two winning coalitions X and Y and a trade between them that renders both losing. Intuitively, X and Y should both be "almost losing" (in the sense that we hope to make both actually losing by a one-for-one trade). Thus, we will try to find appropriate X and Y among the *minimal* winning coalitions. (Results related to this are in Exercise 17.)

The key to showing that the U.S. federal system is not swap robust is the following observation: if one begins with two minimal winning coalitions in the U.S. federal system and swaps a senator for a House member, then both coalitions become losing (as desired) since one of the resulting coalitions has too few senators (although a surplus of House members) and the other has too few House members (although a surplus of senators). If we simply formalize this slightly, we have:

PROPOSITION. *The U.S. federal system is not swap robust.*

PROOF. Let X consist of the president, the 51 shortest senators, and the 218 shortest members of the House. Let Y consist of the president, the 51 tallest senators, and the 218 tallest members of the House. Now let x be the shortest senator and let y be the tallest member of the House. Notice that both X and Y are winning coalitions, and that x is in X but not in Y and y is in Y but not in X. Let X' and Y' be the result of swapping x for y. Then X' is a losing coalition because it has only 50 senators, and Y' is a losing coalition because it has only 217 members of the House. Thus, the U.S. federal system is not swap robust.

Notice that a "swap" cannot involve a voter who is a member of both of the coalitions with which we begin. We avoided this in the above proof by making x the shortest senator whereas Y involved the 51 tallest senators. This is why x was definitely in X but not in Y.

An immediate consequence of the above theorem is the following corollary:

COROLLARY. *The U.S. federal system is not a weighted voting system.*

PROOF. If the U.S. federal system were weighted, then it would be swap robust by the first proposition in this section. But this would then contradict the proposition we just proved.

2.5 TRADE ROBUSTNESS AND THE NONWEIGHTEDNESS OF THE CANADIAN SYSTEM

Section 2.4 provided us with a very nice way to show that the U. S. federal system is not weighted—we simply showed that it is not swap robust. But will this technique always work? That is, if a yes–no voting system is truly not weighted, can we always *prove* it is not weighted (assuming we are clever enough) by showing that it is not swap robust?

Here is another way to ask this same question. The first proposition in **Section 2.4** asserts that *if* a yes–no voting system is weighted, *then* it is swap robust. The converse of this "if-then" statement is the following assertion (which we are not claiming is true): *if* a yes–no voting system is swap robust, *then* it is weighted. (Equivalently, if a yes–no voting system is not weighted, then it is not swap robust.) Just because an if-then statement is true, we cannot conclude that its converse is also true. (For example, the statement: "if a number is larger than 10, then it is larger than 8," is true, but the number 9 shows that its converse is false.) Thus, the question in the previous paragraph can be recast as simply asking if the converse of the first proposition in **Section 2.4** is true. If the converse were true, this would say that weightedness and swap robustness are fully equivalent in the sense that a yes–no voting system would satisfy one of the properties if and only if it satisfied the other property.

It turns out, however, that the converse of the first proposition in **Section 2.4** is false. That is, there are yes–no voting systems that are not weighted, but nevertheless *are* swap robust. Hence, we cannot prove that such a system fails to be weighted by using swap robustness as we did for the U.S. federal system. This raises the question of exactly how one shows that such a system is not weighted. Answering this question is the primary goal of this section, but we begin with the following.

PROPOSITION. *The procedure to amend the Canadian Constitution is swap robust (although we shall show later that it is not weighted).*

PROOF. Suppose that X and Y are winning coalitions in the system for amending the Canadian Constitution, and that x is a province ("voter") in X but not in Y and that y is a province in Y but not in X. Let X' and Y' be the result of swapping x for y. We must show that at least one of X' and Y' is still a winning coalition. That is, we must show that at least one of X' and Y' still satisfies both of the following conditions:

1. It contains at least seven provinces.
2. The provinces it contains represent at least half of the Canadian population.

Note, however, that both X' and Y' certainly satisfy condition 1, since each of the two coalitions started with at least seven provinces, and we simply did a one-for-one swap of provinces to obtain X' and Y'. The rest of the argument is now reminiscent of the proof that a weighted system is swap robust. That is, if x has more population than y, then Y' is a winning coalition since it has more population than Y, and so it satisfies condition 2 since Y satisfied condition 2. If, on the other hand, y has more population than x, then X' is a winning coalition by an analogous argument. This completes the proof.

Our parenthetical remark in the statement of the above proposition promised a proof that the procedure to amend the Canadian Constitution is not weighted. But how do we do this? The answer lies in finding a stronger property than swap robustness that, like swap robustness, holds for every weighted voting system but that does not hold for the procedure to amend the Canadian Constitution. One such property that naturally suggests itself is the following strengthening of swap robustness:

DEFINITION. A yes–no voting system is said to be *trade robust* if an arbitrary exchange of players (that is, a series of trades involving groups of players) among several winning coalitions leaves at least one of the coalitions winning.†

†There are some subtleties here that need not concern a student. For example, we really are working with *sequences* of coalitions instead of *sets* of coalitions. For a thorough treatment, see Taylor-Zwicker, 1999.

Thus, trade robustness differs from swap robustness in two important ways:

1. In trade robustness, the exchanges of players are not necessarily one-for-one as they are in swap robustness.
2. In trade robustness, the trades may involve more than two coalitions.

The following is the expected strengthening of the first proposition in **Section 2.4**.

PROPOSITION. *Every weighted voting system is trade robust.*

PROOF. Notice that a series of trades among several winning coalitions leaves the number of coalitions to which each voter belongs unchanged. Thus, the total weight of all the coalitions added together is unchanged. Moreover, since the total number of coalitions is also unchanged, it follows that the average weight of these coalitions is unchanged as well.

Thus, if we start with several winning coalitions in a weighted voting system, then all of their weights at least meet quota. Hence, their average weight at least meets quota. After the trades, the average weight of the coalitions is unchanged and so it still at least meets quota. Thus, at least one of the coalitions must itself meet quota (since the average of a collection of numbers each less than quota would itself be less than quota). Hence, at least one of the coalitions resulting from the trades is winning, as desired.

To conclude that the system to amend the Canadian Constitution is not weighted, it suffices to establish the following:

PROPOSITION. *The procedure to amend the Canadian Constitution is not trade robust.*

PROOF. Let X and Y be the following winning coalitions (with percentages of population residing in the provinces also given):

X	Y
Prince Edward Island (0%)	New Brunswick (2%)
Newfoundland (2%)	Nova Scotia (3%)
Manitoba (4%)	Manitoba (4%)
Saskatchewan (3%)	Saskatchewan (3%)
Alberta (11%)	Alberta (11%)
British Columbia (13%)	British Columbia (13%)
Quebec (23%)	Ontario (39%)

Now let X' and Y' be obtained by trading Prince Edward Island and Newfoundland for Ontario. It then turns out that X' is a losing coalition because it has too few provinces (having given up two provinces in exchange for one), while Y' is a losing coalition because the eight provinces in Y' represent less than half of Canada's population.

COROLLARY. *The procedure to amend the Canadian Constitution is not a weighted voting system.*

Lani Guinier was nominated by President Clinton to be his assistant attorney general for civil rights. Her nomination was later withdrawn, in part because some of her views were considered radical. One such view involved the desirability (in certain circumstances) of a "minority veto." Interestingly, this idea also leads to a system that is swap robust but not trade robust. See Exercise 17.

2.6 STATEMENT OF THE CHARACTERIZATION THEOREM

In this section, we conclude with what might be described as the "evolution of a theorem." This evolution is worth a moment's reflection, as it serves to illustrate how theorems not only answer questions, but raise new ones as well.

With the observation that the U.N. Security Council is a weighted voting system, the following question naturally suggests itself:

Is every yes–no voting system a weighted system?

The U.S. federal system, however, provides a negative answer to this question, since it is not swap robust while every weighted voting system is swap robust.

With this observation, one might be tempted to think that the only thing preventing a yes–no voting system from being weighted is a lack of swap robustness. Thus, one might conjecture that the following is true (although it turns out not to be):

A yes–no voting system is weighted if and only if it is swap robust.

The procedure to amend the Canadian Constitution, however, shows that this conjecture is false, since it is swap robust but not weighted. But the proof that this system is not weighted suggests that the intuition behind the above conjecture might have been sound, with its failure resulting from the limited kind of trades involved in the notion of swap robustness. This leads quite naturally to the notion of trade robustness, and the conjecture that trade robustness exactly characterizes the weighted voting systems. This conjecture, in fact, turns out to be true, although its proof will not be given here. The result is proven in Taylor and Zwicker (1992,1999), although its precursor goes back several decades to Elgot (1960).

THEOREM. *A yes–no voting system is weighted if and only if it is trade robust.*

It would be nice if trade robustness could be defined in terms of trades between only two coalitions. This turns out not to be the case. In Chapter 8, we consider this particular issue and others related to yes–no voting systems, and we show that if one generalizes the notions of "weights" and "quota" from numbers to vectors (defined later), then every yes–no voting system is a "vector-weighted" system.

2.7 CONCLUSION

In this chapter we considered voting systems in which a single alternative, such as a bill or an amendment, is pitted against the status quo. Four examples of such yes–no voting systems were presented: