600	Iron Age. Assyrian empire. Development of Aramaic language (center of power: Nineveh). Notable rulers: Sennacherib (705–681) and Ashurbanipal (668–627).	Computational and astronomical developments continue.
612–539	Second or New Babylonian empire (Chaldeans) (center of power: Babylon). Notable ruler: Nebuchadnezzer (605–562).	Astronomical observations.
539–311	Persian invasion (539): end of ancient Mesopotamia (centers of power: Babylon and Susa). Notable rulers: Cyrus the Great (c. 525) and Darius (521–485).	Revival of education in mathematics. Great advances in mathematical astronomy.
312–64	Seleucid dynasty, Late Babylonian period (center of power: Antioch).	Work on astronomy and algebra continues: construction of extensive mathematical and astronomical tables.

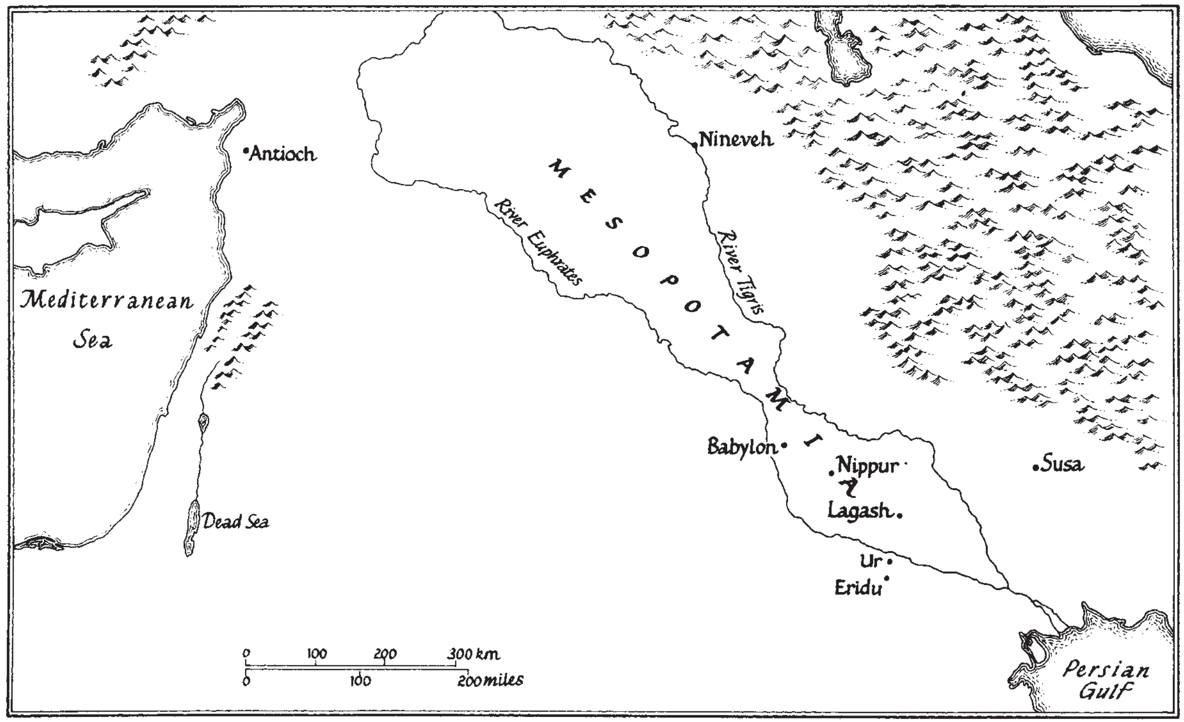


FIGURE 4.1: Map of Mesopotamia

two major and interrelated sciences: mathematics and astronomy. And, the argument continues, the pursuit of intensive agriculture and large-scale breeding of livestock, concentrated in the hands of a central power, necessitated a meticulous control of movements of the goods produced and exchanged. In an attempt to accomplish this task efficiently, writing first developed and was to remain for several centuries its only use. While this argument would seem to be somewhat simplistic today, there is the related point concerning the similarities between ancient Egypt and Mesopotamia in the emergence of priestly bureaucratic structures with both writing and mathematics developed to serve their ends. From the evidence we have so far, "mathematics" preceded writing, in that the earliest records that we have for both Egypt and Mesopotamia relate to inventories (or counting) of objects.[1] It would seem that the bureaucrats in both cultures needed accountants before writers or scribes.

The first city-states, such as Uruk Lagash, Ur, and Eridu, developed in Sumer, the most fertile region of Mesopotamia. They competed vigorously with one another for land, resorting to war at the slightest provocation. As a result, for the first time, there emerged empires, unions of city-states achieved through coercion or persuasion or both, often with a single city dominating the others. The history of Mesopotamia after 3000 BC is the history of one empire or dynasty succeeding another, with each developing its own "style" of dominance and survival.

The Akkadian empire (c. 2375–2225 BC) was the creation of Sargon (2371–2316 BC), who, initially taking advantage of internal dissension in Sumer, conquered most of the Mesopotamian river valley during his reign of fifty-six years. To retain his hold, Sargon arranged the marriage of his daughter, Enheduanna, the high priestess of the Akkadian religion, to the former king of Sumer (the high priest of the Sumerian religion). It was hoped that if two peoples worshipped the same gods, they were unlikely to go to war with one another. Enheduanna set herself the task of writing a text containing the liturgy and rituals from both religions. This text became the earliest-known writing by a woman anywhere in the world.

About 2000 BC, the Akkadian empire was overrun by the Amorites (or Babylonians), who swept down into Mesopotamia from the northern land of Nimrod (present-day upper Iraq). Hammurabi built the city of Babylon as his capital. He arranged for a written legal code, which was among the first in history. His approach to safeguarding his power was two-pronged: increase the prestige of the king and promote the use of organized law.

The royal prestige was safeguarded by centralizing the seat of power: locating the palace and temple within the same group of buildings and so enabling the ruler to perform similar rituals to underpin temporal and secular power. An audience with the king was no different from an audience with a god in a temple. To create an organized system of law, Hammurabi employed legal experts to collect, codify, and disseminate laws across his kingdom. His new Code of Law was carved onto pillars, situated in every city of his empire. All citizens could call upon the protection of Hammurabi's justice; the unity and stability of the empire would thereby be assured by popular support. Hammurabi's dynasty was a high point in Mesopotamian history and a period of flowering of mathematical achievements. Indeed, as indicated later in the chapter, much of the evidence on accomplishment within both disciplines, mathematics as well as law, belongs to this period. The Old Babylonian empire lasted about seven hundred years, finally breaking down as a result of internal disorder and weakness of later rulers.

The Assyrian rulers (c. 900–600 BC) that followed held their empire together through terror, proudly displaying the severed heads and flayed skin of conquered enemies. Like many tyrannies, they were adept in introducing new technology of war, such as iron weapons and horse-drawn chariots. Their power was therefore based on fear, terror, and superior military skill. But once the conquered peoples got over their fears and gained the new technology, they rebelled. The Assyrian capital, Nineveh, was finally destroyed in 612 BC. The Assyrian epoch is marked by relative stagnation in practically all scientific activities with the possible exception of astronomy.

The Chaldean empire (c. 600–550 BC) tried the "restoration" approach. Nebuchadnezzar (630–562 BC) conquered a little, but built a little and spent a lot. He conquered Israel and brought a substantial section of the population back to Babylon as slaves. He developed Babylon, which had fallen into bad times under the Assyrians, including restoring the famous Hanging Gardens. The New Babylonian empire came into existence, and there was a revival of what we would now call algebra. Nebuchadnezzar's approach was, on the whole, successful until the arrival of the Persians.

In his campaigns between 550 and 530 BC, Cyrus united the Persians and the Medes, then moved his show on the road to Mesopotamia. The Persian "style" was tolerance and benevolence to all. They respected local customs and traditions, thereby gaining many supporters. The Jews were allowed to go home; Cyrus even gave them money to rebuild their smashed temple. Many cities just opened their gates to Cyrus and asked to be part of his empire. The Persians gave the region uniform coinage, shared technology, trade, and roads, and asked only for allegiance and taxes in return. There was considerable work on astronomy during this period. Cyrus and his successors ruled in peace for over two hundred years, until they were conquered by the Macedonian Alexander and, in 311 BC, the Seleucid dynasty was established. This was a period when the temples dedicated to the god Marduk in Babylon and the sky god Anu served as the protectors of the Babylonian religion and culture. Babylonian mathematical activity continued, especially the mathematical astronomy relating to the timings of eclipses based on observational data collected in these temples over centuries, and with realization of the full potential for calculation using the sexagesimal (base 60) system. But this is another story that we will not pursue further, except to add that work on astronomy and geometrical algebra continued apace to merge into the swelling stream of Hellenistic mathematics and astronomy.

In discussing the mathematics of Mesopotamia, it is worth raising the same questions we did in the preface to this new (third) edition:

1. *What* was the content of the mathematics known to that culture?
2. *How* was that mathematics thought about and discussed?
3. *Who* was doing the mathematics?

The Material Basis of Mesopotamian Mathematical Culture

The empires that grew out of the city-states of Sumer required a large bureaucracy to carry out their wishes. From the middle of the third millennium this bureaucracy began recruiting scribes. A scribe was a member of a specialized profession, trained in special schools where, increasingly, the curriculum was dominated by applied mathematics.[2] Even when empires collapsed, the two lasting legacies that the scribes had helped to create over the centuries remained: a method of systematic accounting and the introduction of a place-value number system. Both innovations were a result of the ability of scribal schools to respond effectively to the increasingly sophisticated demands made by the administrative apparatus engaged in collecting taxes, conducting land surveys, supervising large-scale building programs, and ensuring the supply of young men for wars.

The centralized Sumerian states collapsed, in part at least, because of the weight of their top-heavy bureaucracy. This led to the emergence of

Hammurabi and the Old Babylonian period. A new economic, social, and ideological order asserted itself. Instead of large-scale agriculture or craft workshops, often owned by the ruler, the emphasis changed to small-scale enterprises owned by private individuals. This decentralization spread into many spheres, including the occupations of scribes. Scribes were no longer small cogs in the large wheel of government. They could be found writing letters for private individuals or tutoring children in private homes. The school for scribes probably continued with the old curriculum for a long time, teaching the student scribes accounting, surveying, and other administratively useful pursuits. The high-status jobs were still to be found in the state bureaucracy. However, one begins to discern from the clay tablets of the period that a new mathematics was developing: a mathematics that was no longer purely utilitarian. This was a period of great interest in what we would describe today as "second- and higher-degree equations" discussed later in this chapter. The Plimpton Tablet (also discussed later in this chapter) belongs to this period. Even the scribal schools were caught up in this wave of new thinking. There are signs that mathematics was developing as a separate discipline, loosened from the coattails of narrow utilitarian preoccupations.

This period of Mesopotamian history, however, came to an end. For about one thousand years, "pure" mathematics took a backseat, to be re-established during the Chaldean period, when there was once again a resurgence of mathematics. But that is another story. However, what we seek to establish in a limited fashion is that mathematical development, howsoever defined, was shaped by institutions such as scribal schools, which in turn were products of the material and social forces driving the society.

To illustrate, consider the evidence in the form of clay tablets of the activities of a scribal school in Nippur (c. 1740 BC) run by a priest in the front courtyard of his house.[3] He had no more than one or two students (possibly his own sons), who began their education by learning how to make wedge-shaped marks in clay with a reed stylus, learning by copying and repetition a set of simple cuneiform signs. The education of the student scribes progressed to writing Sumerian words for different objects, followed by more complex exercises that involved writing and learning multiplication tables and lists of metrological terms. Only after this was the student introduced to writing sentences in Sumerian and learning Sumerian literature. The method of instruction was rote learning, so that where an opportunity to do mathematical calculation was offered, this may have come as a relief even if mistakes were not uncommon. Soon after 1739 BC, the fifteen hundred school tablets that had accumulated were used as bricks and building material to repair the priest's house. However, this mode of instruction continued for a long time, as shown by the discovery of clay tablets in large terra-cotta jars at the home of a family of healers and diviners in Uruk (420 BC), where younger males were taught by their elders to write and calculate in Sumerian and Akkadian. What these tablets and other evidence indicate is that mathematics was rarely pursued in ancient Mesopotamia as a leisure activity, nor was it generally supported by institutional patronage. It was part of the process of providing training in literacy and numeracy, necessary requirements for a future priest or healer or accountant or teacher.

The Persian and Hellenistic periods saw the dethroning of the bureaucratic scribal class, with administration now being carried out by another class in the language of Aramaic or Greek. The place of the scribes was taken by a class of mathematically trained priests known as the *kalu*, located mainly in the temples dedicated to the gods Marduk and Anu, whose ceremonial function was to weep and wail and beat drums during the solar and lunar eclipses. This was to placate the gods and drive away the evil that followed the eclipses. It was the search for accurate methods of predicting these ominous events that led to significant work in mathematical astronomy, which combined observations and calculations. Archaeological evidence (Rocherg 1993 and Robson 2005) indicates that the role of the scribe was taken over by the priest in promoting and preserving mathematical knowledge in general.

Sources of Mesopotamian Mathematics

Of the half a million inscribed clay tablets that have been excavated, fewer than five hundred are of direct mathematical interest. Apart from those in the hands of private collectors, collections of these mathematical tablets are scattered among the museums of Europe in Berlin, London, Paris, and Strasbourg and the universities of Yale, Columbia, Chicago, and Pennsylvania in the United States. Some of the more recent finds, notably from Tell Harmal, Tell Hadad, and Tell Dhibayi in Iraq, were kept in the Iraqi Museum in Baghdad, although the ravages of the recent war have resulted in a number being destroyed or stolen.[4] The tablets vary in size, from as small as a postage stamp to as large as a pillow. Some are inscribed only on one side, others on both sides, and a few even on their edges.

To make a tablet, clay that may have come from the banks of the Tigris or Euphrates was collected and kneaded into shape. It was then ready for recording. The scribe used a piece of reed about the size of a pencil, shaped at one end so that it made wedgelike impressions in the soft, damp clay. He had to work fast, for the clay dried out and hardened quickly, making corrections or additions difficult. Having completed one side, he might turn the tablet over and continue. When he had finished, the tablet was dried in the sun or baked in a kiln, leaving a permanent record for posterity.

The wedge-shaped cuneiform script of the Sumerians was deciphered as early as the middle of the nineteenth century through the pioneering efforts of George Frederick Grotefend (1775–1853) and Henry Creswicke Rawlinson (1810–1895), but only since the 1930s have the mathematical texts been studied seriously.[5] This delay may be partly explained by the different ways in which a mathematician and a philologist approach early literature. The average mathematician, unless presented with a text that falls within the limits of what “mathematics” is perceived to be, has little time for the past; rarely is historical curiosity aroused by mathematical teaching. The philologist seeks to revive the past in order to explore the growth and decline of ancient civilizations; but, probably because of a lack of mathematical training, the philologist rarely takes an interest in ancient mathematics. So the Mesopotamian mathematical texts lay undeciphered and ignored until the pioneering work by Otto Neugebauer, who published his *Mathematische Keilschrift-Texte* in three volumes from 1935 to 1937, and by Francois Thureau-Dangin, whose complete works, titled *Textes mathématiques Babyloniens*, were brought out in 1938. Since then new evidence and interpretations have continued to appear, even in recent years.

There are three main sources for Mesopotamian mathematics. Some of the oldest records, written in Sumerian cuneiform, date back to the last quarter of the fourth millennium BC. From that period, in the temple precincts of the city of Uruk, a single tablet has been discovered of the oldest recorded mathematics. This consists of two exercises on calculating the area of fields. However, much of the information on this period is of a commercial and legal nature, as would now be found on invoices, receipts, and mortgage statements, and details about weights and measures. There are some but not many mathematical records until we come to the Old Babylonian period, during the first half of the second millennium BC. It has been estimated that between two-thirds and three-quarters of all the Mesopotamian mathematical texts that have been found belong to this period; in our subsequent discussion of Mesopotamian mathematics we shall concentrate on the evidence from this period. A very large portion of the remaining texts belong to a period beginning with the establishment of the New Babylonian empire of the Chaldeans, around 600 BC, after the destruction of Nineveh, and continuing well into the Seleucid era. This was also a period of considerable accomplishments in astronomy.

The Origins of Mesopotamian Numeration

From about 8000 BC, a system of recording involving small clay tokens was prevalent in the Near and Middle East. Tokens were small geometric objects, usually in the shape of cylinders, cones, and spheres. They were first identified in societies evolved from a life based on hunting and gathering to one based on agriculture, like the Ishango in central Africa. The earliest tokens were simple in design: they stood for basic agricultural commodities such as grain and cattle. A specific shape of token always represented a specific quantity of a particular item. For example, "the cone . . . stood for a small measure of grain, the sphere represented a large measure of grain, the ovoid (a rough egg-shaped solid with one end being more pointed) stood for a jar of oil" (Schmandt-Besserat 1992, p. 161). Two jars of oil would be represented by two ovoids, three jars by three ovoids, and so on. Thus, the tokens became not only an abstraction for the things being counted but also constituted a system of great specificity and precision.

With the development of city-states and the emergence of empires came a more complex economic and social structure, reflected in both the diversity and the standardization of tokens. This increased the scope for record keeping and commercial contracts in a way that counting using pebbles or twigs could not do. A collection of tokens could represent a future promised transaction or, stored in a temple or palace, a record of a past transaction. Both contracts and archives required secure methods of preserving groups of tokens. The Sumerians devised two main systems of storage: stringing the tokens on a piece of cord and attaching the ends of the strings to a solid lump of clay marked with a security seal called a *bulla*; or storing the tokens inside a clay envelope bearing impressions of the enclosed tokens for identification purposes. "For reasons we do not know, plain tokens were most often secured by envelopes and complex tokens by *bullae*" (Schmandt-Besserat 1992, p. 110).

Of the two systems, the practice of storing tokens in clay envelopes was more significant for the development of mathematics. The last step in the evolution of tokens was a merging of the two systems of *bullae* and envelopes. Simple tokens were pressed to make marks on a solid lump or tablet of clay. Only the clay tablet was then kept. Within a couple of hundred years, this new system was also being used for the complex tokens, but here, because of their complicated shapes and designs, the image of the token did not transfer satisfactorily onto the clay. This new system, in place by about 3000 BC, afforded greater ease of use and storage, at the price of a certain loss of security. These pressed or drawn marks on the clay tablets were the beginnings of the Babylonian numeration system.

From about 3000 BC, among the Sumerians, tokens for different goods began appearing as impressions on clay tablets, represented by different symbols and multiple quantities represented by repetition. Thus three units of grain were denoted by three "grain marks," five jars of oil by five "oil marks," and so on. The limitations of such a system became evident with the increasing complexity of Sumerian economic life: a confusing proliferation of different-style tokens to be learned and the tedium of representing large magnitudes. Recording the sale of five jars of oil or of a limited range of commodities was a simple affair, but an increase in the quantity and range of commodities was a different matter. Temple complexes, such as the temple of the goddess Inanna at Eana in Uruk (3200 BC), were large-scale enterprises, dealing in considerable quantities of goods and labor. A new system of recording and accounting needed to be devised. The accountants at the temple adapted a long-used system of accounting with clay tokens by impressing stylized outlines of tokens to denote numbers, with pictograms and other symbols to denote the objects that were being counted. A number of different numeration and metrological systems were used depending on the objects counted.

The first great innovation, as we saw earlier in chapter 2, was the separation of the quantity of the goods from the symbol for the goods. That is, to represent three units of grain by a symbol for "three" followed by a symbol for "grain unit" in the same way that we would write three goats or three cows or, even more generally, three liters or three kilometers.

Whereas we use the same number signs regardless of their metrological meaning (the "three" for sheep is the same sign as the "three" for kilometers or jars of oil), the Sumerians resorted to a wide variety of different symbols. Nissen et al. (1993) have identified around 60 different number signs, which they group into a dozen or so systems of measurement. For example, the Sumerians used one system for counting discrete objects, such as people, animals, or jars, and other systems for measuring areas. Each system had a collection of signs denoting various quantities.[6]

In each Sumerian metrological system there were a number of different size-units with fixed conversion factors between them, similar to our system, for example, where there are 12 inches in a foot and 3 feet in a yard, and so on. And just as in our old weight and measure systems, Sumerian metrology featured all sorts of conversion factors, although it is notable that they were all simple fractions of 60.[7]

In the early stages, however, there were different systems of numerical representation in Mesopotamia, depending on what was being measured. For a short period, a “bisexagesimal” system (i.e., a system with the units in the ratios 1:10:60:120:1,200:7,200) was used to count products relating to grain and certain other commodities. It operated with conversion factors 10, 6, and 2, so that the symbol for the largest quantity, this time a large circle containing two small circles, represented 7,200 base units. Yet another system was used for measuring grain capacity: the conversion factors were 5, 10, and 3, so that the largest unit, a large cone containing a small circle, was worth 900 base units. To add to the confusion, a single sign could be used in several systems to denote different multiples of the base unit. In particular, the small circle could mean 6, 10, or 18 small cones, depending on context and the system in use.

Gradually, over the course of the third millennium, the round number-signs were replaced by cuneiform equivalents so that numbers could be written with the same sharp stylus that was being used for the words in the text. A detailed account of this innovative system follows in the next section.

The Mesopotamian Number System

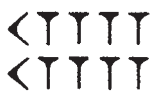
Early clay tablets (c. 3000 BC) show that the Sumerians did not have a systematic positional system for all powers of 60 and their multiples. They used the following symbols:

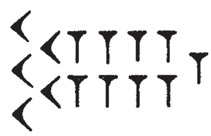
1 10 60 600 3,600

The symbols for the first three numbers were written with the lower end of a cylindrical stylus, held obliquely for 1 and 60 and vertically for 10. The symbol for 600 was a combination of those for 10 and 60; the large circle for 3,600 was written with an extra large round stylus.

One of the most outstanding achievements of Mesopotamian mathematics, and one that helped to shape subsequent developments, was the invention of a place-value number system. From around 2000 BC there evolved a sexagesimal place-value system using only two symbols:  for 1 and  for 10. In this system, the representation of numbers smaller than 60 was as straightforward as it was in the Egyptian notation. Thus

4: 

28: 

59: 

If the Mesopotamians had merely used these symbols on an additive basis (which they did not), their numeration and computations would probably have developed along Egyptian lines. But, from as early as 2500 BC, we find indications that they realized they could double, triple, quadruple (and so on) the two symbols for 1 and 10 by giving them values that depended on their relative positions. Thus the two symbols could be used to form numbers greater than 59:

$60 = 60(1)$: 

$95 = 60(1) + 35$: 

$120 = 60(2)$: 

$4{,}002 = 60^2(1) + 60(6) + 42$: 

It was a relatively simple matter, though one of momentous significance, to extend this principle of positional notation to allow fractions to be represented:

$1/2 = 60^{-1}(30) = 30/60$: 

$1/4 = 60^{-1}(15) = 15/60$: 

$1/8 = 60^{-1}(7) + 60^{-2}(30)$: 

$532\frac{3}{4} = 60(8) + 52 + 60^{-1}(45)$: 

Two important features of Mesopotamian positional notation are highlighted by these examples: unlike our present-day system, there is no symbol for zero, and neither is there a symbol corresponding to our decimal point to distinguish between the integer and fractional parts of a number. There is also the more fundamental question of why the Mesopotamians should have constructed a number system on base 60 rather than the more "natural" base 10 (i.e., the decimal system). However, they used base 10 notation up to 59.

The absence of a symbol for a placeholder could lead to confusion over what number was being recorded. For example,

could be $60(2) + 40 = 160$, or $60^2(2) + 60(0) + 40 = 7{,}240$, or it could be $2 + 60^{-1}(40) = 2\frac{2}{3}$, or even $60^{-1}(2) + 60^{-2}(40) = 2/45$, since there is no "sexagesimal point" placeholder to indicate that the number is a fraction. In the absence of a special symbol for zero, the number might be identifiable from the context in which it appeared, or a space might be left to indicate a missing sexagesimal place. There again, it could have been that the lack of a zero symbol in ancient Mesopotamia was of little practical consequence, for the existence of a large base, 60, would ensure that most numbers of everyday concern could be represented unambiguously. For example, it is unlikely that the prices of commodities in ordinary use would have exceeded 59 "units" (discounting inflation of course!). Moreover, the relative positions of the two  symbols and the four  symbols in the example above would indicate that, if the number were an integer, it would not be less than 160. This was because the Mesopotamians, unlike the Egyptians, wrote their numerals the same way as we do, from left to right.

It was not until the Seleucid period that a separate placeholder symbol was introduced to indicate an empty space between two digits inside a number. Thus the number 7,240 would be written as

where  serves as the placeholder symbol. The problem still remained of how to represent the absence of any units at the end of a number. Nowadays we use the symbol for zero in the terminal position. Without something like that, it is difficult to know whether the number

is $60(3) + 30 = 210$, or $60^2(3) + 60(30) = 12{,}600$, or even $3 + 60^{-1}(30) = 3\frac{1}{2}$. It is therefore clear that while the Mesopotamians were consistent in their use of place-value notation, they never operated with an absolute positional system. When, in the second century AD, Claudius Ptolemy of Alexandria began to use the Greek letter *o* (omicron) to represent zero, even in the terminal position of a number, there was still no awareness that zero was as much a number as any other and so, just like any other, could enter into any computation. Recognition of this fact—"giving to airy nothing, not merely a local inhabitation and a name, a picture, a symbol but also a helpful power" (Halstead 1912)—was not to occur for another thousand years, in India and Central America.

If we are to make any further headway, we need a way of transliterating the Mesopotamian numerical representation into a notation more convenient for us. We shall adopt Neugebauer's convention of using a semicolon (;) to separate the integral part of a number from its fractional part, just as we use the decimal point today—the semicolon is in effect the "sexagesimal point." All other sexagesimal places are separated by a comma (,). Some examples, of numbers whose cuneiform representations have been given above, will make this convention clear:

60	=	$60(1)$:	1,00
95	=	$60(1) + 35$:	1,35
120	=	$60(2)$:	2,00
4,002	=	$60^2(1) + 60(6) + 42$:	1,06,42
1/2	=	$60^{-1}(30) = 30/60$:	0;30
1/8	=	$60^{-1}(7) + 60^{-2}(30)$:	0;07,30
$532\frac{3}{4}$	=	$60(8) + 52 + 60^{-1}(45)$:	8,52;45

With this scheme, the ambiguity in the representation of 7,240 in the Mesopotamian notation disappears: this number is now written as 2,00,40.

Different explanations have been offered for the origins of the sexagesimal system, which, unlike base 10 or even base 20, has no obviously anatomical basis. Theon of Alexandria, in the fourth century AD, pointed to the computational convenience of using the base 60. Since 60 is exactly divisible by 2, 3, 4, 5, 6, 10, 15, 20, and 30, it becomes possible to represent a number of common fractions by integers, thus simplifying calculations: the integers that correspond to the unit fractions 1/2, 1/3, 1/4, 1/5, 1/6, 1/10, 1/15, 1/20, and 1/30 are 30, 20, 15, 12, 10, 6, 4, 3, and 2, respectively. Of the unit fractions with denominators from 2 to 9, only 1/7 is not "regular" (i.e., 60/7 gives a nonterminating number). It is therefore quite a simple matter to work with fractions in base 60. In a decimal base, though, only three of the nine fractions above produce integers, and none of 1/3, 1/6, 1/7, and 1/9 is regular. Indeed, while base 10 may be more "natural," since we have ten fingers, it is computationally more inefficient than base 60, or even base 12.

However, this explanation for the use of base 60 is unconvincing because of its "hindsight" character. It is highly unlikely that such considerations were taken into account when the base was chosen. A second explanation emphasizes the relationship that exists between base 60 and numbers that occur in important astronomical quantities. The length of a lunar month is 30 days. The Mesopotamian estimate of the number of days in a year was 360, based on the zodiacal circle of 360°, divided into twelve signs of the zodiac of 30° each. The argument goes that either 30 or 360 was first chosen as the base, later to be modified to 60 when the advantages of such a change were recognized. Here again, there is a suggestion of deliberate, rational calculation in the choice of the base that is not totally convincing. A more plausible explanation is that the sexagesimal system evolved from metrological systems that used two alternating bases of 10 and 6, favored perhaps by two different groups, which gradually merged, and that the advantages of base 60 for astronomical and computational work then came to be recognized.[8]

The sexagesimal system was used in Mesopotamia in 1800 BC and continued to be used well into the fifteenth century AD. Sexagesimal fractions appeared in Ptolemy's *Almagest* in AD 150. The Alfonsine Tables, astronomical tables prepared from Islamic sources on the instruction of Alfonso X of Castile and written in Latin at the end of the thirteenth century AD, used a consistent sexagesimal place-value system. The Islamic astronomer al-Kashi (d. AD 1429) determined 2π sexagesimally as

6;16,59,28,01,34,51,46,15,50—the decimal equivalent of which is accurate to sixteen places. And Copernicus's influential work in mathematical astronomy during the sixteenth century contained sexagesimal fractions. The current use of the sexagesimal scale in measuring time and angles in minutes and seconds is part of the Mesopotamian legacy.

Before going on to look at operations with Mesopotamian numerals, let us pause to compare the Mesopotamian way of representing numbers with other systems. In assessing a notational system, the following questions are pertinent:

1. Is the system easy to learn and write?
2. Is the system unambiguous?
3. Does the system lend itself readily to computation?

The Mesopotamian system scores well on questions 1 and 3. It is easily learned, being one of the most economical systems in terms of the symbols used. The only other number system that operated with just two symbols (a dot and a dash) was the Mayan, though unlike the Mesopotamian system there was also a special sign for zero. If we compare the Mesopotamian with the Greek number system, which used twenty-seven symbols, the simplicity of the former notation is obvious. But one must contrast this simplicity with the awkwardness of representing a number such as 59, which in the unabridged Mesopotamian notation would require fourteen signs (though some would argue that they *together* represent a single cuneiform sign), as against just two in the Greek notation.[9] The Mesopotamian system is also remarkable for its computational ease, which arises from its place-value principle and its base of 60. Calculating in this base proved a distinct advantage when dealing with fractions. Until the emergence of decimal fraction representation, the Mesopotamian treatment of fractions remained the most powerful computational method available.

But the great disadvantage of Mesopotamian notation was its ambiguity, the consequence of having neither a symbol for zero nor a suitable device for separating the integral part of a number from its fractional part. It was not that the system of notation precluded the incorporation of these additional features, but that the Mesopotamians simply did not use them. (In the time of the New Babylonian empire, though, the placeholder symbol  appeared.) All in all, compared with the Egyptian system, the Babylonian notation was computationally more "productive" and symbolically more economical (since the place-value principle made it unnecessary to invent new symbols for large numbers), but it had the disadvantage of being ambiguous. The Egyptian system had another advantage over the Babylonian: the order in which the symbols representing a number are written is of no consequence in Egyptian notation.

Operations with Mesopotamian Numerals

With a positional system of numeration available, ordinary arithmetical operations with Mesopotamian numerals would follow along the same lines as modern arithmetic. To relieve the tedium of long calculations, the Mesopotamians made extensive use of mathematical tables. These included tables for finding reciprocals, squares, cubes, and square and cube roots, as well as exponential tables and even tables of values of $n^3 + n^2$, for which there is no modern equivalent. These tables account for a substantial portion of the sources of Mesopotamian mathematics available to us.

Multiplication and division were carried out largely as we would today. Division was treated as multiplication of the dividend by the reciprocal of the divisor (obtained from a table of reciprocals). To take a simple example:

EXAMPLE 4.1 Divide 1,029 by 64.

Solution

In Neugebauer's notation, $1{,}029 = 60^1(17) + 60^0(9)$ is written as 17,09. Also, 1/64 becomes 0;00,56,15, since $1/64 = 60^{-2}(56) + 60^{-3}(15)$, found from a table of reciprocals.

Therefore

17,09 multiplied by 0;00,56,15 equals 16;04,41,15.

The long multiplication may have been carried out in the same way as we would today, apart from the sexagesimal base:

```
      0;00,56,15
×          17,09
----------------
         8,26,15
    15;56,15
----------------
    16;04,41,15
```

Continued . . .

Continued . . .

The answer 16;04,41,15 can be converted to the decimal base:

$$16 + 60^{-1}(4) + 60^{2}(41) + 60^{-3}(15) \approx 16.0781.$$

A complete set of sexagesimal multiplication tables was available not only for each number from 2 to 20, and for 30, 40, and 50, but also for many other numbers. This would be sufficient to carry out all possible sexagesimal multiplications, just as present-day multiplication tables for numbers from 2 to 10 are sufficient for all decimal products. Often, the tables of reciprocals were available only for those "regular" integers up to 81 that are multiples of 2, 3, or 5. The reciprocals of "irregular" numbers, or those containing prime numbers that are not factors of 60 (i.e., all prime numbers except 2, 3, and 5), would, in effect, have been nonterminating sexagesimal fractions. For example, the reciprocals of the "regular" numbers 15, 40, and 81 are

$$\frac{1}{15} = 0;04, \quad \frac{1}{40} = 0;01,30, \quad \frac{1}{81} = 0;00,44,26,40.$$

The reciprocals of the "irregular" numbers 7 and 11 are

$$\frac{1}{7} = 0;08,34,17,08,34,17,\ldots, \quad \frac{1}{11} = 0;05,27,16,21,49,\ldots.$$

The tables of reciprocals found on the older tablets are all for "regular" numbers. There is one tablet, from the period just before the Old Babylonian empire, which contains the following problem:

Example 4.2 Divide 5,20,00,00 by 7.

Suggested Solution

Multiply 5,20,00,00 by the approximate reciprocal of 7 (i.e., 0;08,34,17,08) to get the answer: 45,42,51;22,40.

A later tablet from the Seleucid period gives the upper and lower limits on the magnitude of 1/7 as

$$0;08,34,16,59 < \frac{1}{7} < 0;08,34,18$$

Statements such as "approximation given since 7 does not divide" from the earlier periods, and the later estimates of bounds, give us a tantalizing glimpse of the Mesopotamians taking the first step (though it is not clear whether they were fully aware of the implications) in coming to grips with the incommensurability of certain numbers.

A Babylonian Masterpiece

Evidence that the Mesopotamians had no difficulty working with what we now know as irrational numbers is found on a small tablet, belonging to the Old Babylonian period, that forms part of the Yale collection.[10] It contains the diagram shown in figure 4.2a and "translated" in figure 4.2b. The number 30 indicates the length of the side of the square. Of the other two numbers, the upper one (if we assume that the "sexagesimal point" (;) occurs between 1 and 24) is 1;24,51,10, which in decimal notation is

$$1 + 60^{-1}(24) + 60^{-2}(51) + 60^{-3}(10) \approx 1 + 0.4 + 0.01416667 + 0.0000463$$
$$= 1.41421297.$$

To the same number of decimal places, the square root of 2 is 1.41421356, so the Babylonian estimate is correct to five places of decimals. The lower number is easily seen to be the product of 30 (the side of the square) and the estimate of the square root of 2.

The interpretation is now clear, particularly if one notes that on the back of this clay tablet there remains a partly erased solution to a problem involving the diagonal of a rectangle of length 4 and width 3. Let d be the diagonal of the square; applying the Pythagorean theorem then gives

$$d^2 = 30^2 + 30^2;$$
$$d = \sqrt{2}(30) \approx (1;24,51,10)(30;00) = 42;25,35.$$

The number below the diagonal is therefore the length of the diagonal of a square whose side is 30.

The solution to this problem highlights two important features of Mesopotamian mathematics. First, over a thousand years before Pythagoras, the Mesopotamians knew and used the result now known under his name.[11] (In a later section we discuss further applications of this result, as well as evidence that the Mesopotamians may have known the rules for generating Pythagorean triples a, b, c, where $a^2 + b^2 = c^2$.) Second, there is the intriguing question of how the Mesopotamians arrived at their remarkable estimate of the square root of 2, an estimate that would still be in use two thousand years later when Ptolemy constructed his table of chords.