3 Al-Khwārizmī's Algebra

Out of this dual heritage of solutions to problems asking for the discovery of numerical and geometrical unknowns Islamic civilization created and named a science—algebra. The word itself comes from the Arabic word "al-jabr," which appears in the title of many Arabic works as part of the phrase "al-jabr wa al-muqābala." One meaning of "al-jabr" is "setting back in its place" or "restoring," and the ninth century algebraist al-Khwārīzmī, although he is not always consistent, uses the term to denote the operation of restoring a quantity subtracted from one side of the equation to the other side to make it positive. Thus replacing $5x + 1 = 2–3x$ by $8x + 1 = 2$ would be an instance of "al-jabr." The word "wa" just means "and," and it joins "al-jabr" with the word "al-muqābala," which means in this context replacing two terms of the same type, but on different sides of an equation, by their difference on the side of the larger. Thus, replacing $8x + 1 = 2$ by $8x = 1$ would be an instance of "al-muqābala."

Clearly, with the two operations any algebraic equation can be reduced to one in which a sum of positive terms on one side is equal either to a sum of positive terms involving different powers of x on the other, or to zero. In particular, any quadratic equation with a positive root can be reduced to one of three standard forms:

$$px^2 = qx + r,\ px^2 + r = qx, \text{ or } px^2 + qx = r, \text{ with } p,\ q,\ r \text{ all positive,}$$

a condition that runs through the whole medieval period in Islamic mathematics. We shall meet it again in the work of ʿUmar al-Khayyāmī, and it is the rule in Western mathematics as well through the early sixteenth century. Thus the science of *al-jabr wa al-muqābala* was, at its beginning, the science of transforming equations involving one or more unknowns into one of the above standard forms and then solving this form.

3.1 Basic Ideas in Al-Khwārizmī's Algebra

One of the earliest writers on algebra was Muḥammad b. Mūsā al-Khwārizmī, whose treatise on Hindu reckoning we referred to in Chap. 2. His work on algebra, *The Condensed Book on the Calculation of al-Jabr wa al-Muq*ābala, enjoyed wide circulation not only in the Islamic world but in the Latin West as well.

According to al-Khwārizmī there are three kinds of quantities: *simple numbers* like 2, 13, and 101, then *root*, which is the unknown, x, that is to be found in a particular problem, and *wealth*, the square of the root, called in Arabic *māl*. (A possible advantage of thinking of the square term as representing wealth is that al-Khwārizmī can then interpret the number term as *dirhams*, a local unit of currency.) Another word used for "root" by many writers is "thing" (*shay*'). In these terms al-Khwārizmī could list the six basic types of equations as

Roots equal numbers ($nx = m$).
Māl equal roots ($x^2 = nx$).
Māl equal numbers ($x^2 = m$).
Numbers and *māl* equal roots ($m + x^2 = nx$).
Numbers equal roots and *māl* ($m = nx + x^2$).
Māl equals numbers and roots ($x^2 = \mathrm{m} + nx$).

All equations involving all the three kinds of quantities and having a positive solution could be reduced to one of types (4)–(6), the only ones with which al-Khwārizmī concerns himself.

3.2 Al-Khwārizmī's Discussion of $x^2 + 21 = 10x$

In following al-Khwārizmī's discussion of type (4) above we shall use modern notation to render his verbal account. He discusses this type in terms of the specific example $x^2 + 21 = 10x$, which he describes as "māl and 21 equals 10 roots," as follows (translation adapted from F. Rosen):

Halve the number of roots. It is 5. Multiply this by itself and the product is 25. Subtract from this the 21 added to the square (term) and the remainder is 4. Extract its square root, 2, and subtract this from half the number of roots, 5. There remains 3. This is the root you wanted, whose square is 9. Alternately, you may add the square root to half the number of roots and the sum is 7. This is (then) the root you wanted and the square is 49.

Notice that al-Khwārizmī's first procedure is simply a verbal description of our rule

$$\frac{10}{2} - \sqrt{\left(\frac{10}{2}\right)^2 - 21},$$

and his second procedure describes the calculation of $5 + \sqrt{5^2 - 21}$, but since all quantities are named in terms of their role in the problem whenever they appear (For example, "5" is called "the number of roots"), his description of the solution is quite as general, if not so compact, as our

$$\frac{n}{2} \pm \sqrt{\left(\frac{n}{2}\right)^2 - m}.$$

In fact, al-Khwārizmī's generality is reflected in the remarks that continue those quoted above.

"When you meet an instance which refers you to this case, try its solution by addition, and if that does not work subtraction will. In this case, both addition and

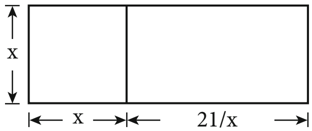

Fig. 2

subtraction can be used, which will not serve in any other of the three cases where the number of roots is to be halved."

"Know also that when, in a problem leading to this case, you have multiplied half the number of roots by itself, if the product is less than the number of dirhams added to māl, then the case is impossible. On the other hand, if the product is equal to the *dirhams* themselves, then the root is half the number of roots."

In the first of the above paragraphs, al-Khwārizmī recognizes that the case we are dealing with is the only one where there can be two positive roots. In the second paragraph, he remarks that there is no solution when what we call the discriminant is less than zero and he says that when $(n/2)^2 = m$ the only solution is $n/2$. Finally, he remarks that in the case $px^2 + m = nx$ it is necessary to divide everything by $\boldsymbol{p}$ to obtain $x^2 + (m/p) = (n/p)x$, which can be solved by the previous method. This shows, by the way, that his coefficients are not restricted to whole numbers.

What distinguishes al-Khwārizmī and his successors from earlier writers on problems of the above sort is that, following the procedures for obtaining the numerical solutions, he gives proofs of the validity of these same procedures, proofs that interpret $x^2 + 21$, for example, as a rectangle consisting of a square (x^2) joined to a rectangle of sides x and $21/x$ (Fig. 2).

4 Thābit's Demonstration for Quadratic Equations

Preliminaries

Al-Khwārizmī presents his proofs in terms of particular equations, but Thābit ibn Qurra in his work gives the demonstrations in general, and for that reason we shall follow him rather than the earlier al-Khwārizmī.

The first two cases, $x^2 + px = q$ and $x^2 + q = px$, sufficiently indicate Thābit's approach. In the proofs he uses two theorems from Euclid's *Elements*, which we now state and prove.

Book II, Prop. 5. *If a line AE is divided at B and bisected at W then the rectangle AB · BE plus the square on BW is equal to the square on* AW (Fig. 3a, b).

Note that in this proposition B may be on either side of the midpoint W. The two parts of Fig. 3 show these two cases and are drawn so that GAEM is a rectangle of sides AE and AG(= AB). The rectangle AB · BE which the theorem speaks of is equal to the shaded rectangle since AB = BD. The foregoing proposition deals with a line segment bisected and divided internally. The next proposition deals with a line segment extended, which we could look on as being bisected and divided externally.