systems in their own right, the U.S. federal system is further complicated by the tie-breaking role of the vice president in the Senate, the veto power of the president, and the possibility of a Congressional override. Somewhat surprisingly, however, these emendations do not drive up the dimension of the U.S. federal system, as we now show.

PROPOSITION. *The U.S. federal system has dimension 2.*

PROOF. We know from Chapter 3 that the U.S. federal system is not weighted. Thus, it suffices to produce two weighted systems, with the same set of voters as the U.S. federal system, whose intersection is the U.S. federal system. The weighted systems that will do the trick are the following.

System I will give:

Weight 0 to each member of the House;
Weight 1 to each member of the Senate;
Weight $\frac{1}{2}$ to the vice president;
Weight $16\frac{1}{2}$ to the president;

and we set the quota at 67.

System II will give:

Weight 1 to each member of the House;
Weight 0 to each member of the Senate;
Weight 0 to the vice president;
Weight 72 to the president;

and we set the quota at 290.

We now want to show that a coalition is winning in the U.S. federal system if and only if it is winning in both System I and in System II. Suppose then that X is a coalition that is winning in the U.S. federal system. Without loss of generality, we can assume that X is a minimal winning coalition (Exercise 11 asks why we lose no generality with this assumption). Thus, X is one of the following three kinds of coalition:

1. X consists of 218 House Members, 51 senators, and the president;

2. X consists of 218 House Members, 50 senators, the vice president, and the president;
3. X consists of 290 House Members and 67 senators.

We leave it to the reader to verify that all three kinds of coalition achieve quota in both System I and in System II (see Exercise 12).

For the converse, assume that X is a winning coalition in both System I and in System II. We consider two cases:

Case 1: X Contains the President

Since X is winning in System I, it must have System I weight at least 67. Since the System I weight of the president is $16\frac{1}{2}$, the other members of X must contribute at least weight $50\frac{1}{2}$ to the total System I weight of X. But House members have weight 0 in System I, so X must contain either 51 (or more) senators or at least 50 senators and the vice president. Now, looking at the System II weight of X, which is at least 290 including the 72 contributed by the president, we see that X must also contain at least $290 - 72 = 218$ members of the House. Thus, in case 1, we see that X is a winning coalition in the federal system, as desired.

Case 2: X does not Contain the President

This is left to the reader (see Exercise 13), and completes the proof.

We conclude this section with the observation that we know of no real-world voting system of dimension 3 or higher.

8.4 VECTOR-WEIGHTED VOTING SYSTEMS

In our early discussions of yes–no voting systems in Chapter 2, we suggested that the observation that the U.N. Security Council is, in fact, a weighted voting system might naturally lead one to conjecture that every yes–no voting system is weighted. We now know that not to be the case, and much of what we have done in Chapter 2 and Chapter 8 has been aimed at exploring the extent to which such a system can fail to be weighted. In this section, however, we show that the intuition provided by the weightedness of the U.N. Security Council is far less naïve than it might now seem.

Generalization has always played an important role in mathematics. For example, our original number system consisted of what we now call positive integers. This system was generalized to include zero, the negative numbers, then fractions, irrationals, and imaginaries. Of course, generalization for its own sake can at least sometimes be pointless. But a natural generalization of an important concept can often shed considerable light. Our goal in this section is to provide such a generalization of the notion of a weighted voting system.

Our starting point will be the observation that one can replace the notion of a real number by one of its generalizations: an ordered pair (x, y) of real numbers. These ordered pairs can be "added" as follows:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

Thus, for example, $(2,4)+(\frac{1}{2},-1) = (\frac{5}{2},3)$. Moreover, we can "compare the size" of ordered pairs as follows:

$$(x_1, y_1) \leq (x_2, y) \text{ if and only if } x_1 \leq x_2 \text{ and } y_1 \leq y_2.$$

Now, let's return to the Canadian system (which we know is not weighted) and show that it is a "generalized weighted system." That is, instead of assigning real numbers as weights, let's assign ordered pairs as weights in the following way:

weight of Prince Edward Island will be (1, 0)
weight of Newfoundland will be (1, 2)
weight of New Brunswick will be (1, 2)
weight of Nova Scotia will be (1, 3)
weight of Manitoba will be (1, 4)
weight of Saskatchewan will be (1, 3)
weight of Alberta will be (1, 11)
weight of British Columbia will be (1, 13)
weight of Quebec will be (1, 23)
weight of Ontario will be (1, 39).

Notice that the first entry of each ordered pair is 1 and the second entry is the percentage of the Canadian population residing in that province. We shall let the ordered pair (7, 50) serve as the "quota."

Given a coalition, it now makes sense to define the weight of the coalition to be the ordered pair obtained by adding up all the ordered pair weights of the provinces in the coalition (just as we obtained the weight of a coalition in a weighted voting system by adding up the weights of all the voters in the coalition). This yields an ordered pair as "weight" for the coalition, which we can then compare (using ≤ as defined above) with the ordered pair that is the quota.

For example, if X is the coalition consisting of Manitoba, Saskatchewan, Alberta, British Columbia, and Ontario, then the "weight" of X is

$$(1,4) + (1,3) + (1,11) + (1,13) + (1,39) = (5,70).$$

If we compare (5, 70) with the quota (7, 50) we find that the weight of this coalition does not meet quota; that is, the statement

$$\text{"}(7, 50) \leq (5, 70)\text{"}$$

is not true since 7 is not less than or equal to 5.

Notice that with these definitions of "weight" and "quota," a coalition's weight meets quota if and only if it contains at least seven provinces (thus guaranteeing the first entry in its weight is at least as large as the first entry in the quota) and the combined population of the provinces in the coalition is at least half the Canadian population (thus guaranteeing that the second entry in its weight is at least as large as the second entry in the quota). Thus, a coalition meets quota if and only if it is a winning coalition in the Canadian system.

In the above discussion of the Canadian system, we used ordered pairs as the "weights" and "quota." As one might imagine, there are other examples where the weights and quota are ordered triples

$$(x, y, z)$$

that are "added" and "compared" in the obvious way. In general, if $x_1, x_2, \ldots, x_n$ are real number, then $(x_1, x_2, \ldots, x_n)$ is called an *ordered n-tuple*. Ordered n-tuples are added and compared as follows:

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n),$$

and

$$(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n)$$

if and only if

$$x_1 \leq y_1 \text{ and } \ldots \text{ and } x_n \leq y_n.$$

All of this leads to the following definition:

DEFINITION. A yes–no voting system is said to be a *vector-weighted system* if, for some positive integer n, there exists an n-tuple "weight" for each voter and an n-tuple "quota" such that a coalition is winning precisely when the sum of the vector weights of the voters in the coalition meets or exceeds quota (in the sense of comparing two n-tuples described above).

Thus, for example, we have shown that the Canadian system is a vector-weighted system. Remarkably, the following turns out to be true.

THEOREM. *Every yes–no voting system is a vector weighted system. Moreover, if a system is of dimension n, then the weights and quota can be taken to be n-tuples but not (n − 1)-tuples.*

PROOF. Suppose $\mathcal{S}$ is an arbitrary yes–no voting system for the set V of voters. By the proposition in the last section, we know that $\mathcal{S}$ has dimension n for some n. Thus, we can choose weighted yes–no voting systems $\mathcal{S}_1, \ldots, \mathcal{S}_n$ so that for every coalition X from V, we have that

X is winning in $\mathcal{S}$

if and only if

X is winning in $\mathcal{S}_1$ and $\ldots$ and X is winning in $\mathcal{S}_n$.

To keep the notation simple, let's assume that $n = 3$. [Exercise 15(b) asks the reader to redo the proof using n in place of 3.]

Let w_1 and q_1 be the weight function and quota associated with $\mathcal{S}_1$, and similarly let w_2 and q_2, and w_3 and q_3 be those for $\mathcal{S}_2$ and $\mathcal{S}_3$ respectively. Thus, if X is a coalition, then

X is winning in $\mathcal{S}_1$ and X is winning in $\mathcal{S}_2$ and X is winning in $\mathcal{S}_3$

if and only if

$w_1(X) \geq q_1$ and $w_2(X) \geq q_2$ and $w_3(X) \geq q_3$.

If v is an arbitrary voter, we can produce a 3-tuple as weight for v by using the three weights he or she is assigned in the three weighted systems $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$ as follows:

$$w(v) = (w_1(v), w_2(v), w_3(v)).$$

Moreover, we can combine the three quotas q_1, q_2, and q_3 into a 3-tuple quota in the obvious way:

$$q = (q_1, q_2, q_3).$$

We must still show that these 3-tuple weights and quota "work" in the sense that a coalition should be winning in $\mathcal{S}$ if and only if its 3-tuple weight meets or exceeds quota (in the sense of comparing 3-tuples). Again, to keep the notation simple, let's assume we have a two-voter coalition $X = \{a, b\}$. Then

$$w_1(X) = w_1(a) + w_1(b);$$

$$w_2(X) = w_2(a) + w_2(b);$$

$$w_3(X) = w_3(a) + w_3(b).$$

Now, putting this together with what we had above yields

X is winning in $\mathcal{S}$

if and only if

X is winning in $\mathcal{S}_1$ and X is winning in $\mathcal{S}_2$ and X is winning in $\mathcal{S}_3$

if and only if

$w_1(X) \geq q_1$ and $w_2(x) \geq q_2$ and $w_3(X) \geq q_3$

if and only if