A winning coalition in the U.N. Security Council must contain all five permanent members (a total weight of 35) and at least four nonpermanent members (an additional weight of 4). Hence, any winning coalition meets or exceeds the quota of 39. A losing coalition, on the other hand, either omits a permanent member, and thus has weight at most

$$(7 \times 4) + (1 \times 10) = 28 + 10 = 38,$$

or contains at most three nonpermanent members, and thus has weight at most

$$(7 \times 5) + (1 \times 3) = 35 + 3 = 38.$$

Hence, any losing coalition falls short of the quota of 39. This completes the proof.

It turns out that if one alters any weighted voting system by giving one or more of the voters veto power, the resulting yes–no voting system is again a weighted voting system. See Exercises 5 and 6 at the end of the chapter.

A natural response to what we have done so far is to conjecture that *every* yes–no voting system is weighted. Perhaps, for example, even for the U.S. federal system, we can do something as clever as what we just did for the U.N. Security Council to find weights and a quota that work. Alas, this turns out not to be the case, as we will see in the next section.

2.4 SWAP ROBUSTNESS AND THE NONWEIGHTEDNESS OF THE FEDERAL SYSTEM

The U.S. federal system, it turns out, is not a weighted voting system. But how does one prove that a system is *not* weighted? Surely we cannot simply say that we tried our very hardest to find weights and a quota and nothing that we tried appeared to work. Asserting that a system is not weighted is saying no one will *ever* find weights and a quota that describe the system. Moreover, we cannot check all possible choices of weights and quota since there are infinitely many such choices.

Here is one answer. To prove that the U.S. federal system is not weighted, it suffices to find a property that we can prove

1. holds for every weighted voting system, and
2. does not hold for the U.S. federal system.

One such property is given in the following definition:

DEFINITION. A yes–no voting system is said to be *swap robust* if a one-for-one exchange of players (a "swap") between two winning coalitions X and Y leaves at least one of the two coalitions winning. One of the players in the swap must belong to X but not Y, and the other must belong to Y but not X.

Thus, to prove that a system is swap robust, we must start with two *arbitrary* winning coalitions X and Y, and an *arbitrary* player x that is in X but not in Y, and an *arbitrary* player y that is in Y but not in X. We then let X' and Y' be the result of exchanging x and y. (Thus, x is now in Y' but not X', while y is in X' but not Y'.) We must then show that either X' or Y' is winning. To illustrate this, consider the following.

PROPOSITION. *Every weighted voting system is swap robust.*

PROOF. Assume we have a weighted voting system and two arbitrary winning coalitions X and Y with X containing at least one voter x not in Y and Y containing at least one voter y not in X. Suppose now that voter x from the winning coalition X is exchanged for voter y from the winning coalition Y to yield X' and Y' as above. If x and y have the same weight then both X' and Y' are winning, since X' weighs the same as X and Y' weighs the same as Y. If, on the other hand, x is heavier than y, it then follows that Y' weighs strictly more than Y, since Y' was obtained by deleting y and adding the heavier x. Thus the weight of Y' certainly exceeds the quota, and thus Y' is winning as desired. (In this latter case, X' may or may not be winning.) If y is heavier than x, then the argument is analogous to what we just gave. This completes the proof.

Our goal now is to show that a particular yes–no voting system—the U.S. federal system—is *not* swap robust. To do this we must *produce* two winning coalitions X and Y and a trade between them that renders both losing. Intuitively, X and Y should both be "almost losing" (in the sense that we hope to make both actually losing by a one-for-one trade). Thus, we will try to find appropriate X and Y among the *minimal* winning coalitions. (Results related to this are in Exercise 17.)

The key to showing that the U.S. federal system is not swap robust is the following observation: if one begins with two minimal winning coalitions in the U.S. federal system and swaps a senator for a House member, then both coalitions become losing (as desired) since one of the resulting coalitions has too few senators (although a surplus of House members) and the other has too few House members (although a surplus of senators). If we simply formalize this slightly, we have:

PROPOSITION. *The U.S. federal system is not swap robust.*

PROOF. Let X consist of the president, the 51 shortest senators, and the 218 shortest members of the House. Let Y consist of the president, the 51 tallest senators, and the 218 tallest members of the House. Now let x be the shortest senator and let y be the tallest member of the House. Notice that both X and Y are winning coalitions, and that x is in X but not in Y and y is in Y but not in X. Let X' and Y' be the result of swapping x for y. Then X' is a losing coalition because it has only 50 senators, and Y' is a losing coalition because it has only 217 members of the House. Thus, the U.S. federal system is not swap robust.

Notice that a "swap" cannot involve a voter who is a member of both of the coalitions with which we begin. We avoided this in the above proof by making x the shortest senator whereas Y involved the 51 tallest senators. This is why x was definitely in X but not in Y.

An immediate consequence of the above theorem is the following corollary:

COROLLARY. *The U.S. federal system is not a weighted voting system.*

PROOF. If the U.S. federal system were weighted, then it would be swap robust by the first proposition in this section. But this would then contradict the proposition we just proved.

2.5 TRADE ROBUSTNESS AND THE NONWEIGHTEDNESS OF THE CANADIAN SYSTEM

Section 2.4 provided us with a very nice way to show that the U. S. federal system is not weighted—we simply showed that it is not swap robust. But will this technique always work? That is, if a yes–no voting system is truly not weighted, can we always *prove* it is not weighted (assuming we are clever enough) by showing that it is not swap robust?

Here is another way to ask this same question. The first proposition in **Section 2.4** asserts that *if* a yes–no voting system is weighted, *then* it is swap robust. The converse of this "if-then" statement is the following assertion (which we are not claiming is true): *if* a yes–no voting system is swap robust, *then* it is weighted. (Equivalently, if a yes–no voting system is not weighted, then it is not swap robust.) Just because an if-then statement is true, we cannot conclude that its converse is also true. (For example, the statement: "if a number is larger than 10, then it is larger than 8," is true, but the number 9 shows that its converse is false.) Thus, the question in the previous paragraph can be recast as simply asking if the converse of the first proposition in **Section 2.4** is true. If the converse were true, this would say that weightedness and swap robustness are fully equivalent in the sense that a yes–no voting system would satisfy one of the properties if and only if it satisfied the other property.

It turns out, however, that the converse of the first proposition in **Section 2.4** is false. That is, there are yes–no voting systems that are not weighted, but nevertheless *are* swap robust. Hence, we cannot prove that such a system fails to be weighted by using swap robustness as we did for the U.S. federal system. This raises the question of exactly how one shows that such a system is not weighted. Answering this question is the primary goal of this section, but we begin with the following.

PROPOSITION. *The procedure to amend the Canadian Constitution is swap robust (although we shall show later that it is not weighted).*

PROOF. Suppose that X and Y are winning coalitions in the system for amending the Canadian Constitution, and that x is a province ("voter")