Country	Votes	Percentage of votes	SSI	Percentage of power
France	4	23.5	14/60	23.3
Germany	4	23.5	14/60	23.3
Italy	4	23.5	14/60	23.3
Belgium	2	11.8	9/60	15.0
Netherlands	2	11.8	9/60	15.0
Luxembourg	1	5.9	0	0

We conclude this section by using the European Economic Community to illustrate a well-known paradox that arises with cardinal notions of power such as those considered in the present chapter (and later in Chapter 9). The setting is as follows: Suppose we have a weighted voting body as set up among France, Germany, Italy, Belgium, the Netherlands, and Luxembourg in 1958. Suppose now that new members are added and given votes, but the percentage of votes needed for passage remains about the same. Intuitively, one would expect the "power" of the original players to become somewhat diluted, or, at worst, to stay the same. The rather striking fact that this need not be the case is known as the "Paradox of New Members." It is, in fact, precisely what occurred when the European Economic Community expanded in 1973.

Recall that in the original European Economic Community, France, Germany, and Italy each had four votes, Belgium and the Netherlands each had two votes, and Luxembourg had one, for a total of seventeen. Passage required twelve votes, which is 70.6 percent of the seventeen available votes. In 1973, the European Economic Community was expanded by the addition of England, Denmark, and Ireland. It was decided that England should have the same number of votes as France, Germany, and Italy, but that Denmark and Ireland should have more votes than the one held by Luxembourg and fewer than the two held by Belgium and the Netherlands. Thus, votes for the original members were scaled up by a factor of $2\frac{1}{2}$, except for Luxembourg, which only had its total doubled. In summary then, the countries and votes stood as follows:

France	10	Belgium	5	England	10
Germany	10	Netherlands	5	Denmark	3
Italy	10	Luxembourg	2	Ireland	3

The number of votes needed for passage was set at forty-one, which is 70.7 percent of the fifty-eight available votes.

The striking thing to notice is that Luxembourg's power—as measured by the Shapley–Shubik index—has increased. That is, while Luxembourg's Shapley–Shubik index had previously been zero, it is clearly greater than zero now since we can produce at least one ordering of the nine countries for which Luxembourg is pivotal. (The actual production of such an ordering is left as an exercise at the end of the chapter.) Notice also that this increase of power is occurring in spite of the fact that Luxembourg was treated worse than the other countries in the scaling-up process. For some even more striking instances of this paradox of new members phenomenon, see the exercises at the end of the chapter where, for example, it is pointed out that even if Luxembourg had been left with one vote, its power still would have increased.

3.4 THE BANZHAF INDEX OF POWER

A measure of power that is similar to (but not the same as) the Shapley–Shubik index is the so-called Banzhaf index of a player. This power index was introduced by the attorney John F. Banzhaf III in connection with a lawsuit involving the county board of Nassau County, New York in the 1960s (see Banzhaf, 1965). The definition takes place via the intermediate notion of what we shall call the "total Banzhaf power" of a player. The definition follows.

DEFINITION. Suppose that p is a voter in a yes–no voting system. Then the total Banzhaf power of p, denoted here by TBP(p), is the number of coalitions C satisfying the following three conditions:

1. p is a member of C.
2. C is a winning coalition.
3. If p is deleted from C, the resulting coalition is not a winning one.

If C is a winning coalition, but the coalition resulting from p's deletion from C is not, then we say that *p's defection from C is critical.*

Notice that TBP(p) is an integer (whole number) as opposed to a fraction between zero and one. To get such a corresponding fraction, we do the following (which is called "normalizing").

DEFINITION. Suppose that p_1 is a player in a yes–no voting system and that the other players are denoted by $p_2, p_3, \ldots, p_n$. Then the *Banzhaf index* of p_1, denoted here by BI(p_1), is the number given by

$$\mathrm{BI}(p_1) = \frac{\mathrm{TBP}(p_1)}{\mathrm{TBP}(p_1) + \cdots + \mathrm{TBP}(p_n)}.$$

Notice that $0 \leq \mathrm{BI}(p) \leq 1$ and that if we add up the Banzhaf indices of all n players, we get the number 1.

Example:

Let's again use the example where the voters are p_1, p_2, and p_3; and p_1 has fifty votes, p_2 has forty-nine votes, p_3 has one vote; and fifty-one votes are needed for passage. We will calculate TBP and BI for each of the three players. Recall that the winning coalitions are

$$C_1 = \{p_1, p_2, p_3\},$$
$$C_2 = \{p_1, p_2\},$$
$$C_3 = \{p_1, p_3\}.$$

For TBP(p_1), we see that p_1 is in each of the three winning coalitions and his defection from each is critical. On the other hand, neither p_2's nor p_1's defection from C_1 is critical, but p_2's is from C_2 and p_3's is from C_3. Thus:

$$\mathrm{TBP}(p_1) = 3 \qquad \mathrm{TBP}(p_2) = 1 \qquad \mathrm{TBP}(p_3) = 1$$

and, thus,

$$\mathrm{BI}(p_1) = \frac{3}{(3+1+1)} = \frac{3}{5}$$
$$\mathrm{BI}(p_2) = \frac{1}{(3+1+1)} = \frac{1}{5}$$
$$\mathrm{BI}(p_3) = \frac{1}{(3+1+1)} = \frac{1}{5}.$$

Recall that for the same example we had $\mathrm{SSI}(p_1) = \frac{2}{3}$, $\mathrm{SSI}(p_2) = \frac{1}{6}$, and $\mathrm{SSI}(p_3) = \frac{1}{6}$.

3.5 TWO METHODS OF COMPUTING BANZHAF POWER

This section presents two new procedures for calculating total Banzhaf power. Both procedures begin with a very simple chart that has the winning coalitions enumerated in a vertical list down the left side of the page, and the individual voters enumerated in a horizontal list across the top. For example, if the yes–no voting system is the original European Economic Community, the chart (with "F" for "France" etc.) will have:

F G I B N L

across the top. Down the left side it will have the fourteen winning coalitions which turn out to be (displayed horizontally at the moment for typographical reasons):

FGI, FGBN, FIBN, GIBN
FGIL, FGBNL, FIBNL, GIBNL
FGIB, FGIN
FGIBL, FGINL
FGIBN
FGIBNL

Notice the order in which we have chosen to list the winning coalitions: the first four are precisely the ones with weight 12, the next four are the ones with weight 13, then the two with weight 14, the two with weight 15, the one with weight 16, and the one with weight 17. If the voting system is weighted, this is a nice way to ensure that no winning coalitions have been missed. In what follows, we shall need the observation that there are fourteen winning coalitions in all.

We now present and illustrate the two procedures for calculating total Banzhaf power. Notice that "critical defection" is not mentioned in either procedure.

PROCEDURE 1. Assign each voter (country) a "plus one" for each winning coalition of which it *is* a member, and assign it a "minus one" for each winning coalition of which it *is not* a member. The sum of these "plus and minus ones" turns out to be the total Banzhaf power of the voter. (The reader wishing to get ahead of us should stop here and contemplate why this is so.) Continuing with the European Economic Community as an example, we have:

	F	G	I	B	N	L
FGI	1	1	1	−1	−1	−1
FGBN	1	1	−1	1	1	−1
FIBN	1	−1	1	1	1	−1
GIBN	−1	1	1	1	1	−1
FGIL	1	1	1	−1	−1	1
FGBNL	1	1	−1	1	1	1
FIBNL	1	−1	1	1	1	1
GIBNL	−1	1	1	1	1	1
FGIB	1	1	1	1	−1	−1
FGIN	1	1	1	−1	1	−1
FGIBL	1	1	1	1	−1	1
FGINL	1	1	1	−1	1	1
FGIBN	1	1	1	1	1	−1
FGIBNL	1	1	1	1	1	1
TBP(sum)	10	10	10	6	6	0

PROCEDURE 2. Assign each voter (country) a "plus two" for each winning coalition in which it appears (and assign it nothing for those in which it does not appear). Subtract the total number of winning coalitions from this sum. The answer turns out to be the total Banzhaf power of the voter. Continuing with the European Economic Community as an example, we have:

	F	G	I	B	N	L
FGI	2	2	2			
FGBN	2	2		2	2	
FIBN	2		2	2	2	
GIBN		2	2	2	2	
FGIL	2	2	2			2
FGBNL	2	2		2	2	2
FIBNL	2		2	2	2	2
GIBNL		2	2	2	2	2
FGIB	2	2	2	2		
FGIN	2	2	2		2	
FGIBL	2	2	2	2		2
FGINL	2	2	2		2	2
FGIBN	2	2	2	2	2	
FGIBNL	2	2	2	2	2	2
(sum)	24	24	24	20	20	14
Minus number of winning coalitions	−14	−14	−14	−14	−14	−14
TBP	10	10	10	6	6	0

The following chart summarizes the Banzhaf indices (arrived at by dividing each country's total Banzhaf power by 10+10+10+6+6+0 = 42). This is analogous to what we did for the Shapley-Shubik indices in **Section 3.2**.

Country	Votes	Percentage of votes	BI	Percentage of power
France	4	23.5	5/21	23.8
Germany	4	23.5	5/21	23.8
Italy	4	23.5	5/21	23.8
Belgium	2	11.8	3/21	14.3
Netherlands	2	11.8	3/21	14.3
Luxembourg	1	5.9	0	0

Why is it that these two procedures give us the number of critical defections for each voter? Let's begin with the following easy observation: Procedure 2 yields the same numbers as does Procedure 1. That is, in going from Procedure 1 to Procedure 2, all the "minus ones" became "zeros" and all the "plus ones" became "twos." Hence, the sum for each voter increased by one for each winning coalition. Thus, when we subtracted off the number of winning coalitions, the result from Procedure 2 became the same as the result from Procedure 1.

So we need only explain why Procedure 1 gives us the number of critical defections that each voter has. (Recall that we are considering only monotone voting systems.) The key to understanding what is happening in Procedure 1 is to have at hand a particularly revealing enumeration of the winning coalitions. Such a revealing enumeration arises from focusing on a single voter p, with different voters in the role of p giving different enumerations. To illustrate such an enumeration, let's let the fixed voter p be the country Belgium in the European Economic Community. The list of winning coalitions corresponding to the fixed voter p will be made up of three "blocks" of coalitions:

Block 1: Those winning coalitions that do not contain p.

Block 2: The coalitions in Block 1 with p added to them.

Block 3: The rest of the winning coalitions.

For example, with Belgium playing the role of the fixed voter p, we would have the fourteen winning coalitions in the European Economic Community listed in the following order:

Block 1:	FGI
	FGIL
	FGIN
	FGINL
Block 2:	FGIB
	FGILB
	FGINB
	FGINLB
Block 3:	FGNB
	FINB
	GINB
	FGNLB
	FINLB
	GINLB

There are several things to notice about the blocks. First, the coalitions in Block 2 are all winning because those in Block 1 are winning and we are only considering monotone voting systems. Second, there are exactly as many coalitions in Block 2 as in Block 1, because if X and Y are two distinct winning coalitions in Block 1, and thus neither contains p, then adding p to each of X and Y will again result in distinct coalitions in Block 2. Moreover, every coalition in Block 2 arises from one in Block 1 in this way. Third, every coalition in Block 3 contains p, since all those not containing p were listed in Block 1.

Finally, and perhaps most importantly, is the observation that p's defection from a winning coalition is critical precisely for the coalitions in Block 3. That is, p does not even belong to the coalitions in Block 1, and p's defection from any coalition in Block 2 gives the corresponding winning coalition in Block 1, and thus is not critical. However, if p's defection from a coalition X in Block 3 were to yield a coalition Y that is winning, then Y would have occurred in Block 1, and so X would have occurred in Block 2 instead of Block 3.

The reason Procedure 1 works is now clear: The minus ones in Block 1 are exactly offset by the plus ones in Block 2, thus leaving a plus one contribution for each coalition in Block 3 and these are precisely the ones for which p's defection is critical.

Other consequences also follow. For example, a monotone yes-no voting system with exactly seventy-one winning coalitions has no dummies as defined in Exercise 18 in Chapter 2. (Exercise 19 asks why.) Notice that the listing of winning coalitions corresponding to the fixed voter p is used only to understand why the procedures work—such listings need not be constructed to actually calculate Banzhaf power using either Procedure 1 or Procedure 2.

Power indices tend to have some paradoxical aspects. For example, Felsenthal and Machover (1994) noticed the following paradoxical result for the Banzhaf index. Consider the weighted voting system in which there are five voters with weights 5, 3, 1, 1, 1 and the quota is 8. We denote this by:

$$[8 : 5, 3, 1, 1, 1].$$

The Banzhaf indicies of the voters turn out to be $\frac{9}{19}$, $\frac{7}{19}$, $\frac{1}{19}$, $\frac{1}{19}$, and $\frac{1}{19}$ (Exercise 32). Now suppose that the voter with weight 5 gives one of his "votes" to the voter with weight 3. This results in the weighted system

$$[8 : 4, 4, 1, 1, 1].$$

It now turns out (Exercise 32 again) that the first voter has Banzhaf index $\frac{1}{2}$. But $\frac{1}{2}$ is greater than $\frac{9}{19}$! (surprise, not factorial). Hence, by giving away a single vote to a single player (and no other changes being made), a player has increased his power as measured by the Banzhaf index. (Part of what is going on here is that the transfer of a vote from the first player to the second makes each of the last three players a dummy. Hence, the first two players together share a larger fraction of the power than they previously did, and—as one would expect—the second player gains more than the first. The trade-off is that the first player is gaining more from the gain caused by the effective demise of the last three voters than he is losing from the transfer of one vote from himself to the second voter.) More on this paradox is found in Exercise 33.

It turns out (as pointed out by Felsenthal and Machover) that the Shapley-Shubik index is not vulnerable to this particular type of paradox. But the Shapley–Shubik index is not immune to such quirks. Exercise 34 presents a paradoxical aspect (due to William Zwicker) of the Shapley–Shubik index that is not shared by the Banzhaf index.

3.6 THE POWER OF THE PRESIDENT

Among the yes-no voting systems we have discussed, perhaps none is of more interest than the United States federal system. This brings us to an obvious question: What do the power indices have to say about the fraction of power held by the president in the U.S. federal system?

The calculations we will be doing (especially for the Shapley–Shubik index of the president) require some mathematical preliminaries. As a simple illustration of the first such preliminary we must confront, suppose we have four objects: a, b, c, and d. In how many ways can we choose two of them (assuming that the order in which we choose them does not matter)? The answer turns out to be six: ab, ac, ad, bc, bd, and cd. In general, if we start with n objects (instead of four) and ask for the number of ways we can choose k of them, where k is between 1 and n, then the following notation is used:

NOTATION. If $1 \leq k \leq n$, then the phrase "n choose k," denoted

$$\binom{n}{k},$$

refers to the number of distinct ways we can choose exactly k objects from a collection of exactly n objects.

The example above shows that "four choose two" equals six. The following proposition gives a relatively easy way to calculate these values.

PROPOSITION. *For $1 \leq k \leq n$ (and the convention that $0! = 1$), the following holds:*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

PROOF. Recall from **Section 3.2** that the number of different ways we can arrange n objects is given by $n!$. With k fixed, we can think of each such arrangement of the n objects as being obtained by the following three-step process:

1. Choose k of the objects to be the initial "block."
2. Arrange these k objects in some order.
3. Arrange the remaining $n - k$ objects in some order.

For example, if the objects are a, b, c, d, e, and f, and $k = 3$, then step 1 might consist of choosing a, d, and f. Step 2 might consist of choosing the following arrangement of the three chosen objects: f followed by a followed by d. Step 3 might consist of choosing the following arrangement of the remaining objects: e follwed by b followed by c. These three steps yield the arrangement: *f a d e b c*.

Step 1 above can be done in n choose k different ways. Step 2 can be done in $k!$ different ways. Step 3 can be done in $(n - k)!$ different ways. Hence, according to the general multiplication principle from

Section 3.2, the number of different ways the three-step process can be done is arrived at by multiplying these three numbers together. That is, the number of distinct arrangements of the n objects arrived at by the three step process is:

$$\binom{n}{k} \times k! \times (n-k)!.$$

number of ways to choose the k objects

number of ways to arrange these k

number of ways to arrange the rest

Moreover, it should be clear that every arrangement of the n objects can be uniquely arrived at by the above three-step process, and we already know there are $n!$ such arrangements. Thus,

$$\binom{n}{k} \times k! \times (n-k)! = n!$$

and so, dividing both sides by $k! \times (n-k)!$ yields

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

as desired. This completes the proof.

With these preliminaries at hand, we can now turn to the task of calculating the power of the president according to the two different power indices that have been introduced. In the version of the U.S. federal system we will consider, the tie-breaking role of the vice president is ignored. We consider each of the two power indices in turn.

The Shapley-Shubik Index of the President

Using the n choose k notation, it takes only a few lines to write down the arithmetic expression giving the Shapley-Shubik index of the president (and the reader who wishes to see it now can flip a few pages ahead and find it there). However, explaining where this expression came from is quite another story (and the one we want to tell). So, let's consider a

simpler version of the federal system—a "mini-federal system"—where there are only six senators and six members of the House and the president. (We choose the number six because it is the smallest positive integer for which half it and two-thirds it are also integers, and these are the fractions involved in the rules for passage.) Passage in the mini-federal system requires either two-thirds of both houses or half of each house and the president.

For the president to be pivotal in an ordering of the thirteen voters in our mini-federal system, he must be preceded by at least three members of the House and at least three members of the Senate, but by fewer than four members of at least one of the two chambers. This can happen in the following seven ways:

1. Three House members and three senators precede the president in the ordering (and, thus, three House members and three senators follow the president in the ordering).
2. Three House members and four senators precede the president in the ordering (and, thus, three House members and two senators follow the president in the ordering).
3. Three House members and five senators precede the president in the ordering (and, thus, three House members and one senator follow the president in the ordering).
4. Three House members and six senators precede the president in the ordering (and, thus, three House members and no senators follow the president in the ordering).
5. Four House members and three senators precede the president in the ordering (and, thus, two House members and three senators follow the president in the ordering).
6. Five House members and three senators precede the president in the ordering (and, thus, one House member and three senators follow the president in the ordering).
7. Six House members and three senators precede the president in the ordering (and, thus, no House members and three senators follow the president in the ordering).

We wish to count how many orderings of each of the seven kinds there are. Consider first the orderings in the first entry on the list. Each such ordering can be built in a four-step process:

Step 1: Choose three of the six House members to precede the president in the ordering. This can be done in six choose three ways.

Step 2: Choose three of the six senators to precede the president in the ordering. This can be done in six choose three ways.

Step 3: Choose an ordering of the six people from steps 1 and 2 who will precede the president. This can be done in 6! ways.

Step 4: Choose an ordering of the six people (the remaining House members and senators) who will come after the president. This can be done in 6! ways.

By the multiplication principle, we know that the total number of orderings that can be constructed by the above four-step process is:

$$\binom{6}{3}\binom{6}{3}6!6!.$$

A similar argument yields a similar expression for the number of orderings that arise in the other six entries on the list. The sum of these seven expressions gives us the total number of orderings of the thirteen voters for which the president is pivotal. Hence, to obtain the Shapley-Shubik index of the president in this mini-federal system, we simply divide that result by 13! This yields:

$$\frac{\binom{6}{3}\binom{6}{3}6!6! + 2\binom{6}{3}\binom{6}{5}7!5! + 2\binom{6}{3}\binom{6}{5}8!4! + 2\binom{6}{3}\binom{6}{6}9!3!}{13!}.$$

This evaluation can be done by hand, and we leave it for the reader. The following expression gives the numerator for the Shapley-Shubik index of the president in the U.S. federal system (with the vice president ignored). The denominator is 536 factorial. We leave it to the

reader (see Exercise 39) to provide an explanation for this expression that is analogous to what we did for the mini-federal system.

$$\binom{435}{218}\left[\binom{100}{51}(218+51)!(535-218-51)! + \cdots + \binom{100}{100}(218+100)!(535-218-100)!\right]$$
$$+ \cdots$$
$$+ \binom{435}{289}\left[\binom{100}{51}(289+51)!(535-289-51)! + \cdots + \binom{100}{100}(289+100)!(535-289-100)!\right]$$
$$+ \binom{435}{290}\left[\binom{100}{51}(290+51)!(535-290-51)! + \cdots + \binom{100}{66}(290+66)!(535-290-66)!\right]$$
$$+ \cdots$$
$$+ \binom{435}{435}\left[\binom{100}{51}(435+51)!(535-435-51)! + \cdots + \binom{100}{60}(435+66)!(435-218-66)!\right]$$

One would not want to simplify such an expression by hand. Fortunately, there are computer programs available—like *Mathematica*—which make things easy. For example, to evaluate the above expression (including the division by 536!) one simply types the following as input for *Mathematica:*

Sum[Binomial [435, h] Binomial[100, s]($s + h$)!(535 − s − h)!,
{h, 218, 289}, {s, 51, 100}]/536!+
Sum[Binomial [435, h] Binomial[100, s]($s + h$)!(535 − s − h)!,
{h, 290, 435}, {s, 51, 66}]/536!

One then simply waits (how long depends upon how fast your computer is) until *Mathematica* responds with:

12059653826881866343910432696016621164465243723811139057 6\
1757128313706451941878271101036003/
751522994006379308440322723474377630024881179500218664184\
4565883705084200048715438500735040

Finding that response a little unsettling, one types "N[%]". This instructs *Mathematica* to express the answer as a nice decimal. The output is then

$$0.16047.$$

Thus, according to the Shapley–Shubik index, the president has about 16 percent of the power in the U.S. federal system.

The Banzhaf Index of the President

The Banzhaf index of power of the president is obtained by dividing his total Banzhaf power by the sum of the Banzhaf powers of all the voters in the U.S. federal system (i.e., the president, the 100 members of the Senate, and the 435 members of the House – for simplicity, we are still ignoring the vice president). Thus, to calculate the Banzhaf index of the president, we need to determine not only his total Banzhaf power, but also that of each member of the Senate and each member of the House.

We will make these calculations of total Banzhaf power by using the second procedure in **Section 3.5**, wherein a voter's total Banzhaf power was shown to be simply twice the number of winning coalitions to which that voter belongs minus the total number of winning coalitions. A little notation will make things easier.

Let S denote the number of coalitions within the Senate that contain at least two-thirds of the members of the Senate. Thus,

$$S = \binom{100}{67} + \cdots + \binom{100}{100}.$$

Let S_p denote the number of coalitions within the Senate that contain at least two-thirds of the members of the Senate and that contain a particular senator p. Such a coalition is arrived at by choosing anywhere from 66 to 99 of the other senators. Thus,

$$S_p = \binom{99}{66} + \cdots + \binom{99}{99}.$$

Let s denote the number of coalitions within the Senate that contain at least one-half of the members of the Senate. Thus,

$$s = \binom{100}{50} + \cdots + \binom{100}{100}.$$

Let s_p denote the number of coalitions within the Senate that contain at least one-half of the members of the Senate and that contain a particular senator p. Such a coalition is arrived at by choosing anywhere from 49 to 99 of the other senators. Thus,

$$s_p = \binom{99}{49} + \cdots + \binom{99}{99}.$$

Let H denote the number of coalitions within the House that contain at least two-thirds of the members of the House. Thus,

$$H = \binom{435}{290} + \cdots + \binom{435}{435}.$$

Let H_p denote the number of coalitions within the House that contain at least two-thirds of the members of the House and that contain a particular member of the House p. Such a coalition is arrived at by choosing anywhere from 289 to 434 of the other members of the House. Thus,

$$H_p = \binom{434}{289} + \cdots + \binom{434}{434}.$$

Let h denote the number of coalitions within the House that contain at least one-half of the members of the House. Thus,

$$h = \binom{435}{218} + \cdots + \binom{435}{435}.$$

Let h_p denote the number of coalitions within the House that contain at least one-half of the members of the House and that contain a particular member of the House p. Such a coalition is arrived at by choosing anywhere from 217 to 434 of the other members of the House. Thus,

$$h_p = \binom{434}{217} + \cdots + \binom{434}{434}.$$

It is now easy to write down expressions involving $S, S_p, s, s_p, H, H_p, h_1$ and h_p (and the n choose k notation) that give us the total Banzhaf power for the president, a member of the House, and a member of the Senate. (Recall that the desired expression is simply two times the number of winning coalitions to which a voter belongs, with the total number of winning coalitions then subtracted from this.) We leave this for the reader. However, as was the case for the Shapley–Shubik index, actual calculations need be done on a computer. The results turn out to be:

BI(the president) = .038.
BI(a senator) = .0033.
BI(a member of the House) = .0015.

A more meaningful way to view these results is in terms of percentages (of power) as opposed to small decimals. It is also more meaningful to consider the power of the Senate as opposed to the power of a single senator, and to do the same for the House (assuming that power is additive—a risky assumption at best). The results then become

(Banzhaf) Power held by the president = 4%
(Banzhaf) Power held by the Senate = 33%
(Banzhaf) Power held by the House = 63%

3.7 THE CHAIR'S PARADOX

Our previous considerations of political power have focused on quantitative measures of influence over outcomes. In this section, we change gears slightly to give quite another view of this somewhat elusive concept of power. We present a classic result known as the Chair's paradox.

The primary purpose of introducing this paradox is to illustrate the extent to which apparent power need not correspond to control over outcomes.

The situation we want to consider is the following. There are three people, A, B, and C, and three alternatives, a, b, and c. The preference lists of the three people are as illustrated below (and replicate those from the voting paradox of Condorcet in Chapter 1).

A	B	C
a	b	c
b	c	a
c	a	b

The social choice procedure being used is somewhat different from those we have considered before. That is, the preference lists will not be regarded as inputs for the procedure, but will only be used to "test" the extent to which each of A, B, and C should be happy with the social choice. The social choice will be determined by a standard voting procedure where voter A (the Chair) also has a tie-breaking vote. The point of not using the preference lists themselves is that we do not want to force any one of the three players to vote for his or her top choice, although it is probably not clear at the moment why anything else would benefit any of them. In fact, voting where everyone is bound to vote for his or her top choice is called *sincere voting*. Anything else (and this is our interest) is called *sophisticated voting*.

In our present situation, a *strategy* for any of the three people A, B, or C is simply a choice for which of the three alternatives a, b, c to vote. A sequence of such votes will be called a *scenario*.

DEFINITION. Fix a player P and consider two strategies $V(x)$ and $V(y)$ for P. [Think of $V(x)$ as being "vote for alternative x."] Then we'll say that $V(x)$ weakly dominates $V(y)$ for player P provided that the following both hold:

1. For every possible scenario (i.e., choice of alternatives for which to vote by the other players), the social choice resulting from $V(x)$ is at least as good for Player P (as measured by his or her preference list) as that resulting from $V(y)$.