Similarly the total number of minimal winning coalitions that do not contain the president is given by

$$B = \binom{100}{67}\binom{435}{290}.$$

Note that every minimal winning coalition of the first type contains 270 voters (and so will contribute $\frac{1}{270}$ to the total Deegan–Packel power of each of its members), and every minimal winning coalition of the second type contains 357 voters (and so will contribute $\frac{1}{357}$ to the total Deegan–Packel power of each of its members).

It follows from the above that the number of minimal winning coalitions is $A + B$ (and so we will be dividing by $A + B$ in passing from total Deegan–Packel power to the Deegan–Packel index of each player). First, however, we note that we immediately have the following:

$$\text{TDPP(president)} = \frac{A}{270}.$$

It also turns out (see Exercise 14) that

$$\text{TDPP(A senator)} = \frac{1}{357}\binom{99}{66}\binom{435}{290} + \frac{1}{270}\binom{99}{50}\binom{435}{218}$$

and

$$\text{TDPP(A House member)} = \frac{1}{357}\binom{100}{67}\binom{434}{289} + \frac{1}{270}\binom{100}{51}\binom{434}{217}.$$

Dividing each of these expressions by $A+B$ (and using *Mathematica* to do the calculations) yields:

$$\begin{aligned}
&\text{DPI(the president)} &&= .0037\\
&\text{DPI(a senator)} &&= .0019\\
&\text{DPI(a member of the House)} &&= .0019
\end{aligned}$$

Again expressing these in terms of percentage of power instead of small decimals, we have:

(Deegan–Packel) Power held by the president = .4%
(Deegan–Packel) Power held by the Senate = 18.9%
(Deegan–Packel) Power held by the House = 80.7%

For more on the Deegan–Packel index and the U.S. federal system, see Packel (1981).

9.4 ORDINAL POWER: INCOMPARABILITY

As we did in Chapter 3, we will assume throughout this section that "yes–no voting system" means "*monotone* yes–no voting system." Thus, winning coalitions remain winning if new voters join them.

Suppose we have a yes–no voting system (and, again, not necessarily a weighted one) and two voters whom we shall call x and y. Our starting point will be an attempt to formalize (that is, to give a rigorous mathematical definition for) the intuitive notion that underlies expressions such as the following:

"x and y have equal power"
"x and y have the same amount of influence"
"x and y are equally desirable in terms of the formation of a winning coalition"

The third phrase is most suggestive of where we are heading and, in fact, the thing we are leading up to is widely referred to as the "desirability relation on individuals" (although we could equally well call it the "power ordering on individuals" or the "influence ordering on individuals"). We shall begin with an attempt to formalize the notion of x and y having "equal influence" or being "equally desirable."

If we think of the desirability of x and of y to a coalition Z, then there are four types of coalitions to consider:

1. x and y both belong to Z.
2. x belongs to Z but y does not.
3. y belongs to Z but x does not.
4. Neither x nor y belongs to Z.

If x and y are equally desirable (to the voters in Z, who want the coalition Z to be a winning one), then for each of the four situations described above, we have a statement that should be true:

1. If Z is a winning coalition, then x's defection from Z should render it losing if and only if y's defection from Z renders it losing.

2. If x leaves Z and y joins Z, then Z should go neither from being winning to being losing nor from being losing to being winning.

3. If y leaves Z and x joins Z, then Z should go neither from being winning to being losing nor from being losing to being winning.

4. x's joining Z makes Z winning if and only if y's joining Z makes Z winning.

In fact, it turns out that condition 4 is strong enough to imply the other three (see Exercises 11 and 12). This leads to the following definition:

DEFINITION. Suppose x and y are two voters in a yes–no voting system. Then we shall say that x and y are *equally desirable* (or, the desirability of x and y is equal, or the same), denoted $x \approx y$, if and only if the following holds:

For every coalition Z containing neither x nor y,
the result of x joining Z is a winning coalition
if and only if
the result of y joining Z is a winning coalition.

For brevity, we shall sometimes just say: "x and y are equivalent" when $x \approx y$.

Example:

Consider again the weighted voting system with three players a, b, and c who have weights 1, 49, and 50, respectively, and with quota $q = 51$. Then the winning coalitions are $\{a,c\}$, $\{b,c\}$, and $\{a,b,c\}$. Notice that $a \approx b$: the only coalitions containing neither a nor b are the empty coalition (call it Z_1) and the coalition consisting of c alone (call it Z_2). The result of a joining Z_1 is the same as the result of b joining Z_1 (a losing coalition) and the result of a joining Z_2 is the same as the result of b joining Z_2 (a winning coalition). On the other hand, a and c are not equivalent, since neither belongs to $Z = \{b\}$, but a joining Z yields $\{a,b\}$ which is losing with 50 votes, while c joining Z yields $\{b,c\}$ which is winning with 51 votes.

This example shows that in a weighted voting system, two voters with very different weights can be equivalent and, thus (intuitively) have the same "power" or "influence."

The relation of "equal desirability" defined above will be further explored in **Section 9.5**. For now, however, we turn our attention to the question of when two voters not only fail to have equal influence, but when it makes sense to say that their influence is "incomparable." What should this mean? Mimicking what we did for the notion of equal desirability, let's say that x and y are incomparable if one coalition Z desires x more than y, and another coalition Z' desires y more than x. Formalizing this yields:

DEFINITION. For two voters x and y in a yes–no voting system, we say that the *desirability of x and y is incomparable*, denoted

$$x \, \mathrm{I} \, y$$

if and only if there are coalitions Z and Z', neither one of which contains x or y, such that the following hold:

1. the result of x joining Z is a winning coalition, but the result of y joining Z is a losing coalition, and

2. the result of y joining Z' is a winning coalition, but the result of x joining Z' is a losing coalition.

For brevity, we shall sometimes just say "x and y are incomparable" when $x \, \mathrm{I} \, y$.

Example:

In the U.S. federal system, let x be a member of the House and let y be a member of the Senate. Then $x \, \mathrm{I} \, y$ (see Exercise 14). On the other hand if x is the vice president and y is a member of the Senate, then x and y are not incomparable (see Exercise 15).

The following proposition characterizes exactly which yes–no voting systems will have incomparable voters. Recall from **Section 2.4** that a yes–no voting system is swap robust if a one-for-one exchange of players between two winning coalitions always leaves at least one of the two coalitions winning.

PROPOSITION. *For any yes–no voting system, the following are equivalent:*

1. *There exist voters x and y whose desirability is incomparable.*
2. *The system fails to be swap robust.*

PROOF. (1 implies 2): Assume that the desirability of x and y is incomparable, and let Z and Z' be coalitions such that:

Z with x added is winning;
Z with y added is losing;
Z' with y added is winning; and
Z' with x added is losing.

To see that the system is not swap robust, let X be the result of adding x to the coalition Z, and let Y be the result of adding y to the coalition Z'. Both X and Y are winning, but the one-for-one swap of x for y renders both coalitions losing.

(2 implies 1): Assume the system is not swap robust. Then we can choose winning coalitions X and Y with x in X but not in Y, and y in Y but not in X, such that both coalitions become losing if x is swapped for y. Let Z be the result of deleting x from the coalition X, and let Z' be the result of deleting y from the coalition Y. Then

Z with x added is X, and this is winning;
Z with y added is losing;
Z' with y added is Y, and this is winning; and
Z' with x added is losing.

This shows that the desirability of x and y is incomparable and completes the proof.

COROLLARY. *In a weighted voting system, there are never voters whose desirability is incomparable.*

PROOF. In **Section 2.4**, we showed that a weighted voting system is always swap robust.

The question of what one can say about voters x and y whose desirability is neither equal nor incomparable is taken up next, but, in the meantime, the reader can try Exercise 17.

9.5 ORDINAL POWER: COMPARABILITY

The emphasis in **Section 9.4** was on formalizing the idea of what it means to say that two voters in a yes–no voting system have incomparable power. Here, however, we switch our emphasis to the question of how we can use ordinal notions to formalize the idea of two voters having comparable power.

The binary relation of "equal desirability" (**Section 9.4**) turns out to be what is called an *equivalence relation*. This means that the relation is reflexive, symmetric, and transitive. These notions are defined in the course of recording the following proposition:

PROPOSITION. *The relation of equal desirability is an equivalence relation on the set of voters in a yes–no voting system. That is, the following all hold:*

1. *The relation is reflexive: if $x = y$ (that is, if x and y are literally the same voter), then x and y are equally desirable.*
2. *The relation is symmetric: if x and y are equally desirable, then y and x are equally desirable.*
3. *The relation is transitive: if x and y are equally desirable and y and z are equally desirable, then x and z are equally desirable.*

REMARK. The reader should avoid letting our use of the phrase *equally desirable* lull him or her into thinking that the theorem is obvious. The only thing that is obvious is that if the theorem could not be rigorously established using the precise formal definition of *equal desirability* that we gave, then we would have been way out of line in choosing this phrase (loaded as it is with intuition) for the mathematical notion presented in the previous definition.

PROOF. We leave 1 and 2 to the reader (see Exercise 15). For 3, assume that x and y are equally desirable and that y and z are equally desirable. We want to show that x and z are equally desirable. Assume then that Z is an arbitrary coalition containing neither x nor z. We must show that the result of x joining Z is a winning coalition if and only if the result of z joining Z is a winning coalition. We consider two cases:

Case 1: y Does not Belong to Z

Since $x \approx y$ and neither x nor y belongs to Z, we know that the result of x joining Z is a winning coalition if and only if the result of y joining Z is a winning coalition. But now, since $y \approx z$ and neither y nor z belongs to Z, we know that the result of y joining Z is a winning coalition if and only if the result of z joining Z is a winning coalition. Thus, the result of x joining Z is a winning coalition if and only if the result of z joining Z is a winning coalition, as desired.

Case 2: y Belongs to Z

This case is quite a bit more difficult than the last one, and the reader should expect to spend several minutes checking to see that each line of the proof follows from previous lines.

We will make use of some set-theoretic notation in what follows. Suppose C is a coalition and v is a voter. Then

1. $C \cup \{v\}$ denotes the coalition resulting from v joining C. Typically, this is used when v does not already belong to C. If v does belong to C, then $C \cup \{v\}$ is the same as C.
2. $C - \{v\}$ denotes the coalition resulting from v leaving C. Typically, this is used when v belongs to C. If v does not belong to C, then $C - \{v\}$ is the same as C.

With this notation at hand, we can proceed with case 2.

Let A denote the coalition resulting from y leaving Z. Thus

$$A = Z - \{y\}$$

and so

$$Z = A \cup \{y\}.$$

Assume that $Z \cup \{x\}$ is a winning coalition. We want to show that $Z \cup \{z\}$ is also a winning coalition. Now,

$$Z \cup \{x\} = A \cup \{y\} \cup \{x\} = A \cup \{x\} \cup \{y\}.$$

Let $Z' = A \cup \{x\}$. Thus $Z' \cup \{y\}$ is a winning coalition. Since $y \approx z$ and neither y nor z belongs to Z', we know that $Z' \cup \{z\}$ is also a winning coalition. But $Z' \cup \{z\} = A \cup \{x\} \cup \{z\} = A \cup \{z\} \cup \{x\}$. Let $Z'' = A \cup \{z\}$. Thus $Z'' \cup \{x\}$ is a winning coalition. Since $x \approx y$ and neither x nor y belongs to Z'', we know that $Z'' \cup \{y\}$ is also a winning coalition. But $Z'' \cup \{y\} = A \cup \{z\} \cup \{y\} = A \cup \{y\} \cup \{z\} = Z \cup \{z\}$. Thus, $Z \cup \{z\}$ is a winning coalition as desired.

A completely analogous argument would show that if $Z \cup \{z\}$ is a winning coalition, then so is $Z \cup \{x\}$. This completes the proof.

For weighted voting systems, a naïve intuition would suggest that x and y are equally desirable precisely when they have the same weight. The problem with this intuition is that a given weighted voting system can be equipped with weights in many different ways. For example, consider the yes–no voting system corresponding to majority rule among three voters. This is a weighted voting system, as can be seen by assigning each of the voters weight 1 and setting the quota at 2. But the same yes–no voting system is realized by assigning the voters weights 1, 100, and 100, and setting the quota at 101. Notice that all three voters have the same weight in one of the weighted systems, but not in the other.

The above intuition, however, is not that far off. In fact, for a weighted voting system, we can characterize exactly when two voters are equally desirable as follows:

PROPOSITION. *For any two voters x and y in a weighted voting system, the following are equivalent:*

1. *x and y are equally desirable.*
2. *There exists an assignment of weights to the voters and a quota that realize the system and that give x and y the same weight.*
3. *There are two different ways to assign weights to the voters and two (perhaps equal) quotas such that both realize the system, but in one of the two weightings, x has more weight than y and, in the other weighting, y has more weight than x.*

PROOF. We first prove that 1 implies 2. Assume that x and y are equally desirable and choose any weighting and quota that realize the system. Let $w(x)$, $w(y)$ and q denote (respectively) the weight of x, the weight of y, and the quota. Expressions like "$w(Z)$" will represent the total weight of the coalition Z. We now construct a new weighting (where we will use nw for "new weight" in place of w for "weight") such that, with the same quota q, this new weighting also realizes the system and $nw(x) = nw(y)$.

The new weighting is obtained by keeping the weight of every voter except x and y the same, and setting both $nw(x)$ and $nw(y)$ equal to the average of $w(x)$ and $w(y)$.

To see that this new weighting still realizes the same system, assume that Z is a coalition. We must show that Z is winning in the new weighting if and only if Z is winning in the old weighting. We consider three cases:

Case 1: Neither x nor y Belongs to Z

In this case, $w(Z) = nw(Z)$ and so $nw(Z) \geq q$ if and only if Z is winning.

Case 2: Both x and y Belong to Z

We leave this for the reader.

Case 3: x Belongs to Z but y Does not Belong to Z

In this case, the new weight of Z is the average of the old weight of Z and the old weight of $Z - \{x\} \cup \{y\}$. That is:

$$nw(Z) = \frac{w(Z) + w(Z - \{x\} \cup \{y\})}{2}$$

Since x and y are equally desirable, either both Z and $Z - \{x\} \cup \{y\}$ are winning or both are losing. If both are winning, then

$$nw(Z) \geq \frac{q+q}{2} = q.$$

If both are losing, then

$$nw(Z) < \frac{q+q}{2} = q.$$

This completes the proof that 1 implies 2.

We now prove that 2 implies 3. Assume that we start with a weighting and quota wherein x and y have the same weight. Let HL denote the weight of the heaviest losing coalition, and let LW denote the weight of the lightest winning coalition. Thus,

$$HL < q \leq LW.$$

Let q' be the average of HL and q. Then q' still works as a quota and

$$HL < q' < LW.$$

Let ϵ be any positive number that is small enough so that

$$HL + \epsilon < q' < LW - \epsilon.$$

We leave it for the reader to check that the system is unchanged if we either increase the weight of x by ϵ or decrease the weight of x by ϵ. This shows that there are two weightings that realize the system, one of which makes x heavier than y and the other of which makes y heavier than x.

Finally, we prove that 3 implies 1. Assume that we have two weightings, w and w', and two quotas, q and q', such that

1. *A coalition Z is winning if and only if $w(Z) \geq q$.*
2. *A coalition Z is winning if and only if $w'(Z) \geq q'$.*

3. $w(x) > w(y)$.

4. $w'(y) > w'(x)$.

To show that x and y are equally desirable, we must start with an arbitrary coalition Z containing neither x nor y and show that $Z \cup \{x\}$ is winning if and only if $Z \cup \{y\}$ is winning. This argument is asked for in Exercise 20. Given this, the proof is complete.

Finally, what can we say about voters x and y whose desirability is neither equal nor incomparable? Looking back at the definition, we see that this happens only if (intuitively) some coalition desires one more than the other, but no coalition desires the other more than this one. Formally:

DEFINITION. For any two voters x and y in a yes–no voting system, we say that *x is more desirable than y*, denoted

$$x > y,$$

if and only if the following hold:

1. for every coalition Z containing neither x nor y, if $Z \cup \{y\}$ is winning then so is $Z \cup \{x\}$, and

2. there exists a coalition Z' containing neither x nor y such that $Z' \cup \{x\}$ is winning, but $Z' \cup \{y\}$ is losing.

We shall also write $x \geq y$ to mean that either $x > y$ or $x \approx y$. (This is analogous to what is done with numbers.) The relation $\geq$ is known in the literature as the *desirability relation on individuals*.

Example:

In the U.S. federal system if x is a senator and y is the vice president, then $x > y$ (see Exercise 21).

The binary relation $\geq$ is called a *preordering* because it is transitive and reflexive. A preordering is said to be *linear* if for every x and y one has either $x \geq y$ or $y \geq x$. Linear preorders are also called *weak orderings* in the literature. With this, we conclude the present discussion with one more definition and one more proposition.

DEFINITION. A yes–no voting system is said to be *linear* if there are no incomparable voters (equivalently, if the desirability relation on individuals is a linear preordering).

PROPOSITION. *A yes–no voting system is linear if and only if it is swap robust.*

COROLLARY. *Every weighted voting system is linear.*

For proofs of these, see Exercise 22.

Finally, for weighted voting systems, we have the following very nice characterization of the desirability relation on individuals.

PROPOSITION. *In a weighted voting system we have $x > y$ if and only if x has strictly more weight than y in every weighting that realizes the system.*

A proof of this (which is quite short, given what we did earlier in this section) is asked for in Exercise 23.

9.6 A THEOREM ON VOTING BLOCS

This section considers a situation that reduces to a kind of weighted voting body that is sufficiently simple so that we can prove a general theorem, taken from Straffin (1980), that allows us to calculate the Shapley–Shubik indices of the players involved in an easy way. We begin with some notation and an example.

NOTATION. Suppose we have a weighted voting system with n players $p_1, \ldots, p_n$ with weights $w_1, \ldots, w_n$ (so, w_1 is the weight of player p_1, w_2 of p_2, etc.) Suppose that q is the quota. Then all of this is denoted by:

$$[q : w_1, w_2, \ldots, w_n].$$