of the House from the available 435. In particular, the "n choose k" expression will involve

$$\binom{99}{66} \text{ and not } \binom{100}{67}$$

With this potential pitfall confronted, the calculations proceed in a way similar to what we've done before, yielding

$$\begin{aligned}\text{TJP(A Senator)} = &\frac{1}{357}\left[\binom{99}{66} \times \binom{435}{290}\right] \\ &+ \frac{1}{67}\binom{99}{66}\left[\binom{435}{291} + \ldots + \binom{435}{435}\right] \\ &+ \frac{1}{270}\left[\binom{99}{50} \times \binom{435}{218}\right] \\ &+ \frac{1}{52}\binom{99}{50}\left[\binom{435}{219} + \cdots + \binom{435}{289}\right] \\ &+ \frac{1}{52}\binom{99}{50}\left[\binom{435}{290} + \cdots + \binom{435}{435}\right].\end{aligned}$$

Expressing the total Johnston power of a member of the House of Representatives in a similar fashion is left to the reader. (See Exercise 8 at the end of the chapter.)

To obtain the desired Johnston indices, we sum the total Johnston power of the 536 players involved, and then divide by the total. The results turn out to be:

JI(The president)	= .77
JI(A senator)	= .0016
JI(A member of the House)	= .00017.

Expressing these in terms of percentages of power instead of small decimals yields:

(Johnston) Power held by the president	= 77%
(Johnston) Power held by the Senate	= 16%
(Johnston) Power held by the House	= 7%

The striking thing to notice is the very different measure of power assigned the president by this index as opposed to those in Chapter 3. A little more on this will be said in the concluding section of this chapter, but there is no substitute for going directly to the literature.

9.3 THE DEEGAN–PACKEL INDEX OF POWER

In 1978, Deegan and Packel introduced a power index based on three assumptions:

1. Only minimal winning coalitions should be considered when determining the relative power of voters.
2. All minimal winning coalitions form with equal probability.
3. The amount or power a player derives from belonging to some minimal winning coalition is the same as that derived by any other player belonging to that same minimal winning coalition.

These assumptions, in fact, uniquely determine a power index. Moreover, the calculation required involves a nice blend of what we did with the two procedures for Banzhaf power (**Section 3.5**) and the calculation of Johnston power (**Section 9.2**). We begin with two definitions.

DEFINITION. Suppose that p is one of the voters in a yes–voting system. Then the *total Deegan–Packel power* of p, denoted here by TDPP(p), is the number arrived at as follows:

Suppose $C_1, \ldots, C_j$ are the *minimal* winning coalitions to which p belongs. Suppose n_1 is the number of voters in C_1, n_2 is the number in C_2, and so on up to n_j being the number of voters in C_j. Then

$$\text{TDPP}(p) = \frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_j}.$$

DEFINITION. Suppose that p_1 is a voter in a yes–no voting system and that the other voters are denoted by $p_2, p_3, \ldots, p_n$. Then the *Deegan-Packel index* of p_1, denoted here by $\text{DPI}(p_1)$, is the number given by

$$\text{DPI}(p_1) = \frac{\text{TDPP}(p_1)}{\text{TDPP}(p_1) + \cdots + \text{TDPP}(p_n)}.$$

Example:

Suppose again that p_1 has fifty votes, p_2 has forty-nine votes, and p_3 has one vote, with passage requiring fifty-one votes. The minimal winning coalitions (subscripted as before) are:

$$C_2 = \{p_1, p_2\}$$
$$C_3 = \{p_1, p_3\}.$$

In calculating total Deegan–Packel power, p_1 receives a contribution of $\frac{1}{2}$ from each of the two minimal winning coalitions, while p_2 and p_3 each receive such a contribution from only one of the two minimal winning coalitions. Thus:

$$\text{TDPP}(p_1) = \frac{1}{2} + \frac{1}{2} = 1$$
$$\text{TDPP}(p_2) = \frac{1}{2} + 0 = \frac{1}{2}$$
$$\text{TDPP}(p_3) = 0 + \frac{1}{2} = \frac{1}{2}$$

and

$$\text{DPI}(p_1) = \frac{1}{1 + \frac{1}{2} + \frac{1}{2}} = \frac{1}{2}$$
$$\text{DPI}(p_2) = \frac{\frac{1}{2}}{1 + \frac{1}{2} + \frac{1}{2}} = \frac{1}{4}$$
$$\text{DPI}(p_3) = \frac{\frac{1}{2}}{1 + \frac{1}{2} + \frac{1}{2}} = \frac{1}{4}.$$

For an additional illustration of how to calculate Deegan–Packel power, we will return to the European Economic Community as set

up in 1958. Notice that the chart we use now only includes the *minimal* winning coalitions in the left hand column.

	F	G	I	B	N	L
FGI	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$			
FGBN	$\frac{1}{4}$	$\frac{1}{4}$		$\frac{1}{4}$	$\frac{1}{4}$	
FIBN	$\frac{1}{4}$		$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	
GIBN		$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	
TDPP	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{3}{4}$	$\frac{3}{4}$	0
DPI	$\frac{5}{24}$	$\frac{5}{24}$	$\frac{5}{24}$	$\frac{3}{16}$	$\frac{3}{16}$	0

Summarizing these results as we did for the previous power indices yields the following:

Country	Votes	Percentage of votes	DPI	Percentage of power
France	4	23.5	$\frac{5}{24}$	20.8
Germany	4	23.5	$\frac{5}{24}$	20.8
Italy	4	23.5	$\frac{5}{24}$	20.8
Belgium	2	11.8	$\frac{3}{16}$	18.8
Netherlands	2	11.8	$\frac{3}{16}$	18.8
Luxembourg	1	5.9	0	0

The Deegan–Packel Index of the President

Any minimal winning coalition in the federal system that contains the president can be constructed by first choosing 51 members of the Senate and then choosing 218 members of the House. Hence, the total number of such minimal winning coalitions is

$$A = \binom{100}{51}\binom{435}{218}.$$

Similarly the total number of minimal winning coalitions that do not contain the president is given by

$$B = \binom{100}{67}\binom{435}{290}.$$

Note that every minimal winning coalition of the first type contains 270 voters (and so will contribute $\frac{1}{270}$ to the total Deegan–Packel power of each of its members), and every minimal winning coalition of the second type contains 357 voters (and so will contribute $\frac{1}{357}$ to the total Deegan–Packel power of each of its members).

It follows from the above that the number of minimal winning coalitions is $A + B$ (and so we will be dividing by $A + B$ in passing from total Deegan–Packel power to the Deegan–Packel index of each player). First, however, we note that we immediately have the following:

$$\text{TDPP(president)} = \frac{A}{270}.$$

It also turns out (see Exercise 14) that

$$\text{TDPP(A senator)} = \frac{1}{357}\binom{99}{66}\binom{435}{290} + \frac{1}{270}\binom{99}{50}\binom{435}{218}$$

and

$$\text{TDPP(A House member)} = \frac{1}{357}\binom{100}{67}\binom{434}{289} + \frac{1}{270}\binom{100}{51}\binom{434}{217}.$$

Dividing each of these expressions by $A+B$ (and using *Mathematica* to do the calculations) yields:

DPI(the president) = .0037
DPI(a senator) = .0019
DPI(a member of the House) = .0019

Again expressing these in terms of percentage of power instead of small decimals, we have:

(Deegan–Packel) Power held by the president = .4%
(Deegan–Packel) Power held by the Senate = 18.9%
(Deegan–Packel) Power held by the House = 80.7%

For more on the Deegan–Packel index and the U.S. federal system, see Packel (1981).

9.4 ORDINAL POWER: INCOMPARABILITY

As we did in Chapter 3, we will assume throughout this section that "yes–no voting system" means "*monotone* yes–no voting system." Thus, winning coalitions remain winning if new voters join them.

Suppose we have a yes–no voting system (and, again, not necessarily a weighted one) and two voters whom we shall call x and y. Our starting point will be an attempt to formalize (that is, to give a rigorous mathematical definition for) the intuitive notion that underlies expressions such as the following:

"x and y have equal power"
"x and y have the same amount of influence"
"x and y are equally desirable in terms of the formation of a winning coalition"

The third phrase is most suggestive of where we are heading and, in fact, the thing we are leading up to is widely referred to as the "desirability relation on individuals" (although we could equally well call it the "power ordering on individuals" or the "influence ordering on individuals"). We shall begin with an attempt to formalize the notion of x and y having "equal influence" or being "equally desirable."

If we think of the desirability of x and of y to a coalition Z, then there are four types of coalitions to consider:

1. x and y both belong to Z.
2. x belongs to Z but y does not.
3. y belongs to Z but x does not.
4. Neither x nor y belongs to Z.

If x and y are equally desirable (to the voters in Z, who want the coalition Z to be a winning one), then for each of the four situations described above, we have a statement that should be true: