The equivalence of differential graded modules and HZ-module spectra, applications, and generalizations

Brooke Shipley

Notes by David Mehrle
dmehrle@math.cornell.edu

Young Topologists Meeting
July 3-7, 2017

Last updated July 27, 2017.
The latest version is online here.

Abstract

This document consists of lecture notes for four lectures given by Brooke Shipley at the 2017 Young Topologists Meeting at the Swedish Royal Institute of Technology (KTH) in Stockholm, Sweden. The original conference abstract for the lecture series is below.

> This sequence of lectures will explore the connections between the differential graded world and the spectral world. There will first be a brief introduction to model categories, stable homotopy, and symmetric spectra. Then we will discuss the equivalence between the homotopy theories of $H\mathbb{Z}$-module (respectively, algebra) spectra and of differential graded modules (respectively, algebras or DGAs), where $H\mathbb{Z}$ here is the Eilenberg-Mac Lane spectrum associated with ordinary homology. We will then use this comparison to develop algebraic models of rational (equivariant) stable homotopy theories and to define topological equivalences of DGAs. In both of these applications we will discuss current on-going work; in the latter case, this uses a variant of Goerss-Hopkins obstruction theory to compute topological equivalences. As time permits we will also discuss extensions of the original comparison to the commutative (or E-infinity) case and to co-modules and co-algebras.

These notes were typed post-mortem from my original handwritten notes, and I am certain that the transcription introduced errors. Additionally, I have attempted to cite original sources for ease of reference, but I may have accidentally left some out. Please let me know if you find any errors or omissions by sending me an email at dmehrle@math.cornell.edu.

Contents

1 The Dold-Kan correspondence

1.1 Simplicial abelian groups

First, recall the idea of a simplicial set.

Definition 1.1. A **simplicial set** is a sequence of sets $X = (X_n)_{n\in\mathbb{N}}$ together with **face maps**

$$d_i \colon X_n \to X_{n-1} \qquad 0 \le i \le n$$

and **degeneracy maps**

$$s_i \colon X_n \to X_{n+1} \qquad 0 \le i \le n$$

satisfying the relations

$$\begin{aligned} d_i d_j &= d_{j-1} d_i & i < j \\ s_i s_j &= s_{j+1} s_i & i \le j \\ d_i s_j &= \begin{cases} s_{j-1} d_i & i < j \\ \mathrm{id} & i = j, j+1 \\ s_j d_{i-1} & i > j+1 \end{cases} \end{aligned}$$

An element $x \in X_n$ is called an n**-simplex**. If $x = s_i(y)$ for some i, then x is called **degenerate**.

Example 1.2. Let $\Delta[0]$ be the simplicial set with $\Delta[0]_n = \{*\}$ for all n. This simplicial set represents a single point.

Example 1.3. Let $\Delta[1]$ be the simplicial set with

$$\begin{aligned} \Delta[1]_0 &= \{0, 1\} \\ \Delta[1]_1 &= \{(0,0), (0,1), (1,1)\} \\ \Delta[1]_2 &= \{(0,0,0), (0,0,1), (0,1,1), (1,1,1)\} \\ &\vdots \end{aligned}$$

This simplicial set represents an interval, with the cell structure of two 0-cells joined by a 1-cell.

In general, there is a simplicial set $\Delta[n]$ for any $n \in \mathbb{N}$, with one non-degenerate n-simplex, two non-degenerate $(n-1)$-simplicies, three non-degenerate $n-2$-simplicies, etc. The notation is supposed to suggest the geometric simplicies: a point, an interval, a triangle, a tetrahedron, etc.

Exercise 1.4. Show that $\Delta[1] \times \Delta[1]$ has exactly two non-degenerate 2-simplicies.

Another, abstract definition of simplicial sets is the following.

Definition 1.5. Let Δ be the category of finite linearly ordered sets and order preserving maps. The **category of simplicial sets** is the category of contravariant functors from Δ to **Set**:

$$\mathbf{sSet} = \mathrm{Fun}(\Delta^{op}, \mathbf{Set}).$$

This definition is easily modified to define simplicial objects in any category. In particular, we will rely on the idea of a simplicial abelian group.

Definition 1.6. A **simplicial abelian group** is a sequence $X = (X_n)_{n\in\mathbb{N}}$ of abelian groups with face and degeneracy maps (as in Definition 1.1) that are homomorphisms.

Example 1.7. Given any simplicial set $Y = (Y_n)_{n\in\mathbb{N}}$, we may define a **free simplicial abelian group** $\mathbb{Z}Y$ on Y where $(\mathbb{Z}Y)_n$ is the free abelian group on Y_n. This gives a functor $\mathbf{sSet} \to \mathbf{sAb}$.

Likewise, for a pointed simplicial set Y, we may define a simplicial abelian group $\widetilde{\mathbb{Z}}Y$, where $(\widetilde{\mathbb{Z}}Y)_n$ is the free abelian group on $Y_n \setminus \{*\}$. This gives a functor $\mathbf{sSet}_* \to \mathbf{sAb}$.

Remark 1.8. The reason for the choice of the name $\widetilde{\mathbb{Z}}$ for the functor $\mathbf{sSet}_* \to \mathbf{sAb}_*$ is Corollary 1.24, which relates the n-th homotopy group of $\widetilde{\mathbb{Z}}Y$ to the n-th reduced homology group of $N\widetilde{\mathbb{Z}}Y$. This is why we add a tilde.

1.2 Chain complexes from simplicial abelian groups

Definition 1.9. Let $\mathbf{Ch}(A)$ be the category of chain complexes of A-modules.

Let $\mathbf{Ch}_{\geq 0}(A)$ be the full subcategory of $\mathbf{Ch}(A)$ consisting of those chain complexes which vanish in negative degree. Likewise, let $\mathbf{Ch}_{\leq 0}(A)$ be the full subcategory of $\mathbf{Ch}(A)$ consisting of those chain complexes which vanish in positive degree.

Notice that a simplicial abelian group $A = (A_n)_{n\in\mathbb{N}}$ looks a lot like a chain complex, but instead of a single differential, there many maps $d_i \colon A_n \to A_{n-1}$. Nevertheless, we may exploit the relations between the degeneracies and face maps to define a functor from simplicial abelian groups to the category of nonnegatively graded chain complexes of $\mathbb{Z}$-modules.

Definition 1.10. Let $C\colon \mathbf{sAb} \to \mathbf{Ch}_{\geq 0}(\mathbb{Z})$ be the functor

$$A \mapsto CA = \left[\cdots \to A_n \xrightarrow{\partial_n} A_{n-1} \to \cdots \to A_2 \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0\right]$$

with differential

$$\partial_n = \sum_{i=0}^{n} d_i.$$

On morphisms of chain complexes, C is the identity.

We have $\partial^2 = 0$ because of the identity $d_i d_j = d_{j-1} d_i$.

Definition 1.11. Let $(DA)_n$ be the subgroup of $(CA)_n$ generated by the image of degenerate simplicies. That is,

$$(DA)_n = \left\{ a \in A_n \;\middle|\; a = s_i(a') \text{ for some } a' \in A_{n-1} \text{ and } 0 \le i \le n-1 \right\}.$$

Then DA is the chain complex of abelian groups with n-th group $(DA)_n$ and differential inherited from CA.

Definition 1.12. Define the **normalized chain complex** associated to a simplicial group $A = (A_n)_{n \in \mathbb{N}}$ by

$$N(A) = {}^{C(A)}\!/_{D(A)}.$$

This defines a functor $N\colon \mathbf{sAb} \to \mathbf{Ch}_{\geq 0}(\mathbb{Z})$.

Example 1.13. Recall the simplicial set $\Delta[0]$, which has one simplex in each dimension. Then $\mathbb{Z}\Delta[0]$ is the simplicial abelian group with a $\mathbb{Z}$ in each dimension, and so $C(\mathbb{Z}\Delta[0])$ is the chain complex with $\mathbb{Z}$ in each nonnegative dimension; the differential ∂_n alternates between the identity when n is odd and the zero map when n is even.

The only nondegenerate simplex is in degree zero, so $(D\mathbb{Z}\Delta[0])_n = \mathbb{Z}$ unless $n = 0$, in which case $(D\mathbb{Z}\Delta[0])_0 = 0$. Therefore,

$$N(\mathbb{Z}\Delta[0])_n = \begin{cases} \mathbb{Z} & n = 0, \\ 0 & n \neq 0. \end{cases}$$

In words, $N(\mathbb{Z}\Delta[0])$ is the chain complex with $\mathbb{Z}$ concentrated in degree zero.

Example 1.14.

$$N(\mathbb{Z}\Delta[1]) = \left[\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{(-1,1)} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots \right]$$

$$N(\widetilde{\mathbb{Z}}\Delta[1]) = \left[\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{\text{id}} \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots \right]$$

Definition 1.15. Let $f\colon A \to B$ be a morphism of chain complexes. We say that f is a **quasi-isomorphism** if A and B have the same homology: $H_*(A) \cong H_*(B)$.

Proposition 1.16. *The chain complex* DA *is quasi-isomorphic to zero.*

Proof. We must show that $H_*(DA) \cong H_*0 = 0$. The differential of DA is inherited from the differential of CA, but all elements of $(DA)_n$ are degenerate n-simplicies, that is, they lie in the image of some $s_i\colon A_{n-1} \to A_n$. One may check that by the relations between face maps and degeneracy maps, the differential $\partial\colon (DA)_n \to (DA)_{n-1}$ is always zero. Hence, the homology of $(DA)_n$ vanishes. □

Definition 1.17. If a chain complex has zero homology, then we say it is **acyclic**.

Corollary 1.18. *The chain complexes* CA *and* NA *are quasi-isomorphic.*

Proof. Consider the short exact sequence of chain complexes:

$$0 \longrightarrow DA \longrightarrow CA \longrightarrow NA \longrightarrow 0.$$

This induces the usual long exact sequence on homology, but $H_*(DA) = 0$. Hence, we get isomorphisms $H_*(CA) \cong H_*(NA)$, induced by the quotient map $CA \to NA$. □

1.3 Simplicial abelian groups from chain complexes

In the previous section, we saw how to build a chain complex from a simplicial abelian group. We can also go the other way, building a simplicial abelian group given a chain complex. This is however a bit more involved.

Since the Dold-Kan correspondence states that the categories **sAb** and $\mathbf{Ch}_{\geq 0}(\mathbb{Z})$ are equivalent via the functor N. Hence, we know *a posteriori* that N has a quasi-inverse, which we will call Γ. As a quasi-inverse to N, Γ is right (and left) adjoint to N. We will construct Γ so that it is indeed a right adjoint.

Definition 1.19. Define $\Gamma\colon \mathbf{Ch}_{\geq 0}(\mathbb{Z}) \to \mathbf{sAb}$ by

$$(\Gamma C)_n = \mathbf{Ch}_{\geq 0}(N\mathbb{Z}\Delta[n], C).$$

Proposition 1.20. Γ *is right adjoint to* N.

Proof. Notice that, for any simplicial abelian group $A = (A_n)_{n\in\mathbb{N}}$,

$$\mathbf{sAb}(\mathbb{Z}\Delta[n], A) \cong A_n.$$

This is a consequence of the Yoneda lemma. Therefore, we have the required isomorphism

$$\mathbf{Ch}_{\geq 0}(N\mathbb{Z}\Delta[n], C) = (\Gamma C)_n \cong \mathbf{sAb}(\mathbb{Z}\Delta[n], \Gamma C)$$

□

Example 1.21. For any chain complex C concentrated in nonnegative degree,

$$(\Gamma C)_1 = C_1 \oplus C_0.$$

Remark 1.22. This is a rather contrived definition of $\Gamma\colon \mathbf{Ch}_{\geq 0}(\mathbb{Z}) \to \mathbf{sAb}$, which we use out of laziness – it makes the proofs easier. Another, more concrete definition is offered in [GJ09, III.2]:

$$(\Gamma C)_n = \bigoplus_{[n]\twoheadrightarrow[k]} C_k$$

where the sum is taken over all epimorphisms from $[n]$ to $[k]$ in the category Δ. The simplicial structure maps are determined from the maps between the objects $[n]$ and $[n \pm 1]$ of Δ.

Theorem 1.23 (Dold-Kan)**.** $N\colon \mathbf{sAb} \to \mathbf{Ch}_{\geq 0}(\mathbb{Z})$ *induces an equivalence of categories.*

Corollary 1.24. *Suppose that* A *is a simplicial abelian group. Then there are natural isomorphisms*

$$\pi_n(A, 0) \cong H_n(NA).$$

1.4 Monoidal structures on sAb and $\mathbf{Ch}_{\geq 0}(\mathbb{Z})$

Definition 1.25. The **tensor product** of two chain complexes C and D is

$$(C \otimes D)_n = \bigoplus_{p+q=n} C_p \otimes D_q$$

The monoidal unit for this tensor product is the chain complex $\mathbb{Z}$ concentrated in degree zero.

Definition 1.26. The **tensor product** of two simiplicial abelian groups A and B is

$$(A \otimes B)_n = A_n \otimes B_n,$$

with monoidal unit $\mathbb{Z}\Delta[0]$.

Example 1.27. Let S^1 be the chain complex which is $\mathbb{Z}$ concentrated in degree one. Then in $\mathbf{Ch}_{\geq 0}(\mathbb{Z})$, $S^1 \otimes S^1 = S^2$, which is $\mathbb{Z}$ concentrated in degree two. Therefore,

$$(N\Gamma S^1 \otimes_{\mathbf{Ch}_{\geq 0}(\mathbb{Z})} N\Gamma S^1)_1 \cong 0$$

But, $(\Gamma S^1)_1 = 1$, so

$$(\Gamma S^1 \otimes_{\mathbf{sAb}} \Gamma S^1)_1 \cong \mathbb{Z}$$

Hence,

$$(N\Gamma S^1 \otimes_{\mathbf{Ch}_{\geq 0}(\mathbb{Z})} N\Gamma S^1)_1 \not\cong N(\Gamma S^1 \otimes_{\mathbf{sAb}} \Gamma S^1)_1$$

Therefore, N cannot be a monoidal functor!

Similarly, we have $\Gamma S^1 \otimes \Gamma S^1 \not\cong \Gamma(S^1 \otimes S^1)$. This again can be seen by checking the value in degree one.

Corollary 1.28. $(\mathbf{sAb}, \otimes)$ *and* $(\mathbf{Ch}_{\geq 0}(\mathbb{Z}), \otimes)$ *are not equivalent as monoidal categories.*

Even though simplicial abelian groups and chain complexes are not equivalent as monoidal categories, we can still hope for the next best thing. Namely, we may wonder if they are monoidally equivalent up to quasi-isomorphism. To make precise this notion, we will need the concept of a monoidal Quillen equivalence from the next section. Nevertheless, we will state the result now.

Proposition 1.29 ([SS03a, Section 2.3])**.** $N\colon \mathbf{sAb} \to \mathbf{Ch}_{\geq 0}(\mathbb{Z})$ *induces a monoidal Quillen equivalence.*

This Quillen equivalence relies on the existence of two natural transformations comparing the two monoidal structures.

Proof Sketch. Define the **Alexander-Whitney map** Φ as the natural transformation

$$\Phi\colon C(A \otimes_{\mathbf{sAb}} B) \to CA \otimes_{\mathbf{Ch}_{\geq 0}(\mathbb{Z})} CB$$

given component-wise by

$$\Phi(a_n \otimes b_n) = \bigoplus_{q+p=n} \widetilde{d}^p a_n \otimes d_0^q b_n,$$

where $\widetilde{d}^p$ is the **front face** map and d_0^q is the **back face** map.

Define the **shuffle map** ∇

$$\nabla\colon CA \otimes_{\mathbf{Ch}_{\geq 0}(\mathbb{Z})} CB \to C(A \otimes_{\mathbf{sAb}} B)$$

by

$$\nabla(a_p \otimes b_q) = \sum_{(p,q)\text{ shuffles}} \operatorname{sgn}(\mu,\nu) s_\nu a_p \otimes s_\mu b_q$$

where $s_\nu a_p \in A_{p+q}$ and $s_\mu b_q \in B_{p+q}$.

These two maps allow us to compare the two monoidal structures on $\mathbf{Ch}_{\geq 0}(\mathbb{Z})$ and **sAb**, and the composite $\Phi \circ \nabla$ is quasi-isomorphic to the identity on $\mathbf{Ch}_{\geq 0}(\mathbb{Z})$. This is one step in showing that these are part of a Quillen equivalence. □

This proposition also has many interesting corollaries. For any simplicial ring R with multiplication $\mu\colon R \otimes R \to R$, the shuffle map ∇ induces a multiplication on NR, making it into a **differential graded algebra**.

$$\begin{array}{ccc} NR \otimes NR & \xrightarrow{\nabla} & N(R \otimes R) \\ & \searrow & \downarrow N(\mu) \\ & & NR \end{array}$$

Corollary 1.30. *There is an adjunction* $L^{\mathrm{mon}} \dashv N$ *inducing a Quillen equivalence between the categories of simplicial rings and differential graded algebras.*

2 Model categories

A model structure on a category codifies an abstract notion of categories up to homotopy equivalence. The idea of a model category is developed by Quillen in his Homotopical Algebra book [Qui06].

Definition 2.1. A **model category** is a category $\mathbf{C}$ with three distinguished classes of maps:

- **Weak equivalences** W, with maps decorated by a tilde ($\xrightarrow{\sim}$).
- **Cofibrations** C, with maps decorated by a hook ($\hookrightarrow$).
- **Fibrations** F, with maps decorated with two heads ($\twoheadrightarrow$).

Each class is closed under composition. Additionally, we describe maps which are both weak equivalences and fibrations as **acyclic fibrations**. Similarly, **acyclic cofibrations**.

These classes satisfy the following axioms:

(MC1) $\mathbf{C}$ has all finite limits and colimits.

(MC2) If any two of f, g and fg are weak equivalences, then so is the third.

(MC3) Each of the three classes W, C and F are closed under retracts.

(MC4) Given a diagram of the form below, if either f or g is in addition a weak equivalence, then the diagonal dashed arrow exists.

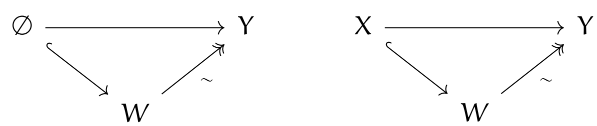
$$\begin{array}{ccc} A & \longrightarrow & X \\ f \big\downarrow & \nearrow & \big\downarrow g \\ B & \longrightarrow & Y \end{array}$$

(MC5) Any morphism f factors either as an acyclic cofibration followed by a fibration or a cofibration followed by an acyclic fibration.

Example 2.2. Some examples of model categories:

- The category of topological spaces has a model structure with weak equivalences $f\colon X \simeq Y$ if and only if $\pi_*(X) \cong \pi_*(Y)$.
- The category of chain complexes has two model structures where the weak equivalences are the quasi-isomorphisms: $f\colon C \simeq D$ if and only if $H_*(C) \cong H_*(D)$. They are the projective and injective model structures defined in Example 2.7 below.
- The category of simplicial abelian groups has a model structure with weak equivalences $f\colon A \simeq B$ if and only if $H_*(NA) \cong H_*(NB)$.

Example 2.3. In **Top**, the factorization of a map $f\colon X \to Y$ as a fibration following an acyclic cofibration is a generalized CW approximation to Y. If $X = \emptyset$, then this is the usual CW approximation



Definition 2.4. The **homotopy category** $\mathrm{Ho}(\mathbf{C})$ of a model category $\mathbf{C}$ is defined by inverting the weak equivalences

$$\mathrm{Ho}(\mathbf{C}) = \mathbf{C}[W^{-1}]$$

Definition 2.5. Given model categories $\mathbf{C}$ and $\mathbf{D}$, and an adjunction $F \dashv G$ between them, we say that (F, G) is a **Quillen pair** if F preserves cofibrations and G preserves fibrations.

$$\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{D}$$

If (F, G) is a Quillen pair, then there is an induced adjunction between homotopy categories.

$$\mathrm{Ho}(\mathbf{C}) \underset{RG}{\overset{LF}{\rightleftarrows}} \mathrm{Ho}(\mathbf{D})$$

Definition 2.6. If (LF, RG) induces an equivalence of homotopy categories, then (F, G) is called a **Quillen equivalence**. We write $\mathbf{C} \simeq_{QE} \mathbf{D}$.

Example 2.7. Given a ring R, the **projective model structure** on $\mathbf{Ch}_{\geq 0}(R)$ is as follows:

- weak equivalences are quasi-isomorphisms;
- fibrations are epimorphisms of R-modules in positive degree;
- cofibrations are degree-wise monomorphisms of R-modules with projective cokernel.

The **injective model structure** on $\mathbf{Ch}_{\leq 0}(R)$ is dual:

- weak equivalences are quasi-isomorphisms;
- fibrations are degree-wise epimorphisms of R-modules with injective kernel;
- cofibrations are monomorphisms of R-modules in negative degree.

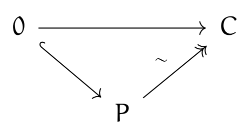
Both of these model structures extend to model structures on all of $\mathbf{Ch}(R)$, which we write as $\mathbf{Ch}(R)_{\text{proj}}$ and $\mathbf{Ch}(R)_{\text{inj}}$.

These two model structures are Quillen equivalent, via the identity functor on $\mathbf{Ch}(R)$:

$$\mathbf{Ch}(R)_{\text{proj}} \simeq_{QE} \mathbf{Ch}(R)_{\text{inj}}.$$

This certainly makes sense: for instance, a cofibration in the projective structure is a degree-wise monomorphism with extra structure, so certainly a degree-wise monomorphism and hence a cofibration in the injective structure.

Example 2.8. In $\mathbf{Ch}(R)_{\text{proj}}$, the factorization of $0 \to C$ as a cofibration followed by an acyclic fibration is a projective resolution of the chain complex C



Note that, however, this projective resolution P need not be a projective object in $\mathbf{Ch}(R)_{\text{proj}}$. We are only guaranteed that maps into C lift with respect to fibrations (epimorphisms in positive degree), but this is not enough for a map to lift with respect to all epimorphisms.

Remark 2.9. There are examples of model categories $\mathbf{C}$ and $\mathbf{D}$ with $\text{Ho}(\mathbf{C}) \simeq \text{Ho}(\mathbf{D})$, but there is no Quillen pair inducing this equivalence. In this case, $\mathbf{C}$ and $\mathbf{D}$ are not Quillen equivalent, even though their homotopy categories are equivalent.

Examples of this phenomena come from stable module categories and Morava K-theory, but we will not discuss that here. For a more concrete example, consider any model category $\mathbf{C}$ whose class of weak equivalences is not simply all isomorphisms of $\mathbf{C}$. Let $\mathbf{D} = \text{Ho}(\mathbf{C})$, endowed with the trivial model structure where the class of weak equivalences is the isomorphisms. Then $\mathrm{Ho}(\mathbf{D}) \cong \mathrm{Ho}(\mathbf{C})$, but there is no Quillen pair inducing this equivalence.

We will return to similar issues when we discuss topological equivalence of differential graded algebras in Section 4.

Recall from Proposition 1.29 that the adjunction $N \dashv \Gamma$ induces a Quillen equivalence between $\mathbf{sAb}$ and $\mathbf{Ch}_{\geq 0}(\mathbb{Z})$ preserving the monoidal structures. We know have the technology to say what this means:

$$\mathrm{Ho}(\mathbf{sAb}) \simeq \mathrm{Ho}(\mathbf{Ch}_{\geq 0}(\mathbb{Z})),$$

and this equivalence is a monoidal equivalence.

Furthermore, we can now expand on the content of Corollary 1.30. The functor $N\colon \mathbf{sAb} \to \mathbf{Ch}_{\geq 0}(\mathbb{Z})$ induces a functor on simplicial rings via the shuffle map. The image of this functor is always a differential graded algebra. Corollary 1.30 says that this is part of a Quillen equivalence:

$$\mathbf{sRing} \simeq \mathbf{DGA}^{+} \quad \text{and} \quad \mathrm{Ho}(\mathbf{sRing}) \simeq \mathrm{Ho}(\mathbf{DGA}^{+}).$$

3 Stable homotopy theory

Stable homotopy theory came out of the study of stable homotopy groups of a space, which are often easier to compute than normal (unstable) homotopy groups. This leads to the study of spectra, the homotopy category of which is called the **stable homotopy category**. Stable homotopy theory, in the modern context, is the study of this category or variations on it.

Definition 3.1 (see Remark 3.5)**.** A **spectrum** is a sequence $X = (X_n)_{n \in \mathbb{N}}$ of pointed topological spaces (or simplicial sets) together with structure maps $\Sigma X_n \to X_{n+1}$ from the suspension of X_n to X_{n+1} for all n.

Spectra correspond to **generalized cohomology theories** – cohomology theories obeying all of the Eilenberg-Steenrod axioms except the dimension axiom. Some examples of generalized cohomology theories are below.

Example 3.2. Some examples of generalized cohomology theories and their corresponding spectra are below.

(a) Ordinary cohomology. If A is an abelian group, then $H^n(X;A)$ is isomorphic to $[X_+, K(A,n)]$, where $K(A,n)$ is an Eilenberg-MacLane space. The corresponding spectrum is the **Eilenberg-Maclane spectrum** $HA_\bullet$, with $HA_n = K(A,n)$ for all $n \geq 0$. The structure maps $\Sigma HA_n \to HA_{n+1}$ are determined by the maps

$$HA_n = K(A,n) \xrightarrow{\mathrm{id}} \Omega K(A, n+1) = HA_{n+1}$$

under the suspention/loop-space adjunction $\Sigma \dashv \Omega$.

(b) Hypercohomology. If $C_\bullet$ is any chain complex of abelian groups, the **hypercohomology** of a space X is

$$\mathbb{H}^s(X; A_\bullet) = \bigoplus_{s=q-p} H^p(X; H_q(A_\bullet)).$$

This is just a direct sum of shifted ordinary cohomology theories, and the corresponding spectrum is denoted HC.

(c) Topological K-theory over $\mathbb{C}$. Given a space X, $K_0(X)$ is the group completion of the monoid of complex vector bundles on X. We may define a generalized cohomology theory via $K_n(X) = K_0(\Sigma^n X)$. The corresponding K**-theory spectrum** is denoted K_n,

$$K_n = \begin{cases} U & n \text{ odd} \\ BU \times \mathbb{Z} & n \text{ even.} \end{cases}$$

(d) Stable cohomotopy. The **sphere spectrum** is the spectrum $\mathbb{S}$ with n-th space $\mathbb{S}_n = S^n$ and structure maps

$$\Sigma S^n = S^1 \wedge S^n \cong S^{n+1}.$$

The associated cohomology theory is known as **stable cohomotopy,** $\pi^*_S(X) = [X, S^n]$. Instead of maps out of S^n as the usual homotopy groups, the stable cohomotopy group is maps into S^n.

Many cohomology theories have ring structures as well. The notion of a ring spectrum captures the idea of a generalized cohomology theory with a product.

Definition 3.3 (Pseudo-definition; see Remark 3.5)**.** A **ring spectrum** is a sequence of pointed topological spaces $R = (R_n)_{n\in\mathbb{N}}$ with compatibly associative and unital products $R_p \wedge R_q \to R_{p+q}$.

Example 3.4. Some examples of ring spectra

(a) For a ring R, HR is a ring spectrum. The cup product gives a graded product $HR^p(X) \otimes HR^q(X) \to HR^{p+q}(X)$. This is induced by the map $K(R,p) \wedge K(R,q) \to K(R, p+q)$.

(b) For a chain complex $A_\bullet$, the hypercohomology spectrum $HA_\bullet$ is a ring spectrum. The product is given by the maps $A_p \otimes A_q \to A_{p+q}$. Note that in this example, the groups $\mathbb{H}(X; A_\bullet)$ are still determined by $H_\bullet(A)$, but the product structure is *not* determined by $H_\bullet(A)$.

(c) Topological K-theory is a ring spectrum, with product induced by the tensor product of vector bundles.

(d) $\mathbb{S}$ is a commutative ring spectrum, with product maps $S^p \wedge S^q \cong S^{p+q}$, but see Remark 3.5

Remark 3.5. We need to be more careful when we say that $\mathbb{S}$ is a commutative ring spectrum. The twist map $S^1 \wedge S^1 \to S^1 \wedge S^1$ is a degree -1 map and a homeomorphism, but this is not the map we're talking about in Example 3.4(d).

This caution brings us to an important point about spectra as we've defined them so far. In 1965, Boardman defined spectra and a smash product $\wedge$ on spectra. But his definition of the smash product is only commutative and associative up to homotopy. This led to A_∞ ring spectra and E_∞ ring spectra, the best approximations to associative and commutative ring spectra, respectively.

In 1991, Lewis gave five reasonable axioms that one might desire for a smash product on the category of spectra, and then proved that no such product exists.

Nevertheless, since 1997 many have defined monoidal categories of spectra by dropping some of the axioms. There are several definitions of spectra, such as **EKMM spectra**, due to Elmendorf-Kriz-Mandell-May, **orthogonal spectra**, due to Mandell-May-Schwede-Shipley, or **symmetric spectra**, due to Smith. Symmetric spectra are nice in particular, because the smash product is commutative and associative.

Theorem 3.6 (Mandell-May-Schwede-Shipley)**.** *All of the models of the monoidal category of spectra define the same homotopy theory.*

3.1 Spectral Algebra

In many ways, the category of spectra is more like the category of chain complexes than it is like the category of topological spaces. Nevertheless are many similarities between studying chain complexes and studying topological spaces. The category of chain complexes has a model structure, and we talk about chain homotopy of complexes, and even refer to some complexes as "spheres" or "disks," as we do with spaces. But there is one feature that chain complexes have that spaces do not: we may shift a complex up or down in degree. The analogue involves replacing a space X by a spectrum.

Definition 3.7. Given a topological space X, its **suspension spectrum** $\Sigma^\infty X$ is the spectrum with n-th space $\Sigma^n X$.

If spectra are analogous to chain complexes, then a suspension spectrum is analogous to the chain complex concentrated in degree zero. We can also shift a spectrum, much as we shift chain complexes in degree.

Definition 3.8. Given a spectrum $X = (X_n)_{n\in\mathbb{N}}$, we may define various shifted spectra. The **suspension** of the spectrum X is the spectrum ΣX with

$$(\Sigma X)_n = X_{n+1}.$$

We can also **desuspend** X to get a spectrum $\Sigma^{-1}X$, with

$$(\Sigma^{-1}X)_n = \begin{cases} * & n = 0 \\ X_{n-1} & n > 0. \end{cases}$$

Given a good category of spectra with a smash product $\wedge$, we may also do algebra.

Definition 3.9. A **ring spectrum** is a spectrum R with a multiplication $\mu\colon R\wedge R \to R$ and a unit $\iota\colon S \to R$ such that the following diagrams commute:

$$\begin{array}{ccc} R\wedge R\wedge R & \xrightarrow{\mu\wedge 1} & R\wedge R \\ {\scriptstyle 1\wedge\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ R\wedge R & \xrightarrow{\mu} & R \end{array} \qquad \begin{array}{ccccc} \mathbb{S}\wedge R & \xrightarrow{\iota\wedge 1} & R\wedge R & \xleftarrow{1\wedge\iota} & R\wedge\mathbb{S} \\ & \underset{\cong}{\searrow} & \downarrow{\scriptstyle\mu} & \underset{\cong}{\swarrow} & \\ & & R & & \end{array}$$

Remark 3.10. In other words, this is a monoid object in our monoidal category of spectra with a smash product. Don't be confused by the terminology! This is called a *ring* spectrum because this structure endows the associated generalized cohomology theory with a ring structure. But this is *not* a ring object in the category of spectra (whatever that would be).

Definition 3.11. An R**-module spectrum** is a spectrum M together with an action $\alpha\colon R\wedge M \to M$ such that the following diagrams commute:

$$\begin{array}{ccc} R\wedge R\wedge M & \xrightarrow{\mu\wedge 1} & R\wedge M \\ {\scriptstyle 1\wedge\alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ R\wedge M & \xrightarrow{\alpha} & M \end{array} \qquad \begin{array}{ccc} R\wedge M & \xrightarrow{\alpha} & M \\ {\scriptstyle \iota\wedge 1}\uparrow & \underset{\cong}{\nearrow} & \\ \mathbb{S}\wedge M & & \end{array}$$

In other words, this is a module over the monoid R in the monoidal category of spectra.

Example 3.12. Suppose that $E = (E_n)_{n\in\mathbb{N}}$ is an $\mathbb{S}$-module. Then E is just the same thing as an ordinary spectra, with no extra structure; iterating the structure maps $S^1 \wedge E_n \to E_{n+1}$ give $S^m \wedge E_n \to E_{m+n}$. These fit together to give an action $\mathbb{S}\wedge E \to E$.

Moreover, $\mathbb{S}$**-algebras** are ring spectra. We will frequently use that terminology.

Definition 3.13. For any ring spectrum R, a **morphism of** R**-module spectra** M and N is a morphism of spectra $f\colon M \to N$ that commutes with the action of R. This yields a category R-**Mod** of modules over R.

Definition 3.14. An R**-algebra spectrum** is a monoid object in R-**Mod**.

To summarize the analogy between homological algebra and spectral algebra, consider the following table.

Category	**Set**	Chain Complexes	**Spectra**	**Spectra**
Monoid Object	$\mathbb{Z}$	$\mathbb{Z}$	$H\mathbb{Z}$	$\mathbb{S}$
Module Category	**Ab**	**dgMod** $=$ **Ch**$(\mathbb{Z})$	$H\mathbb{Z}$-**Mod**	$\mathbb{S}$-**Mod** $=$ **Spectra**
Algebra Category	**Rings**	**dgAlg**	$H\mathbb{Z}$-**Alg**	$\mathbb{S}$-**Alg** $=$ **RingSpectra**
Weak Equivalences	$\cong$	quasi-iso	weak equiv.	weak equiv.
Homotopy Category		$\mathcal{D}(\mathbb{Z})$	Ho$(H\mathbb{Z}$-**Mod**$)$	Ho(**Spectra**)

Remark 3.15. The object $\mathbb{Z}$ is initial in the algebraic cases, and the object $\mathbb{S}$ is initial in the case of spectra. One may then ask where the spectrum $H\mathbb{Z}$, the Eilenberg-MacLane spectrum associated to ordinary cohomology, falls in this analogy. $H\mathbb{Z}$ is not the initial object in the category of **Spectra**. In particular, S^n is not a $K(\mathbb{Z}, n)$ for all n, because $\pi_i(S^n)$ is not necessarily zero for $i > n$. So we add a column to the table above for $H\mathbb{Z}$.

Theorem 3.16 ([Rob87, SS03a, Shi07])**.** *Columns two and three above are equivalent up to homotopy.*

(a) $\mathcal{D}(\mathbb{Z})$ *is equivalent to* Ho$(H\mathbb{Z}$-**Mod**$)$ *as triangulated categories.*

(b) **Ch**$(\mathbb{Z}) \simeq_{QE} H\mathbb{Z}$-**Mod**

(c) **dgAlg** $\simeq_{QE} H\mathbb{Z}$-**Alg**

(d) For any given differential graded algebra A*, there is a Quillen equivalence*

$$A\text{-}\mathbf{dgMod} \simeq_{QE} HA\text{-}\mathbf{Mod}$$

and an equivalence of triangulated categories

$$\mathcal{D}(A) \simeq \mathrm{Ho}(HA\text{-}\mathbf{Mod}).$$

Remark 3.17. The equivalence of columns two and three, especially the equivalences **Ch**$(\mathbb{Z}) \simeq_{QE} H\mathbb{Z}$-**Mod**, and **dgAlg** $\simeq_{QE} H\mathbb{Z}$-**Alg**, may be viewed as a stable homotopy theory version of the Dold-Kan correspondence.

4 Topological equivalence of DGAs

When studying differential graded algebras, or chain complexes, we frequently want to work in the associated homotopy category, where we consider chain complexes the same up to quasi-isomorphism. Yet there may be chain complexes with the same homology, yet no quasi-isomorphism between them (see Example 4.7).

To study this phenomena, we enlarge the category. We saw previously that the category of differential graded algebras is Quillen equivalent to the category of modules over $H\mathbb{Z}$ in **Spectra**. So we may embed the category of differential graded modules inside **Spectra**. This does something like replacing our base ring $\mathbb{Z}$ with $\mathbb{S}$. Previously, $\mathbb{Z}$ was initial, but its image under the functor H is not initial (see Remark 3.15). We now have, in effect, a ring more initial than $\mathbb{Z}$. Moreover, there are now more objects in the stable homotopy theory world than there are in the algebra world of $H\mathbb{Z}$-modules, such as the sphere spectrum $\mathbb{S}$ or the K-theory spectrum K.

Now recall that $\mathbb{S}$ is the initial ring spectrum. Hence, there is a unit map

$$\phi\colon \mathbb{S} \to H\mathbb{Z}.$$

This yields an adjunction

$$\mathbb{S}\text{-}\mathbf{Alg} \underset{\phi^*}{\overset{-\wedge_{\mathbb{S}} H\mathbb{Z}}{\rightleftarrows}} H\mathbb{Z}\text{-}\mathbf{Alg} \quad (\perp)$$

but this adjunction is *not* an equivalence of categories – not all ring spectra ($\mathbb{S}$-algebras) are $H\mathbb{Z}$-algebras.

Definition 4.1. Let A and B be differential graded algebras. We say that A and B are **topologically equivalent** if HA and HB are weakly equivalent as $\mathbb{S}$-algebras, and write $A \simeq_{TE} B$.

Remark 4.2. This is weaker than quasi-isomorphism, which is equivalence of HA and HB as $H\mathbb{Z}$-algebras. But quasi-isomorphism implies topological equivalence.

Example 4.3. An analogy in classical algebra is as follows. Consider $\phi\colon k \to k[X]$. This induces a map $\phi^*\colon k[X]\text{-}\mathbf{Alg} \to k\text{-}\mathbf{Alg}$. Now define $k[Y]$ as a $k[X]$-algebra with trivial X action. Then $k[X] \cong k[Y]$ as k-algebras, but not as $k[X]$-algebras.

From the topological point of view, the notion of topological equivalence is quite reasonable. But from the algebra point of view, the existence of topological equivalence is quite surprising. We have, in effect, a ring smaller than $\mathbb{Z}$.

Example 4.4. If A and B are topologically equivalent differential graded algebras, then A-**Mod** $\simeq_{QE}$ HA-**Mod**, and B-**Mod** $\simeq_{QE}$ HB-**Mod**. Moreover, $HA \cong HB$ as spectra, because A and B are topologically equivalent, hence there is a Quillen equivalence HA-**Mod** $\simeq_{QE}$ HB-**Mod**. This yields a zig-zag of Quillen equivalences demonstrating that A-**Mod** $\simeq_{QE}$ B-**Mod**. However, this may not be a Quillen equivalence of additive categories, because the equivalence HA-**Mod** $\simeq_{QE}$ HA-**Mod** is not necessarily additive.

If the equivalence is additive, then A and B are in fact quasi-isomorphic.

Here is a criterion for two differential graded algebras to be topologically equivalent.

Theorem 4.5 ([DS07]). *Let A and B be two differential graded algebras. The following are equivalent:*

(a) The categories of differential graded A*- and* B*-modules are Quillen equivalent.*

(b) There is a compact generator P *of* $\mathcal{D}(A)$ *such that* $\mathrm{Hom}_A(P, P)$ *is topologically equivalent to* B*.*

Our example of differential graded algebras that are topologically equivalent but not quasi-isomorphic was over $\mathbb{Z}$. Is there an example over any field? In fact, over $\mathbb{Q}$, such an example is impossible.

Theorem 4.6 ([DS07]). *If* A *and* B *are both differential graded* $\mathbb{Q}$*-algebras, then they are topologically equivalent if and only if they are quasi-isomorphic.*

Example 4.7. An example of a non-trivial topological equivalence. Let X be the exterior algebra over $\mathbb{F}_2$ generated by a single element x_2 in degree 2, and let Y be the $\mathbb{Z}$-algebra generated by e_1 such that $de_1 = 2$ and $e_1^4 = 0$. Then $H_*(X) \cong H_*(Y) \cong X$.

To see that these are not quasi-isomorphic, notice that the Eilenberg-MacLane spectra HX and HY are both built from Postnikov extensions of $H\mathbb{F}_2$ by $\Sigma^2 H\mathbb{F}_2$. Such extensions are classified by

$$\mathrm{THH}^4_{H\mathbb{Z}}(H\mathbb{F}_2, H\mathbb{F}_2) \cong \mathbb{F}_2.$$

If X and Y were quasi-isomorphic, they would represent the same class in $\mathrm{THH}^4_{H\mathbb{Z}}(H\mathbb{F}_2, H\mathbb{F}_2)$, but X represents the class of 0, and Y represents the class of 1. Hence, they are not quasi-isomorphic.

Nevertheless, they do represent the same class in

$$\mathrm{THH}^4_{\mathbb{S}}(H\mathbb{F}_2, H\mathbb{F}_2) \cong \mathbb{F}_2,$$

because the map

$$\mathrm{THH}^4_{H\mathbb{Z}}(H\mathbb{F}_2, H\mathbb{F}_2) \to \mathrm{THH}^4_{\mathbb{S}}(H\mathbb{F}_2, H\mathbb{F}_2)$$

induced by $\mathbb{S} \to H\mathbb{Z}$ is zero, so both X and Y land in the same class on the right hand side, and are therefore topologically equivalent.

4.1 Commutative Topological Equivalence

The work in this section is due to Bayindir [Bay17].

Definition 4.8. Let A and B be differential graded algebras. We say that A and B are E_∞**-topologically equivalent** if HA and HB are equivalent as commutative $\mathbb{S}$-algebras.

The notion of E_∞ topological equivalence is stronger than usual topological equivalence over finite fields, insofar as we have the following theorem.

Theorem 4.9 ([Bay17])**.** *Let A and B be co-connective E_∞ differential graded $\mathbb{F}_p$-algebras. Then The following are equivalent:*

(a) A is E_∞-topologically equivalent to B.

(b) A is quasi-isomorphic to B as differential graded R-algebras.

Proof Sketch. A commutative $H\mathbb{F}_p$-algebra structure on a commutative $\mathbb{S}$-algebra X is given by a map of commutative $\mathbb{S}$-algebras $H\mathbb{F}_p \to X$. The set of such maps is denoted $\mathrm{Map}_{\mathbb{S}\text{-}\mathbf{CommAlg}}(HR, X)$. Using Goerss-Hopkins-Miller obstruction theory, there is a spectral sequence converging to

$$\pi_{t-s}(\mathrm{Map}_{\mathbb{S}\text{-}\mathbf{CommAlg}}(HR, X)).$$

If Y is an $H\mathbb{F}_p$-algebra, the E_2 page of this spectral sequence is given by

$$E_2^{0,0} = \mathrm{Hom}_{\mathcal{R}}((H\mathbb{F}_p)_* X, Y_*)$$

and for $t > 0$, by

$$E_2^{s,t} = \mathrm{Der}_{\mathcal{R}}((H\mathbb{F}_p)_* X, Y_*^{S^t}),$$

where

- $\mathcal{R}$ is the **Dyer-Lashof algebra**.
- $\mathrm{Der}^s_{\mathcal{R}}(-,-)$ denotes the s-th André-Quillen cohomology for unstable algebras over $\mathcal{R}$
- Y^{S^t} is the mapping spectrum from the t-sphere to Y
- $\mathrm{Hom}_{\mathcal{R}}((H\mathbb{F}_p)_* X, Y_*)$ denotes morphisms of $\mathcal{R}$-algebras, that is, preserving the Dyer-Lashof operations.

Obstructions to lifting an algebraic map in $E_2^{0,0}$ to a map of commutative $\mathbb{S}$-algebras lie in

$$\mathrm{Der}_{\mathcal{R}}^{t+1}((H\mathbb{F}_p)_*X, Y_*^{S^t})$$

for $t \geq 1$.

In the case of co-connective E_∞ differential graded algebras, these obstructions vanish and there is a unique map from $H\mathbb{F}_p \to X$ for X co-connective. Hence, there is a unique way in which X becomes an $H\mathbb{F}_p$-algebra. □

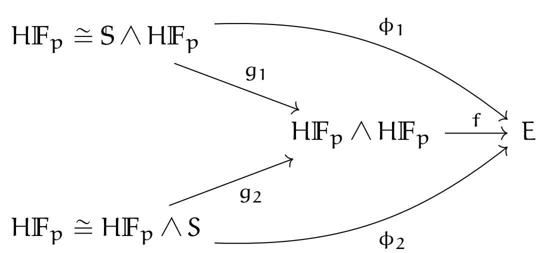
Dyer-Lashof operations are power operations that act on the homology of E_∞ differential graded $\mathbb{F}_p$-algebras. They are preserved by quasi-isomorphism, but are they also preserved by E_∞ topological equivalence?

Theorem 4.10 ([Bay17]). *Let A and B be E_∞ differential graded $\mathbb{F}_p$-algebras such that $H_1(X) = H_1(Y) = 0$. If A and B are E_∞ topologically equivalent, then $H_*(X) \cong H_*(Y)$ as $\mathcal{R}$-algebras, where $\mathcal{R}$ is the Dyer-Lashof algebra.*

This theorem is surprising, because it shows that Dyer-Lashof operations are determined up to homotopy type over $\mathbb{S}$, not over $H\mathbb{F}_p$.

In the above theorem, we really need the first homology to vanish, as the following example demonstrates. It is also an example of E_∞ differential graded $\mathbb{F}_p$-algebras that are E_∞-topologically equivalent but not quasi-isomorphic. This is also an example of two topologically equivalent differential graded algebras over a field that are not quasi-isomorphic.

Example 4.11 ([Bay17]). Consider the diagram



where E is obtained by taking Postnikov sections and attaching a cell to kill an extra class of π_{2p-1}. Then ϕ_1 and ϕ_2 induce different $H\mathbb{F}_p$-algebra structures on E, which we call X and Y respectively.

Note that the two maps g_1 and g_2 induce different $H\mathbb{F}_p$-algebra structures on $H\mathbb{F}_p \wedge H\mathbb{F}_p$ with different Dyer-Lashof operations, but these structures are isomorphic through the switch map of the smash product. Hence, attaching a cell to get E is necessary to make the Dyer-Lashof operations different. Nevertheless, this preserves the E_∞ topological equivalence of the two structures.

We have two E_∞ differential graded $\mathbb{F}_p$-algebras X and Y, such that for $p = 2$, we have graded ring isomorphisms

$$H_*(X) \cong H_*(Y) \cong {}^{\mathbb{F}_2[\xi_1]}\big/_{(\xi_1^4)}$$

with ξ_1 in degree 1. Note in particular that the first homology isn't trivial, which shows the necessity of that assumption in Theorem 4.10.

Even though they are isomorphic as graded rings, the Dyer-Lashof operations on $H_*(X)$ and $H_*(Y)$ are different: $Q^2(\xi_1) = \xi_1^3$ in $H_*(X)$, and $Q^2(\xi_1) = 0$ in $H_*(Y)$. Therefore, X and Y are not quasi-isomorphic, even though they were E_∞ topologically equivalent by construction.

Example 4.12 ([Bay17])**.** A similar, simpler example is to give $H\mathbb{Z} \wedge H\mathbb{F}_p$ two commutative $H\mathbb{Z}$-algebra structures using the maps

$$\begin{array}{ccc} H\mathbb{Z} \cong H\mathbb{Z} \wedge \mathbb{S} & & \\ & \overset{\phi_1}{\searrow} & \\ & & H\mathbb{Z} \wedge H\mathbb{F}_p. \\ & \underset{\phi_2}{\nearrow} & \\ H\mathbb{F}_p \cong \mathbb{S} \wedge H\mathbb{F}_p & & \end{array}$$

This gives two non-equivalent commutative $H\mathbb{Z}$-algebras whose underlying commutative $\mathbb{S}$-algebra structures are the same; that is, two E_∞ DGAs that are not quasi-isomorphic but are E_∞ topologically equivalent.

5 Rational SO(2)-equivariant ring spectra

We saw that in Theorem 4.6 that differential graded $\mathbb{Q}$-algebras are quite simple: any topological equivalence between them is in fact quasi-isomorphism. Rational stable homotopy theory, the study of **rational spectra** (module spectra over $H\mathbb{Q}$), is similarly nice.

Theorem 5.1 ([SS03b, Theorem B.1.11])**.** *There is a Quillen equivalence between the categories of rational spectra and chain complexes of $\mathbb{Q}$-vector spaces.*

$$H\mathbb{Q}\text{-}\mathbf{Mod} \simeq_{QE} \mathbf{Ch}(\mathbb{Q}).$$

We might think of this as a version of the Dold-Kan theorem for stable homotopy theory.

Definition 5.2. We say that a **rational stable homotopy theory** is the homotopy category of a stable model category where the homotopy groups of the mapping spectra are all rational vector spaces.

If we are given a set of weak generators for any rational stable homotopy theory, then it is Quillen equivalent to the category of differential graded modules over a particular differential graded algebra A. We describe this as an **algebraic model** for the given rational stable homotopy theory.

Definition 5.3. The differential graded algebra A providing the algebraic model for a given rational stable homotopy theory may be large and not very explicit. If the number of generators is not finite, then we must instead consider differential graded categories, that is, categories enriched in differential graded algebras.

The general conjecture there is always a nice algebraic model for G-**Spectra** that makes calculations easy.

Conjecture 5.4. *For any compact Lie group G, there is an abelian category $\mathcal{A}(G)$ and a monoidal Quillen equivalence*

$$G\text{-}\mathbf{Spectra} \simeq_{QE} d\mathcal{A}(G),$$

where $d\mathcal{A}(G)$ is the category of differential graded objects of $\mathcal{A}(G)$. Moreover, the category $\mathcal{A}(G)$ is of injective dimension equal to the rank of G.

5.1 Background

Let G be a group and X a based topological space with a G-action. We want to define cohomology theories with an action of G.

Definition 5.5. Let V be a representation of G. Define the **representation sphere** S^V as the one-point compactification of V.

Example 5.6. If $V \cong \mathbb{R}^n$, then $S^V \cong S^n$.

Definition 5.7. A G**-equivariant cohomology theory** F_G^* consists of cohomology theories F_G^V graded by representations V of G such that

$$(F_G^{V\oplus W})^*(S^W \wedge X) \cong (F_G^V)^*(X).$$

Just as generalized cohomology theories are represented by spectra, we would expect that there is a corresponding notion of G-equivariant spectra that represent G-equivariant cohomology theories. Indeed, this is the case.

Theorem 5.8 (Equivariant Brown Representability)**.** *Any G-equivariant cohomology theory F_G^* is represented by a G-spectrum F_G.*

$$F_G^*(X) = [\Sigma^\infty X, F_G]_*^G$$

Definition 5.9 (Pseudo-definition)**.** A G**-spectrum** X is a collection of based G-spaces $X(V)$ indexed by finite-dimensional representations V of G, along with structure maps

$$X(V) \wedge S^W \to X(V \oplus W).$$

Proposition 5.10. *For each representation V of G, the functor*

$$- \wedge S^V \colon \mathrm{Ho}(G\text{-}\mathbf{Spectra}) \to \mathrm{Ho}(G\text{-}\mathbf{Spectra})$$

is an equivalence of categories.

Example 5.11. For any G-space X, the **suspension** G**-spectrum** of X is the spectrum $\Sigma_G^\infty X$ with

$$(\Sigma_G^\infty X)(V) = X \wedge S^V.$$

Proposition 5.12. *There is a model structure on the category G-**Spectra** such that the weak equivalences are those maps f such that $\pi_*^H(f) \otimes \mathbb{Q}$ is an isomorphism for all closed subgroups H of G.*

5.2 An algebraic model of G-spectra for G finite

Definition 5.13. Let G be a finite group and let H be a subgroup. Denote by $W_G(H)$ the **Weyl group** of H in G; this is the quotient of the normalizer of H in G by H,

$$W_G(H) = {}^{N_G(H)}\!/\!_{H}.$$

In this case, we have the following theorem that provides a concrete algebraic model for rational G-spectra.

Theorem 5.14 ([Bar09, Ked15])**.** *The category of rational G-spectra is symmetric monoidally Quillen equivalent to the product category*

$$G\text{-}\mathbf{Spectra} \simeq_{QE} \prod_{(H) \leq G} \mathbf{Ch}(\mathbb{Q}[W_G(H)]),$$

where (H) denotes the conjugacy class of the subgroup H in G.

Proof Sketch. [Bar09] shows that there is a symmetric monoidal Quillen equivalence between rational G-spectra and the category

$$\prod_{(H) \leq G} L_{e_H S}\,(G\text{-}\mathbf{Spectra}),$$

where $L_{e_H S}\,(G\text{-}\mathbf{Spectra})$ is the model category generated by all $e_H(\Sigma^\infty G/H)_+$ with weak equivalences those f such that $e_H \pi_*^K(f) \otimes \mathbb{Q}$ is an isomorphism for all $K \leq G$.

[Ked15] shows that in fact, there is a symmetric monoidal Quillen equivalence between $L_{e_H S}$ (G-**Spectra**) and chain complexes of modules over the rational group ring of the Weyl group of H in G:

$$L_{e_H S}(\text{G-}\mathbf{Spectra}) \simeq_{QE} \mathbf{Ch}(\mathbb{Q}[W_G(H)]) . \qquad \square$$

5.3 An algebraic model of G-spectra for G = SO(2)

Let $\mathbb{T} = SO(2)$. In this section, we describe a simple, concrete algebraic model for the category of rational $\mathbb{T}$-equivariant ring spectra.

Theorem 5.15 ([BGKS17]). *There exists a category $\mathcal{A}(\mathbb{T})$ of injective dimension 1 such that there is a (zig-zag of) symmetric monoidal Quillen equivalences between rational $\mathbb{T}$-equviariant spectra and $d\mathcal{A}(\mathbb{T})$.*

$$\mathbb{T}\text{-}\mathbf{Spectra} \simeq_{QE} d\mathcal{A}(\mathbb{T})$$

So what does the category $\mathcal{A}(\mathbb{T})$ look like? Let

$$\mathcal{O}_{\mathcal{F}} = \prod_{n \geq 1} \mathbb{Q}[c_n]$$

with c_n in degree −2, and let $\mathcal{E}$ be the class of all c_n, so that

$$\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} = \mathrm{colim}_n\, \mathcal{O}_{\mathcal{F}}[c_1^{-1}, \dots, c_n^{-1}].$$

Then $\mathcal{A}(\mathbb{T})$ is the category whose objects are morphisms of $\mathcal{O}_{\mathcal{F}}$-modules of the form

$$\beta \colon M \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathbb{Q}} V$$

such that β is an isomorphism after inverting $\mathcal{E}$.

A morphism of $\mathcal{A}(\mathbb{T})$ is a pair (ψ, ϕ) which makes the following square commute

$$\begin{array}{ccc} M & \xrightarrow{\beta} & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathbb{Q}} V \\ \downarrow \psi & & \downarrow 1 \otimes_{\mathbb{Q}} \phi \\ M' & \xrightarrow{\beta'} & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathbb{Q}} V' \end{array}$$

Then $d\mathcal{A}(\mathbb{T})$ is the associated category with differentials.

References

[Bar09] David Barnes. Classifying rational G-spectra for finite G. *Homology, Homotopy and Applications*, 11(1):141–170, 2009.

[Bay17] Haldun Ozgur Bayindir. Topological equivalences of e-infinity differential graded algebras. *arXiv preprint arXiv:1706.08554*, 2017.

[BGKS17] David Barnes, JPC Greenlees, Magdalena Kedziorek, and Brooke Shipley. Rational $SO(2)$-equivariant spectra. *Algebraic & Geometric Topology*, 17(2):983–1020, 2017.

[DS07] Daniel Dugger and Brooke Shipley. Topological equivalences for differential graded algebras. *Advances in Mathematics*, 212(1):37–61, 2007.

[GJ09] Paul G Goerss and John F Jardine. *Simplicial homotopy theory*. Springer Science & Business Media, 2009.

[Ked15] Magdalena Kedziorek. An algebraic model for rational G-spectra over an exceptional subgroup. *arXiv preprint arXiv:1511.05993*, 2015.

[Qui06] Daniel G Quillen. *Homotopical algebra*, volume 43. Springer, 2006.

[Rob87] Alan Robinson. The extraordinary derived category. *Mathematische Zeitschrift*, 196(2):231–238, 1987.

[Shi07] Brooke Shipley. $H\mathbb{Z}$-algebra spectra are differential graded algebras. *American journal of mathematics*, pages 351–379, 2007.

[SS03a] Stefan Schwede and Brooke Shipley. Equivalences of monoidal model categories. *Algebraic & Geometric Topology*, 3(1):287–334, 2003.

[SS03b] Stefan Schwede and Brooke Shipley. Stable model categories are categories of modules. *Topology*, 42(1):103–153, 2003.