Now you are allowed to put the top-over-bottom pair that is in your moveable column, in this case, penny-over-dime, in another cell in your bank. Your challenge: Can you get all six possible pairs in your bank?

22. From money-mating to cupid's arrow. Explain how the coin coupling challenge in Mindscape 21 connects with the argument behind Arrow's Impossibility Theorem.

In Your Own Words

23. With a group of folks. In a small group, discuss and actively work through the implications of Arrow's Impossibility Theorem and its justification. After your discussion, write a brief narrative describing the ideas of this theorem in your own words.

24. Creative writing. Write an imaginative story (it could be humorous, dramatic, whatever you like) that involves or evokes the ideas of this section.

For the Algebra Lover

Here we celebrate the power of algebra as a powerful way of finding unknown quantities by naming them, of expressing infinitely many relationships and connections clearly and succinctly, and of uncovering pattern and structure.

25. Vote night. There are four candidates running for Student Government. All students vote for one candidate. Carl gets 30% of the votes, Emmy gets 25%, George gets 20%, and Sonya gets the rest. If there were 1200 votes cast, how many votes did Sonya get?

26. Wroof recount. The election in the previous Mindscape is held again due to accusations of voter fraud (one of the candidate's dogs ate some ballots). This time, Carl got twice as many as George and 2/3 as many as Emmy. Sonya got 157 more than Carl. If there were 1200 votes cast, who won this time?

27. Biggest loser? Who was the biggest loser in the election in the previous Mindscape? That is, who received the fewest votes?

28. The *X*-act winner. Your school's math club has 73 members. In a recent election for club president, Ada received x^2 votes and Bernhard received $7x - 5$ votes. If all 73 members voted, who won the election?

29. Borda rules. Candidates A, B, and C are running for an election in which the votes are counted by the Borda method. Suppose the table here shows the percentages of 1st place votes, 2nd place votes, and 3rd place votes for each candidate. Who wins the election? (*Hint*: suppose there are 100 voters. Tally the Borda count total for each candidate. Who has the lowest total?)

	A	B	C
First choice	60%	20%	40%
Second choice	20%	40%	40%
Third choice	40%	40%	20%

10.5 CUTTING CAKE FOR GREEDY PEOPLE

Deciding How to Slice Up Scarce Resources

Can everyone get their fair share?

> *Let them eat cake.*
>
> MARIE ANTOINETTE

One of humanity's biggest challenges is dividing scarce resources among competing people. This process can cause tremendous strife—envy, greed, and justice can all come into play. From one point of view, most of the world's problems arise from the difficulty of allocating scarce resources fairly. What one person views as reasonable and equitable, another may regard as exploitative and grossly biased. When large societal groups carry on these disputes, the results can be anything from a baseball strike to a world war.

Perhaps surprisingly, mathematical thinking can contribute substantial insight to this most human of problems. On our journey through mathematical ideas, we have learned methods to understand difficult questions. We've seen the power of stating clearly what we want to know, identifying essential ingredients of the issue, and understanding simple cases deeply. Surely one of the premiere puzzles for humanity is the fundamental challenge of fair division. We'll find that our strategies of thinking are effective in exploring the contentious question of how to divide scarce resources. To explore this question we'll use an appetizing model: cake.

Let Us Cut Cake

We present this section in terms of cutting a cake because cakes are good battlegrounds for divisive allocations. Cakes also have a variety of features that are important in understanding a basic tenet of fair allocation, that is, different people have different opinions about what is valuable. Some people like icing considerably more than the inside portion of the cake. They may be willing to take a smaller piece of cake if it has a lot of icing on it (a corner piece, for example). Where the balance of value lies is a matter of individual taste. If everyone had the same concept of value, the problem of dividing things would be much simpler. However, we will soon see that individual differences in people's preferences actually may allow us to give everyone *more* than they think they rightfully deserve.

The Two Steps of Allocation

One of the themes of this section is realizing that disputes about allocating resources (such as a cake) often can and should be divided into two steps: (1) agreeing on what proportion of the resource each party deserves and then (2) making a division that realizes that agreement. In this discussion of cake-cutting, we concentrate on the second step.

In some cases the decision about what proportion of the resource each party gets is decided in advance. For example, wills may declare that an estate is to be divided equally among the heirs. Divorce settlements—which often lead to splitting headaches—fall into this category as well. However, the question of what allocation of the resources accomplishes that particular division often represents a considerable obstacle. The principles and methods developed here for cutting a cake can be applied directly or used to provide a perspective to help resolve problems of resource allocation.

One of the principles that will emerge from our discussion is that negotiation among the parties is not necessarily a correct or desirable procedure in determining a good allocation. After the decision has been made regarding what proportion of the cake each person deserves, it is not desirable to have the people enter into disputes about their idiosyncratic value systems. Instead, as we will see, we can be successful simply by asking all individuals to privately convey their preferences regarding various hypothetical divisions of the cake, and then have an outside party find from that information a division that is satisfactory to everyone.

The Setting for the Cutting

To focus our discussion, let's assume that we are dividing a cake among people who all agree that they have equal claim to their proportional share of the cake. And let's assume that each person wants a part of the cake that maximizes its value in his or her eyes and that no part of the cake has negative value to anyone; in other words, no one is on a diet, and there are no Brussels sprouts on the cake that a person might wish to (logically) avoid. So more is better, but different parts of the cake may be of greater relative value to one person than to another. For example, if it is a birthday cake, then perhaps the birthday honoree may value the part of the cake containing her name written in script, while another person may value the portion of the cake that contains that beautiful sugar-loaded candy rose.

We will also assume throughout this section that the cake does not contain any indivisible object. For example, if we are thinking of a wedding cake, the little statues of the bride and groom on top may present a problem because, as much as the bride might want, we cannot cut the groom in half. We will work only with cakes where all parts can be divided.

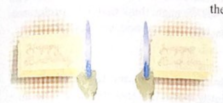
Throughout this book we have seen the power of starting with simple cases to build insights, and we apply that strategy here.

The One Person Case

Suppose we have a cake that we want to divide among one person, say, Alice. This case is not too hard. Alice looks at the cake carefully; assesses its various qualities; weighs the relative value of icing versus size; eyes the candy rose with due appreciation; and, after deep contemplation, takes the whole cake.

Two People—One Cake

Adam and Becky seek an amicable method for dividing a cake in two. In the "I-cut, you-choose" method, Adam cuts the cake into two parts that are equally valuable in his opinion, and then Becky is allowed to choose either piece. Becky is happy because she's given the first choice; so the piece she chooses represents at least half the value of the whole cake in her eyes. Adam is happy because he cut the cake in such a manner that either piece was equally valuable to him.



Is it always *possible* for Adam to cut the cake in such a manner that he would be equally satisfied with either piece? Suppose that Adam holds the knife over the cake and starts with the knife to the left of the cake and slowly moves the knife to the right until finally the entire cake is to the left of the knife. Now if Adam lowers the knife and cuts the cake at any point, then there would be two pieces: the left piece and the right piece.

Notice that at the start, with the knife to the left of the cake, if Adam were to cut it right there, he would actually miss the cake. The entire cake is to the right of the knife. Thus, with this "cut," the left piece is nonexistent and the right piece is the entire cake. Obviously, in this case, Adam would prefer the right piece. Now suppose that he cuts the cake with the knife in the far right position.

Life Lesson
Clarify the question.

Then the entire cake is the left piece. Thus, in this case, Adam would select the left piece. Because he moves the knife continuously from left to right and at the start he would prefer the right piece and at the end he would prefer the left, then there must exist a knife placement somewhere in between such that he would be equally content with either piece.

Cake-Cutting Question.
Given a cake and three people, is there a method of cutting the cake equitably?

The first question is, "What is the question?" Clarifying a question is frequently *the* most important step in resolving the issue. So let's look at the question again carefully.

The difficult word is "equitably." Does equitable mean "fair" in the sense that each person gets a piece that he or she views as having at least one-third of the value of the whole cake? That is a reasonable definition; however, we must face an unfortunate fact: Human beings are typically dissatisfied with just getting their fair share. We want the biggest piece; greed prevails. What we really want is for no one else to get a better piece than we're getting. This greedy attitude leads to a more precise version of the cake-cutting question.

Greedy Division Question.

Given a cake and three people, is there a method for cutting the cake into three pieces so that each person gets the piece that he or she believes has the greatest value? In other words, can the cake be divided into three pieces so that, of the resulting slices, everyone gets their favorite piece?

In the Greedy Division Question, no participant would covet another person's piece. Remember that each of the three people has a different concept of the value of the various cake parts. Why should it be possible to divide the cake into three pieces in such a way that each person's first choice out of the three pieces is different from the first choices of the other two people? We might think that their preferences could be irreconcilable. Perhaps there is no envy-free method, you say? Read on.

All fair, but #1 thinks #2's is better.

A Knife-Moving Method

Let's start with a cake and three people. Suppose someone takes a knife and, starting from the left side of the cake, slowly moves it across the cake. The three potential cake eaters watch intently, probably drooling. As soon as any one of the three believes the knife has reached the one-third mark in her valuation scheme, she yells, "Stop!" The cutter cuts the cake and the yeller gets the left piece. Since the other two people did not yet yell, they each believe that the remaining piece is worth at least two-thirds in their value systems. So they could employ the I-cut, you-choose method to divide the remaining piece or, alternatively, they could let the knife resume its motion, yelling "Stop!" when either of them believes that half the value of the remainder is reached.

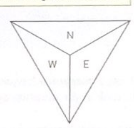
A Challenge—Fair but Not Greedy

Does the preceding knife-moving method always work? When might it *not* satisfy the conditions of the Greedy Division Question? Remember that the Greedy Division Question asks whether it's possible for everyone to get his or her first choice after the entire cake is cut and everyone examines all the pieces.

In fact, the knife-moving method always gives a fair division, but not necessarily an envy-free division of the cake. The failure of this cutting scheme further reinforces our skepticism about finding an envy-free division of a cake for three people. To make progress, we examine the issue in a slightly different manner.

Life Lesson

Choosing a convenient representation of an issue often allows us to see new possibilities.



Life Lesson

Representing questions in different ways is often a valuable step toward finding solutions.

The Point of a Division

Cakes come in all shapes and sizes, and we can imagine many different potential ways to cut those cakes. We'll first consider a triangular cake and delve into other-shaped cakes later.

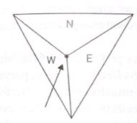
Given a point in a triangular cake, we can cut along the three straight lines from that given point to the vertices of the cake.

These lines divide the cake into three pieces, a North piece, an East piece, and a West piece. Thus, every point in a triangular cake corresponds to a division of the cake into three pieces by cutting from that point to the vertices of the cake.

Suppose Alice, Becky, and Claire wish to share the cake. With the cutting scheme just described, every point on the triangular cake presents each person with a decision, that is, if the cake were cut from that point to the three corners,

which of the three resulting pieces, the North, the East, or the West piece, would be her first choice? To satisfy the conditions of the Greedy Division Question, we seek a point in the triangular cake such that Alice prefers one piece, Becky prefers another, and Claire prefers the third. For example, perhaps at that point Alice prefers the West piece, Becky prefers the East piece, and Claire prefers the North piece. If we cut the cake from that point to the corners of the cake, everyone would get her first-choice piece.

The big question remains: Is there always such a point of universal satisfaction? Notice how we have changed the question from cutting cakes to finding a point in a triangle that possesses certain properties.



If cut from here, Becky tells us that she prefers the North piece (perhaps N contains extra icing).

What's Your Preference?

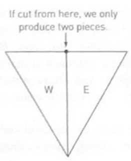
Although it may appear at first counterintuitive, our method for finding an envy-free division of the cake will avoid all negotiations and even discussions among the parties. Instead, we privately ask each person a long list of hypothetical questions. Namely, for each point in the triangular cake, we ask each person to tell what her preference would be (the North, East, or West piece) if the cake were to be cut from that point to the vertices of the cake. We then analyze all that information from each of the three people and deduce that there must be some point in the triangular cake where all three people gave three different preferences—one had declared that she would want the East piece if the cake were cut from that point to the vertices of the triangle; the second person said that she would want the West piece if the cake were cut from there; and the third person said that she would want the North piece if the cake were cut from there.

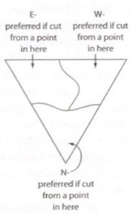
But even from here, we can hear your complaints about the impracticality of this method. You are no doubt saying to yourself, "Great, there are infinitely many points on the cake. So Alice, Becky, and Claire have to make infinitely many decisions about which piece they would each prefer from each point in the triangular cake. So this whole method is totally impractical, and this whole section is a useless crock."

Easy now. Let's try to overcome this infinite obstacle. Suppose we put ourselves in Becky's position. We are given a triangular cake and we are asked to label every point East, West, or North depending on our answer to the question, "Which piece of the cake would you choose if the cake were divided from that point by cutting to the three vertices?"

Preference Diagrams

Faced with labeling trillions and trillions of points (in fact, *infinitely* many), we will not get out our microscope. Instead, we now realize that the points that we want to label East, for example, are not scattered randomly around the triangle. Instead, those points form a region of the triangle on the upper-left side. To understand why, first note that at the upper-left corner of the cake, everyone would definitely pick East. Why? Well, if we "cut" from the upper-left vertex, there would be no northern or western piece at all—only an East piece, that is, the entire cake. Thus, Becky would definitely pick the East piece at that corner. In fact, for any point very near the upper-left vertex of the triangle, she will again choose East. The East piece is dramatically larger than the North or West slivers when the cake





Becky's preference diagram

is cut from any point near that upper-left vertex of the cake to the three vertices of the cake. Likewise, any point in the lower region of the triangular cake will be labeled North and any point near the upper right will be labeled West.

Becky does not have to look at billions of points individually in those particular areas.

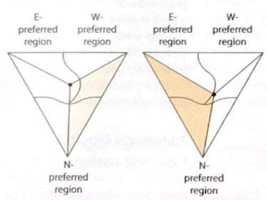
Similarly, as Becky considers points on the upper edge of the cake that runs horizontally from left to right, we are certain that she would never pick the North piece, since any cut from a point on that side produces only two pieces: an East and a West piece.

If the cake is cut from the upper-left vertex, we know Becky will prefer the East piece and if the cake is cut from the upper-right vertex, we know she picks West. Therefore, somewhere along the top edge, her preferences will change from East to West.

At the *exact* point of her change in preference, Becky prefers each of the two equally—they are both her first choice (we have a tie). As Becky considers points as she moves down from the top, her preference will eventually change to North.

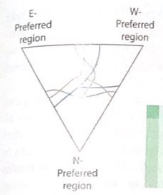
To label all points in the triangle, all Becky really needs to show is where her preferences change. Where are the boundaries between the regions where she prefers East, West, and North? Roughly speaking, her East-West boundary, her East-North boundary, and her West-North boundary will create a possibly wavy Mercedes-Benz symbol as illustrated. Such a Mercedes-Benz symbol creates her *preference diagram* because it conveys the information about what piece she would choose—the North, East, or West—for every possible division point in the triangle.

So if the cake were to be cut from any point within Becky's "E-preferred" region, then we would know that she would select the East piece. If the cake were cut from any point within Becky's "W-preferred" region, then we know that she would find the West piece the most appealing. Similarly for the "N-preferred" region.



If the cake were cut as shown, Becky would like the East piece best—notice that the cut-point is within Becky's "E-preferred" region.

Given this cutting, Becky would pick this West piece—note how the cut-point is from Becky's "W-preferred" region.



Alice = red
Becky = green
Claire = blue

Superimpose the Preference Diagrams

Once Alice, Becky, and Claire each have privately drawn their personal preference diagrams on three identical pictures of the cake, the three diagrams can be superimposed.

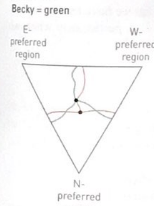
> **The Greedy Division Theorem.**
>
> *Suppose three preference diagrams are superimposed. Then there will be a point where the three people have indicated that they all prefer different pieces.*

In order to answer the Greedy Division Question affirmatively, we must show that there will be some point or points at which Alice, Becky, and Claire have each made a different choice, that is, if the cake were cut from that point, everyone would have a different favorite piece. In fact, such a point can *always* be found.

Proof of the Greedy Division Theorem

Before proceeding with our argument, we would like to make an impassioned plea. What makes the argument ahead challenging is the delicate analysis of the preference diagrams and their boundaries. Thus, we urge you to carefully study the figures together with the prose. Move slowly, think, and stop often to draw your own diagrams in order to help further your understanding. The argument below is definitely *not* a "quick read."

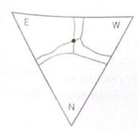
Alice = red
Becky = green



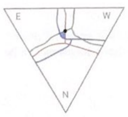
Let's take Alice's preference diagram and superimpose Becky's preference diagram on it. We will first deal with the possibility that the branch point of Becky's preference diagram lies exactly on a boundary of Alice's diagram. Suppose, for example, that Becky's branch point lies on Alice's East-West boundary curve.

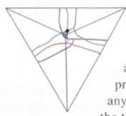
Then that point would be a suitable place from which to cut. We give Claire the piece she most prefers out of the three. Alice takes East or West, whichever is left, because she believes those are equally valuable, and Becky takes the remaining piece since she thinks all three are equally valuable when cut from that point.

In most cases, of course, Becky's branch point will not lie exactly on a boundary of Alice's diagram. Let's suppose that Becky's branch point lies in Alice's West-preferred region, for example.



Then Becky's North-East boundary must go from there to some point on the left side of the triangle. So Becky's North-East boundary must somewhere cross Alice's West-preferred border, which consists of a West-East boundary and a West-North boundary. Suppose Becky's North-East boundary crosses Alice's West-East boundary. Then that point is a suitable place from which to cut the cake. At that point Claire has some preference, say North. Then Claire could have North, Becky could take East, and Alice could take West. If Claire preferred East, then Becky could take North, and Alice could take West.





Finally, if Claire preferred West, then Becky could take North, and Alice could take East. For any of Claire's preferences, she can have what she wants and the other two can take a piece that they have said is a tie for their favorite piece.

Notice that usually we will find near that branch point a whole region of points where Alice, Becky, and Claire all prefer different pieces of the cake. In that case, we may select any point in that region as the place from which to cut the cake to the three corners. Only if one of Claire's boundary lines coincides exactly with Alice's or Becky's near that point will we have to choose the point exactly. Therefore our argument is now complete.

Recapitulation

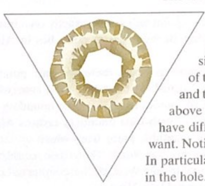
Notice that Alice, Becky, and Claire *never conferred* with one another during the entire cake-cutting process. Instead, they independently conveyed to an outside party their hypothetical preferences for which piece they would prefer if the cake were cut to the vertices from each point in the cake. Those decisions were all made privately. Later we proved that no matter what preference decisions each made, there must be a point from which we can cut the cake to the three vertices such that each of them can have their first choice. So this method of allocating resources emphasizes the power and value of separating the participating people.

How important is the triangular shape? Well, now that we have learned something about triangular cakes, let's explore variations of it, particularly when we have a cake of a different shape.

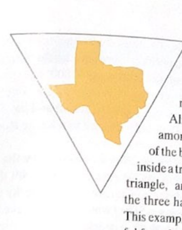
Non-Triangular Cakes and Pies

At real bakeries, occasionally we see a cake that is not triangular. Some peculiarly shaped cakes are rectangular or even round (go figure). Some cakes, often angel food cakes, have holes in the middle. How can we deal with such exotic cakes?

The easiest method for dealing with non-triangular cakes is to take the cake we have and put it inside a big triangle as illustrated.



Now we can use the method for the triangle. Notice that every point in the triangle, whether it's in the cake or not, represents a division of the cake—that is, the part of the cake that is in the North piece of the triangle, the part of the cake that is in the East piece of the triangle, and the part of the cake that is in the West piece of the triangle. The method above will give a point in the triangle where Alice, Becky, and Claire will have different views about their preferences for which piece of the cake they want. Notice that the central cutting point may or may not be in the cake itself. In particular, if we consider an angel food cake, the division point is likely to be in the hole.



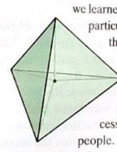
From a Piece of Cake to the Heart of Texas

One time long ago, there was a wealthy Texan who owned all of Texas. He was rich and tall (in fact, he only shopped at rich and tall men's clothing stores). When he died he left Texas to his three children, Alice-Bob, Becky-Bob, and Claire-Bob, saying they should divide Texas among themselves. Having read this book (well, actually just an early draft of the book, since this was quite a while ago), they simply drew a map of Texas inside a triangle, made hypothetical choices for each potential cutting point in the triangle, and found a point from which to cut that made every one of the three happy because they all got their favorite piece of the Lone Star State. This example demonstrates that our method of dividing resources is not only useful for cakes, but for land or other resources as well.

Four or More People

What happens if we want to divide a cake among four people? Faced with this new problem, we'll first try to solve it by using a variation of the method we used for dividing a cake among three people.

We might guess that a way to proceed would be to consider a square cake. However, this great idea does not seem to work. What parts of the cake-cutting method that we learned above fail to work if we try using a square cake and four people? In particular, can you draw four preference diagrams in which no point divides the cake such that each of the four people prefers a different piece? Although this sensible attempt does not work, we do not give up.



Just because one idea fails, we don't stop. There are often several different ways of generalizing an idea. If one fails, we'll try another. Here we seek to take a technique that worked successfully with three people and generalize it to the case with four people. Going from a triangle to a square seems reasonable, but does not work. It turns out to be more effective to go from a triangle to a tetrahedron.

We invite you to discover how to adopt the previous cake-cutting strategy to the case of four people dividing a tetrahedral cake. Notice how each point in a tetrahedron corresponds to dividing the tetrahedron into four sub-tetrahedra by drawing triangles from the point out to the vertices of the big tetrahedron. This generalization to four people requires us to visualize tetrahedra in three dimensions, which is a difficult task.

Splitting the Rent

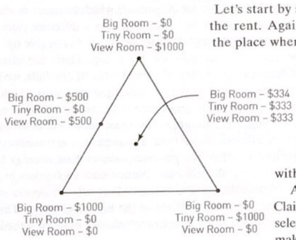
This idea of cake-cutting can be applied to other divisions as well, for example, dividing the rent of an apartment among three roommates. Suppose three people share a three-bedroom apartment. The total rent is $1000 per month. But the three bedrooms are not equally attractive. One is bigger than the others, one has a view, and one is very small. It would not be fair to make all three people pay the same rent. The person who gets the small room should pay less, and people might prefer the room with the view. But how much should each person pay?

The first step is to clarify the question. The three bedrooms will be called the Big Room, the Tiny Room, and the Room with a View, and the three renters will be Alice, Becky, and Claire. The total rent paid must equal \$1000. The question is, "How much should the occupant of each room pay?" As an example, one possible division of the rent would be for Becky to occupy the Big Room and pay \$450, for Alice to stay in the Tiny Room and pay \$150, and for Claire to take the Room with a View and pay \$400.

The question is, "Can the rent and rooms be divided in such a way that no one would want to change rooms if they had to pay the rent associated with the other room?" In other words, we want to divide the rent as above so that Becky does not say to herself, "Well, I like the Big Room, but at \$150, I would prefer to take the Tiny Room and save \$300 per month." We want all three people to feel that they would prefer their room at its price rather than wanting to switch to either of the other rooms and paying its rent.

Let's make the assumption that any renter would prefer a free room if the other two rooms charge rent, and that for any division of the rent, each renter would like some room.

Often the way to solve one question is to realize that it is similar to another question that we do know how to answer. In this case, we may feel that the rental harmony question is similar to the cake-cutting question. Can we answer it in a similar way?



Let's start by seeing if we can represent all possible ways to divide the rent. Again, a triangle does the trick. Each vertex represents the place where the total rent is paid by the occupant of one room, respectively, the Big Room, the Tiny Room, and the Room with a View. Each other point of the triangle is labeled with three dollar values that add up to \$1000 where the first number is the rent for the person in the Big Room, the second number is the rent paid by the person in the Tiny Room, and the third number is the rent paid by the person in the Room with a View.

At each point in the labeled triangle, Alice, Becky, and Claire each has an opinion about which room she would select if the prices were divided as listed. The three renters make their decisions independently. Each person can record her preferences by making a preference diagram as in the proof of The Greedy Division Theorem. In this case, however, all the boundary lines between preferences end up going to the three vertices of the triangles, because everyone will prefer the same room on each side since each such point gives a room away for free.

Now the three diagrams are superimposed and the same arguments similar to those used in cake-cutting demonstrate that there is at least one point where the three renters prefer three different rooms.

Finding a way to represent the situation with renters allows us to see that this situation is very similar to the cake-cutting issue that we already dealt with. So the same method of solution solves a different question as well.

A Look BACK

WE CAN DIVIDE a cake among three people so they all get their favorite piece with regard to that division. Surprisingly, the method does not involve negotiation or psychological issues. Instead it is an unexpected application of mathematical reasoning.

Our strategy is to represent the question in a convenient way. We associate each point of a triangular cake with a division of the cake into three pieces by cutting to the corners. This correspondence of points in the triangle with divisions of the cake lets each person express his or her hypothetical preferences for each possible division point in the triangle. We then superimpose the three diagrams that record each person's preferences from each potential division point. A geometric insight shows that at some point, the three people must have different first choices. By transforming a question about cake into a question about drawing wavy Mercedes-Benz symbols in a triangle, we can find a solution.

The rent-splitting question shows that three renters can divide the rent for an apartment with unequal rooms in such a way that each renter is happy to pay his or her rent rather than switch to another room with its rent. Representing this new question in an effective way allows us to see that at its root the question has the same essential solution as the cake-cutting question. Exploiting successful methods in different settings is an important way to take advantage of insights.

Questions can be formulated in many different ways, so we can sometimes choose the arena in which to fight our battles. Part of successful thinking is to extract the essential ingredients from a complicated situation. That way, we can avoid distractions and get to the heart of the matter. Often the hearts of quite different matters are similar and can be conquered with the same insight.

Life Lessons

Clarify the question.

•

Abstract the essence.

•

Choose a convenient representation of an issue.

MINDSCAPES Invitations to Further Thought

In this section, Mindscapes marked (**H**) *have hints for solutions at the back of the book. Mindscapes marked* (**ExH**) *have expanded hints at the back of the book. Mindscapes marked* (**S**) have solutions.

Developing Ideas

1. **You-cut, you-lose.** In the I-cut, you-choose method, would you rather be the cutter or the chooser? Which role would give you a piece of cake that may be more valuable in your opinion?
2. **Understanding icing (S).** Suppose a person who had not read this section declares that cutting cake fairly for three people is simple—just weigh the cake very carefully and give each person a piece that weighs one-third of the total. What idea is that person missing?
3. **Liquid gold.** Suppose you and your two brothers are dividing a mound of solid gold. You could use the cake-cutting methods to get a fair division, but