Math 6520: Differentiable Manifolds I

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The latest version is online here.

Contents

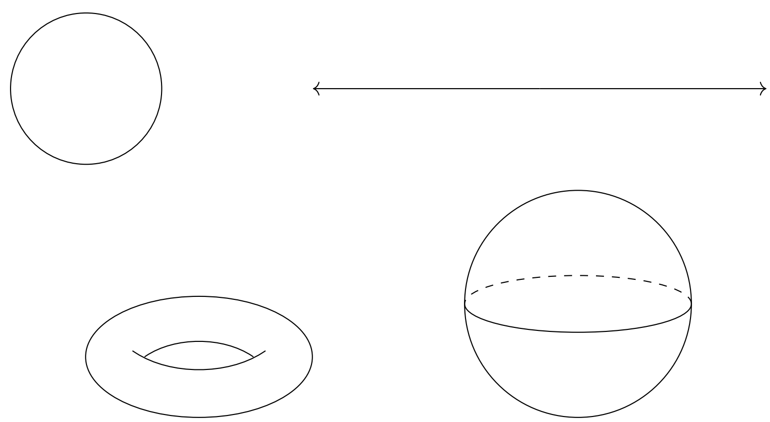
Contents by Lecture

Administrative

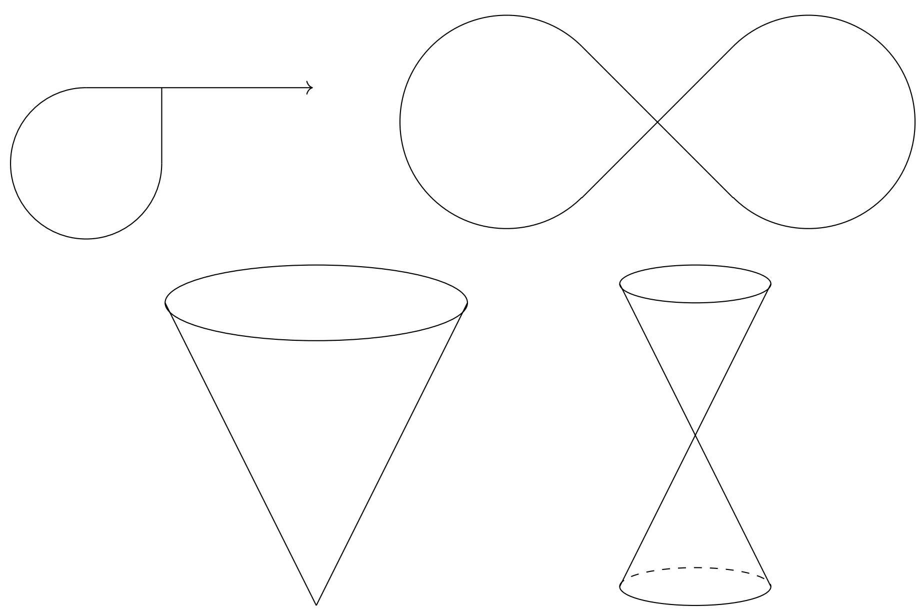
There is a class website. There will be no exams, only homework. The first one will be due on September 7th. There is no textbook, but there are many books that you might want to download. Among them, *Smooth Manifolds* by Lee.

1 Introduction

Example 1.1. Some examples of manifolds:



Example 1.2. Non-examples of manifolds.



Often the non-manifolds are more interesting than the manifolds, but we have to understand the manifolds first. Here are the features of manifolds.

- "Smooth," as in differentiable infinitely many times everywhere.
- "same everywhere," "homogeneous"
- There is a tangent space at every point that is a vector space. The non-examples above have points where we can't define a tangent space, or it isn't a vector space.

Definition 1.3 (Notation)**.** Let A be an open subset of $\mathbb{R}^n$ and $f\colon A \to \mathbb{R}$ a function. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$ be a tuple of non-negative integers (a **multi-index**). We say that the **α-th derivative of** f is

$$\partial^\alpha f := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} f,$$

if it exists. The **order** of α is $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$.

Definition 1.4. We say that f is C^r if $\partial^\alpha f$ exists and is continuous for all multi-indexes α of order $\leq r$. We write

$$C^r(A) = \{f\colon A \to \mathbb{R} \mid f \text{ is } C^r\}$$

for the collection of all C^r functions on open subsets A of $\mathbb{R}^n$.

If f is C^r for every r, then we say that f is C^∞.

$$C^\infty(A) := \bigcap_{r \geq 0} C^r(A).$$

Definition 1.5. A vector valued function $F\colon A \to \mathbb{R}^m$ with components

$$F(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{pmatrix}$$

For $0 \leq r \leq \infty$, we say that F is C^r if f_i is C^r for all $i = 1, \ldots, m$.

Definition 1.6. Let B be open in $\mathbb{R}^m$. Let $F\colon A \to B$ be a map. We say that F is a **diffeomorphism** if F is smooth (i.e. C^∞), bijective, and F^{-1} is smooth as well.

Example 1.7. Smooth bijections need not be diffeomorphisms. $f(x) = x^3$ is smooth (polynomial), and has an inverse $f^{-1}(x) = \sqrt[3]{(x)}$, but f^{-1} fails to be differentiable at $x = 0$.

Remark 1.8. If A, B are open in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and $f: A \to B$ is a diffeomorphism, then $m = n$. Why? Since f is smooth, we can take it's derivative. So the Jacobi matrix of f exists; f is invertible and the derivative of the inverse is the inverse of the derivative (follows from the chain rule, as in the corollary below), so the Jacobi matrix for f must be square.

Proposition 1.9 (Chain Rule)**.** *If A, B, C are open subsets of $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}^\ell$, respectively, and we have C^r functions*

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

then $g \circ f$ is C^r and

$$D(g \circ f)(x) = Dg(f(x)) \circ Df(x).$$

Corollary 1.10. *If $f: A \to B$ is a diffeomorphism, with $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$, then $Df(x)$ is invertible for all x.*

Proof. Let $g = f^{-1}: B \to A$. Then $g \circ f = \mathrm{id}_A$, and $f \circ g = \mathrm{id}_B$. So we have

$$Dg(f(x)) \circ Df(x) = D(g \circ f)(x) = D(\mathrm{id}_A) = I_n$$

And similarly,

$$Df(x) \circ Dg(f(x)) = I_m$$

And moreover, $m = n$. □

Definition 1.11. Let M be a topological space. A **chart** on M is a pair (U, ϕ) where U is an open subset of M (called the **domain**) and $\phi: U \to \mathbb{R}^n$ is a map (called the **coordinate map**) with the properties

(i) $\phi(U)$ is open in $\mathbb{R}^n$,

(ii) $\phi: U \to \phi(U)$ is a homeomorphism (i.e. ϕ is continuous, bijective, and $\phi^{-1}: \phi(U) \to U$ is also continuous).

If $x \in U$, then we say that (U, ϕ) is a **chart at** x. If $x \in U$ and $\phi(x) = 0 \in \mathbb{R}^n$, then we say that the chart is **centered at** x.

Definition 1.12. M is **locally Euclidean** or a **topological manifold** if M admits a chart at every point.

Example 1.13. An example of a topological manifold is the ice cream cone in $\mathbb{R}^3$. A chart might be projection onto the plane. But this isn't a smooth manifold because of the singularity at the apex of the cone (it's pointy, not smooth!).

Definition 1.14. Given two charts (U, ϕ) and (V, ψ) on M, we can form the **transition map**

$$\psi \circ (\phi|_{U\cap V})^{-1}: \phi(U \cap V) \to \psi(U \cap V).$$

The transition map is (by the definition of charts) necessarily a homeomorphism with inverse

$$\phi \circ (\psi|_{U \cap V})^{-1} \colon \psi(U \cap V) \to \phi(U \cap V).$$

It eventually becomes really dreary to write the restriction every time, so we will abbreviate $\psi \circ \phi^{-1}$, respectively $\phi \circ \psi^{-1}$. The charts are **compatible** if $\psi \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ are smooth (equivalently, if either is a diffeomorphism). This is trivially true if $U \cap V = \emptyset$.

By Remark 1.8, if $(U, \phi \colon U \to \mathbb{R}^n)$ and $(V, \psi \colon V \to \mathbb{R}^m)$ are compatible and $U \cap V \neq \emptyset$, then $m = n$.

Definition 1.15. An **atlas** $\mathcal{A}$ on M is a collection of charts

$$\mathcal{A} = \{(U_\alpha, \phi_\alpha) \mid \alpha \in I\}$$

with the properties

(i) $\bigcup_{\alpha \in I} U_\alpha = M$

(ii) every pair of charts (U_α, ϕ_α), (U_β, ϕ_β) is compatible, i.e. the transition map
$$\phi_{\beta\alpha} = \phi_\beta \circ \phi_\alpha^{-1} \colon \phi_\alpha(U_{\alpha\beta}) \to \phi_\beta(U_{\alpha\beta})$$
is smooth, where $U_{\alpha\beta} = U_\alpha \cap U_\beta$.

Given any atlas, we can always make it bigger. For example, to a world atlas we could add maps of each city, and then to that we could add all naval maps, etc. The set of atlases on M is partially ordered by inclusion.

Definition 1.16. If $\mathcal{A}, \mathcal{B}$ are atlases on a topological space M, we say that $\mathcal{A} \leq \mathcal{B}$ if $\mathcal{A} \subseteq \mathcal{B}$.

A **smooth structure** on M is a maximal atlas.

Definition 1.17. A **(smooth) manifold** is a pair $(M, \mathcal{A})$ where $\mathcal{A}$ is a maximal atlas (smooth structure) on M.

To emphasize: maximality of $\mathcal{A}$ means that if $\mathcal{B}$ is another atlas on M, and if $\mathcal{A} \subseteq \mathcal{B}$, then $\mathcal{A} = \mathcal{B}$.

Lemma 1.18. *Let $\mathcal{A}$ be an atlas on M. Then $\mathcal{A}$ is contained in a unique maximal atlas.*

Proof. Define

$$\overline{\mathcal{A}} = \big\{(U, \phi) \;\big|\; (U, \phi) \text{ is a chart on } M \text{ and compatible with every chart in } \mathcal{A}\big\}.$$

Then if $\mathcal{B}$ is an atlas on M and $\mathcal{A} \leq \mathcal{B}$, then $\mathcal{B} \leq \overline{\mathcal{A}}$.

We also need to check that $\overline{\mathcal{A}}$ is an atlas itself. Clearly the union of all of the charts in $\overline{A}$ is M, since $\mathcal{A} \leq \overline{\mathcal{A}}$. So remains to show that each pair of charts in $\overline{A}$ are compatible. The idea is that each chart in $\overline{\mathcal{A}}$ is compatible with one in $\mathcal{A}$, and smoothness is a local property.

Let $c_0 = (U_0, \phi_0)$ and $c_1 = (U_1, \phi_1)$ be charts in $\overline{\mathcal{A}}$. We need to show that c_0 and c_1 are compatible, that is,

$$\phi_{10} = \phi_1 \circ \phi_0^{-1} \colon \phi_0(U_{01}) \to \phi_1(U_{01})$$

is smooth. Enough to show that for each $x \in \phi_0(U_{01})$, ϕ_{10} is smooth in a neighborhood of x. Choose a chart $c_2 = (U_2, \phi_2) \in \mathcal{A}$ at $\phi_0^{-1}(x)$. Then c_0, c_1 are compatible with c_2 by construction of $\overline{\mathcal{A}}$. Therefore, ϕ_{12} and ϕ_{20} are smooth. So

$$\phi_{10} = \phi_1 \circ \phi_0^{-1} = \phi_1 \circ \phi_2^{-1} \circ \phi_2 \circ \phi_0^{-1} \colon \phi_0(U_0 \cap U_1 \cap U_2) \to \phi_1(U_0 \cap U_1 \cap U_2)$$

is the composition of two smooth maps, and so ϕ_{10} is smooth at $x \in U_{01}$. □

A consequence of this lemma is that to specify a manifold structure on a topological space M, you need only specify a single atlas on M. Then this guarantees that there is a unique smooth structure that comes from that atlas.

1.1 Dimension

Definition 1.19. Let $(M, \mathcal{A})$ be a (smooth) manifold. Let $(U, \phi\colon U \to \mathbb{R}^n)$ and $(V, \psi\colon V \to \mathbb{R}^m)$ be two charts at $x \in M$. By compatibility, we have that $m = n$. Define $\dim_x(M) = n$, the **dimension of** M **at** x.

Remark 1.20. Note that $\dim_y(M) = n$ for any $u \in U$. So for each n, in the set

$$M_n := \{x \in M \mid \dim_x(M) = n\}$$

is open. Therefore,

$$\{x \in M \mid \dim_x(M) \neq n\} = \bigcup_{m \in \mathbb{N} \setminus \{n\}} \{x \mid \dim_x(M) = m\}$$

is also open, as the union of open sets. Hence, M_n is open and closed, so M_n is a union of connected components of M.

Definition 1.21. We say that M is **pure of dimension** n if each connected component has the same dimension n.

Remark 1.22 (Notation)**.** If M is an n-dimensional manifold, then we often say that M is an n**-manifold** and use the notation M^n.

Theorem 1.23 ((Kervaire, 1960)). *Not every topological manifold has a smooth structure. Kervaire gave a 10-dimensional example.*

Remark 1.24 (Convention). Usually we denote a manifold $(M, \mathcal{A})$ by just M, omitting the atlas $\mathcal{A}$. When we say "a chart on M," we mean a chart in the smooth structure $\mathcal{A}$.

1.2 Lots of Examples

Example 1.25.

(1) Let E be a finite-dimensional real vector space. Choose a linear isomorphism $\phi \colon E \to \mathbb{R}^n$. This makes E into a topological space by declaring $U \subseteq E$ to be open if its image $\phi(U) \subseteq \mathbb{R}^n$ is open. This defines a topology on E, and (E, ϕ) is a chart. Let $\mathcal{A}$ be the smooth structure defined by this chart.

This smooth structure is independent of ϕ. Reason: if ψ is another choice of linear isomorphism $E \cong \mathbb{R}^n$, then $\psi \circ \phi^{-1} \colon \mathbb{R}^n \to \mathbb{R}^n$ is linear, and hence smooth. So (E, ψ) also extends to the same smooth structure.

(2) Any set M equipped with the discrete topology is a zero-dimensional manifold. The charts are of the form $(\{x\}, \phi \colon \{x\} \to \mathbb{R}^0)$.

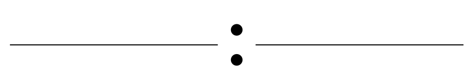
(3) Let $(M, \mathcal{A})$ be a manifold and $U \subseteq M$ an open subset. Let $\mathcal{A}_U$ be the collection of charts $(V, \psi) \in \mathcal{A}$ with $V \subseteq U$. Then $\mathcal{A}_U$ is a smooth structure on U, called the **induced smooth structure**. $(U, \mathcal{A}_U)$ is an **open submanifold**.

(4) The product $M = M_1 \times M_2$ of two manifolds M_1 and M_2 is a manifold. Given charts $(U_1, \phi_1 \colon U_1 \to \mathbb{R}^{n_1})$ on M_1 and $(U_2, \phi_2 \colon U_1 \to \mathbb{R}^{n_2})$ on M_2, we can form their **product chart** (U, ϕ) with $U = U_1 \times U_2$ and $\phi = \phi_1 \times \phi_2$, that is,

$$\phi(x_1, x_2) = (\phi_1(x_1), \phi_2(x_2)) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} = \mathbb{R}^{n_1+n_2}.$$

(5) The line with two origins. Let $\widehat{M} = \mathbb{R} \times \{0, 1\}$. Define an equivalence relation on $\widehat{M}$ generated by $(x, 0) \sim (x, 1)$ for all $x \neq 0$. Let $M = \widehat{M}/\sim$ be the quotient space. Denote by $[x, 0]$ the equivalence class of $(x, 0)$.

Let $\pi \colon \widehat{M} \to M$ be the map that takes a point to its equivalence class. M has the topology that a set $U \subseteq M$ is open if $\pi^{-1}(U)$ is open. The picture goes like this:



$M = U_0 \cup U_1$ is the union of two open sets U_0, U_1, where $U_0 = \pi(\mathbb{R} \times \{0\})$ and $U_1 = \pi(\mathbb{R} \times \{1\})$. Define charts $\phi_0 \colon U_0 \to \mathbb{R}$ by $\phi_0([x, 0]) = x$ and $\phi_1 \colon U_1 \to \mathbb{R}$ by $\phi_1([x, 1]) = x$.

What's the point of this? Well, "it has two origins which is in some people's opinion undesirable." M is furthermore not Hausdorff! The two origins are in each other's closure.

Remark 1.26 (Convention)**.** Henceforth in this class we consider only manifolds which are

(1) **pure**: each connected component is the same dimension

(2) Hausdorff

(3) **second-countable**: there is a sequence $U_1, U_2, \ldots, U_n, \ldots$ such that every open set U in M is the union of some subcollection of the U_i's. Equivalently, the topology on M has a countable basis.

Example 1.27 (Continued from Example 1.25)**.**

(6) The n**-sphere** is

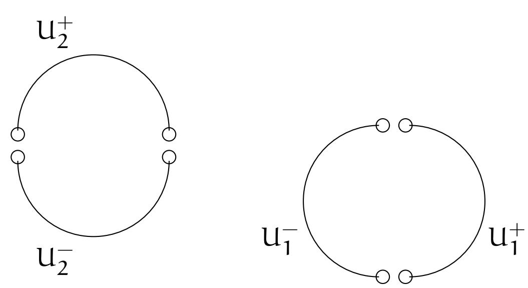
$$S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\},$$

where $\|x\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$ is the Euclidean norm. Give S^n the subspace topology. Define open subsets

$$U_i^+ = \{x \in S^n \mid x_i > 0\}$$

$$U_i^- = \{x \in S^n \mid x_i < 0\}$$

for $i = 1, \ldots, n+1$. For example, when $n = 1$, we have



For charts, define $\phi_i^\pm \colon U_i^\pm \to \mathbb{R}^n$ by

$$\phi_i^\pm(x) = (x_1, \ldots, x_{i-1}, \widehat{x_i}, x_{i+1}, \ldots, x_{n+1})$$

where the hat indicates that the i-th coordinate should be omitted. $\phi_i^\pm$ is a homeomorphism from $U_i^\pm$ onto $\{y \in \mathbb{R}^n \mid \|y\| < 1\}$. The transition map for $i < j$ from $\phi_i^\pm$ to $\phi_j^\pm$ is

$$\phi_j^\pm \circ (\phi_i^\pm)^{-1}(y) = \left(y_1, \ldots, y_{i-1}, \pm\sqrt{1 - \|y\|^2}, y_i, \ldots, \widehat{y_j}, \ldots y_n\right).$$

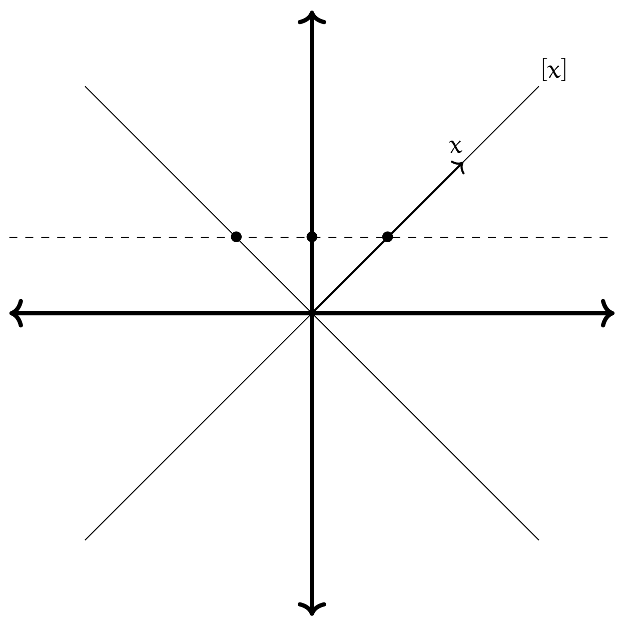
This is smooth for $\|y\| < 1$.

(7) The n**-dimensional real projective space** $\mathbb{P}^n(\mathbb{R})$ or $\mathbb{RP}^n$ or $\mathbb{P}^n_\mathbb{R}$ is the set of all lines (i.e. 1-dimensional linear subspaces) of $\mathbb{R}^{n+1}$.

For $x \in \mathbb{R}^{n+1}$, let $[x] = \mathbb{R}x$ be the line spanned by x. Define

$$\begin{array}{rccc} \tau\colon & \mathbb{R}^{n+1} \setminus \{0\} & \to & \mathbb{P}^n(\mathbb{R}) \\ & x & \mapsto & [x] \end{array}$$

Notice that τ is surjective and $\tau(x) = \tau(y)$ if and only if $y = \lambda x$ for some $\lambda \neq 0$. Given $\mathbb{P}^n(\mathbb{R})$ the quotient topology with respect to τ. That is, $U \subseteq \mathbb{PR}^n$ is open if and only if $\tau^{-1}(U)$ is open in $\mathbb{R}^{n+1} \setminus \{0\}$.



For $i = 1, 2, \ldots, n$, let

$$U_i = \{[x] \in \mathbb{P}^n(\mathbb{R}) \mid x_i \neq 0\}.$$

This is open. Define $\phi_i\colon U_i \to \mathbb{R}^n$ by

$$\phi_i([x]) = \frac{1}{x_i}(x_1, \ldots, \widehat{x_i}, \ldots, x_{n+1}).$$

Also define

$$\begin{aligned} \rho_i\colon \quad \mathbb{R}^n &\to \mathbb{R}^{n+1} \\ y &\mapsto (y_1, \ldots, y_{i-1}, 1, y_i, \ldots, y_n) \end{aligned}$$

Then

$$\rho_i(\phi_i([x])) = \left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, 1, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{n+1}}{x_i}\right).$$

So we get that

$$\tau(\rho_i(\phi_i([x]))) = [x],$$

and

$$\phi_i(\tau(\rho_i(y))) = [y_1, \ldots, y_{i-1}, 1, y_i, \ldots, y_n] = y.$$

So $\tau \circ \rho_i = \phi_i^{-1} \colon \mathbb{R}^n \to U_i$. This lets us compute the transition maps.

$$\phi_j \circ \phi_i^{-1}(y) = \left(\frac{y_1}{y_j}, \ldots, \widehat{y_j}, \ldots, \frac{y_{i-1}}{y_j}, \frac{1}{y_j}, \frac{y_i}{y_j}, \ldots, \frac{y_n}{y_j}\right)$$

is smooth on it's domain (which is $y_j \neq 0$).

This shows that $\mathbb{P}^n\mathbb{R}$ is an n-manifold.

We could have just as well used $\mathbb{C}$ instead of $\mathbb{R}$. Then $\mathbb{P}^n\mathbb{C}$ is the space of all lines (i.e. 1-dimensional *complex* subspaces of $\mathbb{C}^{n+1}$). This would be a manifold of dimension $2n$ instead of n.

If you're feeling adventurous, you could make projective space over the quaternions instead of $\mathbb{R}$ or $\mathbb{C}$. $\mathbb{P}^n(\mathbb{H})$ is the space of all lines (i.e. 1-dimensional quaternionic subspaces in $\mathbb{H}^{n+1}$). This is a $4n$-manifold.

Why is $\mathbb{P}^n(\mathbb{R})$ Hausdorff and second countable?

Lemma 1.28.

(i) The quotient map $\tau\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{P}^n(\mathbb{R})$ *is an open map.*

(ii) $\mathbb{P}^n(\mathbb{R})$ *is second countable.*

Proof.

(i) Let $V \subseteq \mathbb{R}^{n+1} \setminus \{0\}$. Is $\tau(V)$ open? To answer this, we want to know if $\tau^{-1}(\tau(V))$ is open. But $\tau^{-1}(\tau(V)) = \bigcup_{\lambda \neq 0} \lambda V$ is the union of open sets and therefore open.

(ii) Let $\{V_i\}$ be a countable basis of the topology on $\mathbb{R}^{n+1} \setminus \{0\}$. Then by (i), $\{\tau(V_i)\}$ is a countable basis for the topology on $\mathbb{P}^n(\mathbb{R})$. □

Lemma 1.29. *Let* X *be any topological space and* R *an equivalence relation on* X. *Let* $Y = X/R$ *with the quotient topology. Then* Y *is Hausdorff if*

(i) the graph of R *is closed in* $X \times X$*, and*

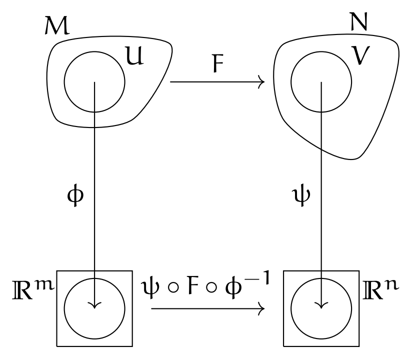
(ii) the quotient map $X \to Y$ *is open.*

1.3 The Smooth Category

Definition 1.30. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be (smooth) manifolds and $F\colon M \to N$ a map. F is called **smooth** if

(i) F is continuous

(ii) the expression $\psi \circ F \circ \phi^{-1}\colon \phi(U \cap F^{-1}(V)) \to \psi(V)$ is smooth for any pair of charts (U, ϕ) of M and (V, ψ) of N.



Definition 1.31. We call the charts (U, ϕ), (V, ψ) **adapted to** F if $U \subseteq F^{-1}(V)$.

To know that smooth manifolds with smooth maps between them forms a category, we need to know that composition of smooth maps is smooth. But this follows from the chain rule. Hence, smooth manifolds form a **category** $\mathcal{C}^\infty$ with

- objects: smooth manifolds
- arrows: smooth maps.

Definition 1.32. We denote by $\mathcal{C}^\infty(M, N)$ the set of all smooth maps $M \to N$, and $\mathbb{C}^\infty(M) = \mathbb{C}^\infty(M, \mathbb{R})$ (or sometimes $\mathbb{C}^\infty(M, \mathbb{C})$).

Remark 1.33. To check that $F\colon M \to N$ is smooth, we need only check condition (ii) for all pairs of adapted charts C, C' where C ranges over *some* atlas $\mathcal{A}_0 \subseteq \mathcal{A}$.

Example 1.34. Let $M = \mathbb{R}^{n+1} \setminus \{0\}$, and let $N = \mathbb{P}^n(\mathbb{R})$. Let $F = \tau\colon M \to N$ be the quotient map , $\tau(x) = [x]$. Then F is continuous. Let $V_i = \{x \in \mathbb{R}^{n+1} \setminus \{0\} \mid x_i \neq 0\}$. Then $\mathcal{A}_0 = \{(V_i, \mathrm{id}_{V_i}) \mid i = 1, \ldots, n\}$ is an atlas on $\mathbb{R}^{n+1} \setminus \{0\}$.

$$\tau(V_i) = U_i = \{[x] \in N \mid x_i \neq 0\}.$$

Recall that

$$\phi_i([x]) = \frac{1}{x_i}(x_1, \ldots, \widehat{x_i}, \ldots, x_{n+1}) \in \mathbb{R}^n.$$

$\{(U_i, \phi_i) \mid i = 1, \ldots, n\}$ is an atlas on N. The expression for F in these charts is

$$\phi_i \circ F \circ \mathrm{id}(x) = \phi_i([x]) \colon V_i \to \mathbb{R}^n.$$

This is smooth.

Example 1.35. If $M = \mathbb{S}^n$, $N = \mathbb{R}^{n+1}$, then the inclusion map $\mathbb{S}^n \to \mathbb{R}^{n+1}$ is smooth.

2 Tangent Vectors

Definition 2.1. Given a manifold M, consider the set of triples (c, x, h) where $c = (U, \phi\colon U \to \mathbb{R}^n)$ is a chart, $x \in U$ and $h \in \mathbb{R}^n$. Call two triples equivalent,

$$(c_1, x_1, h_1) \sim (c_2, x_2, h_2)$$

if $x_1 = x_2$ and

$$h_2 = D(\phi_2 \circ \phi_1^{-1})_{\phi_1(x)} h_1 = D(\phi_{21})_{\phi_1(x)} h_1.$$

An equivalence class $[c, x, h]$ is a **tangent vector to** M **at** x.

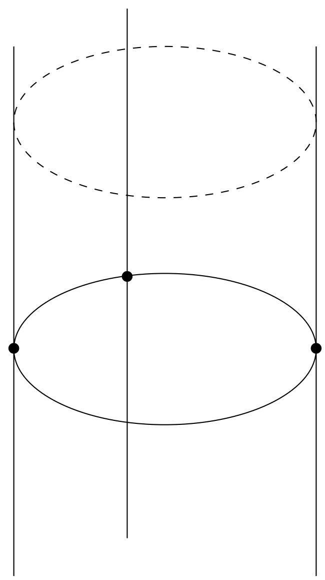
Definition 2.2. The **tangent bundle** TM is the collection of all of all tangent vectors to M, with the projection $\pi = \pi_M \colon TM \to M$, $\pi([c, x, h]) = x$.

Definition 2.3. The **tangent space to** M **at** x is $T_xM = \pi^{-1}(x)$.

Notice that

$$TM = \coprod_{x \in M} T_xM.$$

Example 2.4. For $M = \mathbb{S}^1$, the picture of the tangent bundle looks like this:



Lemma 2.5. *Let $x \in M$ and $c = (U, \phi)$ a chart at x. Then*

(i) The map $\mathbb{R}^n \to T_x M$ defined by $h \mapsto [c, x, h]$ is a bijection.

(ii) Let $d_x\phi \colon T_x M \to \mathbb{R}^n$ be the inverse of this bijection. Let $c_1 = (U_1, \phi_1)$ be another chart at x. Then

$$\begin{array}{ccc} & T_x M & \\ {}^{d_x\phi}\swarrow & & \searrow^{d_x\phi_1} \\ \mathbb{R}^n & \xrightarrow{\qquad\qquad} & \mathbb{R}^n \end{array}$$

$$d_x\phi_1 \circ (d_x\phi)^{-1} = D(\phi_1 \circ \phi^{-1})_{\phi(x)}$$

Proof.

(i) If $[c, x, h_1] = [c, x, h_2]$, then $h_2 = D(\phi \circ \phi^{-1})h_1 = h_1$, so the map is injective. If $v \in T_x M$ then $v = [c_0, x, h_0]$ for some chart $c_0 = (U_0, \phi_0)$ at x. Then $v = [c, x, h]$ with $h = D(\phi \circ \phi_0^{-1})_{\phi_0(x)} h_0$. So the map is surjective.

(ii)

$$\begin{aligned} d_x\phi_1 \circ (d_x\phi)^{-1}(h) &= d_x\phi_1([c, x, h]) \\ &= d_x\phi_1([c_1, x, D(\phi_1 \circ \phi^{-1})_{\phi(x)} h]) \\ &= D(\phi_1 \circ \phi^{-1})_{\phi(x)} h \end{aligned}$$

□

Now we can endow T_xM with a vector space structure by declaring $d_x\phi$ to be a linear isomorphism. This is independent of the chart by the previous lemma.

Let V be an open submanifold of M. Then each chart $c = (U, \phi)$ on V is a chart on M. So for $x \in U$, $h \in \mathbb{R}^n$, we have tangent vectors $[c, x, h]_V \in T_xV$ and $[c, x, h]_M \in T_xM$. There is no real distinction between them, and we'll treat them as if they were the same. Both T_xM and T_xV are isomorphic to $\mathbb{R}^n$, with isomorphism given by $d_x\phi$.

$$\begin{array}{ccccccc}
[c,x,h]_V & \in & T_xV & \xrightarrow{\cong} & T_xM & \ni & [c,x,h]_M \\
& \searrow & \downarrow{\scriptstyle d_x\phi} & & \downarrow{\scriptstyle d_x\phi} & \swarrow & \\
& & h \in & \mathbb{R}^n & \ni h & &
\end{array}$$

We identify T_xM with T_xV, and the tangent bundle of V is $TV = \pi_M^{-1}(V)$.

Definition 2.6. We call the isomorphism $d_x\phi\colon T_xM \xrightarrow{\sim} \mathbb{R}^n$ the **derivative of** ϕ **at** x.

Example 2.7. If $M = U$ is an open subset of $\mathbb{R}^n$, then let $c = (U, \mathrm{id}_U)$ be the identity chart. Then $d_x(\mathrm{id}_U)\colon T_xU \xrightarrow{\sim} \mathbb{R}^n$. So the map $TU \to U \times \mathbb{R}^n$ given by $[c, x, h] \mapsto (x, h)$ is a bijection.

We again *identify* $TU = U \times \mathbb{R}^n$. Then $\pi\colon TU \to U$ is given by $\pi(x, h) = x$.

Definition 2.8. Let $F\colon M^m \to N^n$ be a smooth map. Choose F-adapted charts (U, ϕ) and (V, ψ). The $x \in U$, $h \in \mathbb{R}^n$, $v = [c, x, h]$, we define the **tangent map** by $T_xF\colon T_xM \to T_{F(x)}N$

$$T_xF(v) = TF([c, x, h]) = \left[c', F(x), D(\psi \circ F \circ \phi^{-1})_{\phi(x)}h\right].$$

Lemma 2.9. $T_xF\colon TM \to TN$ *maps* T_xM *linearly to* $T_{F(x)}N$.

Proof. By the definition of T_xF, the following diagram commutes.

$$\begin{array}{ccc}
T_xM & \xrightarrow{T_xF} & T_{F(x)}N \\
\downarrow{\scriptstyle d_x\phi} & & \downarrow{\scriptstyle d_{f(x)}\psi} \\
\mathbb{R}^m & \xrightarrow{D(\psi\circ F\circ\phi^{-1})_{\phi(x)}} & \mathbb{R}^n
\end{array}$$

Hence, T_xF is the composite of linear maps. □

Lemma 2.10. *The tangent map is well-defined, that is, it doesn't depend on the choice of charts.*

Proof. Let $x \in M$. Consider two pairs of adapted charts $c_1 = (U_1, \phi_1), c_1' = (V_1, \psi_1)$ and $c_2 = (U_2, \phi_2), c_2' = (V_2, \psi_2)$. With respect to each pair, we get an expression for F. Let $\widetilde{F}_i = \psi_i \circ F \circ \phi_i^{-1}$ be this expression for F for $i = 1, 2$. Then we can compare the two expressions via

$$\widetilde{F}_2 = \psi_{21} \circ \widetilde{F}_1 \circ \phi_{21}^{-1}.$$

Let $h_1, h_2 \in \mathbb{R}^n$. Suppose that $[c_1, x, h_1] = [c_2, x, h_2]$. Then

$$h_1 = D(\phi_{12})_{\phi_2(x)} h_2 \tag{1}$$

So

$$\begin{aligned}
\left[c_2', F(x), D\widetilde{F_2}_{\phi_2(x)} h_2\right] &= \left[c_2', F(x), D(\psi_{21} \circ \widetilde{F}_1 \circ \phi_{21}^{-1})_{\phi_2(x)} h_2\right] \\
&= \left[c_2', F(x), D(\psi_{21})_{\psi_1(F(x))} \circ D(\widetilde{F}_1)_{\phi_1(x)} h_1\right] \quad \text{by (1), chain rule} \\
&= \left[c_2, F(x), D(\widetilde{F}_1)_{\phi_1(x)} h_1\right] \\
&= TF(v)
\end{aligned}$$

□

Example 2.11. Let $U \subseteq \mathbb{R}^m$ be open and $f\colon U \to \mathbb{R}^n$ smooth. Then $Tf\colon U \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^n$. If c is the identity chart on U and c' is the identity chart on $\mathbb{R}^n$, then

$$Tf(x, h) = Tf([c, x, h]) = [c', f(x), Df_x h] = (f(x), Df_x h).$$

So Tf records both f and Df.

Let $c = (U, \phi\colon U \to \mathbb{R}^n)$ be a chart on a manifold M. Then

$$T\phi\colon TU \to \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$$

is given by $T\phi([c, x, h]) = (\phi(x), h)$. Writing $v = [c, x, h]$,

$$T\phi(v) = (\phi(x), d_x\phi(v)).$$

For each x, the map $T_x\phi\colon T_xU \xrightarrow{\sim} \{\phi(x)\} \times \mathbb{R}^n$ is bijective. So we conclude $T\phi\colon TU \to \phi(U) \times \mathbb{R}^n$ is a bijection. The codomain $\phi(U) \times \mathbb{R}^n$ is an open subset of $\mathbb{R}^{2n}$.

Definition 2.12. The pair $Tc = (TU, T\phi)$ is a chart (in the sense of homework 1) on TM, called a **tangent chart**.

Theorem 2.13.

(i) *The tangent charts* $\mathsf{T}c$ *form an atlas (again in the sense of homework 1) on* $\mathsf{T}M$*, and hence* $\mathsf{T}M$ *is a* $2n$*-manifold. (The topology on* $\mathsf{T}M$ *is that we declare* $V \subseteq \mathsf{T}M$ *to be* ***open*** *if* $\mathsf{T}\phi(V \cap U)$ *is open in* $\mathbb{R}^{2n}$ *for every chart* (U, ϕ) *on* M*.)*

(ii) *For every smooth* $f\colon M \to N$*, the tangent map* $\mathsf{T}f\colon \mathsf{T}M \to \mathsf{T}N$ *is smooth.*

Proof.

(i) Let $c_1 = (U, \phi)$, $c_2 = (U_2, \phi_2)$ be charts on M. What is the transition map $\mathsf{T}c_1 \to \mathsf{T}c_2$?

For $x \in U_1 \cap U_2$, $v \in T_xM$, we have the tangent charts

$$T_x\phi_i(v) = (\phi_i(x), d_x\phi_i(v))$$

for $i = 1, 2$. So

$$T_x\phi_2 \circ (T_x\phi_1)^{-1}\colon \phi_1(U_1 \cap U_2) \times \mathbb{R}^n \longrightarrow \phi_2(U_1 \cap U_2) \times \mathbb{R}^n$$

$$T_x\phi_2 \circ (T_x\phi_1)^{-1}(y, h) = \left(\phi_2 \circ \phi_1^{-1}(y), D(\phi_2 \circ \phi_1^{-1})_{\phi(x)}h\right)$$

Both $\phi_2 \circ \phi_1^{-1}$ and $D(\phi_2 \circ \phi_1^{-1})$ are smooth, so this defines a smooth map between subsets of $\mathbb{R}^{2n}$. Hence, the transition maps are smooth, and this checks that the tangent charts define an atlas.

(ii) Let $c = (U, \phi\colon U \to \mathbb{R}^n)$. Let $c' = (V, \psi\colon V \to \mathbb{R}^n)$ be adapted charts on M and N, respectively. Then $\mathsf{T}c$ is a chart on $\mathsf{T}M$, and $\mathsf{T}c'$ is a chart on $\mathsf{T}N$. We can express $\mathsf{T}f$ in these charts:

$$\begin{array}{rccc} \mathsf{T}\psi \circ \mathsf{T}f \circ (\mathsf{T}\phi)^{-1}\colon & \phi(U) \times \mathbb{R}^n & \longrightarrow & \psi(V) \times \mathbb{R}^n \\ & (y, h) & \longmapsto & \left(\psi \circ f \circ \phi^{-1}(y), D(\psi \circ f \circ \phi^{-1})_y h\right) \end{array}$$

This is again a smooth pair of maps, so it is smooth again. □

Theorem 2.14 (Chain rule)**.** T *is a functor from* $\mathcal{C}^\infty$ *to itself. More precisely, this means*

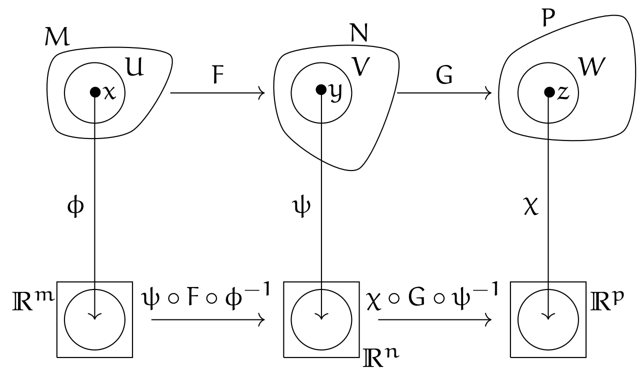
(1) *for each smooth manifold* M*, we get a new smooth manifold* $\mathsf{T}M$*, and*

(2) *for each smooth map* $F\colon M \to N$*, we get a smooth map* $\mathsf{T}F\colon \mathsf{T}M \to \mathsf{T}N$

in such a way that T *respects identity and composition.*

Proof. We've already seen that $\mathsf{T}M$ is a smooth manifold, and that the tangent map $\mathsf{T}F$ is smooth. It's clear that T respects identities, so we will only check that T respects composition.

Let M, N, P be smooth manifolds of dimensions m, n and p, respectively. Let $F\colon M \to N$ and $G\colon N \to P$. Let $x \in M$ and $y = F(x) \in N$, $z = G(y) \in P$. Choose charts $c = (U, \phi)$ at x, $c' = (V, \psi)$ at y, and $c'' = (W, \chi)$ at z in M, N, and P, respectively.



We may assume that $F(U) \subseteq V$, $G(V) \subseteq W$. Express F and G in coordinates as

$$\widetilde{F} = \psi \circ F \circ \phi^{-1}, \qquad \widetilde{G} = \chi \circ G \circ \psi^{-1}.$$

Let $H = G \circ F$; in coordinates, the expression for H is

$$\begin{aligned} \widetilde{H} &= \chi \circ H \circ \phi^{-1} \\ &= \chi \circ G \circ F \circ \phi^{-1} \\ &= \chi \circ G \circ \psi^{-1} \circ \psi \circ F \circ \phi^{-1} \\ &= \widetilde{G} \circ \widetilde{F} \end{aligned}$$

So for $v = [c, x, k] \in T_x M$, we have on one hand

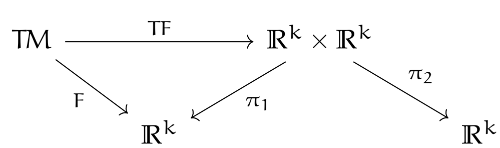
$$TH(v) = [c'', z, D(\widetilde{H})_{\phi(x)} k]$$

but on the other hand,

$$TG(TF(v)) = TG([c', y, D(\widetilde{F})_{\phi(x)} k]) = [c'', z, D(\widetilde{G})_{\psi(y)} D(\widetilde{F})_{\phi(x)} k]$$

and these are equal by the usual chain rule for $\mathbb{R}^n$. □

Example 2.15. In the special case when $N = \mathbb{R}^k$, then if $F\colon M \to \mathbb{R}^k$ is smooth, then $TF\colon TM \to T\mathbb{R}^k = \mathbb{R}^k \times \mathbb{R}^k$ sends $T_x M$ to $T_{F(x)}\mathbb{R}^k = \{F(x)\} \times \mathbb{R}^k$ and the following commutes



In other words, $\pi_1(TF(v)) = F(x)$ if $v \in T_x M$.

Define $dF(v) = \pi_2(TF(v))$. For $v \in T_x M$, $T_x F(v) = (F(x), d_x F(v))$. So for each x,

$$d_x F \colon T_x M \to \mathbb{R}^k$$

is linear. Then $d_x F(v)$ is the **directional derivative** of f at x along v.

If $c = (U, \phi)$ is a chart at x, and if $v = [c, x, h]$, with $h \in \mathbb{R}^n$, and $\widetilde{F} = F \circ \phi^{-1}$, then

$$T_x F(v) = (F(x), D\widetilde{F}_{\phi(x)} h).$$

So

$$d_x F(v) = D\widetilde{F}_{\phi(x)} h = \lim_{t \to 0} \frac{\widetilde{F}(\phi(x) + th) - \widetilde{F}(\phi(x))}{t}.$$

This explains the name directional derivative.

Definition 2.16. A linear map $T_x M \to \mathbb{R}$ is a **cotangent vector** to M at x. The **cotangent space** at x is $(T_x M)^*$, usually written $T_x^* M$.

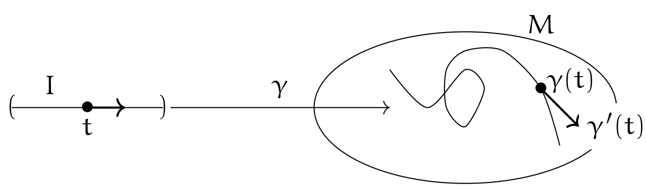
So if $f \colon M \to \mathbb{R}$ is smooth, for each $x \in M$ we have $d_x f \in T_x^* M$.

Example 2.17. For a special case, let $I \subseteq \mathbb{R}$ be an open interval and let $\gamma \colon I \to M$ be a smooth map, called a **smooth path** in M.

Then $T\gamma \colon TI = I \times \mathbb{R} \to TM$. For each $t \in I$,

$$T_t \gamma \colon \{t\} \times \mathbb{R} \to T_{\gamma(t)} M$$

is linear, so determined by it's value $T_t \gamma(t, 1) = \gamma'(t) \in T_{\gamma(t)} M$. This is often called the **velocity vector** at time t.



Lemma 2.18. *For every $x \in M$ and $v \in M$, there is a path $\gamma \colon I \to M$ with $0 \in I$, $\gamma(0) = x$, and $\gamma'(0) = v$.*

2.1 Derivations

Definition 2.19. Let $x \in M$ and let U, V be open neighborhoods of x, with $f \colon U \to \mathbb{R}$, $g \colon V \to \mathbb{R}$ smooth functions. Call f and g equivalent if there is an open neighborhood of x, $W \subseteq U \cap V$, such that $f|_W = g|_W$. The equivalence class of f is called the **germ** of f at x. We use the notation $[f]$ or $[f]_x$ to denote the germ of f at x.

Definition 2.20. The set of all germs at x is denoted $C^\infty_{M,x}$.

If $[f]_x$, $[g]_x$ are germs at x, then $[f+g]_x$ and $[fg]_x$ and $[cf]_x$ are well-defined germs, for $c \in \mathbb{R}$. Hence, $C^\infty_{M,x}$ is a commutative unital $\mathbb{R}$-algebra.

Remark 2.21. Said slightly differently, $C^\infty_{M,x}$ is the colimit over all open U containing x of $C^\infty(U)$,

$$C^\infty_{M,x} = \operatorname*{colim}_{U \ni x} C^\infty(U).$$

Definition 2.22. The **evaluation map** $\mathrm{ev} = \mathrm{ev}_x$, is defined by

$$\begin{aligned} \mathrm{ev}_x \colon \quad C^\infty_{M,x} &\longrightarrow \mathbb{R} \\ [f]_x &\longmapsto f(x) \end{aligned}$$

$\mathrm{ev}\colon C^\infty_{M,x} \to \mathbb{R}$ is a morphism of $\mathbb{R}$-algebras with unit.

Definition 2.23. A **derivation** of M at x is an $\mathbb{R}$-linear map $\ell\colon C^\infty_{M,x} \to \mathbb{R}$ satisfying the **Leibniz rule**:

$$\ell([fg]) = \ell([f])g(x) + f(x)\ell([g]).$$

Definition 2.24. Let $\mathcal{D}_x M$ be the set of all derivations of M at x. Then

$$\mathcal{D}_x M \subseteq \mathrm{Hom}_{\mathbb{R}}(C^\infty_{M,x}, \mathbb{R})$$

is a linear subspace.

Lemma 2.25. *Let $\ell \in \mathcal{D}_x M$. Then $\ell(1) = 0$.*

Proof. $\ell(1) = \ell(1 \cdot 1) = \ell(1)\cdot 1 + 1 \cdot \ell(1) = 2\ell(1) \implies \ell(1) = 0$ □

Example 2.26. Let $M = \mathbb{R}^n$, and $x = 0$. For $i = 1, 2, \ldots, n$, define

$$\ell_i([f]_0) = \frac{\partial f}{\partial x_i}(0).$$

For shorthand, we write that

$$\ell_i = \frac{\partial}{\partial x_i}\bigg|_{x=0}.$$

Then $\ell_i \in \mathcal{D}_0\mathbb{R}^n$. Hence,

$$\ell = \sum_{i=1}^n c_i \ell_i \in \mathcal{D}_0\mathbb{R}^n$$

is a derivation at 0 as well, for $c_1, \ldots, c_n \in \mathbb{R}$. Note

$$\ell([f]_0) = \sum_{i=1}^n c_i \frac{\partial f}{\partial x_i}(0) = Df_0\vec{c},$$

where $\vec{c}$ is the vector with components $c_i \in \mathbb{R}$. This means that ℓ is the directional derivative operator along $\vec{c}$ evaluated at zero.

Lemma 2.27. *Keep the same notation as in Example 2.26. Then $\ell_1, \ldots, \ell_n$ form a basis of $\mathcal{D}_0\mathbb{R}^n$. Hence every derivation $\ell \in \mathcal{D}_0\mathbb{R}^n$ is a directional derivative, that is, $\ell([f]_0) = Df_0\vec{v}$, for a unique $\vec{v} \in \mathbb{R}^n$.*

Proof. Let $\ell = \mathcal{D}_0\mathbb{R}^n$. Let $[f] \in C^\infty_{\mathbb{R}^n,0}$. To show that ℓ_i are spanning, write

$$f(x) = f(0) + \sum_{i=0}^{n} x_i f_i(x)$$

using Taylor's Theorem, where $f_i \in C^\infty(U_i)$ for some $U_i \ni 0$ open, and

$$f_i(0) = \frac{\partial f}{\partial x_i}(0).$$

So now we have that

$$\begin{aligned}
\ell([f]_0) &= \ell([f(0)]) + \sum_{i=1}^{n} \ell([x_i f_i]) \\
&= 0 + \sum_{i=1}^{n} \left(\ell([x_i]) \frac{\partial f}{\partial x_i}(0) + 0\ell([f_i]) \right) \\
&= \sum_{i=1}^{n} \ell([x_i]) \ell_i([f]) \\
&= \sum_{i=1}^{n} v_i \ell_i([f]) \\
&= \mathcal{D}f_0\vec{v}
\end{aligned}$$

where $v_i = \ell([x_i])$. This demonstrates that the ℓ_i are spanning.

To verify that the ℓ_i are independent, suppose that

$$\ell = \sum_{i=1}^{n} v_i \ell_i = 0.$$

Then $\mathcal{D}f_0\vec{v} = 0$ for all $[f] \in C^\infty_{\mathbb{R}^n,0}$. In particular, for $f(x) = x_i$, then we see that $v_i = 0$. □

Now for M an arbitrary n-manifold with $x \in M$, $v \in T_xM$, define the derivation

$$\ell_v \colon C^\infty_{M,x} \to \mathbb{R}$$

by $\ell_v([f]_x) = d_x f(v)$. What does this mean? If we choose a chart $c = (U, \phi)$ at x, such that $v = [c, x, h]$ and set $\widetilde{f} = f \circ \phi^{-1}$, then

$$d_x f(v) = D\widetilde{f}_{\phi(x)} h.$$

So for each $v \in T_xM$, ℓ_v is a derivation.

Theorem 2.28. *The map* $\mathcal{L}_x \colon T_xM \to \mathcal{D}_xM$ *defined by* $\mathcal{L}_x(v) = \ell_v$ *is an isomorphism of vector spaces.*

To prove this theorem, we first need a few definitions and lemmas.

Definition 2.29. For a smooth map $F \colon M \to N$ and $x \in M$ and $y = F(x) \in N$, define the **pullback** of germs

$$\begin{array}{rccc} F^*\colon & C^\infty_{N,y} & \longrightarrow & C^\infty_{M,x} \\ & [g]_y & \longmapsto & [g \circ F]_x. \end{array}$$

So the map on germs goes backwards compared to the map of smooth manifolds. But on derivations, we get a map in the forward direction.

Definition 2.30. If $F \colon M \to N$ is smooth, define the **pushforward** of derivations $F_* \colon \mathcal{D}_xM \to \mathcal{D}_yN$ by

$$F_*(\ell)([g]_y) = \ell(F^*[g]_y)$$

Then $F_*(\ell)$ is a derivation of N at y. We will sometimes use the alternative notation $F_* = \mathcal{D}_xF \colon \mathcal{D}_xM \to \mathcal{D}_yN$.

Pushbacks and pullforwards come with their own versions of the chain rule.

Lemma 2.31. *If* $F \colon M \to N$ *and* $G \colon N \to P$ *are smooth maps of manifolds, and* $x \in M$, $y = F(x)$, $z = G(y)$, *then*

$$(G \circ F)^* = F^* \circ G^*.$$

Proof.

$$\begin{aligned} (F^* \circ G^*)([h]_z) &= F^*(G^*[h]_z) \\ &= F^*([h \circ G]_y) \\ &= [h \circ G \circ F]_x = (G \circ F)^*[h]_z \end{aligned}$$

□

Lemma 2.32. *If* $F \colon M \to N$ *and* $G \colon N \to P$ *are smooth maps of manifolds, and* $x \in M$, $y = F(x)$, $z = G(y)$, *then*

$$(G \circ F)_* = G_* \circ F_*$$

Proof.

$$\begin{aligned} (G \circ F)_*(\ell) &= \ell \circ (G \circ F)^* \\ &= \ell \circ F^* \circ G^* \\ &= G_*(\ell \circ F^*) = (G_* \circ F_*)(\ell) \end{aligned}$$

□

Lemma 2.33. *Let* $F\colon M \to N$ *be smooth,* $x \in M$*,* $y = F(x)$*. Then the diagram*

$$\begin{array}{ccc} T_x M & \xrightarrow{T_x F} & T_y N \\ \downarrow{\scriptstyle \mathcal{L}_x} & & \downarrow{\scriptstyle \mathcal{L}_y} \\ \mathcal{D}_x M & \xrightarrow{F_* = \mathcal{D}_x F} & \mathcal{D}_y N \end{array}$$

commutes.

Proof. Let $v \in T_x M$, $[g]_y \in C^\infty_{N,y}$. Then let's chase the diagram. First counterclockwise starting in the top left.

$$\begin{aligned} \mathcal{D}_x F(\mathcal{L}_x(v))([g]_y) &= \mathcal{L}_x(v)(F^*([g]_y)) \\ &= \mathcal{L}_x(v)([g \circ F]_x) \\ &= d_x(g \circ F)(v) \end{aligned} \tag{2}$$

Second, clockwise starting in the top left.

$$\mathcal{L}_y(T_x F(v))([g]_y) = d_x g(T_x F(v)) \tag{3}$$

We have that (2) and (3) are equal by the chain rule. □

Now we can prove the theorem.

Proof of Theorem 2.28. Chose a chart $c = (U, \phi)$ centered at x. Then apply Lemma 2.33 to $F = \phi$ to get a commutative diagram

$$\begin{array}{ccc} T_x M & \xrightarrow{d_x \phi} & \mathbb{R}^n \\ \downarrow{\scriptstyle \mathcal{L}_x} & & \downarrow{\scriptstyle \mathcal{L}_0} \\ \mathcal{D}_x M & \xrightarrow{\phi_* = D_x \phi} & \mathcal{D}_0 \mathbb{R}^n \end{array}$$

Then the top arrow and the right arrow are isomorphisms, so $T_x M \cong D_0 \mathbb{R}^n$. Claim that ϕ_* is also an isomorphism.

To see that, notice that $\phi^*\colon C^\infty_{\mathbb{R}^n,0} \to C^\infty_{M,x}$ is also an isomorphism, so

$$\phi^*\colon \mathcal{D}_x M \xrightarrow{\sim} \mathcal{D}_0 \mathbb{R}^n$$

is also an isomorphism. Hence, $\mathcal{L}_x$ is an isomorphism as well because the diagram commutes. □

Now that we've proved Theorem 2.28, we can think of tangent vectors as derivations instead of equivalence classes of triples. We will in fact identify $T_x M = \mathcal{D}_x M$ and $T_x F = \mathcal{D}_x F$. The following remark explains why this works.

Remark 2.34. The isomorphism

$$\mathcal{L}_x \colon T_x M \xrightarrow{\sim} \mathcal{D}_x M = \mathrm{Der}(C^\infty_{M,x}, \mathbb{R})$$

defined by $\mathcal{L}_x(v)([f]_x) = d_x f(v)$ (the directional derivative) is a **natural isomorphism**, that is, a natural transformation between the functors $\mathcal{D}_x(-)$ and $T_x(-)$ that is an isomorphism.

Naturality means that for any smooth map $F \colon M \to N$, for any $x \in M$ with $y = F(x)$, then the following commutes

$$\begin{array}{ccc} T_x M & \xrightarrow{\mathcal{L}_x} & \mathcal{D}_x M \\ \downarrow{\scriptstyle T_x F} & & \downarrow{\scriptstyle F_* = \mathcal{D}_x F} \\ T_y N & \xrightarrow{\mathcal{L}_y} & \mathcal{D}_y N \end{array}$$

commutes.

Remark 2.35. On the next homework, you will prove that

$$\mathcal{D}_x M \cong \left({}^{\mathfrak{m}_x}/_{\mathfrak{m}_x^2} \right)^*,$$

where $\mathfrak{m}_x$ is the unique maximal ideal of $C^\infty_{M,x}$, given by the germs that vanish at x.

$$\mathfrak{m}_x = \ker(\mathrm{ev}_x \colon C^\infty_{M,x} \to \mathbb{R}) = \{[f]_x \in C^\infty_{M,x} \mid f(x) = 0\}.$$

Hence the cotangent space $T^*_x M$ is defined by

$$T^*_x M = \mathrm{Hom}_{\mathbb{R}}(T_x M, \mathbb{R}) \cong \mathcal{D}^*_x M \cong {}^{\mathfrak{m}_x}/_{\mathfrak{m}_x^2}.$$

This is often called the **Zariski cotangent space**.

3 Submanifolds

We still don't have a satisfactory way of constructing many manifolds. Most manifolds come as submanifolds of some other one, so talking about submanifolds will give us a good tool to construct more examples.

For $k \leq n$, we identify $\mathbb{R}^k$ with the subspace

$$\{x \in \mathbb{R}^n \mid x_{k+1} = x_{k+2} = \ldots = x_n = 0\} \subseteq \mathbb{R}^n.$$

This is the prototype for defining k-submanifolds of an n-manifold.

Definition 3.1. Let A be a subset of M with $x \in A$. A chart $c = (U, \phi)$ at x is a **submanifold chart** of dimension k for A at x if

$$A \cap U = \phi^{-1}(\mathbb{R}^k).$$

Definition 3.2. Let $c = (U, \phi)$ be a submanifold chart of dimension k for A at x. Let $\phi_A = \phi_{A \cap U}$. Then $c_A = (A \cap U, \phi_A)$ is the **restriction** of the chart to A.

Definition 3.3. Let $A \subseteq M$ and $x \in A$. We call A a **submanifold** (or **embedded submanifold**) of dimension k if there exists a collection of submanifold charts of dimension k for A whose domains cover A.

The restrictions of these charts to A define an atlas, and hence a smooth structure on A, called the **induced smooth structure**. The induced topology is the subspace topology.

Definition 3.4. If A is a k-dimensional submanifold of an n-dimensional manifold M, then the **codimension of** A is

$$\operatorname{codim}_M(A) = n - k.$$

Example 3.5. Open subsets of M are submanifolds of codimension zero.

Definition 3.6. Let X be a topological space, and $Y \subseteq X$. Then Y is **locally closed** if for every $y \in Y$, there is a neighborhood $U \ni y$, open in X, such that $Y \cap U$ is closed in U.

Remark 3.7. An equivalent definition of locally closed is that Y is of the form $Y = C \cap V$ with C closed in X and V open in X.

Lemma 3.8. *Let A be a k-dimensional submanifold of an n-manifold M. Then let $i\colon A \to M$ be the inclusion. Then*

(i) A is a locally closed subset of M;

(ii) the inclusion map $i\colon A \to M$ is smooth and $Ti\colon TA \to TM$ is injective.

(iii) Let $c = (U, \phi)$ be a submanifold chart for A. Then $Tc = (TU, T\phi)$ is a submanifold chart for $Ti(TA)$. For $x \in A$, we have $T_x i(T_x A) = (d_x\phi)^{-1}(\mathbb{R}^k)$.

(iv) $Ti(TA)$ is a submanifold of TM of dimension $2k$.

Proof.

(i) Taking a submanifold chart (U, ϕ) at $x \in A$, we have $U \cap A = \phi^{-1}(\mathbb{R}^k)$. And $\phi\colon U \to \phi(U)$ is a homeomorphism, so $\phi^{-1}(\mathbb{R}^k)$ is closed in U.

(ii) i is continuous. Let $c = (U, \phi)$ be a submanifold chart for A. Then the pair of charts $c_A = (U \cap A, \phi_A)$, $c = (U, \phi)$ is adapted for i, and

$$\begin{array}{ccc} U \cap A & \xhookrightarrow{\ i\ } & U \\ \downarrow{\scriptstyle \phi_A} & & \downarrow{\scriptstyle \phi} \\ \phi(U \cap A) \cap \mathbb{R}^k & \xhookrightarrow{\ \tilde{i}\ } & \phi(U) \end{array} \tag{4}$$

The map $\tilde{i}$ is just the inclusion of $\mathbb{R}^k$ into $\mathbb{R}^n$. It is linear, and hence smooth. Now by the chain rule, the diagram

$$\begin{array}{ccc} T(U \cap A) & \xhookrightarrow{Ti} & TU \\ \big\downarrow {\scriptstyle T\phi_A} & & \big\downarrow {\scriptstyle T\phi} \\ \phi(U \cap A) \times \mathbb{R}^k & \xhookrightarrow{T\tilde{i}} & \phi(U) \times \mathbb{R}^n \end{array}$$

commutes. (Apply the functor T to the diagram (4).) Here, $T\tilde{i}$ is the restriction to $\phi(U \cap A) \times \mathbb{R}^k$ of the inclusion $\mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^n \times \mathbb{R}^n$. So for $x \in A \cap U$,

$$\begin{array}{ccc} T_x A & \xrightarrow{T_x i} & T_x M \\ \big\downarrow {\scriptstyle d_x \phi_A} & & \big\downarrow {\scriptstyle d_x \phi} \\ \mathbb{R}^k & \longrightarrow & \mathbb{R}^n \end{array}$$

So for each x, $T_x i \colon T_x A \hookrightarrow T_x M$ is injective.

(iii) & (iv) By the previous part, we have that $Ti(T(A \cap U)) = (T\phi)^{-1}(\mathbb{R}^k \times \mathbb{R}^k)$. This shows that $(TU, T\phi)$ is a submanifold chart for $Ti(TA)$, and therefore that $Ti(TA)$ is a submanifold of TM of dimension $2k$. □

Henceforth, we identify TA with $Ti(TA) \subseteq TM$, and identify $T_x A$ with $T_x i(TA) \subseteq TM$.

The next theorem gives us an alternative way of looking at charts.

Theorem 3.9. *Let V be an open neighborhood of $x \in M$, let $\phi_1, \ldots, \phi_n \in C^\infty(V)$. Suppose that $d_x\phi_1, \ldots, d_x\phi_n \in T_x^*M$ form a basis for T_x^*M. Let*

$$\phi = (\phi_1, \ldots, \phi_n) \colon V \to \mathbb{R}^n.$$

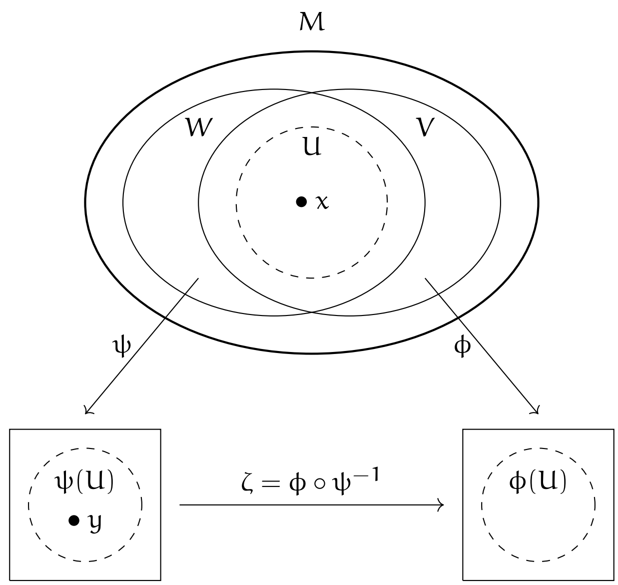
Then there is an open neighborhood $U \subseteq V$ of x such that $(U, \phi|_U)$ is a chart at x.

Proof. To check that this is a chart, we must produce a neighborhood $U \supseteq V$ of x such that $\phi|_U$ is a diffeomorphism onto an open subset of $\mathbb{R}^n$.

Choose any chart (W, ψ) at $x \in M$. It's enough to show that

$$\zeta = \phi \circ \psi^{-1} \colon \psi(U) \to \phi(U)$$

is a diffeomorphism for a small enough open neighborhood $U \subseteq V \cap W$ of x.



Notice that because the $d_x\phi_i$ are independent, then $d_x\phi \colon T_xM \to \mathbb{R}^n$ is an isomorphism.

Now let $y = \psi(x)$. By the chain rule, then

$$d_y\zeta = d_x\phi \circ (d_x\psi)^{-1}.$$

so $d_y\zeta$ is an isomorphism, because it's the composition of two isomorphisms. So by the inverse function theorem, there is an open neighborhood $U' \subseteq \psi(W)$ of y such that $\zeta(U')$ is open and

$$\zeta \colon U' \to \zeta(U')$$

is a diffeomorphism. Now take $U = \psi^{-1}(U')$, and we have the desired chart. □

Corollary 3.10. *A subset A of an n-manifold M is a submanifold of codimension ℓ if and only if for all $x_0 \in A$, there is an open $U \ni x$ and $\zeta_1, \ldots, \zeta_\ell \in C^\infty(U)$ satisfying*

(i) $d_{x_0}\zeta_1, d_{x_0}\zeta_2, \ldots, d_{x_0}\zeta_\ell \in T^*_{x_0}M$ *are linearly independent*

(ii) for all $x \in U$, $x \in A \iff \zeta_1(x) = \zeta_2(x) = \ldots = \zeta_\ell(x) = 0$.

Proof. ($\Longrightarrow$). Choose a submanifold chart (U, ϕ) at x. Then $d_x\phi \colon T_xM \xrightarrow{\sim} \mathbb{R}^n$, so $d_x\phi_1, \ldots, d_x\phi_m \in T^*_xM$ are independent, and also $A \cap U = \phi^{-1}(\mathbb{R}^k)$. Let

$\zeta_1 = \phi_{k+1}$, $\zeta_2 = \phi_{k+2}$, …, $\zeta_\ell = \phi_n$. It can be verified that this choice of functions works.

($\Longleftarrow$). Conversely, put $\phi_{k+1} = \zeta_1, \dots, \phi_n = \zeta_\ell$. Then complete the covectors

$$d_x\phi_{k+1} = d_x\zeta_1, \dots, d_x\phi_n = d_x\zeta_\ell \in T_x^*M$$

to a basis of T_x^*M, say

$$\alpha_1, \dots, \alpha_k, \alpha_{k+1} = d_x\phi_{k+1}, \dots, \alpha_n = d_x\phi_n \in T_x^*M.$$

So by Lemma 3.12 there are functions $\phi_1, \dots, \phi_k \in C^\infty(U)$ such that for a sufficiently small enough open $U \ni x$, we have

$$d_x\phi_1 = \alpha_1, \dots, d_x\phi_k = \alpha_k.$$

By the previous theorem, $\phi = (\phi_1, \dots, \phi_n)\colon U \to \mathbb{R}^n$ is a chart (after possibly shrinking U), and $U \cap A = \phi^{-1}(\mathbb{R}^k)$. □

Remark 3.11.

(1) Keep the same setup as in Corollary 3.10. Let $\zeta = (\zeta_1, \dots, \zeta_\ell) \in C^\infty(U, \mathbb{R}^\ell)$. Then Item (i) is equivalent to $d_{x_0}\zeta\colon T_{x_0}M \to \mathbb{R}^\ell$ surjective, and Item (ii) is equivalent to $U \cap A = \zeta^{-1}(0)$.

(2) $T_{x_0}A$ consists of all $v \in T_{x_0}M$ with

$$d_{x_0}\zeta_1(v) = \dots = d_{x_0}\zeta_\ell(v) = 0.$$

So $T_{x_0}A = \{v \in T_{x_0}M \mid d_{x_0}\zeta(v) = 0\} = \ker(d_{x_0}\zeta)$.

In the special case of $M = \mathbb{R}^n$, $d_{x_0}\zeta$ manifests itself as the Jacobian matrix $D\zeta_{x_0}$, which is an $\ell \times n$ matrix. Then $d_{x_0}\zeta_j = (D\zeta_j)_{x_0}$ is the j-th row of $D\zeta_{x_0}$. So

$$\begin{aligned} T_{x_0}A = \ker(D\zeta(x_0)) &= \ker((D\zeta_1)_{x_0}) \cap \dots \cap \ker((D\zeta_\ell)_{x_0}) \\ &= \langle \nabla_{x_0}\zeta_1 \rangle^\perp \cap \dots \cap \langle \nabla x_0\zeta_\ell \rangle^\perp, \end{aligned}$$

where $\langle v \rangle$ denotes the linear subspace generated by the vector $v \in \mathbb{R}^\ell$.

Lemma 3.12. *For every $\alpha \in T_x^*M$, there is a germ $[f]_x \in C^\infty_{M,x}$ such that $d_xf = \alpha$. That is, $d_x\colon C^\infty_{M,x} \to T_x^*M$ is surjective.*

Proof. On the next homework. □

3.1 Rank

Definition 3.13. Let M and N be manifolds of dimension m and n, respectively. Let $F\colon M \to N$ be smooth. The **rank** of F at $x \in M$ is

$$\operatorname{rank}_x(F) = \operatorname{rank}(T_xF\colon T_xM \to T_{F(x)}N) = \dim(T_xF(T_xM)).$$

Remark 3.14 (Recall)**.** A linear map $T\colon \mathbb{R}^m \to \mathbb{R}^n$ of rank $r \leq \min(m,n)$ has an $m \times n$ matrix representation

$$\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$$

with respect to suitable bases of $\mathbb{R}^m$ and $\mathbb{R}^n$. Then in coordinates relative to these bases,

$$T(x_1, \ldots, x_m) = (x_1, \ldots, x_r, \underbrace{0, \ldots, 0}_{n-r}).$$

Theorem 3.15. *Let* $f\colon M \to N$ *be a smooth map of smooth manifolds,* $m = \dim(M)$*,* $n = \dim(N)$*. Let* $a \in M$*,* $r = \operatorname{rank}_a(f) = \operatorname{rank}(T_af) \leq \min(m,n)$*. Then there exist adapted charts* (U, ϕ) *centered at* a *and* (V, ψ) *centered at* $f(a)$ *and a smooth map* $g\colon \phi(U) \to \mathbb{R}^{n-r}$ *such that*

(i) $\psi \circ f \circ \phi^{-1}(x_1, \ldots, x_n) = (x_1, \ldots, x_r, g(x_1, \ldots, x_m))$

(ii) $g(0) = 0$

(iii) $Dg_0 = 0$

Proof. Let $E = T_aM$ and $F = T_{f(a)}N$, $T = T_af\colon E \to F$, and $F_1 = \operatorname{im}(T)$. Let $r = \dim F_1$. Choose a basis $v_1, \ldots, v_r$ of F_1 and complete it to a basis $v_1, \ldots, v_r$, $v_{r+1}, \ldots, v_n$ of F. Choose vectors $u_1, \ldots, u_r \in E$ with $Tu_i = v_i$ for $1 \leq i \leq r$.

Since the images of the u_i's are independent, then the u_i's must be independent themselves. So $E_1 = \operatorname{span}\{u_1, \ldots, u_r\} \subseteq E$ has dimension r, and

$$T|_{E_1}\colon E_1 \to F_1$$

is an isomorphism.

Now if $u \in E_1 \cap \ker(T)$, then $T(u) = 0$, so $u = 0$ because $u \in E_1$. Hence, $E_1 \cap \ker(T) = \{0\}$. So if we choose a basis $u_{r+1}, \ldots, u_m$ of $\ker(T)$, then $u_1, \ldots, u_r$, $u_{r+1}, \ldots, u_m$ is a basis of E by the rank-nullity theorem.

We have that

$$T(u_j) = \begin{cases} v_j & \text{for } j \leq r \\ 0 & \text{for } j > r \end{cases}$$

The matrix of T with respect to these bases is

$$A = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$$

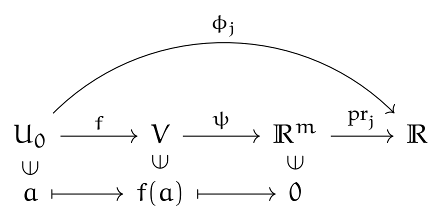
Let $u_1^*, \dots, u_m^* \in E^*$ and $v_1^*, \dots, v_n^* \in F^*$ be the dual bases, with $u_i^*(u_j) = \delta_{ij}$ and $v_i^*(v_j) = \delta_{ij}$. Then the dual map $T^*\colon F^* \to E^*$ satisfies

$$T^*(v_j^*) = v_j^* \circ T = u_j^*.$$

By Theorem 3.9, there is a chart (V, ψ) centered at $f(a)$ such that $d_{f(a)}\psi_j = v_j^*$ for $j = 1, \dots, n$. Let $U_0 = f^{-1}(V)$, this is an open neighborhood of a. So for $j = 1, \dots, r$, define

$$\phi_j := \psi_j \circ f$$

This choice guarantees that the following commutes.



Then for $j = 1, \dots, r$,

$$d_a\phi_j = d_{f(a)}\psi_j \circ T_a f = v_j^* \circ T = u_j^*$$

and also, $\phi_j(a) = 0$ by the choice of charts. Now to make a chart at $a \in M$, choose C^∞ functions $\phi_{r+1}, \dots, \phi_m \in C^\infty(U)$ with

$$\phi_j(a) = 0, \qquad d_a\phi_j = u_j^*$$

for all $j \geq r+1$.

So by Corollary 3.10, there is an open neighborhood $U \subseteq U_0$ of x such that the restriction of $\phi = (\phi_1, \dots, \phi_r, \phi_{r+1}, \dots, \phi_m)$ to U is a coordinate map on U. Hence, (U, ϕ) is a chart.

So let $g_j = \psi_j \circ f \circ \phi^{-1}$ for $j = r+1, \dots, n$ and set

$$g = (g_{r+1}, \dots, g_n)\colon \phi(U) \to \mathbb{R}^{n-r}.$$

We will verify that this choice of g is the map that we want.

It's easily seen that $g(0) = 0$ by the choice of charts. Moreover, for $x \in \phi(U)$, we have that

$$\psi_j \circ f \circ \phi^{-1}(x) = \phi_j \circ \phi^{-1}(x) = x_j$$

for $1 \leq j \leq r$, and

$$\psi_j \circ f \circ \phi^{-1}(x) = g_j(x)$$

for $r+1 \leq j \leq n$. So

$$\psi \circ f \circ \phi^{-1}(x) = (x_1, \ldots, x_r, g(x))$$

as desired.

Finally, $d_a \phi_j = u_j^*$ for all j implies that

$$d_a \phi_j(u_i) = \delta_{ij}.$$

So $d_a \phi(u_i) = e_i$ is the standard basis vector of $\mathbb{R}^n$. Hence for $j \geq r+1$,

$$\begin{aligned} D(g_j)_0 e_i &= \left(d_{f(a)} \psi_j \circ T \circ (d_a \phi)^{-1} \right) e_i \\ &= d_{f(a)} \psi_j (T(u_i)) \\ &= \begin{cases} d_{f(a)} \psi_j(v_i) = v_j^*(v_i) = \delta_{ij} & \text{if } i \leq r \\ d_{f(a)} \psi_j(0) = 0 & \text{if } i \geq r+1 \end{cases} \end{aligned}$$

If $i \leq r$, we get $D(g_j)_0 e_i = \delta_{ij}$. But $j \geq r+1$ and $i \leq r$, so the Kronecker delta vanishes here and we see that $D(g_j)_0 e_i = 0$ for all i.

If $i \geq r+1$, then $T(u_i) = 0$, so $D(g_j)_0 e_i = d_{f(a)} \psi_j(T(u_i)) = 0$.

Hence, $D(g_j)_0$ kills all standard basis vectors, so $Dg_0 = 0$. □

Corollary 3.16. *Every* $a_0 \in M$ *has a neighborhood* U *such that* $\operatorname{rank}_a(f) \geq \operatorname{rank}_{a_0}(f)$ *for all* $a \in U$.

This is called **semicontinuity** of the rank function.

Proof. Choosing charts as in Theorem 3.15, we have

$$\psi \circ f \circ \phi^{-1}(x) = \begin{bmatrix} x_1 \\ \vdots \\ x_r \\ g(x) \end{bmatrix}.$$

$$D(\psi \circ f \circ \phi^{-1})_x = \begin{bmatrix} I_r & 0 \\ D_1 g_x & D_2 g_x \end{bmatrix},$$

where $Dg_x = (D_1 g_x, D_2 g_x)$, $D_1 g_x$ is the partials of g with respect to $x_1, \ldots, x_r$, and $D_2 g_x$ is the partials of g with respect to $x_{r+1}, \ldots, x_m$.

With this in mind, we can see that

$$\operatorname{rank}(f) = \operatorname{rank}(\psi \circ f \circ \psi^{-1}) = \operatorname{rank} D(\psi \circ f \circ \phi^{-1}) \geq r. \qquad \square$$

Definition 3.17. $f\colon M \to N$ has **constant rank** at $a \in M$ (or is a **subimmersion**) if $\operatorname{rank}_b(f) = \operatorname{rank}_a(f)$ for b in a neighborhood of a.

Theorem 3.18 (Constant Rank Theorem)**.** *Suppose* f *has constant rank at* a*. Then there exist adapted charts* (U, ϕ) *centered at* a *and* (V, ψ) *centered at* $f(a)$ *such that*

$$\psi \circ f \circ \phi^{-1}(x_1, \ldots, x_m) = (x_1, \ldots, x_r, \underbrace{0, \ldots, 0}_{n-r}).$$

Proof. Choose charts (U, ϕ) and (V, ψ) as in Theorem 3.15. Then f is constant rank near a if and only if $\widetilde{f}$ has constant rank near $0 \in \mathbb{R}^m$, where $\widetilde{f} = \psi \circ f \circ \phi^{-1}$. Therefore, since

$$D\widetilde{f}_0 = \begin{bmatrix} I_r & 0 \\ D_1 g_0 & D_2 g_0 \end{bmatrix},$$

and $g(0) = 0$, then we must have that $D_2 g_x = 0$ for x close to 0 in $\mathbb{R}^m$. Hence, $g(x)$ is independent of $x_{r+1}, \ldots, x_m$ for $x \in W = W_1 \times W_2$, where W_1 is a neighborhood of $0 \in \mathbb{R}^r$ and W_2 is a neighborhood of $0 \in \mathbb{R}^{m-r}$. So there is a smoooth $h\colon W_1 \to \mathbb{R}^{n-r}$ with $g(x_1, \ldots, x_m) = h(x_1, \ldots, x_r)$. Hence,

$$\psi \circ f \circ \phi^{-1}(x_1, \ldots, x_m) = (x_1, \ldots, x_r, h(x_1, \ldots, x_r)).$$

Define a shear transformation

$$\begin{aligned} \sigma\colon W_1 \times \mathbb{R}^{n-r} &\longrightarrow W_1 \times \mathbb{R}^{n-r} \\ (u, v) &\longmapsto (u, v - h(u)) \end{aligned}$$

This is a diffeomorphism with inverse

$$\sigma^{-1}(y, z) = (y, z + h(y)).$$

We have that

$$\sigma \circ \psi \circ f \circ \phi^{-1}(x_1, \ldots, x_m) = (x_1, \ldots, x_r, 0, \ldots, 0).$$

So finally, replace ψ with $\sigma \circ \psi$. Need also to restrict ϕ to $U \cap \phi^{-1}(W)$ and ψ to $V \cap \psi^{-1}(W \times \mathbb{R}^{n-r})$ for everything to remain well-defined. □

The previous theorem is a workhorse of differential topology. There are two extreme cases that come up quite often: namely when the rank of f is either that of the domain or of the codomain.

3.2 Submersions and Immersions

Definition 3.19. Let $f\colon M \to N$ be smooth. Then f is called

(a) an **immersion** at $a \in M$ if $T_a f\colon T_a M \to T_{f(a)} N$ is injective.

(b) a **submersion** at $a \in M$ if $T_a f\colon T_a M \to T_{f(a)} N$ is surjective.

Corollary 3.20 (Immersion Theorem). *If f is an immersion at a, then $\operatorname{rank}_b(f) = m$ for b in a neighborhood of a, and there exist adapted charts (U, ϕ) centered at a and (V, ψ) centered at $f(a)$ with*

$$\psi \circ f \circ \phi^{-1}(x_1, \ldots, x_m) = (x_1, \ldots, x_m, \underbrace{0, \ldots, 0}_{n-m}).$$

Proof. If f is an immersion at a, then $\operatorname{rank}_a(f) = m = \dim(M)$. So $m \leq n$ and for b close to a, the rank is bounded below by m by Theorem 3.18, but also above by the number of columns, so we see that

$$m \leq \operatorname{rank}_b(f) \leq m.$$

Hence, $\operatorname{rank}_b(f)$ is constant near a. Now take $r = m$ in Theorem 3.18. ☐

Corollary 3.21 (Submersion Theorem). *If f is a submersion at a, then $\operatorname{rank}_b(f) = m$ for b in a neighborhood of a, and there exist adapted charts (U, ϕ) centered at a and (V, ψ) centered at $f(a)$ with*

$$\psi \circ f \circ \phi^{-1}(x_1, \ldots, x_m) = (x_1, \ldots, x_n).$$

Proof. If f is a submersion at a, then $\operatorname{rank}_a(f) = n = \dim(N)$. Similarly, we see that

$$n \leq \operatorname{rank}_b(f) \leq n$$

for b near a. So $\operatorname{rank}_b(f)$ is constant near a. Now take $r = n$ in Theorem 3.18. ☐

Definition 3.22. If $f\colon X \to Y$, then the **fiber of** f **over** $y \in Y$ is

$$f^{-1}(y) = \{x \in X \mid f(x) = y\}.$$

Theorem 3.23 (Fiber Theorem). *Let $f\colon M \to N$, $\dim(M) = m$, $\dim(N) = n$. Let $c \in \mathbb{N}$ and $A = f^{-1}(c)$. Further assume that f is of constant rank r at all points of A.*

Then the fiber $f^{-1}(c)$ is a closed submanifold of M of dimension $\dim(A) = m - r$ and the tangent bundle is equal to

$$TA = \ker(Tf) = \bigsqcup_{a \in A} \ker(T_a f).$$

Proof. Let $a \in A$. Choose charts (U, ϕ) and (V, ψ) as in Theorem 3.15, and let $F \subseteq \mathbb{R}^m$ be the linear subspace

$$F = \{x \in \mathbb{R}^m \mid x_1 = x_2 = \ldots = x_r = 0\} \cong \mathbb{R}^{m-r}.$$

Then $x \in \phi(U \cap A) \iff \widetilde{f}(x) = 0$. But

$$\widetilde{f}(x) = (x_1, \ldots, x_r, \underbrace{0, \ldots, 0}_{n-r}),$$

so $\widetilde{f}(x) = 0 \iff x \in F$. So $U \cap A = \phi^{-1}(F)$. This shows that A is a submanifold of M at a; up to renumbering the coordinates, (U, ϕ) is a submanifold chart for A. We have that $\dim(A) = \dim(F) = m - r$.

Now the following diagram commutes by the definition of $T_a f$.

$$\begin{array}{ccc} T_a M & \xrightarrow{T_a f} & T_c N \\ \Big\downarrow{\scriptstyle d_a \phi} & & \Big\downarrow{\scriptstyle d_c \psi} \\ \mathbb{R}^m & \xrightarrow{\widetilde{f} = D\widetilde{f}_0} & \mathbb{R}^n \end{array}$$

Moreover, the vertical maps are isomorphisms. Hence,

$$\begin{aligned} T_a A &= (d_a \phi)^{-1}(F) \\ &= (d_a \phi)^{-1}(\ker(\widetilde{f})) \\ &= \ker\left(\widetilde{f} \circ d_a \phi\right) \qquad\qquad = \ker(D\widetilde{f}_0 \circ d_a \phi) \\ &= \ker(d_c \psi \circ T_a f) \\ &= \ker(T_a f), \end{aligned}$$

the last equality because $d_c \psi$ is an isomorphism. □

Definition 3.24. The number $e(f, A) = n - r$ is called the **excess** of f at $A = f^{-1}(c)$.

Remark 3.25. Interpretation of excess. Near $a \in A$, $\phi(A)$ is the solution set of n equations in m variables,

$$\begin{aligned} \widetilde{f}_1(x_1, \ldots, x_m) &= 0 \\ \widetilde{f}_2(x_1, \ldots, x_m) &= 0 \\ &\vdots \\ \widetilde{f}_n(x_1, \ldots, x_m) &= 0 \end{aligned}$$

If the equations were functionally independent, then we would have $\dim(A) = m - n$. Instead, only r equations are independent, so $\dim(A) = m - r = m - n + e(f, A)$.

In the case where the excess is zero, then we have the following important definition.

Definition 3.26. Call $c \in N$ a **regular value** of f if for all $a \in A = f^{-1}(c)$, f is a submersion at a.

Theorem 3.27 (Regular value theorem)**.** *If $c \in N$ is a regular value of f, then $A = f^{-1}(c)$ is a closed submanifold of dimension $m - n$, and $TA = \ker(Tf)$.*

Example 3.28. Let $M = M(n \times n, \mathbb{R}) \cong \mathbb{R}^{n^2}$ be the $n \times n$ matrices with entries in $\mathbb{R}$. Then define $f\colon M \to M$ by $f(X) = XX^T$. Then

$$f^{-1}(I) = \{X \in M \mid XX^T = I\}$$

is the **orthogonal group** of $n \times n$ orthogonal matrices, denoted $O(n)$ or $O(n, \mathbb{R})$.

Claim 3.29. $O(n)$ is a submanifold of M of dimension $n(n-1)/2$ and $T_I O(n) = \mathfrak{o}(n)$, the Lie algebra of skew-symmetric matrices.

Proof. Let $N = \{Y \in M \mid Y = Y^T\}$ be the subspace of symmetric matrices. View f as a map $f\colon M \to N$. We can check that I is a regular value of f. Let $A \in M$. Then $Df_A\colon M \to N$ is linear. For $H \in M$, what is $Df_A H$?

$$\begin{aligned} f(A+H) &= (A+H)(A+H)^T \\ &= AA^T + AH^T + HA^T + HH^T \\ &= f(A) + L_A(H) + R(H) \end{aligned}$$

where $L(H) = AH^T + HA^T$ is linear and $R(H) = HH^T$. So, considering the matrix norms,

$$\frac{\|R(H)\|}{\|H\|} = \frac{\|HH^T\|}{\|H\|} \le \frac{\|H\|^2}{\|H\|} = \|H\| \xrightarrow[H \to 0]{} 0$$

So we have that

$$Df_A(H) = L_A(H) = AH^T + HA^T.$$

To check that f is a submersion at $A \in O(n)$, we need to know that L_A is surjective. So let $B \in N$. We want to solve

$$AH^T + HA^T = B$$

for H. To do this, rewrite the above as

$$AH^T + HA^T = \tfrac{1}{2}B + \tfrac{1}{2}B.$$

Then half of the above equation is easy to solve. If we set $HA^T = \frac{1}{2}B$, then

$$H = HA^T A = \tfrac{1}{2}BA.$$

This $H = \frac{1}{2}BA$ satisfies

$$AH^T + HA^T = \tfrac{1}{2}AA^TB + \tfrac{1}{2}BAA^T = B$$

because B is symmetric.

Hence, I is a regular value of f, because $f^{-1}(I) = O(n)$ and f is a submersion at each point in $O(n)$.

Therefore,

$$\dim(O(n)) = \dim(M) - \dim(N) = n^2 - \tfrac{1}{2}n(n+1) = \tfrac{1}{2}n(n-1).$$

Then $T_I O(n) = \ker(Df(I)) = \ker(H \mapsto H^T + H)$, which is all of the skew-symmetric matrices. □

Remark 3.30. $O(n)$ is our first example of a **Lie group**. For any $A \in O(n)$,

$$1 = \det(AA^T) = \det(A)^2 \implies \det(A) = \pm 1.$$

So $O(n)$ has two connected components, namely the preimage under det of $+1$ and the preimage under det of -1.

The **special orthogonal group** is

$$SO(n) = \{A \in O(n) \mid \det(A) = 1\},$$

the connected component of $O(n)$ of determinant 1 matrices is one coset, and the other connected component of $O(n)$ is the coset

$$A_0\, SO(n) = \{A \in O(n) \mid \det(A) = -1\},$$

where A_0 is the matrix

$$A_0 = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

Remark 3.31. Another method we could use to prove Claim 3.29 would be to check that f is a subimmersion of appropriate rank. We'll do this later when we talk about Lie groups.

3.3 Embeddings

Definition 3.32. $f\colon M \to N$ is an **embedding** if it is an immersion (that is, an immersion at every point of M) and a homeomorphism onto its image.

Remark 3.33. Equivalently, $f\colon M \to N$ is an embedding if

(i) f is injective,

(ii) $T_a f$ is injective for all $a \in M$, and

(iii) $f^{-1}\colon f(M) \to M$ is continuous.

Theorem 3.34 (Embedding Theorem)**.**

(i) The inclusion of a submanifold $i\colon A \to M$ *is an embedding.*

(ii) If $f\colon P \to M$ *is an embedding, then* $A - f(P)$ *is a submanifold of* M *and* $f\colon P \to A$ *is a diffeomorphism and* $TA = Tf(TP)$.

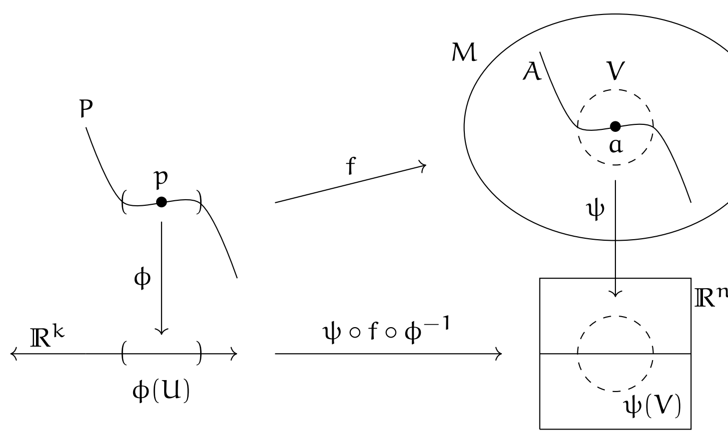
Proof.

(i) Recall that Ti is injective, so i is an immersion. By definition, the inclusion is injective. Also the topology on A is the subspace topology induced by its smooth structure is the subspace topology inherited from M. In particular, this means that i is a homeomorphism onto its image.

(ii) Let $k = \dim(P)$ and $n = \dim(M)$. Then let $p \in P$, $a = f(p) \in A$. The immersion theorem gives adapted charts (U, ϕ) centered at p and (V, ψ) centered at a such that

$$\psi \circ f \circ \phi^{-1}(x_1, \dots, x_k) = (x_1, \dots, x_k, 0, \dots, 0)$$

is the inclusion $\mathbb{R}^k \to \mathbb{R}^n$.



We would like that $f(U) = A \cap V$ and $f(U) = \psi^{-1}(\mathbb{R}^k)$. This may however fail, so first replace (V, ψ) by (V_0, ψ_0) where $V_0 = V \cap \psi^{-1}(\phi(U) \times \mathbb{R}^k)$ and $\psi_0 = \psi|_{V_0}$.

To get $f(U) = A \cap V_0$ use the fact that f is a homeomorphism onto A. So tehre is open $V' \subseteq M$ with$f(U) = A \cap V'$. Replace (V_0, ψ_0) with (V_1, ψ_1) where $V_1 = V_0 \cap V'$ and $\psi_1 = \psi_0|_{V_1}$. Now we have that $f(U) = A \cap V_1$ and $f(U) = \psi_1^{-1}(\mathbb{R}^k)$.

Then (V_1, ψ_1) is a submanifold chart for A at a. The map $f^{-1}\colon A \to P$ is represented near a by

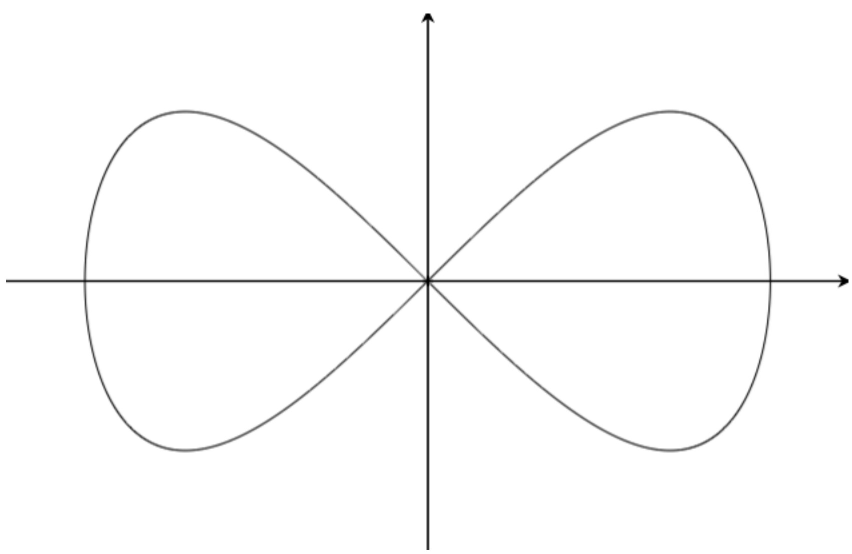
$$\begin{aligned} \phi \circ f^{-1} \circ \psi^{-1} \colon \psi(A \cap V_1) &\longrightarrow \phi(U) \\ x &\longmapsto x \end{aligned}$$

noting that both $\psi(A \cap V_1)$ and $\phi(U)$ are contained in $\mathbb{R}^k$. So f^{-1} is smooth, so f is a diffeomorphism and $Tf\colon TP \to TM$ sends TP onto TA. □

Example 3.35 (Non-example)**.** Let $P = (-\frac{\pi}{2}, \frac{3\pi}{2})$, $M = \mathbb{R}^2$, and

$$f(t) = \begin{pmatrix} \cos(t) \\ \cos(t)\sin(t) \end{pmatrix}.$$

We have chosen the domain such that this is injective. The graph of this function for $t \in P$ is the **lemniscate**.



f is an injective immersion, and

$$f'(t) = \begin{pmatrix} -\sin(t) \\ -\sin^2(t) + \cos^2(t) \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then $f^{-1}\colon f(P) \to P$ is not continuous: f is not an embedding, f(P) is not a submanifold.

Example 3.36 (Non-example)**.** Let $P = \mathbb{R}$ and $M = S^1 \times S^1$ the 2-torus. Let

$$f(t) = \begin{pmatrix} e^{it} \\ e^{i\alpha t} \end{pmatrix}.$$

Hence we think of S^1 as the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$. Then

$$f'(t) = \begin{pmatrix} ie^{it} \\ i\alpha e^{i\alpha t} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

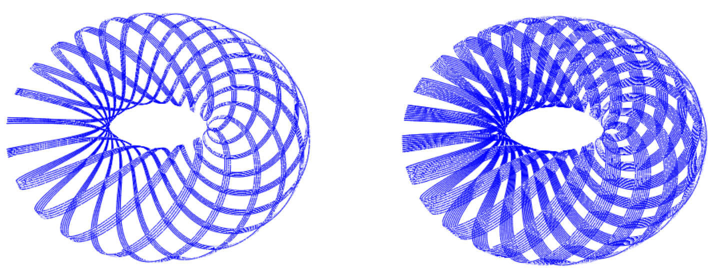
so f is an immersion.

If $\alpha = \frac{p}{q} \in \mathbb{Q}$, then f is not injective: $f(t) = f(t + 2\pi q)$. But f descends to a map

$$\bar{f} \colon \mathbb{R}/_{2\pi q\mathbb{Z}} \to S^1 \times S^1$$

called a **torus knot**. $\bar{f}$ is an injective immersion, and in fact $\bar{f}$ is an embedding because the inverse is continuous. To check that $\bar{f}^{-1} \colon f(\mathbb{R}) \to \mathbb{R}/_{2\pi q\mathbb{Z}}$ is continuous, one must check that if $C \subseteq \mathbb{R}/_{2\pi q\mathbb{Z}}$ is closed, then $\bar{f}(C)$ is closed in $S^1 \times S^1$. True because $\mathbb{R}/_{2\pi q\mathbb{Z}} \cong S^1$ is compact.

Fact 3.37 (Kronecker)**.** *If $\alpha \notin \mathbb{Q}$, then f is injective and $A = f(P)$ is **dense** in M, so it is certainly not a submanifold.*



The plot of the torus knot $f(t) = (e^{it}, e^{i\alpha t})$ for α irrational and $0 \leq t \leq 200$ (left) or $0 \leq t \leq 500$ (right). Notice that the as the upper bound on t grows larger, more of the torus is filled in.

Definition 3.38. An **immersed submanifold** of a manifold M is a pair (P, f) where P is a manifold and $f \colon P \to M$ is an injective immersion.

Then $f \colon P \to A = f(P)$ is a continuous bijection, but $f^{-1} \colon A \to P$ is not necessarily continuous for the subspace topology on A.

Remark 3.39. Identifying P with A, we see an immersed submanifold is a subset A of M equipped with a smooth structure such that the inclusion $i \colon A \to M$ is an immersion.

But the topology on A induced by this smooth structure may be finer (bigger, stronger) than the subspace topology.

4 Vector Fields

Let M be an n-manifold, and let $\pi = \pi_M \colon TM \to M$ the tangent bundle projection. Then for any $a \in M$, $\pi^{-1}(a) = T_aM$ for all $a \in M$.

Definition 4.1. A **vector field** on M is a smooth section of π, that is, a map $\xi \colon M \to TM$ satisfying $\pi \circ \xi = \mathrm{id}_M$. So $\xi(a) \in T_aM$ for all $a \in M$. We often write $\xi(a) = \xi_a$.

Definition 4.2. A point $a \in M$ is a **zero** or **equilibrium** of ξ if $\xi_a = 0 \in T_aM$.

Let $c = (U, \phi)$ be a chart on M. Then $\xi(U) \subseteq TU = \pi^{-1}(U)$, so c and Tc are adapted for ξ. An expression for ξ in coordinates is then

$$T\phi \circ \xi \circ \phi^{-1} \colon \phi(U) \longrightarrow T\phi(U) = \phi(U) \times \mathbb{R}^n$$

So this defines a vector field on $\phi(U) \subseteq \mathbb{R}^n$. For each $x \in \phi(U)$,

$$T\phi \circ \xi \circ \phi^{-1}(x) = (x, h)$$

for some $h \in \mathbb{R}$. We write $h = \widetilde{\xi}(x)$, with $\widetilde{\xi} \colon \phi(U) \to \mathbb{R}^n$.

Definition 4.3. A vector field ξ is **smooth** if and only if $\widetilde{\xi}$ is smooth for all charts c on M.

Remark 4.4. This meshes with our definition of smooth morphisms of manifolds: $f \colon M \to N$ is smooth if and only if it's expression in any pair of charts is smooth as a map $\mathbb{R}^m \to \mathbb{R}^n$. We see that ξ is smooth if and only if it's expression $T\pi \circ \xi \circ \phi^{-1}$ is smooth, if and only if $\widetilde{\xi}$ is smooth.

Definition 4.5. For vector fields ξ, η and a function $f \colon M \to \mathbb{R}$, we define vector fields $\xi + \eta$ and $f\xi$ by $(\xi + \eta)(a) = \xi(a) + \eta(a)$ and $(f\xi)(a) = f(a)\xi(a)$.

With respect to a chart c we have $\widetilde{(\xi + \eta)} = \widetilde{\xi} + \widetilde{\eta}$ and $\widetilde{(f\xi)} = \widetilde{f}\widetilde{\xi}$.

Definition 4.6 (Notation)**.** $\mathcal{T}(M) = \{\xi \colon M \to TM \mid \pi \circ \xi = \mathrm{id}_M, \xi \text{ is smooth}\}$

Remark 4.7. $\mathcal{T}(M)$ is a module over the algebra $C^\infty(M)$.

Definition 4.8. A k**-frame** on M is an ordered k-tuple $(\xi_1, \ldots, \xi_k)$ of smooth vector fields on M such that for all $a \in M$, $(\xi_1)_a, \ldots, (\xi_k)_a \in T_aM$ are linearly independent.

Example 4.9. A 1-frame on M is a nowhere vanishing vector field ξ. That is, $\xi_a \neq 0$ for all a.

Definition 4.10. An n-manifold M is called **parallelizable** if it has an n-frame.

Let $k_M = \max\left\{k \mid M \text{ has a } k\text{-frame}\right\} \leq \dim(M)$. M is parallelizable if and only if $k_M = \dim(M)$.

Example 4.11. Let $M = U$ be an open subset of $\mathbb{R}^n$. A smooth vector field on U is a smooth map $\xi \colon U \to TU = U \times \mathbb{R}^n$ of the form $\xi(x) = (x, \widetilde{\xi}_x)$, where $\widetilde{\xi} \colon U \to \mathbb{R}^n$ is smooth.

U is parallelizable: the constant vector fields $e_1, \ldots, e_n \colon U \to \mathbb{R}^n$ serve as an n-frame for U.

Example 4.12. A Lie groups G is parallelizable by a basis of the associated Lie algebra $\mathfrak{g} = T_0 G$, left-or-right-translated around G.

4.1 Independent vector fields on spheres

Theorem 4.13 (F. Adams, 1962)**.** *Let* $M = S^{n-1}$. *Let* $m = \max\{\ell \mid 2^\ell \text{ divides } n\}$. *Write* $m = 4b + a$, *with* $b \in \mathbb{Z}$, $a \in \{0, 1, 2, 3\}$. *Let* $\rho(n) = 2^a + 8b$. *This is the* ***Radon-Hurwitz number.*** *Then*

$$k_{S^{n-1}} = \rho(n) - 1.$$

Here are the first 10 values of $\rho(n)$. This counts the number of independent vector fields on a sphere. If n is odd, then $m = 0$, $a = b = 0$, so $k_M = 0$, and every smooth vector field on an even dimensional sphere has zeros.

n	1	2	3	4	5	6	7	8	9	10	$\cdots$
$\rho(n)$	1	2	1	4	1	2	1	8	1	2	$\cdots$

S^{n-1} is parallelizable if and only if $k_{S^{n-1}} = n - 1$ if and only if $\rho(n) = n$.

Theorem 4.14 (Kervaire, 1956)**.** $\rho(n) = n$ *if and only if* $n = 1, 2, 4, 8$. *Therefore,* S^{n-1} *is parallelizable if and only if* $n = 1, 2, 4, 8$.

The reason that this only works for $n = 1, 2, 4, 8$ is that there are only four real division algebras: $\mathbb{R}$, $\mathbb{C}$, the quaternions, and the octonions. This will be on your homework.

Remark 4.15. Some of these spheres have Lie group structures. S^1 is the Lie group $U(1)$, and S^3 can be identified with the unit quaternions, which is also $SU(2)$. S^7 is the unit octonions, but they are not associative, so there is no Lie group structure on S^7.

4.2 Flows

Recall that for $\gamma\colon J \to M$, where $J \subseteq \mathbb{R}$ is an open interval, the tangent map is

$$T_t\gamma\colon T_t\mathbb{R} = \{t\} \times \mathbb{R} \to T_{\gamma(t)}M.$$

We put $\gamma'(t) = T_t\gamma(t, 1)$.

Definition 4.16. Let $\xi \in T(M)$. An **integral curve** or **trajectory** of ξ is a smooth map $\gamma\colon J \to M$, which satisfies $\gamma'(t) = \xi_{\gamma(t)}$ for all $t \in J$.

We say that γ **starts at** $a \in M$ if $0 \in J$ and $\gamma(0) = a$.

For $s \in \mathbb{R}$ let $J + s = \{t + s \mid t \in J\}$.

Lemma 4.17 (Time Translation Lemma)**.** *Let $\gamma\colon J \to M$ be a trajectory for ξ. Define $\delta\colon J - s \to M$ by $\delta(t) = \gamma(t + s)$. Then δ is a trajectory. If $s \in J$, then δ starts at $a = \gamma(s)$.*

Proof. By the chain rule, $\delta'(t) = \gamma'(t + s) = \xi_{\gamma(t+s)} = \xi_{\delta(t)}$. If $s \in J$, then $0 \in J - s$ and $\delta(0) = \gamma(s) = a$. □

What if there's a hole in our manifold, and a mouse running along a trajectory would fall into that hole? Then we cannot extend the map $\gamma\colon J \to M$ to all of $\mathbb{R}$. The question is when we can extend a trajectory to all of $\mathbb{R}$.

Definition 4.18. Define a partial ordering on trajectories as follows. If $\gamma_1\colon J_1 \to M$ and $\gamma_2\colon J_2 \to M$ are trajectories, then $\gamma_1 \leq \gamma_2$ if $J_1 \subseteq J_2$ and $\gamma_2|_{J_1} = \gamma_1$.

A trajectory $\gamma\colon J \to M$ of ξ is **maximal** if it is maximal with respect to this partial ordering. That is, if for any other trajectory $\gamma_1\colon J_1 \to M$ with $J_1 \supseteq J$ and $\gamma_1|_J = \gamma$, we have $J = J_1$.

Essentially, maximality of a trajectory means that it cannot be extended any further. The **initial value problem** for ξ is to find for each $a \in M$ a trajectory $\gamma\colon J \to M$ such that $\gamma'(t) = \xi_{\gamma(t)}$ starting at a, $\gamma(0) = a$, which is maximal. The next theorem says that the initial value problem has a solution for any $a \in M$.

Theorem 4.19 (Existence and Uniqueness)**.** *For each $a \in M$, there is a unique maximal trajectory starting at a.*

Proof. To settle existence, choose a chart (U, ϕ) at a. Then write the vector field in coordinates: for any path $\gamma\colon J \to M$, we get a path

$$\widetilde{\gamma} = \phi \circ \gamma\colon J \longrightarrow \phi(U) \subseteq \mathbb{R}^n.$$

Also, write

$$\begin{aligned} T\phi \circ \xi \circ \phi^{-1}\colon \phi(U) &\longrightarrow \phi(U) \times \mathbb{R}^n \\ x &\longmapsto (x, \widetilde{\xi}_x) \end{aligned}$$

where $\widetilde{\xi}\colon \phi(U) \to \mathbb{R}^n$ is the expression for ξ in the chart (U, ϕ). Then γ is a trajectory of ξ starting at a if and only if

$$\begin{cases} \widetilde{\gamma}'(t) = \xi_{\widetilde{\gamma}(t)} \\ \widetilde{\gamma}(0) = \phi(a), \end{cases} \tag{5}$$

which is a vector-valued ordinary differential equation, and $\widetilde{\xi}_{\widetilde{\gamma}(t)}$ is smooth.

By the existence theorem for solutions to ODE's, a solution $\widetilde{\gamma}\colon J \to \phi(U)$ exists. Compose with ϕ^{-1} to the desired $\gamma = \phi^{-1} \circ \widetilde{\gamma}\colon J \to U$ starting at a.

To settle uniqueness, let $\gamma_1\colon J_1 \to M$ and $\gamma_2\colon J_2 \to M$ be two trajectories starting at a solving the ODE (5).

Let $I = \{t \in J_1 \cap J_2 \mid \gamma_1(t) = \gamma_2(t)\}$. Then $0 \in I$ because $\gamma_1(0) = a = \gamma_2(0)$. Let $\Gamma(t) = (\gamma_1(t), \gamma_2(t))$. Notice that $\Gamma\colon J_1 \cap J_2 \to M \times M$ is smooth.

We have that $I = \Gamma^{-1}(\Delta_M)$ where $\Delta_M = \{(x,x) \mid x \in M\}$. Since M is Hausdorff, Δ is closed. Hence, I is closed in $J_1 \cap J_2$ as the inverse image of a closed set under a smooth map.

The uniqueness theorem for solutions to ODE's, applied to (5), so I is open. Hence, $I = J_1 \cap J_2$ as a connected, nonempty component of $J_1 \cap J_2$. Therefore, $\gamma_1 = \gamma_2$ on $J_1 \cap J_2$.

Finally, to show maximality, let $\{\gamma_\alpha\colon J_\alpha \to M \mid \alpha \in A\}$ be the collection of trajectories starting at a. For all $\alpha, \beta \in A$, we have $\gamma_\alpha = \gamma_\beta$ on $J_\alpha \cap J_\beta$ by the argument for uniqueness. So define $J = \bigcup_{\alpha \in A} J_\alpha$. This is an open interval containing a, and $\gamma(t) = \gamma_\alpha(t)$ for all $t \in J$ and $\alpha \in A$ such that $t \in J_\alpha$. Then γ is well-defined, smooth, and maximal by construction. □

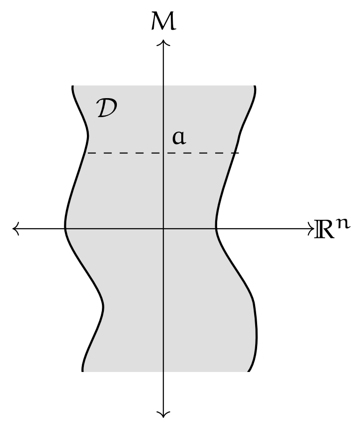
Corollary 4.20. *Let $\gamma\colon J \to M$ be the maximal trajectory of ξ starting at $a \in M$. Let $s \in J$, and $b = \gamma(s)$ and $\delta(t) = \gamma(t+s)$. Then $\delta\colon J - s \to M$ is the unique maximal trajectory starting at b.*

Proof. Combine the Time Translation Lemma (Lemma 4.17) and the previous theorem (Theorem 4.19). □

Definition 4.21. For $a \in M$, let $0 \in \mathcal{D}^a = \mathcal{D}^a(\xi) \subseteq \mathbb{R}$ be the domain of the maximal trajectory starting at a. The **flow domain of** ξ is

$$\mathcal{D} = \mathcal{D}(\xi) = \{(t,a) \mid t \in \mathcal{D}^a\} \subseteq \mathbb{R} \times M\}.$$

The **flow** of ξ is the map $\theta\colon \mathcal{D} \to M$ defined by $\theta(t,a) = \gamma(t)$, where γ is the maximal trajectory of ξ starting at a.



Remark 4.22 (Notation). Also put $\mathcal{D}_t = \{a \in M \mid (t,a) \in \mathcal{D}\}$, and $\theta_t(a) = \theta^a(t) = \theta(t,a)$ for $t \in \mathbb{R}$, $a \in M$. Define

$$\theta^a : \mathcal{D}^a \to M$$
$$\theta_t : \mathcal{D}_t \to M$$

by $\theta(t,a) = \theta_t(a) = \theta^a(t)$. Finally, note that

$$t \in \mathcal{D}^a \iff (t,a) \in \mathcal{D} \iff a \in \mathcal{D}_t.$$

Example 4.23. If $M = \mathbb{R}$, and $\xi(x) = x^2$, then what is the flow of ξ? We solve the initial value problem for ξ:

$$\begin{cases} x'(t) = x(t)^2 \\ x(0) = x_0 \end{cases}$$

This is an ODE that we can solve. We have

$$\frac{x'(t)}{x(t)^2} = 1$$

so integrating, we see that

$$-\frac{1}{x(t)} = t + C$$

for some constant C. Hence,

$$x(t) = -\frac{1}{t+C}.$$

Substituting the inital value $x(0) = x_0$, we see that $C = -\frac{1}{x_0}$. Therefore,

$$x(t) = \frac{1}{-t + \frac{1}{x_0}} = \frac{x_0}{1 - tx_0}$$

is the solution. The flow is

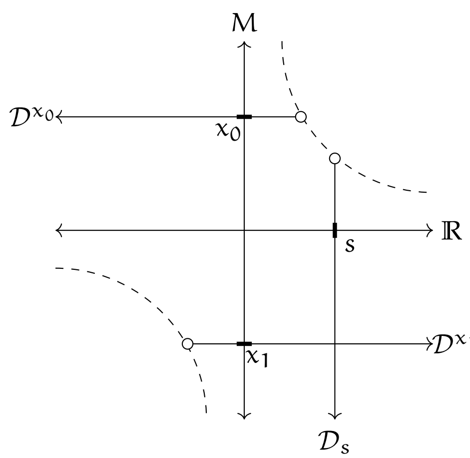
$$\theta(t, x) = \frac{x}{1 - tx}.$$

The domain of this flow is for $x_0 = 0$, $x(t) = 0$ for all t, so $\mathcal{D}^0 = \mathbb{R}$. This is the **equilibrium solution.** For $x_0 > 0$, $x(t)$ exists for $t < \frac{1}{x_0}$, so

$$\mathcal{D}^{x_0} = (-\infty, \tfrac{1}{x_0}).$$

For $x_0 < 0$, we have that

$$\mathcal{D}^{x_0} = \left(\tfrac{1}{x_0}, \infty\right).$$



Theorem 4.24.

(i) The flow domain $\mathcal{D} \subseteq \mathbb{R} \times M$ of ξ is open and the flow $\theta\colon \mathcal{D} \to M$ is smooth. In particular for each $t \in \mathbb{R}$ the set $\mathcal{D}_t \subseteq M$ is open and $\theta_t\colon \mathcal{D}_t \to M$ is smooth.

(ii) $\mathcal{D}^{\theta(t,a)} = \mathcal{D}^a - t$ for all $(t, a) \in \mathcal{D}$.

*(iii) We have that $\mathcal{D}_{s+t} \supseteq \mathcal{D}_s \cap \theta_s^{-1}(\mathcal{D}_t)$ and on $\mathcal{D}_s \cap \theta_s^{-1}(\mathcal{D}_t)$ the **flow law***

$$\theta_{s+t} = \theta_t \circ \theta_s \tag{6}$$

holds.

(iv) $\mathcal{D}_0 = M$ and $\theta_0 = \mathrm{id}_M$.

Proof.

(i) We will only sketch the proof of this, because it's analysis that we don't need to think about. Use charts to reduce to the case of $M = U \subseteq \mathbb{R}^n$ open. Then the result is true by the theory of ODE's: for each $a \in U$, there is an open neighborhood V of a in U and $\varepsilon > 0$ such that for all $x \in V$, the trajectory $\theta(t, x)$ exists for all $t \in (-\varepsilon, \varepsilon)$. This shows $\mathcal{D}$ is open. Also, $\theta(t, x)$ depends smoothly on t and x.

(ii) Let $\gamma(t) = \theta(t, a)$. For some $s \in \mathcal{D}^a$, let $\delta(t) = \gamma(t+s)$. The domain of δ is $\mathcal{D}^a - s$ by time translation; δ starts at $\gamma(s) = \theta(s, a)$. That is, $\delta(t) = \theta(t, \theta(s, a))$. The definition interval of $\mathcal{D}$ is $\mathcal{D}^{\theta(s,a)}$.

(iii) Fix $a \in M$; let γ, δ be as in (ii). We get

$$\gamma(t+s) = \theta(t, \theta(s, a)).$$

That is,

$$\theta(t+s, a) = \theta(t, \theta(s, a)).$$

We can rewrite this as

$$\theta_{s+t}(a) = \theta_t(\theta_s(a)).$$

For this to hold we must have $s \in \mathcal{D}^a$ and $t \in \mathcal{D}^a - s = \mathcal{D}^{\theta(s,a)}$. This is true if and only if $a \in \mathcal{D}_s$ and $\theta(s, a) \in \mathcal{D}_t$. Again, this is true if and only if $a \in \mathcal{D}_s$ and $a \in \theta_s^{-1}(\mathcal{D}_t)$.

If t, s satisfy these conditions, then $s + t \in \mathcal{D}^a$. That is, $a \in \mathcal{D}_{s+t}$.

(iv) Clear from the results of the other parts. □

Corollary 4.25. *$\theta_t(\mathcal{D}_t) = \mathcal{D}_{-t}$ and $\theta_t \colon \mathcal{D}_t \to \mathcal{D}_{-t}$ is a diffeomorphism with inverse θ_{-t}.*

Proof. By Theorem 4.24(iii), for all $(s, t, a) \in \mathbb{R} \times \mathbb{R} \times M$,

$$s + t \in \mathcal{D}^a \iff s \in \mathcal{D}^a - t \overset{\text{(iii)}}{=} \mathcal{D}^{\theta(t,a)} \iff \theta(t, a) \in \mathcal{D}_s \iff \theta_t(a) \in \mathcal{D}_s.$$

Set $s = -t$. Then

$$0 \in \mathcal{D}^a \iff \theta_t(a) \in \mathcal{D}_{-t}.$$

But we always have that $0 \in \mathcal{D}^a$, so $\theta_t(\mathcal{D}_t) \subseteq \mathcal{D}_{-t}$.

Then the flow law (6) tells us that

$$\theta_{-t}(\theta_t(a)) = \theta_0(a) = a$$

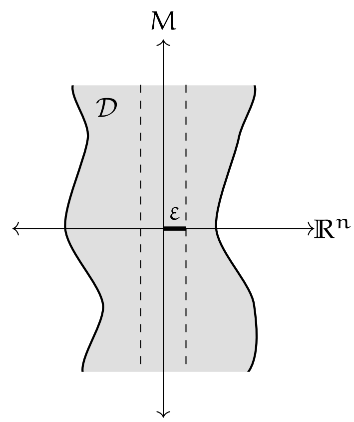
and replacing t by $-t$, we get that $\theta_t \circ \theta_{-t} = \mathrm{id}_{\mathcal{D}_{-t}}$. □

Definition 4.26. ξ is **complete** if $\mathcal{D} = \mathcal{D}(\xi) = \mathbb{R} \times M$, that is, $\mathcal{D}^a = \mathbb{R}$ for all a, or $\mathcal{D}_t = M$ for all t.

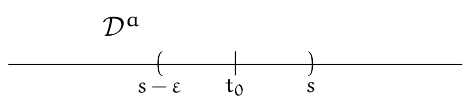
Remark 4.27. In the case that the flow of ξ is complete, then each θ_t is a diffeomorphism $M \to M$. That is, $\theta_t \in \mathrm{Diff}(M)$, the group of diffeomorphisms $M \to M$, and in this case $t \mapsto \theta_t$ is a homomorphism $\mathbb{R} \to \mathrm{Diff}(M)$ by the flow law (6). It also defines an action of $\mathbb{R}$ on M.

When is a vector field complete? Are there easy criteria for this?

Lemma 4.28 (Uniform Time Lemma)**.** *If there is $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \times M \subseteq \mathcal{D}$, then ξ is complete.*



Proof. Let $a \in M$. Let $s = \sup(\mathcal{D}^a)$. Suppose $s < \infty$.



Then let $t_0 \in (s-\varepsilon, s) \in \mathcal{D}^a$. Let $b = \theta(t_a, a)$. Since $(-\varepsilon, \varepsilon) \subseteq \mathcal{D}^b = \mathcal{D}^{\theta(t_0, a)} = \mathcal{D}^a - t_0$, we have $(t_0 - \varepsilon, t_0 + \varepsilon) \subseteq \mathcal{D}^a$. But $t_0 + \varepsilon > s$, which is a contradiction. So $s = \infty$. Similarly, $\inf(\mathcal{D}^a) = -\infty$. □

Definition 4.29. The **support** of ξ is

$$\mathrm{supp}(\xi) = \overline{\{a \in M \mid \xi_a \neq 0_a\}}$$

Theorem 4.30. *If* $\mathrm{supp}(\xi)$ *is compact, then* ξ *is complete.*

Proof. Let $K = \mathrm{supp}(\xi)$. If $a \notin K$, then $\xi_a = 0$, so $\theta(t, a) = a$ for all t and $\mathcal{D}^a = \mathbb{R}$. For every $a \in K$ there is $\varepsilon_a > 0$ and a neighborhood U_a of a such that $(-\varepsilon, \varepsilon) \times U_a \subseteq \mathcal{D}$. That is to say that for all $b \in U_a$, we have $(-\varepsilon_a, \varepsilon_a) \subseteq \mathcal{D}^b$.

Cover K by finitely many $U_{a_1}, \ldots, U_{a_p}$; put $\varepsilon = \min\{\varepsilon_{a_1}, \ldots, \varepsilon_{a_p}\}$. Then $(-\varepsilon, \varepsilon) \subseteq \mathcal{D}^a$ for all $a \in M$.

Now apply the uniform time Lemma 4.28. □

Corollary 4.31. *On a compact manifold every vector field is complete.*

Example 4.32. An example of a complete vector field: linear vector fields on vector spaces.

Let $\xi\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear map with matrix A. The flow of ξ is given by $\theta\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, that is, the solution to the differential equation

$$\begin{cases} x'(t) &= \xi(x(t)) = Ax(t) \\ x(0) &= x_0 \end{cases}$$

The solution to this equation is

$$x(t) = \exp(tA)x_0,$$

where exp is the matrix exponential. This is defined for all t.

A **time-dependent linear vector field** on $\mathbb{R}^n$ is a vector field ξ on $\mathbb{R} \times \mathbb{R}^n$ of the form

$$\xi(t,x) = \left(\frac{d}{dt}, A(t)x\right)$$

where $A\colon \mathbb{R} \to M(n,\mathbb{R})$ is a smooth map. Such a vector field is also complete.

4.3 Derivations, Revisited

Let A be a commutative unital ring and let B be a commutative A-algebra with identity. Let C be a B-module. Think of A as the scalars, (e.g. $A = \mathbb{R}$) and B as the functions (e.g. $B = C^\infty(M,\mathbb{R})$), and C as the vector valued functions (e.g. $C^\infty(M,\mathbb{R}^k)$ or $\mathcal{T}(M)$).

Definition 4.33. An A**-derivation of** B **into** C is a map $\ell\colon B \to C$ satisfying

(a) A-linearity: $\ell(a_1b_1 + a_2b_2) = a_1\ell(b_1) + a_2\ell(b_2)$, and

(b) the **Leibniz rule:** $\ell(b_1b_2) = \ell(b_1)b_2 + b_1\ell(b_2)$.

for all $a_i \in A$ and $b_i \in B$.

Remark 4.34. We should really write $b_2\ell(b_1)$ instead of $\ell(b_1)b_2$ because C is a (left) B-module, but if we were working in the non-commutative case then this would be *wrong*. But in our case, B is commutative so we may pretend that C is a B-bimodule with the left and right actions coinciding. In the noncommutative case, we must add the assumption that C is a B-bimodule.

Definition 4.35. The set $\mathrm{Der}_A(B,C)$ is the set of all A-derivations of B into C. It is a B-module as well as an A-module; if $b_1, b_2 \in B$ and $\ell_1, \ell_2 \in \mathrm{Der}_A(B,C)$, then $b_1\ell_1 + b_2\ell_2$ is also a derivation for B commutative.

Lemma 4.36. *If* $\ell \in \mathrm{Der}_A(B, C)$*, then* $\ell(a 1_B) = 0$ *for all* $a \in A$.

Proof. $\ell(1_B) = \ell(1_B 1_B) = \ell(1_B)1_B + 1_B \ell(1_B) = 2\ell(1_B) \implies \ell(1_B) = 0$. □

Example 4.37. Let $a \in M$ and let $B = C^\infty_{M,a}$ be the algebra of germs at a, and $C = \mathbb{R}$. Here $A = \mathbb{R}$. Recall that we have an evaluation map

$$\begin{aligned} C^\infty_{M,a} &\xrightarrow{\mathrm{ev}_a} C = \mathbb{R} \\ [f] &\longmapsto f(a) \end{aligned}$$

This is an algebra homomorphism. This makes C into a B-module, via

$$[f] \cdot c = f(a)c.$$

We called $\mathrm{Der}_\mathbb{R}(C^\infty_{M,a}, \mathbb{R})$ the derivations of M at a. The map

$$\begin{aligned} T_a M &\xrightarrow{\mathcal{L}_a} \mathrm{Der}_\mathbb{R}(C^\infty_{M,a}, \mathbb{R}) \\ v &\longmapsto \mathcal{L}_a(v) \end{aligned}$$

is an isomorphism given by

$$\mathcal{L}_a(v)([f]) = d_a f(v).$$

Definition 4.38. Let $B = C^\infty(M) = C$. We call $\mathrm{Der}_\mathbb{R}(C^\infty(M))$ the **derivations of** M**.**

Definition 4.39. Each $\xi \in \mathcal{T}(M)$ defines a derivation $\mathcal{L}_\xi$ of M, given by

$$\mathcal{L}_\xi(f) = df(\xi).$$

This is the **Lie derivative** or **directional derivative** of f along ξ.

That is, $\mathcal{L}_\xi(f)$ is the function defined by

$$\mathcal{L}_\xi(f)(a) = d_a f(\xi_a),$$

and $\mathcal{L}_\xi$ satisfies the Leibniz rule.

Remark 4.40. We need to check that $\mathcal{L}_\xi(f)$ is smooth. We check this in a chart (U, ϕ), where we may write

$$\widetilde{f} = f \circ \phi^{-1}.$$

Recall that

$$(T\phi \circ \xi \circ \phi^{-1})(x) = (x, \widetilde{\xi}(x)),$$

so that

$$T\phi \circ \xi \circ \phi^{-1} = \mathrm{id}_{\phi(U)} \times \widetilde{\xi},$$

where $\widetilde{\xi}\colon \phi(U) \to \mathbb{R}^n$ is smooth. Then

$$\mathcal{L}_\xi(f)(a) = d_a f(\xi_a) = D(\widetilde{f})_{\phi(a)}(\widetilde{\xi}_{\phi(a)})$$

for all $a \in U$. Since $\widetilde{f}, \widetilde{\xi}$ are smooth, then so is $\mathcal{L}_\xi(f)$.

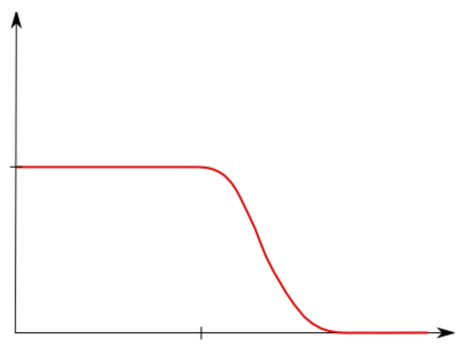
Theorem 4.41. *The map $\mathcal{L}\colon \mathcal{T}(M) \to \mathrm{Der}_{\mathbb{R}}(C^\infty(M))$ defined by $\xi \mapsto \mathcal{L}_\xi$ is an isomorphism.*

To prove this theorem, we will first need a few analysis lemmas.

Lemma 4.42 (Existence of Smooth Step Functions)**.** *For all $0 < p < q$ there is a smooth function $\lambda\colon \mathbb{R}^n \to [0,1]$ with*

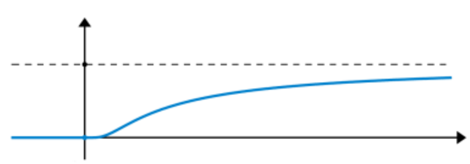
$$\lambda(x) = \begin{cases} 1 & \textit{for } \|x\| < p \\ 0 & \textit{for } \|x\| > q. \end{cases}$$

The function λ looks like this:



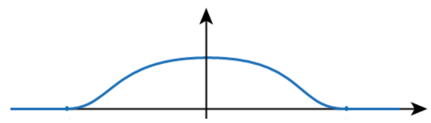
Proof of Lemma 4.42. Define

$$\alpha(x) = \begin{cases} e^{-1/x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}.$$



We can show that α is in fact smooth! Then $\alpha^{(k)}(0) = 0$ for all $k \geq 0$; we say that α is **flat** at 0. The k-th Taylor polynomial of α at $x = 0$ is the zero polynomial.

Define $\beta\colon \mathbb{R} \to [0,\infty)$ by $\beta(x) = \alpha(x-p)\alpha(q-x)$.



Define $\gamma\colon \mathbb{R} \to [0,1]$ by

$$\gamma(x) = \frac{\int_x^q \beta(t)\,dt}{\int_p^q \beta(t)\,dt}.$$

Then γ is C^∞.

Finally, let $\lambda(x) = \gamma(\|x\|)$. This works. □

Recall from Definition 4.29 that the **support** of $f \in C^\infty(M)$ is

$$\operatorname{supp}(f) = \overline{\{x \in M \mid f(x) \neq 0\}}$$

Lemma 4.43 (Extension Lemma)**.** *For every* $a \in M$*, the restriction map*

$$\begin{aligned} C^\infty(M) &\longrightarrow C^\infty_{M,a} \\ f &\longmapsto [f]_a \end{aligned}$$

is surjective.

Proof. Let $[g]_a \in C^\infty_{M,a}$. Then there is $g \in C^\infty(U)$, where U is an open neighborhood of a. We will cook up a global function f on M that has the same germ as g at a.

Without loss, we may assume that U is the domain of a chart (U,ϕ) centered at a. Choose $0 < p < q$ such that $B_p(0) \subseteq B_q(0) \subseteq \phi(U)$. Choose λ as in the previous laemma, and let $\rho = \lambda \circ \phi \in C^\infty(U)$. Then

$$\operatorname{supp}(\rho) = \phi^{-1}(B_q(0))$$

is compact and hence closed in M (because M is Hausdorff).

Hence $V = M \setminus \operatorname{supp}(\rho)$ is open and U, V form an open cover of M. Now define

$$f = \begin{cases} 0 & \text{on } V \\ \rho g & \text{on } U \end{cases}$$

On $U \cap V$, $\rho g = 0$, so f is well-defined, and f is smooth on U and on V, so f is smooth. Finally, $f = g$ on a sufficiently small neighborhood of a, namely $\phi^{-1}(B_q(0))$, so $[f]_a = [g]_a$. □

Remark 4.44. Taking $g = 1$ in the Extension Lemma (Lemma 4.43), we get the existence of **smooth bump functions** on M, corresponding to ρ in the previous proof. This leads to the next lemma.

Lemma 4.45 (Bump Function Lemma)**.** *For each $a \in M$ and every neighborhood U of a, there is a smooth $\rho \in C^\infty(M)$ with $\operatorname{supp}(\rho) \subseteq U$ and $\rho = 1$ in a neighborhood of a.*

Lemma 4.46 (Locality of Derivations)**.** *Let $\ell \in \operatorname{Der}_{\mathbb{R}}(C^\infty(M))$ and $f \in C^\infty(M)$. Let $U \subseteq M$ be open. If $f = 0$ on U, then $\ell(f) = 0$ on U.*

Proof. Let $a \in U$, $\rho \in C^\infty(M)$ a bump function at a, supported on U. Then $\rho f = 0$, so

$$f = (\rho + (1 - \rho))f = (1 - \rho)f.$$

So

$$\ell(f) = \ell((1 - \rho)f) = \ell(1 - \rho)f + (1 - \rho)\ell(f).$$

This vanishes on a neighborhood of a. But the argument holds for all $a \in U$, so $\ell(f) = 0$ on U. □

Proof Sketch of Theorem 4.41. The map $\xi \mapsto \mathcal{L}_\xi$ is evidently $\mathbb{R}$-linear.

To check injectivity, suppose $\mathcal{L}_\xi = 0$. Then for all $f \in C^\infty(M)$,

$$\mathcal{L}_\xi(f) = df(\xi) = 0.$$

Let $a \in M$, and evaluate this at a. By Lemma 4.43, $\operatorname{ev}_a \colon C^\infty(M) \to C^\infty_{M,a}$ is surjective, so for all $[f]_a \in C^\infty_{M,a}$,

$$\mathcal{L}_a(\xi_a)([f]_a) = d_a f(\xi_a) = 0.$$

Now using the pointwise version of the isomorphism $\mathcal{L}_a \colon T_aM \cong \operatorname{Der}_a(M) = \operatorname{Der}_{\mathbb{R}}(C^\infty_{M,a}, \mathbb{R})$, we see that $\xi_a = 0 \in T_aM$. So $\xi = 0$.

To check surjectivity, use locality of derivations and using charts, reduce to the case where $M = U$ is open in $\mathbb{R}^n$. Then the isomorphism $\mathcal{L} \colon \mathcal{T}(M) \to \operatorname{Der}(M)$ is another case of Taylor's theorem. (This is on the homework). □

5 Intermezzo: Point-set topology of Manifolds

This section is some annoying business that we've been postponing.

5.1 Paracompactness

Definition 5.1. A topological space X is **paracompact** if it is Hausdorff and every open cover $\mathcal{U}$ of X admits a **locally finite refinement** $\mathcal{U}'$.

$\mathcal{U}'$ is **locally finite** if each $x \in X$ has a neighborhood which intersects only finitely many members of $\mathcal{U}'$.

$\mathcal{U}'$ is a **refinement** of U if every member of $\mathcal{U}'$ is a subset of a member of $\mathcal{U}$.

A subcover is a type of refinement, and a compact Hausdorff space is paracompact.

In the next statement, "manifold" means merely a topological space with a smooth structure. In particular, we do not demand that it is Hausdorff or second countable.

Theorem 5.2. *Let M be a Hausdorff manifold. Then the following are equivalent:*

(i) Every connected component of M is second countable;

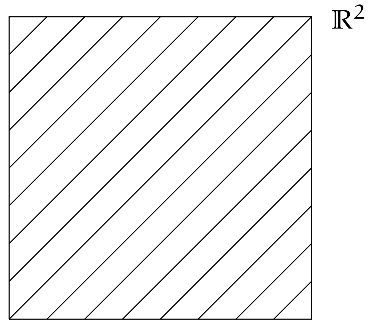
(ii) M is metrizable;

(iii) M is paracompact.

This theorem tells us that paracompactness is a bit weaker than the property second countable. Can we construct a manifold which is paracompact but not second countable? It has to have a ridiculous number of connected components.

Example 5.3. Let M be the disjoint union of uncountably many copies of $\mathbb{R}$ (with its standard smooth structure). Then M is paracompact but not second countable.

This is not too weird, although it might look like it. If we consider the plane $\mathbb{R}^2$ but with a different topology: draw a line, and all the lines parallel to it, and say that intervals of these lines are the basic open sets. This makes M a one-dimensional manifolds.



This is an example of a **foliation**.

Proof Sketch of Theorem 5.2. (i) $\implies$ (ii). Let C be a connected component of M. Then C is Hausdorff and second countable. M is also locally compact, so M is **regular**. For each $x \in M$ and each closed subset $A \subseteq M$ with $x \notin A$, there are open sets U, V with $x \in A$, $A \subseteq V$, and $U \cap V = \emptyset$.

So now by Urysohn's Metrization Theorem C is metrizable. So every component C of M is metrizable, so M is metrizable. Why? For each connected component C, let d_C be a metric on C. Let

$$\delta_C(x,y) = \frac{d_C(x,y)}{1 + d_C(x,y)},$$

then δ_C is a metric on C that defines the same topology. Define

$$d(x,y) = \begin{cases} \delta_C(x,y) & \text{if } x, y \text{ are in the same connected component } C \\ 1 & \text{otherwise.} \end{cases}$$

Then d is a metic compatible with the topology on M.

(ii) $\implies$ (iii). Every metrizable space is paracompact (look it up!)

(iii) $\implies$ (i). We may assume without loss that M is connected. Every point in M has a second countable neighborhood by the paracompact assumption. To get global second countability, we show that M is **σ-compact**, that is, a countable union of compact sets.

M has an open cover $\mathcal{U}$ whose members U have compact closure. By paracompactness, we may replace $\mathcal{U}$ by a refinement that is locally finite. So assume $\mathcal{U}$ is locally finite. Let $\emptyset \neq U_0 \in \mathcal{U}$. Then $\overline{U}_0$, being compact, intersects only finitely many members of $\mathcal{U}$, say $U_1, U_2, \ldots, U_{n_1}$. Likewise, $\overline{U}_0 \cup \overline{U}_1 \cup \ldots \cup \overline{U}_n$ intersects only finitely many members of $\mathcal{U}$, say $U_{n_1+1}, \ldots, U_{n_2}$.

Continuing in this way, we get a sequence $U_0, U_1, U_2, \ldots$ in $\mathcal{U}$, with

$$\bigcup_{j=0}^{\infty} \overline{U}_j = \bigcup_{j=0}^{\infty} U_j.$$

The collection $\{U_j\}_{j \in \mathbb{N}}$ is locally finite, and therefore $\{\overline{U}_j\}_{j \in \mathbb{N}}$ is locally finite. The union of a locally finite family of closed sets is closed, so this implies that

$$\bigcup_{j=0}^{\infty} \overline{U}_j$$

is closed. On the other hand,

$$\bigcup_{j=0}^{\infty} \overline{U}_j = \bigcup_{j=0}^{\infty} U_j$$

is open, so

$$M = \bigcup_{j=0}^{\infty} \overline{U}_j.$$

Hence, M is σ-compact. Global second-countability follows. □

Example 5.4. Let ω_1 be the **first uncountable ordinal**. The elements of ω_1 are all countable ordinals. The **long ray** is

$$R = \omega_1 \times [0, 1)$$

equipped with **lexicographic order**

$$\begin{cases} (a, s) < (b, t) & \text{if } a < b \\ (a, s) < (a, t) & \text{if } s < t \end{cases}$$

and the order topology (subbasis consisting of all segments $R_{<x}$ and $R_{>x}$).

R is a connected 1-manifold with boundary point $(0, 0)$. The **long line** is obtained by gluing two copies of R together along their boundary.

The long line is not second countable, so it is not paracompact.

For the purposes of differential geometry, however, this example is generally useless.

5.2 Partitions of Unity

Definition 5.5. Let $\mathcal{U} = \{U_i \mid i \in I\}$ be an open cover of M. A **partition of unity subordinate to** $\mathcal{U}$ is a family of smooth functions $\{\lambda_i \colon M \to [0, 1] \mid i \in I\}$ such that

(a) $\operatorname{supp}(\lambda_i) \subseteq U_i$,

(b) $\{\operatorname{supp}(\lambda_i) \mid i \in I\}$ is locally finite,

(c) $\displaystyle\sum_{i \in I} \lambda_i = 1$.

Remark 5.6. Note that (b) implies that for any $a \in M$, there is a neighborhood V of a such that only finitely many λ_i are nonzero on V, so the sum $\sum_{i \in I} \lambda_i(a)$ make sense. Moreover, for any $J \subseteq I$ the sum $\sum_{i \in J} \lambda_i$ is well-defined and smooth.

Theorem 5.7. *Let M be paracompact and let $\mathcal{U}$ be an open cover. Then there exists a partition of unity subordinate to $\mathcal{U}$.*

In particular, this theorem holds when M is Hausdorff and second countable.

Lemma 5.8. *Let M be paracompact. Then*

(i) M is regular, and

(ii) *every open cover* $\mathcal{U} = \{U_i \mid i \in I\}$ *of* M *has a* ***shrinking:*** *a locally finite refinement* $\mathcal{V} = \{V_i \mid i \in I\}$*, indexed by the same set, with* $\overline{V}_i \subseteq U_i$.

Proof.

(i) Let $A \subseteq M$ be closed, $x \in X \setminus A$. We want to separate A and x by open sets. For each $y \in A$, choose open sets U_y, V_y with $x \in U_y$, $y \in V_y$, $U_y \cap V_y = \emptyset$.

Then $\{U_y, V_y \mid y \in A\} \cup \{M \setminus (A \cup \{x\})\}$ is an open cover of M. Let $\{W_i\}_{i \in I}$ be a locally finite refinement. Let

$$J = \{i \in I \mid W_i \cap V_y \neq \emptyset \text{ for some } y \in A\}.$$

Choose open $U \ni x$ that meets only finitely many W_j for $j \in J$, say $W_{j_1}, \ldots, W_{j_n}$. Let

$$U' = U \setminus (\overline{W}_{j_1} \cup \ldots \cup \overline{W}_{j_n}), \qquad W = \bigcup_{j \in J} W_j.$$

Then $U' \cap W = \emptyset$, $U' \ni x$ is open, and W is an open neighborhood of A.

(ii) Let $A_i = M \setminus U_i$. Let $x \in U_i$. Use regularity to find a neighborhood $W_{i,x} \ni x$ such that $W_{i,x} \cap A = \emptyset$. So $\overline{W}_{i,x} \subseteq U_i$. Then $\mathcal{W} = \{W_{i,x} \mid i \in I, x \in M\}$ is an open cover of M and a refinement of $\mathcal{U}$. Choose a locally finite refinement $\mathcal{W}'$ of $\mathcal{W}$.

Then $\mathcal{W}' = \{W'_j \mid j \in J\}$ for some index set J. Choose $f\colon J \to I$ such that $W'_j \subseteq W_{f(j),x}$. for some $x \in U_{f(j)}$. Then $\overline{W}'_j \subseteq U_{f(j)}$ since $\overline{W}_{i,x} \subseteq U_i$.

For $i \in I$, set

$$V_i = \bigcup_{j \in f^{-1}(i)} W'_j.$$

Because $\mathcal{W}'$ is locally finite,

$$\overline{V}_i = \bigcup_{j \in f^{-1}(i)} \overline{W}'_j \subseteq U_i.$$

So $\mathcal{V} = \{V_i \mid i \in I\}$ is a shrinking of $\mathcal{U}$. □

Proof of Theorem 5.7. Choose a locally finite atlas $\{(V_j, \phi_j) \mid j \in J\}$ of M, such that $\mathcal{V} = \{V_j \mid j \in J\}$ refines $\mathcal{U}$ and $\overline{V}_j$ is compact with $\phi_j(V_j)$ is bounded in $\mathbb{R}^n$. It suffices to define a partition of unity subordinate to $\mathcal{V}$.

Choose a shrinking $\mathcal{W} = \{W_j \mid j \in J\}$ of $\mathcal{V}$. Then $\overline{W}_j \subseteq V_j$, so $\overline{W}_j$ is compact. Cover $\phi(W_j)$ with finitely many closed balls

$$B_{j,1}, B_{j,2}, \ldots, B_{j,m(j)} \subseteq \phi(W_j).$$

Choose smooth functions $\nu_{j,k}\colon \mathbb{R}^n \to [0,1]$ such that

$$\nu_{j,k}(x) > 0 \iff x \in \operatorname{int}(B_{j,k});$$

these exist by Lemma 4.45). Let

$$\nu_j = \sum_{k=1}^{m(j)} \nu_{j,k}\colon \mathbb{R}^n \to [0,\infty).$$

Then

$$\nu_j(x) \begin{cases} > 0 & \text{for } x \in \phi(W_j) \\ = 0 & \text{for } x \in \mathbb{R}^n \setminus \bigcup_{j=1}^{m(j)} B_{j,k}. \end{cases}$$

Let

$$\mu_j = \begin{cases} \nu_j \circ \phi_j & \text{on } V_j \\ 0 & \text{on } M \setminus \phi_j^{-1}\left(\bigcup_{j=1}^{m(j)} B_{j,k}\right). \end{cases}$$

Then μ_j is well-defined and smooth; $\mu_j > 0$ on W_j, $\operatorname{supp}(\mu_j) \subseteq V_j$. The collection $\{\operatorname{supp}(\nu_j) \mid j \in J\}$ is locally finite.

Finally, put

$$\lambda_j = \frac{\mu_j}{\sum_{i \in J} \mu_i} \qquad \square$$

5.3 Some applications

Assume throughout this section that M is paracompact.

Lemma 5.9 (Smooth Urysohn's Lemma)**.** *Let A, B be disjoint closed subsets of M. Then there exists a smooth $f\colon M \to [0,1]$ with $f = 0$ on A and $f = 1$ on B. Hence, there exist open sets $U \supseteq A$ and $V \supseteq B$ and $U \cap V = \emptyset$.*

Proof. Let $X = M \setminus B$, $Y = M \setminus A$. Then $\{X, Y\}$ is an open cover of M, so let $\{f, g\}$ be a partition of unity subordinate to $\{X, Y\}$. Then $\operatorname{supp}(f) \subseteq X$, so $f = 0$ on A, and $\operatorname{supp}(g) \subseteq Y$, so $g = 0$ on B. We have that $f = g = 1$, so $f = 1 - g = 1$ on B. Now take $U = f^{-1}([0,\varepsilon])$ and $V = f^{-1}([1-\varepsilon, 1])$. $\square$

Definition 5.10. Let $A \subseteq M$ be closed. A **germ** at A is an equivalence class $[f]_A$ of smooth functions where $f \in C^\infty(U)$ for some $U \supseteq A$ open, and $f \in C^\infty(U)$, $g \in C^\infty(V)$ are equivalent, $f \sim_A g$, if $f = g$ for some $W \supseteq A$ open, $W \subseteq U \cap V$.

Definition 5.11. Let $C^\infty_{M,A}$ be the algebra of germs $[f]_A$ at A.

Lemma 5.12 (Generalized Extension Lemma)**.** *Let $A \subseteq M$ be closed and $U \subseteq M$ an open neighborhood of A. Let $f\colon A \to \mathbb{R}$ be smooth. Then there exists a smooth $\widetilde{f}\colon M \to \mathbb{R}$ satisfying $\widetilde{f}|_A = f$ and $\operatorname{supp}(f) \subseteq U$.*

Recall that if A is closed, we say $f\colon A \to \mathbb{R}$ is smooth if for all $a \in A$, there is an open neighborhood V_a of a and a smooth function $f_a \in C^\infty(V_a)$ such that $f_a|_{V \cap A} = f$.

Proof. For each $a \in A$ we choose V_a and f_a as above. Without loss, we may assume that $V_a \subseteq U$. Let

$$\mathcal{U} = \{V_a \mid a \in A\} \cup \{M \setminus A\};$$

this is an open cover of M. Let

$$\{\lambda_a \mid a \in A\} \cup \{\lambda_0\}$$

be a partition of unity subordinate to U. This means that each $\lambda_a\colon M \to [0,1]$ is smooth with $\operatorname{supp}(\lambda_a) \subseteq V_a$ and $\operatorname{supp}(\lambda_0) \subseteq M \setminus A$, and the collection of supports is locally finite, and $\sum_{a \in A} \lambda_a + \lambda_0 = 1$.

Define $\widetilde{f}_a\colon M \to \mathbb{R}$ by

$$\widetilde{f}_a = \begin{cases} \lambda_a f_a & \text{on } V_a \\ 0 & \text{on } M \setminus \operatorname{supp}(\lambda_a) \end{cases}$$

and $\widetilde{f}_0 = 0$. Then $\widetilde{f}_a, \widetilde{f}_0$ are smooth. Put

$$\widetilde{f} = \sum_{a \in A} \widetilde{f}_a.$$

Then for $x \in A$, $\lambda_0(x) = 0$, so

$$\begin{aligned}
\widetilde{f}(x) &= \sum_{a \in A} \widetilde{f}_a(x) + \widetilde{f}_0(x) \\
&= \sum_{a \in A} \lambda_a(x) f_a(x) + \lambda_0(x) f(x) \\
&= \sum_{a \in A} \lambda_a(x) f(x) + \lambda_0(x) f(x) \\
&= \left(\sum_{a \in A} \lambda_a(x) + \lambda_0(x) \right) f(x) \\
&= f(x)
\end{aligned}$$

Moreover,

$$\begin{aligned}
\operatorname{supp}(\widetilde{f}) &\subseteq \bigcup_{a \in A} \operatorname{supp}(\widetilde{f}_a) \\
&\subseteq \bigcup_{a \in A} \operatorname{supp}(\lambda_a) \\
&\subseteq \bigcup_{a \in A} V_a \\
&\subseteq U.
\end{aligned}$$

□

Theorem 5.13. *Let $A \subseteq M$ be closed. Let $C^\infty_{M,A}$ be the algebra of germs of smooth functions at A. Then the restriction map $C^\infty(M) \to C^\infty_{M,A}$ given by $f \mapsto [f]_A$ is surjective.*

Proof. Let $[g]_A \in C^\infty_{M,A}$ be represented by a smooth $g\colon U \to \mathbb{R}$, where $U \supseteq A$ is open. Then $M \setminus U$ and A are nonintersecting closed subsets, so by Urysohn's Lemma, there is an open $V \supseteq A$ with $\overline{V} \subseteq U$. By the generalized extension lemma, the constant function 1 on $\overline{V}$ extends to a smooth $\rho\colon M \to \mathbb{R}$ with $\operatorname{supp}(\rho) \subseteq U$. Put

$$f = \begin{cases} \rho g & \text{on } U \\ 0 & \text{on } M \setminus \operatorname{supp}(\rho). \end{cases}$$

Then f is well-defined and smooth and $f = g$ on V, so $[f]_A = [g]_A$. □

Remark 5.14. This theorem is one of the dividing lines between algebraic or complex geometry and real differential geometry. Locally defined polynomial or holomorphic functions on an open set on a complex manifold or variety don't tend to extend globally without acquiring poles – they extend meromorphically, but not holomorphically.

There are also applications of partitions of unity to Riemannian geometry.

Definition 5.15. A **Riemannian metric** on M is a collection

$$g = \{g_a \mid a \in M\}.$$

where for each $a \in M$,

$$g_a\colon T_aM \times T_aM \to \mathbb{R}$$

is an inner product (positive definite symmetric bilinear form) with the property that for each chart (U, ϕ), the expression $\widetilde{g}$ for g

$$\widetilde{g}\colon \phi(U) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

$$\widetilde{g}_x(h,k) = g_{\phi^{-1}(x)}((d_x\phi)^{-1}(h), (d_x\phi)^{-1}(k))$$

is smooth.

Definition 5.16. A **Riemannian manifold** is a pair (M, g) of a Hausdorff manifold M and a Riemannian metric g on M.

Remark 5.17. We can also express a metric as a matrix. For each $x \in \phi(U)$, let

$$A_x = \left[\widetilde{g}_x(e_i, e_j)\right]_{1 \le i,j \le n}$$

be the matrix of the inner product $\widetilde{g}_x$ on $\mathbb{R}^n$ with respect to the standard basis. Then $\widetilde{g}_x(h,k) = h^T A_x k$, and $\widetilde{g}$ is smooth if and only if the map $A\colon \phi(U) \to M_{n\times n}(\mathbb{R})$, $x \mapsto A_x$ is smooth.

Proposition 5.18. *A manifold* M *has a Riemannian metric.*

Proof. Choose an atlas $\{(U_i, \phi_i) \mid i \in I\}$ on M. For each $i \in I$, $a \in U_i$, and $v, w \in T_a M$, define a metric using the ordinary dot product in $\mathbb{R}^n$.

$$(g_i)_a(v, w) = \langle d_a\phi(v), d_a\phi(w)\rangle = d_a\phi(v)^T d_a\phi(w).$$

The matrix of $\widetilde{g}_i$ is the identity matrix, so on U_i, g_i defines a smooth Riemannian metric.

Choose a partition of unity $\{\lambda_i \mid i \in I\}$ subordinate to the atlas $\{(U_i, \phi_i) \mid i \in I\}$ and put

$$(h_i)_a = \begin{cases} \lambda_i(a)(g_i)_a & \text{for } a \in U_i \\ 0 & \text{for } a \in M \setminus \operatorname{supp}(\lambda_i). \end{cases}$$

Then h_i is a smooth symmetric bilinear positive semidefinite form on TM. Let

$$g = \sum_{i \in I} h_i.$$

This g is smooth, symmetric, bilinear, and positive semidefinite. We just need to check that g is positive definite. Given $a \in M$ and $v \in T_a M$, then

$$g_a(v, v) = \sum_{\substack{i \in I \\ a \in \operatorname{supp}(\lambda_i)}} \lambda_i(a)(g_i)_a(v, v) \geq 0.$$

Since $(g_i)_a(v, v) > 0$ for at least one i, we get $g_a(v, v) > 0$. □

Proposition 5.19. *Every Riemannian manifold* (M, g) *is paracompact.*

Proof. We may assume that M is connected, so let $\gamma\colon [0, 1] \to M$ be a smooth path. Then **length** of γ is

$$\operatorname{length}(\gamma) = \int_0^1 \|\gamma'(t)\| \, dt$$

where $\|\cdot\|$ is the length with respect to g. Then define for $x, y \in M$

$$d(x, y) = \inf\{\operatorname{length}(\gamma) \mid \gamma(0) = x, \gamma(1) = y\}.$$

Then d is a metric on M in the usual topological sense, and d is compatible with the topology. So M is metrizable, and therefore paracompact. □

6 Lie Groups

Definition 6.1. A **Lie group** is a set G equipped with a smooth structure and a group structure which are **compatible** in the sense that multiplication

$$\begin{aligned} \mu\colon G \times G &\longrightarrow G \\ (g,h) &\longmapsto gh \end{aligned}$$

and inversion

$$\begin{aligned} \iota\colon G &\longrightarrow G \\ g &\longmapsto g^{-1} \end{aligned}$$

are both smooth. The unit of G is denoted by 1.

Example 6.2. Let $\mathbb{F}$ be a finite-dimensional associative unital division algebra with unit over $\mathbb{R}$. Although this sounds very general, a theorem of Frobenius implies that $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Let $M(m \times n, \mathbb{F})$ be the set of $m \times n$ matrices with entries in $\mathbb{F}$.

Under matrix multiplication, $M(n, \mathbb{F}) := M(n \times n, \mathbb{F})$ is a $\mathbb{R}$-algebra. The **general linear group** of dimension n over $\mathbb{F}$ is its group of units,

$$GL(n, \mathbb{F}) = \{X \in M(n, \mathbb{F}) \mid X \text{ invertible }\}.$$

$M(n, \mathbb{F})$ is a vector space over $\mathbb{R}$ of dimension $n^2 \dim_{\mathbb{R}}(\mathbb{F})$. A matrix X is invertible if and only if $\det(X) \neq 0$, so $G = GL(n, \mathbb{F})$ is an open submanifold of $M(n, \mathbb{F})$. (There's a bit of subtlety here in defining the determinant of a quaternionic matrix, because multiplication of quaternions is not commutative. However, we can use the embedding $M(n, \mathbb{H}) \hookrightarrow M(n, \mathbb{C})$.)

Multiplication $M(n, \mathbb{F}) \times M(n, \mathbb{F}) \to M(n, \mathbb{F})$ is $\mathbb{R}$-bilinear, and therefore smooth. Inversion is smooth because there's a formula for inversion of a matrix, known as **Cramer's rule:** (which is really only useful here)

$$X^{-1} = \frac{1}{\det(X)} \operatorname{adj}(X),$$

where adj(X) is the **adjugate matrix**.

For $n = 1$, we have

$$GL(1, \mathbb{F}) = \mathbb{F}^{\times} = \mathbb{F} \setminus \{0\}.$$

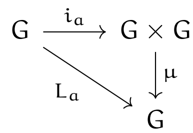
Example 6.3. $\mathbb{F}^n$, equipped with addition, is a Lie group of dimension $n \dim_{\mathbb{R}}(\mathbb{F})$.

Definition 6.4. Let G be a fixed Lie group. For fixed $a \in G$, define maps

$$\begin{aligned} L_a\colon G &\longrightarrow G, \\ g &\longmapsto ag \end{aligned} \qquad \begin{aligned} R_a\colon G &\longrightarrow G, \\ g &\longmapsto ga \end{aligned} \qquad \begin{aligned} \mathrm{Ad}_a\colon G &\longrightarrow G. \\ g &\longmapsto aga^{-1} \end{aligned}$$

Claim 6.5. L_a, R_a, Ad_a are diffeomorphisms from G to G.

Proof. Let $i_a(g) = (a, g)$. Then the following diagram commutes.



and $L_a = \mu \circ i_a$ is the composition of two smooth maps. Hence, L_a is smooth. Also $(L_a)^{-1} = L_{a^{-1}}$, so L_a is a diffeomorphism. Similarly, R_a is a diffeomorphism.

Finally $\mathrm{Ad}_a = L_a \circ R_a^{-1} = R_a^{-1} \circ L_a$, so Ad_a is the composition of two diffeomorphisms and therefore itself a smooth diffeomorphism. □

Definition 6.6. For subsets $U, V \subseteq G$, define

$$\begin{aligned} aU &= L_a(U), \\ Ua &= R_a(U), \\ aUa^{-1} &= \mathrm{Ad}_a(U), \\ U^{-1} &= \iota(U), \\ UV &= \mu(U \times V). \end{aligned}$$

Definition 6.7. The **identity component** G_0 of G is the connected component of G containing the identity.

Lemma 6.8.

(i) Suppose G is connected. Let U be an open neighborhood of the identity. Then U generates G.

(ii) G is second-countable.

(iii) G_0 is a subgroup of G

(iv) The connected components of G are the (left) cosets of G_0.

(v) G is Hausdorff and paracompact.

Proof.

(i) Let H be the subgroup generated by U, that is,

$$H = \bigcap \{K \mid K \text{ is a subgroup of } G \text{ containing } U\} = \bigcup_{k \in \mathbb{N}} (UU^{-1})^{(k)},$$

where $V^{(k)} = V \cdot V \cdots V$ is the set of k-fold products of elements of V. So H is open, as the union of open sets. Hence $gH = L_g(H)$ is open, so

$$H = G \setminus \bigcup_{g \notin H} gH$$

is also closed. Hence, the subgroup H generated by U is both open and closed in G, and G is connected, so $H = G$.

(ii) If we take U to be second-countable, then as in the previous part, we see that

$$G = \bigcup_{k \in \mathbb{N}} (UU^{-1})^{(k)}$$

is also second-countable.

(iii) Note that G_0 is both an open and closed in G, and moreover, because μ and ι are smooth, $\mu(G_0 \times G_0) \subseteq G_0$ and $\iota(G_0) \subseteq G_0$, so G_0 is an open and closed subgroup of G.

(iv) Every coset gG_0 of G_0 is an open and closed connected submanifold, and hence a connected component of G. The cosets cover G, so the decomposition of G into cosets decomposes G into connected components as well. Moreover, $\dim(gG_0) = \dim(L_g(G_0)) = \dim(G_0)$, so G is pure.

(v) Let

$$\Delta_G = \{(g, g) \mid g \in G\}$$

be the diagonal. Then $\Delta_G = f^{-1}(1)$, where

$$\begin{aligned} f\colon G \times G &\longrightarrow G \\ (g, h) &\longmapsto gh^{-1} \end{aligned}$$

G is a manifold, so $\{1\}$ is closed. And f is continuous, so Δ_G is closed, which implies G is Hausdorff. To show that G is paracompact, we know that G is both second-countable and Hausdorff. □

Definition 6.9. Let H, G be Lie groups. A **Lie Group homomorphism** is a map $f\colon G \to H$ such that f is both smooth and a group homomorphism.

Example 6.10.

(a) For all $g \in G$, $\mathrm{Ad}_g\colon G \to G$ is a Lie group automorphism, called the **inner automorphism** defined by g.

(b) Morphisms $G \to GL(n, \mathbb{F})$ are called **representations** of G over $\mathbb{F}$.

(c) The determinant maps $\det\colon GL(n, \mathbb{R}) \to \mathbb{R}^\times$ and $\det\colon GL(n, \mathbb{C}) \to \mathbb{C}^\times$ and $\det\colon GL(n, \mathbb{H}) \to \mathbb{R}^\times$ are morphisms of Lie groups.

Definition 6.11. A **Lie subgroup** (or **embedded Lie subgroup**) of G is a subset H that is both a subgroup of G and an embedded submanifold.

Lemma 6.12. *Let* M, N *be manifolds, and* $f \colon M \to N$ *smooth. Let* $A \subseteq M$, $B \subseteq N$ *be submanifolds, and suppose that* $f(A) \subseteq B$. *Let* $g \colon A \to B$ *be the restriction of* f *to* A. *Then* g *is smooth.*

Proof. First, restrict the domain of f, $f|_A \colon A \to N$. Then

$$\begin{array}{ccc} A & \xhookrightarrow{i_A} & M \\ & \searrow^{f|_A} & \downarrow f \\ & & N \end{array}$$

commutes, so $f|_A$ is the composition of smooth maps and therefore smooth.

Next, restrict the codomain. If we have a global smooth retraction $r_B \colon N \to B$ of the inclusion $i_B \colon B \hookrightarrow N$, then $g = r_B \circ f|_A$ is smooth.

$$\begin{array}{ccc} A & \xrightarrow{f|_A} & N \\ & \searrow^{g} & \downarrow r_B \\ & & B \end{array}$$

Although we may not always have a global smooth retraction, the implicit function theorem guarantees that it exists locally, and this is sufficient. □

Remark 6.13. This is an easy way to generate maps between smooth manifolds, because they may often be embedded in ambient Euclidean space, so if a map between manifolds is smooth on the whole ambient Euclidean space, then it is smooth on the manifolds.

Lemma 6.14. *Let* H *be a Lie subgroup of* G. *Then the smooth structure and group structure on* H *are compatible. Hence,* H *is a Lie group.*

Proof. Multiplication and inversion on H, $\mu_H \colon H \times H \to H$ and $\iota_H \colon H \to H$ are obtained by restricting the domain and codomain of μ_G and ι_G. Hence, by Lemma 6.12, μ_H and ι_H are smooth. □

Example 6.15.

(a) The identity component G_0 of any Lie group G is a Lie subgroup of G.

(b) $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ is a Lie subgroup of $\mathbb{C}^\times = GL(1, \mathbb{C})$.

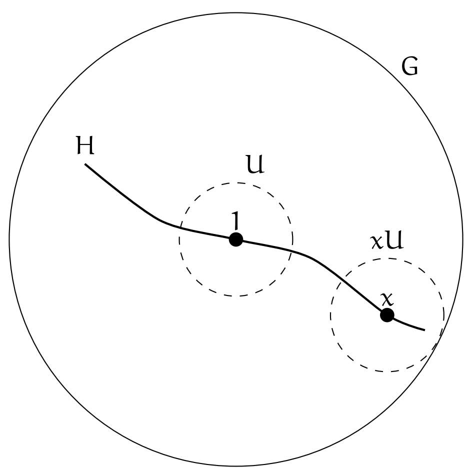
(c) S^3 is a Lie subgroup of $\mathbb{H}^\times = GL(1, \mathbb{H})$.

Example 6.16 (A non-example)**.** Let $G = \mathbb{S}^1 \times \mathbb{S}^1$, and let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. The map $f\colon \mathbb{R} \to G$ given by $f(t) = (e^{it}, e^{i\alpha t})$ is a morphism of Lie groups. It's also an injective immersion with dense image (proof due to Kronecker). However, $H = f(\mathbb{R})$ is not a Lie subgroup (f is not an embedding). But we say that H is an **immersed** Lie subgroup.

Lemma 6.17. *Let H be a Lie subgroup of G. Then H is closed in G.*

Proof. H is a submanifold, so H is locally closed in G. Choose an open neighborhood U of $1 \in H$ such that $U \cap H$ is closed in U. After replacing U with $U \cap U^{-1}$, we may assume that $U = U^{-1}$. Let $x \in \overline{H}$. The open neighborhood xU of x intersects H, since $x \in \overline{H}$, so let $y \in xU \cap H$.

Then $x \in yU^{-1} = yU$, and we have $y(U \cap H) = L_y(U \cap H) = yU \cap H$, the last equality from the fact that $y \in H$. This is closed in yU. But $x \in yU \cap \overline{H}$, and $yU \cap \overline{H}$ is the closure of $yU \cap H$ in yU. So $x \in yU \cap H$. Therefore, $x \in H$. □



Fact 6.18 (Converse of Lemma 6.17)**.** *IF H is a subgroup of G and H is a closed subset of G, then H is a Lie subgroup.*

6.1 Vector fields on Lie groups

$$\begin{aligned} G \times G &\longrightarrow G \\ (g, h) &\longmapsto gh = L_g(h) \end{aligned}$$

The multiplication $\mu\colon G \times G \to G$ is smooth, so the partial tangent map with respect to the second factor is also smooth.

$$\begin{aligned} G \times TG &\longrightarrow TG \\ (g, \xi) &\longmapsto TL_g(\xi). \end{aligned}$$

Put $\mathfrak{g} = T_1 G$, the tangent space at the identity. The restriction to $G \times \mathfrak{g}$ is also smooth.

$$\begin{array}{rcl} G \times \mathfrak{g} & \longrightarrow & TG \\ (g, \xi) & \longmapsto & T_1 L_g(\xi) \in T_g G \end{array}$$

Hence, for each $\xi \in \mathfrak{g}$ the map

$$\begin{array}{rcl} \xi_L \colon G & \longrightarrow & TG \\ g & \longmapsto & T_1 L_g(\xi) \in T_g G \end{array}$$

is smooth: $\xi_L \in \mathcal{T}(G)$. Hence, if $\xi_1, \ldots, \xi_n$ is a basis of $\mathfrak{g}$, then $(\xi_1)_L(g), \ldots, (\xi_n)_L(g)$ is a basis of $T_g G$. So we have shown the following.

Lemma 6.19. *Every Lie group is parallelizable.*

Similarly, we have $\xi_R \in \mathcal{T}(G)$ given by $\xi_R(g) = T_1 R_g(\xi)$.

Lemma 6.20. ξ_L *is complete for every* $\xi \in \mathfrak{g}$.

Proof. Let $\gamma \colon (-\varepsilon, \varepsilon) \to G$ be a trajectory of ξ_L starting at 1_G:

$$\begin{cases} \gamma'(t) = \xi_L(\gamma(t)) = T_1 L_{\gamma(t)}(\xi) \\ \gamma(0) = 1_G. \end{cases}$$

Let $g \in G$ and let $\delta = L_g \circ \gamma \colon (-\varepsilon, \varepsilon) \to G$. Then $\delta(t) = g\gamma(t)$. Then $\delta(0) = g \cdot 1 = g$ and

$$\begin{aligned} \delta'(t) &= T_{\gamma(t)} L_g(\gamma'(t)) \\ &= T_{\gamma(t)} L_g(\xi_L(\gamma(t))) \\ &= T_{\gamma(t)} L_g \circ T_1 L_{\gamma(t)}(\xi) \\ &= T_1(L_g \circ L_{\gamma(t)})(\xi) \\ &= T_1 L_{g\gamma(t)}(\xi) = T_1 L_{\delta(t)}(\xi) = \xi_L(\delta(t)). \end{aligned}$$

So δ is a trajectory of ξ_L starting at g. So ξ_L is complete by the Uniform Time Lemma (Lemma 4.28). □

Definition 6.21. For $\xi \in \mathfrak{g} = T_1 G$, let $\gamma \colon \mathbb{R} \to G$ be the maximal trajectory of ξ_L starting at 1_G. Define $\exp(\xi) = \gamma(1) \in G$. We have a map $\exp \colon \mathfrak{g} \to G$.

Example 6.22. Let $G = GL(n, \mathbb{F})$ where $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Then $\mathfrak{g} = T_1 G = M(n, \mathbb{F})$. Let $A \in \mathfrak{g}$. To find the trajectory of A_L, solve

$$\begin{cases} \gamma'(t) = L_{\gamma(t)} = \gamma(t) A \\ \gamma(0) = I. \end{cases}$$

The solution here is $\gamma(t) = e^{tA}$. So $\exp(A) = e^A$. Notice that A_R has the same trajectory starting at 1.

6.2 Actions of Lie groups on manifolds

Definition 6.23. Let M be a manifold and G a Lie group. A **(left) action** of G on M is a smooth map

$$\begin{aligned} \theta\colon G\times M &\longrightarrow M \\ (g,a) &\longmapsto g\cdot a \end{aligned}$$

with the properties

(a) $(gh)\cdot a = g\cdot(h\cdot a)$

(b) $1\cdot a = a$

for $g,h\in G$ and $a\in M$. A G**-manifold** is a manifold M equipped with a G-action.

Remark 6.24 (Notation)**.** Write

$$\theta(g,a)=\theta_g(a)=\theta^a(g)=ga=g\cdot a.$$

Then $\theta_{gh}=\theta_g\circ\theta_h$.

For each $g\in G$, θ_g is invertible with inverse $(\theta_g)^{-1}=\theta_{g^{-1}}$. So each θ_g is a diffeomorphism and $g\mapsto\theta_g$ is a group homomorphism $G\to \mathrm{Diff}(M)$.

Definition 6.25. The set

$$Ga = G\cdot a = \{ga \mid g\in G\} = \theta^a(G)$$

is the G**-orbit** of $a\in M$.

The action of G on M is **transitive** if $M = Ga$ for some $a\in M$ (and hence for every $a\in M$).

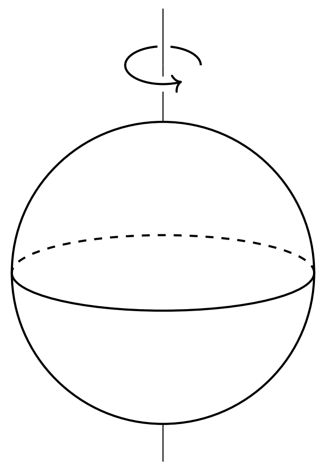
Definition 6.26. The set

$$G_a = \{g\in G \mid ga = a\} = (\theta^a)^{-1}(a)$$

is a closed subgroup of G, called the **isotropy** or **stabilizer subgroup** of a.

The action is **free at** a if $G_a=\{1\}$, and **free** if it is free at all $a\in M$.

Example 6.27. If $M=\mathbb{S}^2$ and $G=\mathbb{S}^1$, then G spins M about the vertical axis with uniform angular velocity.



The orbit Ga of a is the circle of latitude a, and the action is free at each point $x \in \mathbb{S}^2$ except for the north pole N and south pole S.

$$G_a = \begin{cases} \{1\} & \text{if } a \notin \{N, S\} \\ G & \text{if } a \in \{N, S\}. \end{cases}$$

Example 6.28. (a) A complete vector field $\xi \in \mathcal{T}(M)$ has a flow $\theta \colon \mathbb{R} \times M \to M$, which, by the flow law, defines an $\mathbb{R}$-action on M.

(b) $L \colon G \times G \to G$ given by $L(g, h) = gh$ is the **left-translation action** of G on itself.

(c) $R \colon G \times G \to G$ given by $R(g, h) = hg$ is the **right-translation action** of G on itself. It is not a (left) action, but it is a right action: $(gh)a = h(ga)$. However, $(g, h) \mapsto h^{-1}g$ *is* a left-action of G on itself.

(d) $\mathrm{Ad} \colon G \times G \to G$ given by $(g, h) \mapsto ghg^{-1}$ is the **adjoint** or **conjugation** action of G on itself.

(e) Let H be another Lie group and $f \colon G \to H$ a Lie group homomorphism. Then the map $G \times H \to H$ given by $(g, h) \mapsto f(g)h$ is a G-action on H.

(f) A **representation of** G over $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ is a Lie group homomorphism $G \to \mathrm{GL}(n, \mathbb{F})$. This defines an action of G on $M = \mathbb{F}^n$ with the property that each $\theta_g \colon M \to M$ is $\mathbb{F}$-linear.

Remark 6.29. For $\mathbb{F} = \mathbb{H}$, we regard $\mathbb{H}^n$ as a *right* vector space. For $q \in \mathbb{H}$, $x \in \mathbb{H}^n$,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

we put

$$xq = \begin{bmatrix} x_1 q \\ x_2 q \\ \vdots \\ x_n q \end{bmatrix}.$$

Matrices $A \in M(n, \mathbb{F})$ act on $\mathbb{F}^n$ by *left* multiplication:

$$Ax = \begin{bmatrix} a_1 1 x_1 + \ldots + a_{1n} x_n \\ a_{21} x_1 + \ldots + a_{2n} x_n \\ \vdots \\ a_{n1} x_1 + \ldots + a_{nn} x_n \end{bmatrix}$$

Matrix multiplication is *not* left-linear:

$$A(qx) \neq q(Ax)$$

because multiplication is not commutative. However, it is right-linear

$$A(xq) = (Ax)q$$

because $\mathbb{H}$ is associative.

Lemma 6.30. *Let* $a \in M$. *Then*

(i) $\theta^a \colon G \to M$ *has constant rank.*

(ii) The stabilizer G_a *is a Lie subgroup of* G *with* $T_1 G_a = \ker(T_1 \theta^a)$.

(iii) If the action is free at a*, then* θ^a *is an immersion and* Ga *is an immersed submanifold.*

Proof.

(i) Let's compare the rank of θ^a at two points. Through some notational trickery,

$$(gh)a = g(ha) \implies \theta^a(gh) = g\theta^a(h) \implies \theta^a(L_g(h)) = \theta_g(\theta^a(h)).$$

So the following diagram commutes, expressing that θ^a is G-**equivariant**.

$$\begin{array}{ccc} G & \xrightarrow{\theta^a} & M \\ {\scriptstyle L_g}\downarrow & & \downarrow{\scriptstyle \theta_g} \\ G & \xrightarrow{\theta^a} & M \end{array}$$

So now take derivatives of both sides of this equation, and use the chain rule. We get the following commutative diagram.

$$\begin{array}{ccc} T_1 G & \xrightarrow{T_1\theta^a} & T_a M \\ {\scriptstyle T_1 L_g}\downarrow & & \downarrow{\scriptstyle T_a\theta_g} \\ T_g G & \xrightarrow{T_g\theta^a} & T_{ga} M \end{array}$$

But in the diagram above, L_g is a diffeomorphism of G, and θ_g is a diffeomorphism of M, so $T_1 L_g$ and $T_a\theta_g$ are linear isomorphisms. Hence, $T_1\theta^a$ and $T_g\theta^a$ have the same rank, for each $g \in G$. So θ^a has the same rank at each $g \in G$.

(ii) Combine part (i) and Theorem 3.18 to see that $G_a = (\theta^a)^{-1}(a)$ is a submanifold of G with tangent space $T_1 G_a = \ker(T_1\theta^a)$.

(iii) If the action is free at a, then θ^a is injective. By (i), θ^a is a map of constant rank, so $T_g\theta^a$ is also injective for all $g \in G$. So θ^a is an injective immersion, and therefore $\theta^a(G) = Ga$ is an immersed submanifold. □

Remark 6.31. In Lemma 6.30(iii), the conclusion still holds if the action is not free at a. In fact, the orbit $Ga = \theta^a(G)$ is always an immersed submanifold.

Corollary 6.32. *Let H be a Lie group and $f\colon G \to H$ a Lie group homomorphism. Then*

(i) f has constant rank.

(ii) The subgroup $N = \ker(f) = f^{-1}(1_H)$ is a (normal) Lie subgroup of G and $T_1 N = \ker(T_1 f)$.

(iii) If f is injective, then $f(G)$ is an immersed Lie subgroup of H.

Proof. Apply Lemma 6.30 to the action of G on H given by $g \cdot h = f(g)h$. □

6.3 Classical Lie Groups

Example 6.33.

(a) Let $G = GL(n, \mathbb{F})$ where $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, and

$$H = Z(\mathbb{F})^\times = \begin{cases} \mathbb{R}^\times & \text{if } \mathbb{F} = \mathbb{R}, \\ \mathbb{C}^\times & \text{if } \mathbb{F} = \mathbb{C}, \\ \mathbb{R}^\times & \text{if } \mathbb{F} = \mathbb{H}. \end{cases}$$

Let $f = \det\colon G \to H$. The **special linear group**

$$GL(n, \mathbb{F}) = \ker(\det) = \{X \in M(n, \mathbb{F}) \mid \det(X) = 1\}.$$

(b) Let $G = GL(n, \mathbb{F})$ act on $M = M(n, \mathbb{F})$ via

$$g \cdot X = gXg^T,$$

where g^T is the transpose of $g \in G$. Let's verify that this is an action.

$$g \cdot X = ghX(gh)^T = ghXh^Tg^T = g \cdot (h \cdot X).$$
$$1 \cdot X = x$$

(Note that this only defines an action in the case that $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$; see Remark 6.34.)

The stabilizer of $X = I$ under this action is the **orthogonal group**

$$O(n, \mathbb{F}) = \{g \in GL(n, \mathbb{F}) \mid gg^T = I\}.$$

This is a Lie subgroup of $GL(n, \mathbb{F})$.

(c) The tangent space to $O(n, \mathbb{F})$ at the identity is

$$T_I O(n, \mathbb{F}) = \{X \in M(n, \mathbb{F}) \mid X^T = -X\}.$$

Why is this? We have defined an action $\theta(g, X) = gXg^T$, and at the identity, this is the map

$$\begin{aligned} \theta^I\colon G &\longrightarrow M \\ g &\longmapsto gg^T \end{aligned}$$

We want to find the stabilizer of the derivative of θ^I at $A = I$.

$$T_I\theta^I\colon T_I G \cong M \longrightarrow T_I M \cong M$$

We have that if H is a small matrix,

$$\begin{aligned} \theta^I(I + H) &= (I + H)(I + H)^T = I + H + H^T + HH^T \\ &= \theta^I(I) + (H + H^T) + HH^T \end{aligned}$$

Therefore, the derivative of θ_I at I is the linear portion of the above map, which is $H \mapsto H + H^T$.

(d) To get a version of the orthogonal group over any $\mathbb{F}$, consider $\theta(g, X) = gXg^*$ where $g^* = \overline{g}^T$ is the conjugate transpose. Then

$$G_I = U(n, \mathbb{F}) = \{g \in G \mid gg^* = I\}$$

is the **unitary group** over $\mathbb{F}$.

$$\begin{aligned} U(n, \mathbb{R}) &= O(n, \mathbb{R}) \\ U(n, \mathbb{C}) &\neq O(n, \mathbb{C}) \\ U(n, \mathbb{H}) &= Sp(n) \end{aligned}$$

The last of these, $U(n, \mathbb{H})$ is often written $Sp(n)$, called the **compact symplectic group**.

Remark 6.34 (The "Socks and Shoes Rule")**.** When you dress yourself in the morning, you put on socks first and then shoes. But in the evening, to undo this operation, you first take off your shoes and then your socks:

$$(AB)^T = B^T A^T.$$

Note that the socks and shoes rule only works if your matrices have entries in a commutative ring! To prove that identity, you need to commute the scalars.

6.4 Smooth maps on Vector Fields

Let M, N be manifolds and $f\colon M \to N$ smooth. Then f induces a map $Tf\colon TM \to TN$ on the tangent spaces. Does f induce a map $\mathcal{T}(M) \to \mathcal{T}(N)$?

Example 6.35. Let $M = \mathbb{R}$, $N = \mathbb{R}^2$. Then if

$$f(t) = \begin{pmatrix} \cos t \\ \cos t \sin t \end{pmatrix}, \qquad \xi = \frac{d}{dt},$$

there is no vector field η on N with the property that $\eta(f(t)) = T_a f(\xi(t))$.

Definition 6.36. $\xi \in \mathcal{T}(M)$ and $\eta \in \mathcal{T}(N)$ are f**-related**, $\xi \sim_f \eta$, if $\eta_{f(a)} = T_a f(\xi_a)$ for all $a \in M$, that is,

$$\eta \circ f = Tf \circ \xi,$$

and the following diagram commutes.

$$\begin{array}{ccc} TM & \xrightarrow{Tf} & TN \\ \xi \uparrow\downarrow \pi_M & & \eta \uparrow\downarrow \pi_N \\ M & \xrightarrow{f} & N \end{array}$$

Equivalently, f maps trajectories of ξ to trajectories of η, or the following commutes.

$$\begin{array}{ccc} C^\infty(M) & \xleftarrow[f^*]{} & C^\infty(N)C^\infty(N) \\ \downarrow{\mathcal{L}_\xi} & & \downarrow{\mathcal{L}_\eta} \\ C^\infty(M) & \xleftarrow[f^*]{} & C^\infty(N) \end{array}$$

Given ξ, when can we guarantee the existence of a related η?

Lemma 6.37. *The following are equivalent for* $\xi \in \mathcal{T}(M), \eta \in \mathcal{T}(N)$.

(i) $\xi \sim_f \eta$,

(ii) $\xi(g \circ f) = \eta(g) \circ f$ *for all* $g \in C^\infty(N)$,

(iii) for any trajectory γ *of* ξ, $f \circ \gamma$ *is a trajectory of* η.

Remark 6.38 (Notation)**.** In Lemma 6.37(ii) we identify $\xi \in \mathcal{T}(M)$ with $\mathcal{L}_\xi \in \mathrm{Der}(M)$ given by

$$\xi(h) = \mathcal{L}_\xi(h) = dh(\xi)$$

for $h \in C^\infty(M)$.

We also write $g \circ f = f^*(g)$. Then Lemma 6.37(ii) reads

$$\xi \circ f^* = f^* \circ \eta.$$

Proof of Lemma 6.37.

(i) ⇔ (ii). If $a \in M$:

$$\xi(g \circ f)(a) = d_a(g \circ f)(\xi_a) = d_{f(a)} g \circ T_a f(\xi_a) \tag{7}$$

$$(\eta(g) \circ f)(a) = d_{f(a)} g(\eta_{f(a)}) \tag{8}$$

So (7) = (8) for all $g \in C^\infty(N)$ if and only if

$$\alpha(T_a f(\xi_a)) = \alpha(\eta_{f(a)})$$

for all $\alpha \in T^*_{f(a)}N$ if and only if

$$T_a F(\xi_a) = \eta_{f(a)}$$

if and only if $\xi \sim_f \eta$.

(ii) ⇒ (iii) Let $\delta(t) = f(\gamma(t))$. Then

$$\begin{aligned} \delta'(t) &= (f \circ \gamma)'(t) \\ &= T_{\gamma(t)} f(\gamma'(t)) \\ &= T_{\gamma(t)} f(\xi_{\gamma(t)}) \\ &= \eta_{f(\gamma(t))} = \eta_{\delta(t)}. \end{aligned} \tag{9}$$

So δ is a trajectory of η.

(iii) $\Rightarrow$ (ii) Let $a = \gamma(0)$, $\delta = f \circ \gamma$. Then by (9), δ is a trajectory of η starting at $f(a)$. So

$$\eta_{f(a)} = \delta'(0) = T_a f(\gamma'(0)) = T_a f(\xi_a).$$

This tells us that $\xi \sim_f \eta$.

$\square$

Corollary 6.39. *Suppose that $\xi \sim_f \eta$. Let*

$$\theta_\xi \colon \mathcal{D}_\xi \to M$$
$$\theta_\eta \colon \mathcal{D}_\eta \to N$$

be the flows of ξ and η, respectively. For $t \in \mathbb{R}$, let

$$M_t = \mathcal{D}_\xi \cap (\{t\} \times M)$$
$$N_t = \mathcal{D}_\eta \cap (\{t\} \times N)$$

Then $f(M_t) \subseteq N_t$ and

$$\begin{array}{ccc} M_t & \xrightarrow{f} & N_t \\ \downarrow{\scriptstyle \theta_{\xi,t}} & & \downarrow{\scriptstyle \theta} \\ M_{-t} & \xrightarrow{f} & N_{-t} \end{array}$$

We can restate Lemma 6.37 in a more friendly way. This gives us conditions under which, given ξ, we can guarantee the existence of a related η.

Lemma 6.40 (Lemma 6.37, restated)**.** *If f is a diffeomorphism, for every $\xi \in \mathcal{T}(M)$ there is a unique $\eta = f_*(\xi) = Tf \circ \xi \circ f^{-1}$ which is f-related to η.*

Conversely, we have the following.

Lemma 6.41. *If f is a local diffeomorphism ($T_x f$ invertible for all x), for every $\eta \in \mathcal{T}(N)$ there is a unique f-related $\eta = f^*(\eta) \in \mathcal{T}(M)$, namely*

$$\xi_x = (T_x f)^{-1}(\eta_{f(x)}).$$

This is C^∞ because f locally has smooth inverses.

Now suppose f is an embedding. Given $\eta \in \mathcal{T}(N)$, when is there an f-related $\xi \in \mathcal{T}(M)$? Identify M with the submanifold $A = f(M)$. The we want

$$\xi_a = \eta_a$$

for $a \in A$, so $\eta_a \in T_a A$.

Definition 6.42. We say that η is **tangent** to A if $\eta_a \in T_a A$ for all $a \in A$.

Lemma 6.43. *Let $f \colon M \to N$ be an embedding and identify M with $A = f(M) \subseteq N$. For each $\eta \in \mathcal{T}(N)$ which is tangent to A, there is a unique $\xi = \eta|_A$ which is f-related to η.*

6.5 Lie algebras and the Lie bracket

Let k be a commutative ring (for example $k = \mathbb{R}$) and let A be a k-module (perhaps $A = C^\infty(M)$). Let $\mathrm{End}_k(A)$ be the k-algebra of k-linear endomorphisms of A, that is, k-linear maps $f\colon A \to A$. This is a k-algebra under composition.

Definition 6.44. The **commutator** of $f, g \in \mathrm{End}_k(A)$ is

$$[f, g] = f \circ g - g \circ f.$$

Definition 6.45. A **Lie algebra** L over k is a k-module with a k-bilinear product $[\cdot\,,\cdot]\colon L \times L \to L$ satisfying

(a) **anti-symmetry:** $[x, x] = 0$ for all $x \in L$,

(b) the **Jacobi identity:** $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

Remark 6.46 (Warning!)**.** Note that the definition of a Lie algebra doesn't demand that L has a unit, nor does it demand associativity! Instead of associativity, we have the Jacobi identity, which is an infinitesimal form of associativity.

Example 6.47. Let $L = M(n, k)$. The $n \times n$ matrices over k with the $[x, y] = xy - yx$ is a Lie algebra.

More generally, $L = \mathrm{End}_k(A)$ for any k-module A.

Example 6.48. $\mathfrak{g} = T_1 G$ is a Lie algebra, where G is a Lie group. Details to come later.

Definition 6.49. Suppose that A is a k-algebra. The k**-derivations** of A,

$$\mathrm{Der}_k(A) \subseteq \mathrm{End}_k(A),$$

is the set of those k-linear $\ell\colon A \to A$ satisfying the **Leibniz rule**:

$$\ell(a, b) = \ell(a)b + a\ell(b).$$

Lemma 6.50 ("One of nature's little miracles."). *$\mathrm{Der}_k(A)$ is a* ***Lie subalgebra*** *of $\mathrm{End}_k(A)$, that is, a k-submodule closed under the Lie bracket $[\cdot\,,\cdot]$.*

Proof. Let $\ell_1, \ell_2 \in \mathrm{Der}_k(A)$. Let $a_1, a_2 \in A$. Then

$$\begin{aligned}(\ell_1\ell_2)(a_1a_2) &= \ell_1(\ell_2(a_1)a_2 + a_1\ell_2(a_2)) \\ &= \ell_1(\ell_2(a_1))a_2 + \ell_2(a_1)\ell_1(a_2) + \ell_1(a_1)\ell_2(a_2) + a_1\ell_1(\ell_2(a_2))\end{aligned}$$

Similarly,

$$\begin{aligned}(\ell_2\ell_1)(a_1a_2) &= \ell_2(\ell_1(a_1)a_2 + a_1\ell_1(a_2)) \\ &= \ell_2(\ell_1(a_1))a_2 + \ell_1(a_1)\ell_2(a_2) + \ell_2(a_1)\ell_1(a_2) + a_1\ell_2(\ell_1(a_2))\end{aligned}$$

In the difference of these two, the second-order terms cancel and we are left with

$$[\ell_1, \ell_2](a_1a_2) = [\ell_1, \ell_2](a_1)a_2 + a_1[\ell_1, \ell_2](a_2). \qquad \square$$

A consequence of this lemma is that for every manifold M,

$$\mathrm{Der}(M) = \mathrm{Der}_{\mathbb{R}}(C^\infty(M))$$

is a Lie algebra under the commutator bracket. Hence, $\mathcal{T}(M) \cong \mathrm{Der}(M)$ is a Lie algebra: for $\xi, \eta \in \mathcal{T}(M)$, define

$$[\xi, \eta] = \mathcal{L}^{-1}([\mathcal{L}_\xi, \mathcal{L}_\eta]).$$

This definition makes

$$\mathcal{L} \colon \mathcal{T}(M) \to \mathrm{Der}(M)$$

a Lie algebra isomorphism.

Restricting ξ, η to a chart (U, ϕ), write

$$\partial_i = \phi^* \left(\frac{d}{dx_i} \right)$$

for the frame on U induced by the standard basis on $\mathbb{R}^n$. Then

$$\xi = \sum_{i=1}^n \xi_i \partial_i, \qquad \eta = \sum_{j=1}^n \eta_j \partial_j,$$

for some coefficients $\xi_i, \eta_j \in \mathbb{R}$. For $f \in C^\infty(U)$, then we have

$$\begin{aligned}
[\xi, \eta](f) &= \xi(\eta(f)) - \eta(\xi(f)) \\
&= \xi \left(\sum_j \eta_j \partial_j f \right) - \eta \left(\sum_i \xi_i \partial_i f \right) \\
&= \sum_{i,j} \xi_i (\partial_i \eta_j \partial_j f + \eta_j \partial_i \partial_j f) - \sum_{i,j} \eta_j \left(\partial_j \xi_i \partial_i f + \xi_i \partial_i \partial_j f \right) \\
&= \sum_{i,j} \xi_i (\partial_i \eta_j)(\partial_j f) - \eta_j (\partial_j \xi_i)(\partial_i f)
\end{aligned}$$

So as a differential operator,

$$[\xi, \eta] = \sum_{i,j=1}^n \left(\xi_i \partial_i \eta_j - \eta_i \partial_i \xi_j \right) \partial_j.$$

Proposition 6.51 (Naturality of the Lie bracket)**.** *Let* $f \colon M \to N$ *be smooth, and* $\xi_1, \xi_2 \in \mathcal{T}(M)$*,* $\eta_1, \eta_2 \in \mathcal{T}(N)$*. Suppose that* $\xi_1 \sim_f \eta_1$*,* $\xi_2 \sim_f \eta_2$*. Then*

$$[\xi_1, \xi_2] \sim_f [\eta_1, \eta_2].$$

Proof. We know that $\xi_i \circ f^* = f^* \circ \eta_i$ for $i = 1, 2$. Hence $\xi_i \circ \xi_j \circ f^* = f^* \eta_i \circ \eta_j$. Therefore, $[\xi_1, \xi_2] \circ f^* = f^* \circ [\eta_1, \eta_2]$. □

Example 6.52. Special cases of the above.

(a) If f is an embedding, and $\eta_1, \eta_2 \in \mathcal{T}(N)$ are tangent to the submanifold M, then so is $[\eta_1, \eta_2]$.

(b) If f is a diffeomorphism, then $f_*[\xi_1, \xi_2] = [f_*\xi_1, f_*\xi_2]$.

(c) If f is a local diffeomorphism, then $f^*[\eta_1, \eta_2] = [f^*\eta_1, f^*\eta_2]$.

Remark 6.53. Let $F\colon M \to N$ be a diffeomorphism. We defined

$$\begin{aligned} F^*\colon \mathcal{T}(N) &\longrightarrow \mathcal{T}(M) \\ \eta &\longmapsto (TF)^{-1} \circ \eta \circ F \end{aligned}$$

$$\begin{aligned} F_*\colon \mathcal{T}(M) &\longrightarrow \mathcal{T}(N) \\ \xi &\longmapsto TF \circ \xi \circ F^{-1} \end{aligned}$$

Then according to these definitions, we have that $F^* = (F_*)^{-1}$.

Remark 6.54. Let $F\colon M \to N$ be a diffeomorphism. We can also define $F^*\colon \operatorname{Der}(N) \to \operatorname{Der}(M)$ by

$$F^*(\ell)(f) = F^*\ell((F^*)^{-1}(f)) = \ell(f \circ F^{-1}) \circ F.$$

$$\begin{array}{ccc} C^\infty(N) & \xrightarrow{F^*} & C^\infty(M) \\ \big\downarrow \ell & & \big\downarrow F^*(\ell) \\ C^\infty(N) & \xrightarrow{F^*} & C^\infty(M). \end{array}$$

But because $\operatorname{Der}(M) \cong \mathcal{T}(M)$, we can identify the two, and see that the following also commutes.

$$\begin{array}{ccc} \mathcal{T}(N) & \xrightarrow{F^*} & \mathcal{T}(M) \\ \mathcal{L}_{\cong} \big\downarrow & & \mathcal{L}_{\cong} \big\downarrow \\ \operatorname{Der}(N) & \xrightarrow{F^*} & \operatorname{Der}(M) \end{array}$$

6.6 Brackets and Flows

Remark 6.55. There are three different notations for the derivative of $f \in C^\infty(M)$ along a vector field $\xi \in \mathcal{T}(M)$:

$$df(\xi) = \mathcal{L}_\xi(f) = \xi(f)$$

We will define a fourth notation for this concept to muddy the waters even further.

Definition 6.56 (Notation)**.** Let $\theta = \theta_\xi \colon \mathcal{D}_\xi \to M$ be the flow of ξ, and let $a \in M$. Then $\xi_a = {}^{d}/_{dt}\theta^a(0)$, so according to Remark 6.55,

$$d_a f(\xi_a) = d_a f(\frac{d}{dt}\theta^a(0)) = d(f\circ\theta^a)(0) = \frac{d}{dt}(f\circ\theta_t(a))\bigg|_{t=0} = \frac{d}{dt}\theta_t^*(f)(a)\bigg|_{t=0}.$$

We write

$$df(\xi) = \frac{d}{dt}\theta_t^*(f)\bigg|_{t=0}.$$

This shows us that we may substitute for f any other type of object that can be pulled back under diffeomorphism. So we can take deriatives of vector fields along vector fields, for example.

Definition 6.57. For $\xi, \eta \in \mathcal{T}(N)$, define

$$\mathcal{L}_\xi(\eta) = \frac{d}{dt}\theta_t^*(\eta)\bigg|_{t=0},$$

where $\theta = \theta_\xi$ is the flow of ξ.

Remark 6.58. Some authors use the sign convention

$$\frac{d}{dt}(\theta_{\xi,t})_*(\eta)\bigg|_{t=0}$$

with

$$(\theta_{\xi,t})_* = (\theta_{\xi,t}^*)^{-1} = \theta_{\xi,-t}^*.$$

This gives the opposite sign of our $\mathcal{L}_\xi(\eta)$.

What's the interpretation of $\mathcal{L}_\xi(\eta)$? Well, it makes more sense pointwise. For each $a \in M$ and for $t \in \mathcal{D}^a$, we have

$$\theta_t^*(\eta)(a) = (T_a\theta_t)^{-1}\eta_{\theta_t(a)} \in T_aM.$$

This is a smooth curve $\mathcal{D}^a \to T_aM$. Hence, it'd derivative at $t = 0$ exists and is again in T_aM. What is that vector field?

Theorem 6.59. $\mathcal{L}_\xi(\eta) = [\xi, \eta]$

Proof. We will show that $\mathcal{L}_\xi(\eta)(f) = [\xi,\eta](f)$ for any $f \in C^\infty(M)$.

Let $f \in C^\infty(M)$. Let θ be the flow of ξ, and $\mathcal{D} = \mathcal{D}_\xi$ be the flow domain. Let $a \in M$. Choose an open $U \ni a$ and $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon)\times U \subseteq \mathcal{D}$.

Define

$$F\colon (-\varepsilon,\varepsilon)\times U \longrightarrow \mathbb{R}$$
$$F(t,x) = f(\theta_t(x)) - f(x)$$

Then $F(0, x) = 0$, so there is a smooth $G\colon (-\varepsilon, \varepsilon) \times U \to \mathbb{R}$ with

$$\begin{cases} F(t,x) = tG(t,x) \\ {}^{dF}/_{dt}(0,x) = G(0,x). \end{cases} \tag{10}$$

Namely, we can take

$$G(t,x) = \int_0^1 \partial_1 F(ts, x)\, ds.$$

Then (10) is equivalent to

$$\begin{cases} \theta_t^*(f) = f + tG_t \quad \text{(where } G_t(x) = G(t,x)\text{)} \\ \xi(f) = G_0. \end{cases} \tag{11}$$

Pulling back η (or rather $\mathcal{L}_\eta$) along θ_t gives

$$\begin{aligned} \theta_t^*(\eta)(f) &= \theta_t^*\left(\eta(\theta_{-t}^*(f))\right) \\ &= \theta_t^*(\eta(f \circ \theta_{-t})) \\ &= \theta_t^*(\eta(f - tG_{-t})) && \text{by (11)} \\ &= \theta_t^*(\eta(f)) - t\theta_t^*(\eta(G_{-t})) \end{aligned}$$

So

$$\begin{aligned} \mathcal{L}_\xi(\eta)(f) &= \frac{d}{dt}\theta_t^*(\eta)(f)\bigg|_{t=0} \\ &= \xi(\eta(f)) - \theta_0^*(\eta(G_{-0})) - \left[t\frac{d}{dt}\theta_t^*(\eta(G_{-t}))\right]_{t=0} \\ &= \xi(\eta(f)) - \theta_0^*(\eta(G_{-0})) \\ &= \xi(\eta(f)) - \eta(G_{-0} \circ \theta_0) \\ &= \xi(\eta(f)) - \eta(G_0) \\ &= \xi(\eta(f)) - \eta(\xi(f)) \\ &= [\xi, \eta](f) \end{aligned}$$

□

Corollary 6.60. *For* $\xi, \eta \in \mathcal{T}(M)$, $f \in C^\infty(M)$,

(i) $\mathcal{L}_\xi(\eta) = -\mathcal{L}_\eta(\xi)$

(ii) $\mathcal{L}_\xi(f\eta) = \mathcal{L}_\xi(f)\eta + f\mathcal{L}_\xi(\eta)$

(iii) $F_*(\mathcal{L}_\xi(\eta)) = \mathcal{L}_{F_*(\xi)}(F_*(\eta))$

for any diffeomorphism $F\colon M \to N$.

Proof.

(i) Use $[\xi, \eta] = \mathcal{L}_\xi(\eta)$ by Theorem 6.59. The Lie bracket is antisymmetric.

(ii) This is a general fact about derivations. If A is a k-algebra over a commutative ring k, and $\ell_1, \ell_2 \in \mathrm{Der}_k(A)$, then for all $a \in A$,

$$[\ell_1, a\ell_2] = \ell_1(a)\ell_2 + a[\ell_1, \ell_2].$$

(iii) By Proposition 6.51, we have $F_*[\xi, \eta] = [F_*\xi, F_*\eta]$ and by Theorem 6.59, we have $\mathcal{L}_\xi(\eta) = [\xi, \eta]$. Combining these gives the desired result. □

Remark 6.61 (Note to reader)**.** If you're reading these notes online and spot any errors, please send any corrections to dmehrle@math.cornell.edu[1].

Remark 6.62 (Recall)**.** Given $F\colon M \to N$ smooth, $\xi \in \mathcal{T}(M), \eta \in \mathcal{T}(N)$, we say that $\xi \sim_F \eta$ if for all $x \in M$,

$$T_x F(\xi_x) = \eta_{F(x)}.$$

This condition is equivalent to the following diagram commuting for all t.

$$\begin{array}{ccc} M & \xrightarrow{F} & N \\ \downarrow{\scriptstyle \theta_{\xi,t}} & & \downarrow{\scriptstyle \theta_{\eta,t}} \\ M & \xrightarrow{F} & N \end{array}$$

The domain of $\theta_{\xi,t}$ is not M, but see Remark 6.63 below.

Remark 6.63 (Notation)**.** Notice that the domain of $\theta_{\xi,t}$ is not M, but instead $M_t = \{x \in M \mid (t,x) \in \mathcal{D}_\xi\}$, and its target is M_{-t}. Therefore M_t is open in M and every $x \in M$ is in M_t for some sufficiently small t. So by abuse of notation, we consider M to be the domain of $\theta_{\xi,t}$.

Lemma 6.64. $\left.\dfrac{d}{dt}\theta^*_{\xi,t}(\eta)\right|_{t=s} = \theta^*_{\xi,s}(\mathcal{L}_\xi(\eta))$

Proof. Put $\theta_\xi = \theta$. Then

$$\begin{aligned}
\left.\frac{d}{dt}\theta^*_t(\eta)\right|_{t=s} &= \lim_{t\to s}\frac{1}{t-s}\left(\theta^*_t(\eta) - \theta^*_s(\eta)\right) && \text{defn of derivative} \\
&= \lim_{t\to 0}\frac{1}{t}\left(\theta^*_{s+t}(\eta) - \theta^*_s(\eta)\right) \\
&= \theta^*_s\left(\lim_{t\to 0}\frac{1}{t}(\theta^*_t(\eta) - \eta)\right) && \text{flow law} \\
&= \theta^*_s(\mathcal{L}_\xi(\eta))
\end{aligned}$$
□

[1]Especially those of you who skip manifolds class and read these notes instead!

The following lemma is often read as "F commutes with the flow of ξ."

Lemma 6.65. *Let* $F\colon M \to N$ *be a diffeomorphism. Then*

$$\xi \sim_F \xi \iff F_*(\xi) = \xi \iff \forall t, \theta_{\xi,t} \circ F = F \circ \theta_{\xi,t}.$$

Proof. Recall that $\xi \sim_F \eta \iff F_*(\xi) = \eta \iff \theta_{\eta,t} \circ F = F \circ \theta_{\xi,t}$. Now take $\eta = \xi$. □

There are two tribes of mice on a manifold, and they're friendly. Call them the ξ-mice and the η-mice. They give each other rides around the manifold, and take turns carrying each other on their back. But they're stupid mice, so if they start in the same place and the ξ-mice carry the η-mice first, they don't always end up in the same place as if the η-mice carry the ξ-mice first. When do they end up in the same place? The next theorem tells us the answer.

Theorem 6.66 (Commuting Flow Theorem)**.** *Let* $\xi, \eta \in \mathcal{T}(M)$. *Then for all* s, t,

$$\theta_{\xi,s} \circ \theta_{\eta,t} = \theta_{\eta,t} \circ \theta_{\xi,s} \iff [\xi, \eta] = 0.$$

Proof. ($\Longrightarrow$). Apply Lemma 6.65 to $F = \theta_{\xi,s}$ and the vector field η:

$$\theta^*_{\xi,s}(\eta) = \eta.$$

Hence,

$$[\xi, \eta] = \frac{d}{ds}\theta^*_{\xi,s}(\eta)\bigg|_{s=0} = 0.$$

($\Longleftarrow$). Let $a \in M$. Put $\gamma(s) = \theta^*_{\xi,s}(\eta)(a) \in \mathcal{T}_a(M)$. Then $\gamma\colon (-\varepsilon, \varepsilon) \to T_a M$ is smooth and for all s,

$$\begin{aligned} \gamma'(s) &= \theta^*_{\xi,s}(\mathcal{L}_\xi(\eta)) && \text{Lemma 6.64} \\ &= \theta^*_{\xi,s}([\xi,\eta]) && \text{Theorem 6.59} \\ &= 0. \end{aligned}$$

This holds for all s, so for all s, $\gamma(s) = \gamma(0)$, that is,

$$\theta^*_{\xi,s}(\eta) = \eta.$$

so by Lemma 6.65,

$$\theta_{\xi,s} \circ \theta_{\eta,t} = \theta_{\eta,t} \circ \theta_{\xi,s}.$$ □

The previous theorem also holds for sets of k vector fields on M.

Theorem 6.67. *Let $\xi_1, \dots, \xi_k \in \mathcal{T}(M)$. Then suppose that*

(a) for all $i, j \leq k$, $[\xi_i, \xi_j] = 0$, and

(b) $\xi_1, \dots, \xi_k$ are linearly independent at $a \in M$.

Then there exists a chart (U, ϕ) centered at a such that $\xi_i|_U = \phi^*\left({}^{\partial}/_{\partial x_i}\right)$ for $i = 1, 2, \dots, k$.

Proof. Using a preliminary chart (V, ψ) centered at a, the vector fields $\eta_i = \psi_*(\xi_i|_V)$ on $\psi(V) \subseteq \mathbb{R}^n$ satisfy

(a) $[\eta_i, \eta_j] = 0$ by naturality of the Lie bracket (Proposition 6.51),

(b) $\eta_1(0) = 2_1, \dots, \eta_k(0) = e_k$, where $e_1, \dots, e_k$ are standard basis vectors on $\mathbb{R}^k$.

Now we may assume without loss of generality that M is an open neighborhood of $a = 0 \in \mathbb{R}^n$ and $\xi_i(0) = e_i$ for $i = 1, 2, \dots, k$. Now put $\theta_i = \theta_{\xi_i}$ and

$$F(x) = F(x_1, \dots, x_n) = \theta_{1,x_1} \circ \theta_{2,x_2} \circ \cdots \circ \theta_{k,x_k}(0, \dots, 0, x_{k+1}, \dots, x_n).$$

Then $F(0) = 0$ and $F\colon W \to \mathbb{R}^n$ is well-defined and smooth on an open neighborhood W of $0 \in \mathbb{R}^n$.

For $i > k$,

$$F(0, \dots, 0, x_i, 0, \dots, 0) = (0, \dots, 0, x_i, 0, \dots, 0).$$

so

$$DF_0 e_i = e_i \tag{12}$$

for $i > k$.

For $i \leq k$,

$$\begin{aligned} F(x_1, \ldots, x_i + h, \ldots, x_n) &= \theta_{1,x_1} \circ \cdots \circ \theta_{i,x_{i+h}} \circ \cdots \circ \theta_{k,x_k}(0, \ldots, 0, x_{k+1}, \ldots, x_n) \\ &= \theta_{i,h} \circ F(x_1, \ldots, x_i, \ldots, x_n) \end{aligned}$$

by the flow law and Theorem 6.66. This can be rewritten as

$$F(\vartheta_{i,h}(x)) = \theta_{i,h}(F(x)),$$

where $\vartheta_{i,h}(x) = x + he_i$, that is, ϑ_i is the flow of ${}^{\partial}/_{\partial x_i}$. Hence,

$$\frac{\partial}{\partial x_i} \sim_F \xi_i.$$

Therefore,

$$T_x F\left(\frac{\partial}{\partial x_i}\right) = (\xi_i)_{F(x)}. \tag{13}$$

Now put $x = 0$, to see that

$$T_0 F\left(\frac{\partial}{\partial x_i}\right) = (\xi_i)_0.$$

In particular, we conclude that for $i \leq k$,

$$DF_0 e_i = e_i. \tag{14}$$

Finally, (12) and (14) imply that $DF_0 = I_n$, so F is a local diffeomorphism. Hence by (13),

$$F_*\left(\frac{\partial}{\partial x_i}\right) = \xi_i. \qquad \square$$

7 Vector bundles

7.1 Variations on the notion of manifolds

Let $\mathbb{M}$ be a fixed topological space. An $\mathbb{M}$-valued chart on a set M is a pair (U, ϕ) ,w here $U \subseteq M$ and $\phi \colon U \to \mathbb{M}$ is a bijection to an open subset of $\mathbb{M}$. A transition between two charts (U, ϕ), (V, ψ) is

$$\psi \circ \phi^{-1} \colon \phi(U \cap V) \to \psi(U \cap V).$$

These transitions might satisfy various notions of compatibility; see Example 7.6 below.

We might not only demand that transitions are smooth, but also analytic. Then we get **analytic manifolds**.

Definition 7.1. A function $f\colon U \to \mathbb{R}$ is **real-analytic** if for all $x \in U$, the Taylor series $T(f, x)$ at x converges to f uniformly in a neighborhood of x.

Definition 7.2 (Notation)**.** We denote by $C^\omega(U)$ the class of real-analytic functions on a domain U.

Example 7.3 (Counterexample)**.** The function

$$f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0. \end{cases}$$

is smooth but not real analytic. The Taylor series for f at 0 is just $T(f, 0) = 0$, so this converges, but not to f, so f is not analytic at $x = 0$.

Alternatively, if the transition maps are affine maps, then we get **affine manifolds**.

Definition 7.4. An **affine map** $\mathbb{R}^n \to \mathbb{R}^n$ is a linear map composed with translation. An **affine structure** on M is a maximal atlas in which the transitions are (restrictions of) affine maps.

For any affine manifold, there are well-defined notions of straight lines and affine submanifolds.

Example 7.5. Examples of affine manifolds include $\mathbb{R}^n, \mathbb{P}^n(\mathbb{F})$, and $\mathbb{T}^n = (\mathbb{S}^1)^n$, because $\mathbb{T}^n$ is the quotient of $\mathbb{R}^n$ by a lattice.

Example 7.6.

	Model	Transitions	Type of manifold
(1)	$\mathbb{R}^n$	smooth diffeomorphisms	smooth manifold
(2)	$\mathbb{R}^n$	C^r-diffeomorphisms $\big(r \in \mathbb{N} \cup \{\infty, \omega\}\big)$	C^r-manifold
(3)	$\mathbb{C}^n$	biholomorphic maps	complex manifold
(4)	$(\mathbb{Q}_p)^n$ p-adic numbers	analytic diffeomorphisms	p-adic manifolds
(5)	$\mathbb{R}^{n-1} \times [0, \infty)$	smooth diffeomorphisms	manifold with boundary
(6)	$[0, \infty)^n$	smooth diffeomorphisms	manifold with corners
(7)	$\mathbb{R}^n$	affine diffeomorphisms	affine manifold
(8)	$B \times F$ B top. space F vector space	vector bundle transition	vector bundle over B with fiber F
(9)	$B \times F$ B manifold F vector space	smooth vector bundle transitions	smooth vector bundle

In (2), C^0-diffeomorphism means homeomophism. Moreover, if $1 \leq r \leq s$, then every C^r-structure on M contains a C^s-structure. This, however, is false for $r = 0$; that was the contents of several fields medals in the mid 20-th century.

7.2 Vector Bundles

Let $\mathbb{F}$ be $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, and let $r \in \mathbb{N}$. Let E, B be manifolds with a smooth surjection $\pi = \pi_E : E \to B$.

Definition 7.7. A **local triavialization** of E (alternatively, an $\mathbb{F}$**-vector bundle chart**) is a pair (U, ϕ) with $U \subseteq B$ open and a diffeomorphism

$$\phi : \pi^{-1}(U) \to U \times \mathbb{F}^r$$

such that

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\ \phi\ } & U \times \mathbb{F}^r \\ {\scriptstyle \pi|_U}\downarrow & \swarrow{\scriptstyle \mathrm{pr}_1} & \\ U & & \end{array}$$

commutes; that is, $\pi|_U = \mathrm{pr}_1 \circ \phi$.

Definition 7.8 (Notation)**.** For $b \in B$, we define $\phi_b \colon \pi^{-1}(b) \to \mathbb{F}^r$ by

$$\pi^{-1}(b) \xrightarrow{\phi} \{b\} \times \mathbb{F}^r \xrightarrow{pr_2} \mathbb{F}^r, \qquad \phi_b = pr_2 \circ \phi$$

Remark 7.9. $\pi \colon E \to B$ is a submersion, so each $\pi^{-1}(b) = E_b$ is a submanifold of E and $\phi_b \colon E_b \to \mathbb{F}^r$ is a diffemorphism.

Definition 7.10. Two local trivializations (vector bundle charts) (U_1, ϕ_1), (U_2, ϕ_2) are **compatible** if

$$\mathbb{F}^r \xrightarrow{\phi_{1,b}^{-1}} \pi^{-1}(b) \xrightarrow{\phi_{2,b}} \mathbb{F}^r$$

is $\mathbb{F}$-linear for all $b \in U_1 \cap U_2$.

Fact 7.11. *If two local trivializations (U_1, ϕ_1), (U_2, ϕ_2) are compatible, then*

$$\begin{aligned} g_{12} \colon U_1 \cap U_2 &\longrightarrow GL(r, \mathbb{F}) \\ b &\longmapsto g_{12}(b) \end{aligned}$$

is smooth.

Definition 7.12. A **vector bundle atlas** is a compatible set of local trivializations $\{(U_\alpha, \phi_\alpha) \mid \alpha \in A\}$ whose domains cover B.

Definition 7.13. A **vector bundle of rank** r over B is a manifold E with a smooth surjection $\pi \colon E \twoheadrightarrow B$ and a maximal vector bundle atlas. E is called the **total space** of the vector bundle and B is called the **base space**.

Definition 7.14 (Notation)**.**

(1) For any open $U \subseteq B$, write $E|_U = \pi^{-1}(U)$; this is called the **restriction of** E **to** U. Then $\pi \colon E|_U \to U$ is a vector bundle of rank r over U.

(2) For $b \in B$, write $E_b = \pi^{-1}(b)$. We make E_b into an r-dimensional vector space over $\mathbb{F}$ by declaring $\phi_b \colon E_b \to \mathbb{F}^r$ to be a linear isomorphism. This is independent of the choice of local trivialization because $g_{\alpha\beta}(b)$ is linear for all local trivializations (U_α, ϕ_α), (U_β, ϕ_β) in the atlas.

Example 7.15. The tangent bundle of a smooth manifold M with the map $\pi \colon TM \to M$ of a manifold M is a real vector bundle of rank $r = \dim(M)$. For (U, ϕ) a chart on M, we have a tangent chart $(TU, T\phi)$, where

$$T\phi \colon TU \xrightarrow{\sim} \phi(U) \times \mathbb{R}^n.$$

Since $\phi(U)$ is diffeomorphic to U, composing with $\phi^{-1} \times \mathrm{id}_{\mathbb{R}^n}$ gives

$$\overline{\phi} = (\phi^{-1} \times \mathrm{id}_{\mathbb{R}^n}) \circ T\phi \colon TU = \pi^{-1}(U) \to U \times \mathbb{R}^n.$$

Then $(U, \overline{\phi})$ is a local trivialization on TM. The transitions from a vector bundle chart $(U_1, \overline{\phi}_1)$ to $(U_2, \overline{\phi}_2)$ is

$$\begin{aligned} g_{12} \colon U_1 \cap U_2 &\longrightarrow GL(n, \mathbb{R}) \\ a &\longmapsto D_{\phi(a)}(\phi_2 \circ \phi_1^{-1}). \end{aligned}$$

Example 7.16. Let B be any manifold, and $E = B \times \mathbb{F}^r$. Then $\pi \colon E \to B$ is projection onto the first factor, with the vector bundle atlas generated by the global vector bundle chart $(U = B, \phi = \mathrm{id}_E)$. This makes E into a vector bundle called the **trivial vector bundle** over B of rank r.

Definition 7.17 (Notation)**.** For local trivializations $(U_{i_1}, \phi_{i_1}), \ldots, (U_{i_s}, \phi_{i_s})$ in a vector bundle atlas $\{(U_i, \phi_i) \mid i \in I\}$, define

$$U_{i_1 i_2 \ldots i_s} := U_{i_1} \cap U_{i_2} \cap \ldots \cap U_{i_s}$$

Given a vector bundle $\pi \colon E \to B$ and a vector bundle atlas $\{(U_i, \phi_i) \mid i \in I\}$, consider the collection of smooth maps

$$g_{ij} \colon U_i \cap U_j \to GL(r, E) \tag{15}$$

given by the $r \times r$ matrices

$$g_{ij}(b) = \phi_{j,b} \circ \phi_{i,b}^{-1} \colon \mathbb{F}^r \to \mathbb{F}^r.$$

Definition 7.18. We call each g_{ij} a **cocycle**. These g_{ij} satisfy the **cocycle condition:**

$$\begin{cases} g_{ii} = \mathrm{id}_{U_i} & \text{on } U_i \\ g_{jk} \circ g_{ij} = g_{ik} & \text{on } U_i \cap U_j \cap U_k. \end{cases} \tag{16}$$

These two cases imply that $g_{ij} = g_{ji}^{-1}$ on $U_i \cap U_j$.

Given a collection of cocycles $g = \{g_{ij}\}$, that is, an open cover $\{U_i \mid i \in I\}$ and a collection of maps as in (15) satisfying the cocycle condition (16), define

$$\widetilde{E} = \bigsqcup_{i \in I} (U_i \times \mathbb{F}^r).$$

For $(a, u) \in U_i \times \mathbb{F}^r$ and $(b, v) \in U_j \times \mathbb{F}^r$, define an equivalence relation

$$(a, u) \sim (b, v) \iff a = b \text{ and } g_{ij}(a) u = v.$$

Fact 7.19. *This equivalence relation is regular and*

$$E_g := \widetilde{E} / \sim$$

is a vector bundle over B *with cocycles given by* $\{g_{ij}\}$*.*

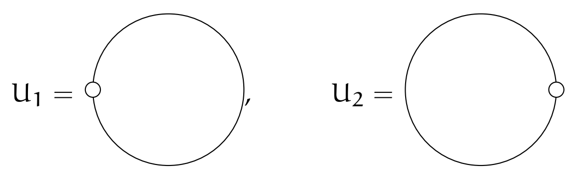
Fact 7.20. *If $g = \{g_{ij}\}$ is the collection of cocycles of a vector bundle E, then there is an isomorphism of vector bundles $E \to E_g$.*

Definition 7.21. Given two vector bundles $\pi_1 \colon E_1 \to B$ and $\pi_2 \colon E_2 \to B$ over the same base B, a **vector bundle morphism** $F \colon E_1 \to E_2$ is a smooth map such that

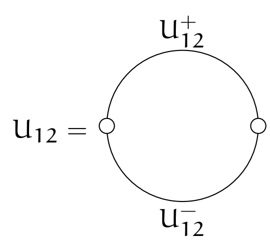
$$\begin{array}{ccc} E_1 & \xrightarrow{\quad F \quad} & E_2 \\ & {\scriptstyle \pi_1} \searrow \quad \swarrow {\scriptstyle \pi_2} & \\ & B & \end{array}$$

commutes and F restricts to linear maps between the fibers of π_1 and π_2.

Example 7.22. Let $B = \mathbb{S}^1$ with open cover given by



Then



Then define a single cocycle $g_{12} \colon U_{12} \to GL(1, \mathbb{R}) = \mathbb{R}^\times$.

$$g_{12} = \begin{cases} +1 & \text{on } U_{12}^+ \\ -1 & \text{on } U_{12}^-. \end{cases}$$

Definition 7.23. A **section** of $\pi \colon E \to B$ is a map $\sigma \colon B \to E$ satisfying $\pi \circ \sigma = \mathrm{id}_B$, that is, $\pi(\sigma(b)) = b$, so $\sigma(b) \in E_b$ for all $b \in B$.

Composing σ with a vector bundle chart (U, ϕ) gives a smooth map

$$\widetilde{\sigma} = \mathrm{pr}_2 \circ \phi \circ \sigma \colon U \to \mathbb{F}^r.$$

$$\begin{array}{ccc} E|_U & \longrightarrow & U \times \mathbb{F}^r \\ {\scriptstyle \sigma} \uparrow \downarrow {\scriptstyle \pi} & \swarrow {\scriptstyle \mathrm{pr}_1} & \downarrow {\scriptstyle \mathrm{pr}_2} \\ U & & \mathbb{F}^r \end{array}$$

We have that $\phi \circ \sigma(b) = (b, \widetilde{\sigma}(b))$ for $b \in U$. We call $\widetilde{\sigma}$ the **expression** for σ in the chart (U, ϕ). Since σ is smooth, then $\widetilde{\sigma}$ is smooth as well.

Sections of a vector bundle can be added:

$$(\sigma_1 + \sigma_2)(b) = \sigma_1(b) + \sigma_2(b).$$

They can also be multiplied by functions $f\colon B \to \mathbb{F}$

$$(f\sigma)(b) = f(b)\sigma(b).$$

This corresponds to scalar multiplication in E_b.

We have

$$\begin{cases} \widetilde{(\sigma_1 + \sigma_2)} = \widetilde{\sigma}_1 + \widetilde{\sigma}_2, \\ \widetilde{(f\sigma)} = \widetilde{f}\,\widetilde{\sigma}. \end{cases}$$

for any vector bundle chart (U, ϕ). So if $\sigma_1, \sigma_2, \sigma$ and f are smooth, then $\sigma_1 + \sigma_2$ and $f\sigma$ are smooth as well.

Definition 7.24 (Notation)**.** We denote by $\Gamma(E)$ the set of smooth sections of $\pi\colon E \to B$. For $U \subseteq B$ open, $\Gamma(U, E) = \Gamma(E|_U)$ is the smooth sections of $E_| U$.

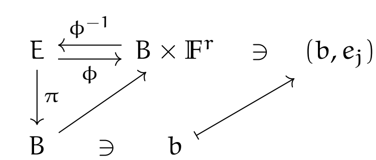
For each U, $\Gamma(U, E)$ is a $C^\infty(U)$-module.

Example 7.25. The set of smooth vector fields on a manifold M is the set of smooth sections of the tangent bundle: $\mathcal{T}(M) = \Gamma(TM)$.

Definition 7.26. A k**-frame** on E is a k-tuple $(\sigma_1, \ldots, \sigma_k)$ of sections $\sigma_i \in \Gamma(E)$ such that $\sigma_1(b), \ldots, \sigma_k(b) \in E_b$ are linearly independent in the $\mathbb{F}$-vector space E_b for all $b \in B$.

If E has a k-frame, then $k \leq \operatorname{rank}(E)$. A trivialization $\phi\colon E \to B \times \mathbb{F}^r$ gives rise to an r-frame $(\sigma_1, \ldots, \sigma_r)$ given by the standard basis $e_1, \ldots, e_r$ of $\mathbb{F}^r$

$$\sigma_j(b) = \phi^{-1}(b, e_j) \in E_b.$$



Conversely, given an r-frame $(\sigma_1, \ldots, \sigma_r)$, we define

$$\begin{aligned} \Psi\colon B \times \mathbb{F}^r &\longrightarrow E \\ (b, y) &\longmapsto \textstyle\sum_{j=1}^r y_j \sigma_j(b). \end{aligned}$$

Lemma 7.27. Ψ *is a diffeomorphism and* Ψ^{-1} *is a trivialization (global vector bundle chart) of* E

Proof. Since $\sigma_1, \ldots, \sigma_r$ are smooth, then Ψ is smooth as well. Since $\sigma_1, \ldots, \sigma_r$ are linearly independent, $r = \operatorname{rank}(E)$, then Ψ is bijective.

So it remains to show that $T_{(b,y)}\Psi$ is bijective for all $(b, y) \in B \times \mathbb{F}^r$. Choose a vector bundle chart (U, ϕ_E) at $b \in B$ and also a chart (U, ϕ_B) on B at b. We have

$$\underbrace{\phi_B(U) \times \mathbb{F}^r \xrightarrow[\text{diffeo}]{\phi\times\text{id}} U \times \mathbb{F}^r \xrightarrow{\Psi} E|_U \xrightarrow[\text{diffeo}]{\phi_E} U \times \mathbb{F}^r \xrightarrow[\text{diffeo}]{\phi_B\times\text{id}} \phi_B(U) \times \mathbb{F}^r}_{f}.$$

f is a smooth map $f\colon \phi_B(U) \times \mathbb{F}^r \to \phi_B(U) \times \mathbb{F}^r$ of the form

$$f(x, y) = (x, A(x, y))$$

where $A\colon \phi_B(U) \times \mathbb{F}^r \to \mathbb{F}^r$ is smooth, $\mathbb{F}$-linear for each $x \in \phi_B()$ and $y \mapsto A(x, y)$ is invertible for each x. Hence,

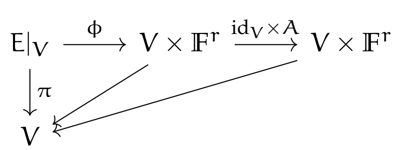
$$Df(x, y) = \begin{bmatrix} I_n & 0 \\ D^{(1)}A(x, y) & A(x, \cdot) \end{bmatrix}.$$

For all x, $A(x, y)$ is invertible, so $Df(x, y)$ is invertible and therefore $T_{(b,y)}\Psi$ is invertible. □

For any vector bundle, there is a one-to-one correspondence between trivializations and r-frames, and for any $U \subseteq B$ open, there is a one-to-one correspondence between vector bundle charts with domain U and r-frames on $E|_U$.

Lemma 7.28. *Let* $(\sigma_1, \ldots, \sigma_k)$ *be a* k*-frame on* E *and* $b_0 \in B$*. Then there exists an open neighborhood* U *of* b_0 *and sections* $\sigma_{k+1}, \ldots, \sigma_r$ *such that* $(\sigma_1, \ldots, \sigma_r)$ *is an* r*-frame over* U*.*

Proof. Choose a vector bundle chart (V, ϕ) on E at b_0. The vectors $\sigma_1(b_0, \ldots, \sigma_k(b_0) \in E_{b_0}$ are independent, so $\phi(\sigma_1(b_0)), \ldots, \phi(\sigma_k(b_0)) \in \mathbb{F}^r$ are independent. So after composing ϕ with an invertible $r \times r$ matrix A, we may assume $\phi(\sigma_j(b_0)) = e_j$ is the standard basis vector, with $1 \le j \le k$.



For $j > k$ and $b \in V$, set $\sigma_j(b) = \phi^{-1}(b, e_j)$. Then $(\sigma_1, \ldots, \sigma_r)$ are independent at b_0 and therefore independent for all b in a neighborhood U of b_0. □

Remark 7.29. Assume $\pi\colon E \to B$ is a smooth vector bundle over $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. E has a zero section $0_E : B \to E$, right inverse to the projection $\pi\colon E \to B$. The zero section is defined by $0_E(b) = 0 \in E_b$ for all b. $0_E : B \to E$ is an embeddding, and by an abuse of language, $0_E(B) \cong B$ is often called the zero section of E.

7.3 Subbundles

Let $F \subseteq E$.

Definition 7.30. A vector bundle chart (U, ϕ) on E is a **subbundle chart** for $F \subseteq E$ of rank s if

$$\pi^{-1}(U) \cap F = \phi^{-1}(U \times \mathbb{F}^s),$$

where we regard $\mathbb{F}^s$ as an $\mathbb{F}$-linear subspace of $\mathbb{F}^r$.

Definition 7.31. $F \subseteq E$ is a **subbundle** if for all $b \in B$, there is a vector bundle chart (U, ϕ) on E with $b \in U$ and

$$F \cap \pi^{-1}(U) = \phi^{-1}(U \times \mathbb{F}^s)$$

for some $s \leq r$, where we regard $\mathbb{F}^s$ as a subspace of $\mathbb{F}^r$.

For each $b \in B$, F_b is a linear subspace of E_b, and moreover, F is a submanifold of E.

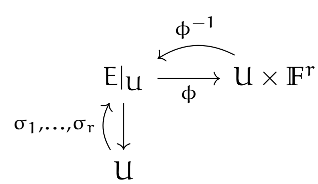
Proposition 7.32. *The following are equivalent.*

(i) $F \subseteq E$ *is a subbundle of rank* s*.*

(ii) F *is a submanifold and* $F_b = E_b \cap F$ *is an* s*-dimensional* $\mathbb{F}$*-linear subspace of* E_b *for all* $b \in B$*.*

(iii) For all $b_0 \in B$*, the fiber* F_b *is an* $\mathbb{F}$*-linear subspace of* E_b *and there is an open neighborhood* U *of* b_0 *and an* s*-frame* $\sigma_1, \ldots, \sigma_s$ *of* $E|_U$ *such that for all* $b \in U$*,* $\sigma_1(b), \ldots, \sigma_s(b) \in E_b$ *is a basis of* F_b*.*

Proof. (iii) $\implies$ (i). By Lemma 7.28, after shrinking U we can extend $\sigma_1, \ldots, \sigma_s$ to a frame $\sigma_1, \ldots, \sigma_r$ of $E|_U$. Then the trivialization given by $\sigma_1, \ldots, \sigma_r$ is a subbundle chart for F.



$$\phi^{-1}(x, y) = y_1\sigma_1(x) + y_2\sigma_2(x) + \ldots + y_r\sigma_r(x)$$

(i) $\implies$ (ii). Immediate from definition.

(ii) $\implies$ (iii). Let $b_0 \in B$. After a preliminary choice of vector bundle chart (U, ϕ_E) on E at b_0 and a chart (U, ϕ_B) on B at b_0,

$$E|_U \xrightarrow{\phi_E} U \times \mathbb{F}^r \xrightarrow{\phi_B \times \mathrm{id}} \phi_B(U) \times \mathbb{F}^r$$

we can assume that $B = U = \phi_B(U)$ is an open neighborhood of $b_0 = 0 \in \mathbb{R}^n$ and $E = U \times \mathbb{F}^m$ is the trivial bundle; and $F_{b_0} = F_0 = F \cap E_0 = \{0\} \times \mathbb{F}^s$. Let $\mathrm{pr}\colon \mathbb{F}^r \to \mathbb{F}^s$ be the projection,

$$p = \mathrm{id}_U \times \mathrm{pr}\colon E \to U \times \mathbb{F}^s$$

and

$$f = p|_F\colon F \to U \times \mathbb{F}^s.$$

Then f is smooth and for each $x \in U$, f restricts to a linear map

$$f_x\colon F_x \to \{x\} \times \mathbb{F}^s \to \mathbb{F}^s.$$

$f_0 = \mathrm{id}_{\mathbb{F}^s}$, so after shrinking U we may assume that f_x is a linear isomorphism for all $x \in U$.

F is a submanifold of E containing $U \times \{0\}$ and $\{0\} \times \mathbb{F}^s$ so

$$T_0F = T_0U \oplus \mathbb{F}^s = T_0(U \times \mathbb{F}^s)$$

and

$$T_0 f\colon T_0F \xrightarrow{\mathrm{id}} T_0(U \times \mathbb{F}^s).$$

By the inverse function theorem, there is a neighborhood of $0 \in \mathbb{R}^n$ (which we may assume to be U) and a neighborhood V of $0 \in \mathbb{F}^s$ such that f has a local inverse

$$g\colon U \times V \to F.$$

Choose s vectors $v_1, \ldots, v_s \in V$ which span $\mathbb{F}^s$. For $x \in U$ put $\sigma_j(x) = g(x, v_j)$. Then $\sigma_1, \ldots, \sigma_s$ are smooth, $\sigma_j(x) \in F_x$, and $\sigma_1(x), \ldots, \sigma_s(x)$ span F_x. □

Example 7.33. Let $B = \mathbb{P}(V)$, the set of lines through the origin in an $\mathbb{F}$-vector space V. Let $E = \mathbb{P}(V) \times V$, with fibers V. Let F be determined from E by the incidence relation of a point lying on a line.

$$F = \widetilde{V} = \{(\ell, v) \in E \mid v \in \ell\}.$$

Then F is a submanifold of E, and $\ell \in \mathbb{P}(V)$,

$$F_\ell = E_\ell \cap F = \{\ell\} \times \ell \subseteq \{\ell\} \times V = E_\ell.$$

Then by Proposition 7.32(iii), this is the **tautological bundle** over $\mathbb{P}(V)$.

$$\begin{array}{ccc} & \widetilde{V} & \\ {}^{\pi}\swarrow & & \searrow^{\beta} \\ \mathbb{P}(V) & & V \end{array}$$

where π is the $\mathbb{F}$-line bundle projection and β is the blow-down map.

8 Foliations

Definition 8.1. Let M be a manifold. A **tangent sub-bundle** or **distribution** over M is a subbundle of TM.

Example 8.2.

(1) Let $(\xi_1, \dots, \xi_r)$ be an r-frame on M, with $r \leq n = \dim M$. Let E be the span of $\{\xi_1, \dots, \xi_r\}$, that is,

$$E_x = \operatorname{span}\{\xi_1(x), \dots, \xi_r(x)\} \subseteq T_x M$$

for any $x \in M$. By the proposition, E is a distribution on M.

(2) Let $f\colon M \to N$ be a map of constant rank. Let $E = \ker(Tf)$, that is,

$$E_x = \ker(T_x f)$$

for $x \in M$. For all $x \in M$, the subspace $E_x \subseteq T_x M$ has dimension $\dim(M) - r$. Choosing charts, (U, ϕ) on M, (V, ψ) on N such that

$$\widetilde{f}(x) = (x_1, \dots, x_r, 0, \dots, 0)$$

we see that the tangent chart $(TU, T\phi)$ also defines a subbundle chart for $E = \ker(Tf)$. So E is a tangent subbundle.

Let $\mathrm{Gr}(r, \mathbb{R}^n)$ be the Grassmannian of r-planes in $\mathbb{R}^n$.

Definition 8.3. Let E be a subbundle of rank r of the trivial rank n bundle $B \times \mathbb{R}^n$. The **Gauss map** of E is the map $f\colon B \to \mathrm{Gr}(r, \mathbb{R}^n)$ defined by $f(b) = E_b$.

Definition 8.4. $\mathrm{Fr}(r, \mathbb{R}^n)$ is the **Stiefel manifold** of all r-frames in $\mathbb{R}^n$.

Lemma 8.5. f *is smooth.*

Proof. We can cover B with open subsets U for which there exists an r-frame $\sigma_1, \dots, \sigma_r \in \Gamma(U, \mathbb{R}^n)$ such that $\sigma_1, \dots, \sigma_r$ span $E|_U$. Define

$$\begin{aligned} \bar{f} \colon U &\longrightarrow \mathrm{Fr}(r, \mathbb{R}^n) \subseteq M_{n\times r}(\mathbb{R}) \\ b &\longmapsto (\sigma_1(b), \dots, \sigma_r(b)). \end{aligned}$$

Then $\bar{f}$ is smooth and $f = p \circ \bar{f}$, where

$$\begin{aligned} p \colon \mathrm{Fr}_r(\mathbb{R}^n) &\longrightarrow \mathrm{Gr}(r, \mathbb{R}^n) \\ (x_1, \dots, x_r) &\longmapsto \mathrm{span}\{x_1, \dots, x_r\} \end{aligned}$$

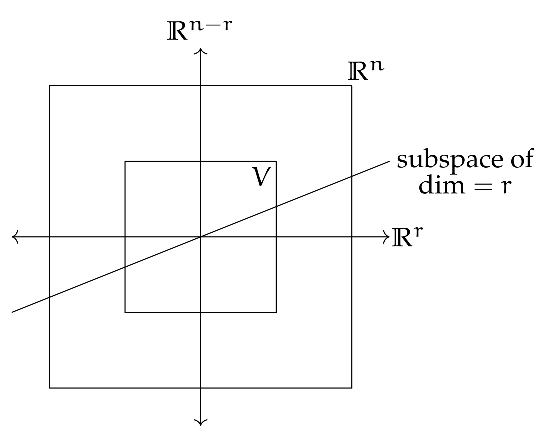
p is smooth because it is the quotient map for the action θ of $GL(r, \mathbb{R})$ on $\mathrm{Fr}(r, \mathbb{R}^n)$ defined by

$$\theta(g)(x_1, \dots, x_r) = (x_1, \dots, x_r)g^{-1}.$$

This action is proper and free, so p is a smooth submersion. □

Remark 8.6. "It's so easy to get bogged down in these graduate courses by checking all the details, but I'd rather give you an idea of the theory. So let's move on at this point."

Example 8.7.



Let $0 \le r \le n$. Let V_1 be an open neighborhood of $0 \in \mathbb{R}^r$ and V_2 an open neighborhood of $0 \in \mathbb{R}^{n-r}$, and $V = V_1 \times V_2 \subseteq \mathbb{R}^r \times \mathbb{R}^{n-r} \cong \mathbb{R}^n$. Let $f \colon V \to M_{(n-r)\times r}(\mathbb{R})$ be smooth. For $(x, y) \in V$, let

$$E^{f(x,y)} = \left\{ (\vec{a}, f(x,y)\vec{a}) \mid (x,y) \in V, \vec{a} \in \mathbb{R}^r \right\} \subseteq \mathbb{R}^n.$$

Let

$$E^f = \bigcup_{(x,y)\in V} \{(x,y)\} \times E^{f(x,y)} \subseteq V \times \mathbb{R}^n = TV.$$

This is the tangent subbundle on V of rank r. E^f has a global frame: let

$$\partial_i = \frac{\partial}{\partial x_i}$$

be the standard vector fields corresponding to the standard basis vectors $e_1, \ldots, e_n \in \mathbb{R}^n$. So a basis of $E^{f(x,y)}$ is

$$(e_i, f(x,y)e_i) \qquad i = 1, \ldots, r.$$

So the vector fields

$$\xi_i(x,y) = (\partial_i, f(x,y)\partial_i) \qquad i = 1, \ldots, r$$

form an r-frame that spans E^f.

Let $\pi_i \colon V \to V_i$ be the projection for $i = 1, 2$. The first component of ξ_i is ∂_i, that is, $\xi_i \sim_{\pi_1} \partial_i$. Therefore,

$$[\xi_i, \xi_j] \sim_{\pi_1} [\partial_i, \partial_j] = 0$$

that is

$$T_{(x,y)}\pi_1[\xi_i, \xi_j](x,y) = 0.$$

Also, if $f(0,0) = 0$, then $E^{f(0,0)} = \mathbb{R}^r$ and

$$\xi_i(0,0) = (\partial_i, 0)$$

is just the first r standard basis vectors of $\mathbb{R}^n$.

This example is important because any distribution (tangent subbundle) locally looks like the one above.

Proposition 8.8. *Let E be a tangent subbundle of rank r over M. For every $a \in M$, there is a chart $(U, \phi \colon U \to \mathbb{R}^n)$ centered at a and a smooth*

$$f \colon \phi(U) \to M_{(n-r)\times r}(\mathbb{R})$$

such that $f(0) = 0$ and $E|_U = (T\phi)^{-1}(E^f)$.

Proof. Choose a basis $v_1, v_2, \ldots, v_n \in T_aM$ such that $E_a = \mathrm{Span}\{v_1, \ldots, v_r\}$. Let $\alpha_1, \ldots, \alpha_n \in T_a^*M$. Then

$$d_a\phi \colon T_aM \to \mathbb{R}^n$$

maps $E_a \to \mathbb{R}^r \subseteq \mathbb{R}^n$. Let

$$F = (T\phi)(E|_{U'}),$$

a subbundle of $TV' = V' \times \mathbb{R}^n$ where $V' = \phi(U')$. Let $f' \colon V' \to \mathrm{Gr}(r, \mathbb{R}^n)$ be the Gauss map of F.

On $\mathrm{Gr}(r, \mathbb{R}^n)$, we have a chart (O, ψ), centered at the standard hyperplane $\mathbb{R}^r$, with $\psi \colon O \to M_{(n-r)\times r}$. Let $V = (f')^{-1}(0)$ and $U = \phi^{-1}(V)$, and $f = f'|_V$. Then $F = E^f$. ∎

Definition 8.9. An **integral manifold** of a distribution $E \subseteq TM$ is an immersed submanifold $A \subseteq M$ such that $T_a A = E_a$ for all $a \in A$.

Picture E as a series of lines at each point on the manifold M, and A as a set of curves that are tangent to these lines everywhere.

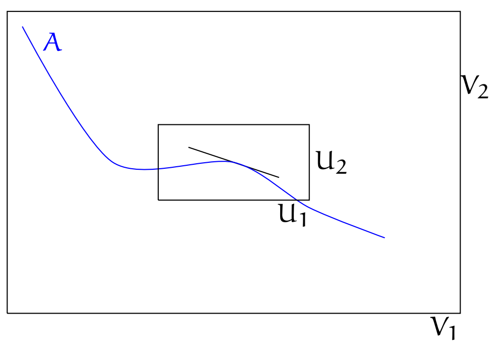
Example 8.10.

(1) If $E = \mathrm{Span}\{\xi\}$, $\xi_a \neq 0$ for all a, then integral manifolds are images of trajectories $\gamma\colon I \to M$. Because ξ is nowhere zero, γ is an injective immersion.

(2) $E = \ker(Tf)$ where $f\colon M \to N$ is any constant rank map. The fibers $f^{-1}(y)$ are integral manifolds.

Example 8.11. $E = E^f$ for some smooth $f\colon V \to M_{(n-r)\times r}(\mathbb{R})$, where $V = V_1 \times V_2 \subseteq \mathbb{R}^r \times \mathbb{R}^{n-r} \cong \mathbb{R}^n$ is open is a product of open sets $V_1 \subseteq \mathbb{R}^r$, $V_2 \subseteq \mathbb{R}^{n-r}$. E^f is the tangent subbundle over V given by

$$E^f_{(x,y)} = \text{graph of } f(x,y)\colon \mathbb{R}^r \to \mathbb{R}^{n-r} \subseteq \mathbb{R}^n.$$



For every $(x_0, y_0) \in V_1 \times V_2$, and every integral manifold A of E containing (x_0, y_0), by the implicit function theorem, there are open $U_1 \subseteq V_1$, $U_2 \subseteq V_2$ such that $A \cap U$ is the graph of a unique smooth function $u\colon U_1 \to U_2$.

Conversely, suppose that $u\colon V_1 \to V_2$ is a C^∞ map. Then

$$A_u = \text{graph of } u \subseteq V_1 \times V_2 = V$$

is an n-dimensional submanifold of V. Then is A_u an integral submanifold?

A_u is the image of an embedding

$$\tilde{u}\colon V_1 \to V \qquad \tilde{u}(x) = \begin{bmatrix} x \\ u(x) \end{bmatrix}.$$

Let $(x, y) \in A_u$, that is $y = u(x)$. A basis of $T_{(x,y)}A_u$ is

$$D\tilde{u}(x)e_i = \begin{bmatrix} I_r \\ Du(x) \end{bmatrix} e_i = \begin{bmatrix} e_i \\ Du(x)e_i \end{bmatrix}.$$

A basis of $E^f_{(x,y)}$ are the columns of

$$\begin{bmatrix} I_r \\ f(x,y) \end{bmatrix}.$$

That is, the vectors

$$\begin{bmatrix} e_i \\ f(x,y)e_i \end{bmatrix}, \qquad i = 1, \dots, r.$$

We want $T_{(x,y)}A_u = E^f_{(x,y)}$ in order for this to be integral. That is,

$$Du(x)e_i = f(x,y)e_i, \qquad i = 1, \dots, r.$$

Or more concisely,

$$Du(x) = f(x,y).$$

We have determined the following.

Proposition 8.12. *A_u is an integral submanifold if and only if u satisfies the system of first-order partial differential equations*

$$Du(x) = f(x, u(x)). \tag{17}$$

The next theorem tells us that we cannot have integral submanifolds that meet in a point or anything like that.

Theorem 8.13. *Let A_1, A_2 be integral manifolds of E. Then $A_1 \cap A_2$ is an open submanifold of A_1 and A_2.*

Proof. Let $E \subseteq TM$ be a distribution. Let $a \in A_1 \cap A_2$. Without loss of generality, $M = V = V_1 \times V_2$, where $V_1 \subseteq \mathbb{R}^r$, $V_2 = V^{n-r}$ are open neighborhoods of 0, and $a = (0,0) \in \mathbb{R}^r \times \mathbb{R}^{n-r} = \mathbb{R}^n$, $E = E^f$ for some

$$f \colon V \to M_{(n-r)\times r}(\mathbb{R}),$$

and $A_1 = \operatorname{graph}(u_1)$, $A_2 = \operatorname{graph}(u_2)$ for two solutions $u_1, u_2 \colon V_1 \to V_2$ of the PDE (17).

We have that $u_1(0) = 0 = u_2(0)$. Without loss, we may assume that V_1 is connected. Let $x \in V_1$. Then choose a smooth path

$$\gamma \colon [0,1] \longrightarrow V_1 \underset{u_2}{\overset{u_1}{\rightrightarrows}} V_2$$

with $\gamma(0) = 0 \in \mathbb{R}^r$, $\gamma(1) = x$. Put $v_i = u_i \circ \gamma$ for $i = 1, 2$. Then v_i satisfies an ODE

$$v_i'(t) = Du_i(\gamma(t))\gamma'(t) \overset{(17)}{=} f(\gamma(t), u_i(\gamma(t)))\gamma'(t) = f(\gamma(t), v_i(t))\gamma'(t)$$

for $0 \le t \le 1$.

We have $v_1(0) = 0 = v_2(0)$. Hence, by uniqueness of ODE solutions, $v_1 = v_2$. In particular, $v_1(1) = v_2(1)$, which means that

$$u_1(x) = u_2(x)$$

for all $x \in V_1$. Therefore, $u_1 = u_2$, and it follows that $A_1 = A_2$.

The theorem follows because we reduced to the case of successively smaller neighborhoods; we have shown that on a sufficiently small open neighborhood, $A_1 = A_2$. □

Corollary 8.14. *Let $\{A_i\}_{i \in I}$ be a family of integral manifolds of E. Then $\bigcup_{i \in I} A_i$ is an integral manifold as well. Moreover, if I is finite, then $\bigcap_{i \in I} A_i$ is an integral manifold of E.*

8.1 Integrability

Definition 8.15. E is **integrable** if for every $a \in M$, there is an integral manifold of E containing a.

The union of all connected integral manifolds containing a is the unique largest connected integral manifold of E containing a, and is called the **leaf** of a.

So M is a disjoint union of leaves. Each leaf is an r-dimensional immersed submanifold.

Remark 8.16 (Caution!)**.** Let $\{A_i\}_{i \in I}$ be a family of integral manifolds of E. Let $A = \bigcup_{i \in I} A_i$; the topology on A may be finer than the subspace topology inherited from M.

Example 8.17. Let $M = \mathbb{R}^3$ with coordinates (x_1, x_2, y). Let E be the vector bundle spanned by

$$\xi = \frac{\partial}{\partial x_1} + f(x_1, x_2)\frac{\partial}{\partial y},$$
$$\eta = \frac{\partial}{\partial x_2} + g(x_1, x_2)\frac{\partial}{\partial y},$$

where $f, g \colon \mathbb{R}^2 \to \mathbb{R}$ are smooth. That is, for $x \in \mathbb{R}^3$,

$$E_x = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ f(x_1, x_2) \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ g(x_1, x_2) \end{pmatrix} \right\}.$$

This is the graph of a linear map $\mathbb{R}^2 \to \mathbb{R}$ with matrix

$$\big(f(x_1, x_2), \quad g(x_1, x_2)\big).$$

An integral manifold A through x in $\mathbb{R}^3$ is, near x, the graph of a function $u \colon U \to \mathbb{R}$, where U is an open neighborhood of $(x_1, x_2) \in \mathbb{R}^2$.

Let

$$\widetilde{u}(x_1, x_2) = \begin{pmatrix} x_1 \\ x_2 \\ u(x_1, x_2) \end{pmatrix}.$$

Then A is the image of $\widetilde{u} \colon U \to \mathbb{R}^3$, so $T_x A$ is the image of $D\widetilde{u}(x_1, x_2)$, which is the column space of the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ {}^{\partial u}/_{\partial x_1} & {}^{\partial u}/_{\partial x_2} \end{bmatrix}.$$

So we must have that

$$\frac{\partial u}{\partial x_1} = f, \qquad \frac{\partial u}{\partial x_2} = g.$$

The integrability condition is that

$$\frac{\partial f}{\partial x_2} = \frac{\partial g}{\partial x_1}.$$

Let E be a distribution on M. For each open $U \subseteq M$, $\Gamma(U, E)$ is a subspace of $\Gamma(U, TM) = \mathcal{T}(U)$.

Definition 8.18. E is **involutive** if $\Gamma(U, E)$ is a Lie subalgebra of $\mathcal{T}(U)$ for all open U.

Lemma 8.19. *If E is integrable, then E is involutive.*

Proof. Assume that E is integrable, and let $\xi, \eta \in \Gamma(U, E)$. For every $a \in U$, ξ, η are tangent to an integral manifold A containing a. Hence, $[\xi, \eta]$ is tangent to A as well, so $[\xi, \eta] \in \Gamma(U, E)$. ☐

Theorem 8.20 (Frobenius Integrability Theorem)**.** *The following are equivalent.*

(i) *E is integrable.*

(ii) *E is involutive.*

(iii) *For all $a_0 \in M$, there is a chart (U, ϕ) centered at a_0 such that $E|_U$ is spanned by the frame $\phi^*(\partial_1), \phi^*(\partial_2), \dots, \phi^*(\partial_r)$.*

Remark 8.21. Knowing (iii), we see that the submanifolds

$$\phi_{r+1}(x) = c_1, \qquad \phi_{r+2}(x) = c_2, \qquad \cdots \qquad \phi_n(x) = c_{n-r}$$

are integral submanifolds of $E|_U$.

Proof of Theorem 8.20.

(i) $\implies$ (ii). This is proven in Remark 8.21.

(ii) $\implies$ (iii). Let $a_0 \in M$. Without loss of generality, assume $M = V = V_1 \times V_2$, where V_1 is an open neighborhood of $0 \in \mathbb{R}^r$ and V_2 is an open neighborhood of $0 \in \mathbb{R}^{n-r}$, $a = (0,0)$, $E = E^f$ for some smooth $f\colon V \to M_{(n-r)\times r}(\mathbb{R})$.

E^f has a global frame

$$\xi_i = \begin{pmatrix} \partial_i \\ f(x,y)\partial_i \end{pmatrix}, \qquad i = 1, \ldots, r.$$

and $\xi_i \sim_{\pi_1} \partial_i$, where $\pi_1 : \mathbb{R}^n \to \mathbb{R}^r$. That is,

$$T\pi_1(\xi_i) = \partial_i.$$

So

$$T\pi_1([\xi_i, \xi_j]) = [\partial_i, \partial_j] = 0.$$

We know that $\Gamma(E)$ is a Lie subalgebra of $\mathcal{T}(M)$, so $[\xi_i, \xi_j] \in \Gamma(E)$, which means

$$[\xi_i, \xi_j](x,y) \in E_{(x,y)} = E^f_{(x,y)}$$

for all $(x,y) \in V_1 \times V_2$. Now

$$T\pi_1 : E^f_{(x,y)} \xrightarrow{\cong} T_x V_1 \cong \mathbb{R}^r.$$

Therefore, $[\xi_i, \xi_j](x,y) = 0$ for all $(x,y) \in V$, so

$$[\xi_i, \xi_j] = 0.$$

Then by Proposition 8.8 and Example 8.7, there is a chart (U, ϕ) centered at $a_0 \in M$ with $\xi_i = \phi^*(\partial_i)$.

(iii) $\implies$ (i). By Remark 8.21, above, there is an integral submanifold through every point in M. So E is integrable. □

Fact 8.22. *If M is paracompact, then the leaves of an integrable distribution are also paracompact.*

8.2 Distributions on Lie Groups.

Let $M = G$ be a Lie group.

Definition 8.23. A vector field $\xi \in \mathcal{T}(G)$ is **left-invariant** if $TL_g(\xi_h) = \xi_{gh}$ for all $g \in G$. Write $\mathcal{T}(G)_L$ for the space of all left-invariant vector fields.

So this vector field is determined by it's value in the Lie algebra $\mathfrak{g} = T_{id}G$. Recall that the left trivialization

$$\begin{aligned} \phi_L \colon G \times \mathfrak{g} &\longrightarrow TG \\ (g, \xi) &\longmapsto T_{id}L_g(\xi) \end{aligned}$$

is a diffeomorphism. For $\xi \in \mathfrak{g}$, let $\xi_{L,g} = \phi_L(g, \xi)$. Then $\xi_L \in \mathcal{T}(G)_L$. There is a one-to-one correspondence

$$\mathfrak{g} = T_{id}G \longleftrightarrow \mathcal{T}(G)_L.$$

Lemma 8.24. *$\mathcal{T}(G)_L$ is a Lie subalgebra of $\mathcal{T}(G)$.*

Proof. $\xi, \eta \in \mathcal{T}(G)_L \implies L_g^*([\xi,\eta]) = [L_g^*(\xi), L_g^*(\eta)] = [\xi, \eta]$ for all $g \in G$. □

Via the isomorphism $\phi_L \colon \mathfrak{g} \to \mathcal{T}(G)_L$, this makes $\mathfrak{g} = \mathcal{T}_{id}G$ a Lie algebra.

Definition 8.25. $\mathfrak{g}$ is the **Lie algebra of** G.

Remark 8.26. If we had done this with right-invariant vector fields instead, we would end up with the opposite Lie algebra $\mathfrak{g}^{op}$, which is the same except the bracket has a negative sign.

9 Differential Forms

9.1 Operations on Vector Bundles

Let $\pi \colon E \to B$ be a smooth vector bundle over $\mathbb{F} = \mathbb{R}, \mathbb{C}$. We want to perform algebraic operations on the fibers of E.

Example 9.1. E^* is the dual of E. Define

$$E^* = \coprod_{b \in B} (E_b)^*$$

where $E_b^* = \mathrm{Hom}_{\mathbb{F}}(E_b, \mathbb{F})$ is the dual vector space. Define $\pi = \pi_{E^*} \colon E^* \to B$ by sending $(E_b)^*$ to b. To get a vector bundle atlas on E^*, take a chart (U, ϕ) for E:

$$\begin{array}{ccc} E|_U & \xrightarrow{\ \phi\ } & U \times \mathbb{F}^r \\ \downarrow & \swarrow_{pr_1} & \\ U. & & \end{array}$$

We have $\mathbb{F}$-linear isomorphisms

$$\underbrace{E_b \xrightarrow{\ \phi\ } \{b\} \times \mathbb{F}^r \longrightarrow \mathbb{F}^r}_{\phi_b}$$

We have transpose maps $\phi_b^T \colon (\mathbb{F}^r)^* \cong (E_b)^*$. Define a chart (U, ϕ^*) on E^*

$$\begin{array}{ccc} E^*|_U & \xrightarrow{\ \phi^*\ } & U \times (\mathbb{F}^r)^* \\ \downarrow \pi & \swarrow \mathrm{pr}_1 & \\ U & & \end{array}$$

by letting the restriction of ϕ^* be the inverse transpose map

$$(\phi_b^T)^{-1} \colon E_b^* \cong (\mathbb{F}^r)^* \cong \{b\} \times (\mathbb{F}^r)^*.$$

To verify that this is an atlas, we need to check compatibility. Given two charts (U_1, ϕ_1) and (U_2, ϕ_2) on E with transition map $g_{12} \colon U_1 \cap U_2 \to GL(r, \mathbb{F})$, where $g_{12}(b) = \phi_{2,b} \circ \phi_{1,b}^{-1}$. The transition map for (U_1, ϕ_1^*) and (U_2, ϕ_2^*) is

$$g_{12}^*(b) = (\phi_{2,b}^T)^{-1} \circ (\phi_{1,b}^T)^{-1} = (g_{12}(b)^T)^{-1}.$$

That is,

$$\begin{array}{ccc} U_1 \cap U_2 & \xrightarrow{\ g_{12}\ } & GL(r, \mathbb{F}) \\ & \searrow g_{12}^* & \downarrow f \\ & & GL(r, \mathbb{F}) \end{array}$$

$g_{12}^* = f \circ g_{12}$, where $f(A) = (A^T)^{-1}$. In particular, the transition map $g_{12}^* = f \circ g_{12}$ is smooth since f is a Lie group homomorphism, and therefore the dual charts define an atlas. Hence E^* is a vector bundle.

If $\sigma_1, \dots, \sigma_r$ is an r-frame on $E|_U$ corresponding to a chart (U, ϕ) on E, then take the r-frame on $E^*|_U$ corresponding to (U, ϕ^*) is the **dual frame** $\sigma_1^*, \dots, \sigma_r^* \in \Gamma(U, E^*)$ characterized by

$$\sigma_i^*(\sigma_j) = \delta_{ij},$$

that is,

$$\sigma_1^*(b)(\sigma_j(b)) = \delta_{ij}$$

for all $b \in U$.

Example 9.2. A special case of the previous example is when $E = TM$ is the tangent bundle of a manifold M. Then $E^* = T^*M$ is the **cotangent bundle**of M. Choosing a chart (U, ϕ) on M gives rise to an n-frame on TU

$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}.$$

The dual frame on T^*U is denoted by

$$dx^1, dx^2, \ldots, dx^n.$$

Definition 9.3. Sections of T^*M are called **differential forms of degree** 1.

For a more general construction of new vector bundles from old ones, let **Vect** be the category of finite-dimensional vector spaces over $\mathbb{F}$. Let $\mathcal{F}$: **Vect** $\to$ **Vect** be a functor.

For a vector bundle $\pi\colon E \to B$, define

$$\mathcal{F}(E) = \coprod_{b \in B} \mathcal{F}(E_b)$$

as a set, and define $\pi\colon \mathcal{F}(E) \to B$ by sending $\mathcal{F}(E_b)$ to $b \in B$.

A chart (U, ϕ) on E gives rise to a chart $(U, \mathcal{F}(\phi))$ on $\mathcal{F}(E)$ as follows.

$$\begin{array}{ccc} \mathcal{F}(E)|_U & \xrightarrow{\mathcal{F}(\phi)} & U \times \mathcal{F}(\mathbb{F}^r) \\ \downarrow & \swarrow_{pr_1} & \\ U & & \end{array}$$

For each $b \in U$, we have $\phi_b\colon E_b \cong \mathbb{F}^r$.

If $\mathcal{F}$ is covariant, then apply $\mathcal{F}$:

$$\mathcal{F}(\phi_b)\colon \mathcal{F}(E_b) \cong \mathcal{F}(\mathbb{F}^r).$$

Define the restriction of $\mathcal{F}(\phi)$ to $\mathcal{F}(E_b)$ to be the map

$$\mathcal{F}(E_b) \xrightarrow{\mathcal{F}(\phi_b)} \mathcal{F}(\mathbb{F}^r) \longrightarrow \{b\} \times \mathcal{F}(\mathbb{F}^r)$$

If $\mathcal{F}$ is contravariant, then apply F to ϕ_b to get

$$\mathcal{F}(\phi_b)\colon \mathcal{F}(\mathbb{F}^r) \cong \mathcal{F}(E_b).$$

Define the restriction of $\mathcal{F}(\phi)$ to $\mathcal{F}(E_b)$ to be the map

$$\mathcal{F}(E_b) \xrightarrow{\mathcal{F}(\phi_b)^{-1}} \mathcal{F}(\mathbb{F}^r) \longrightarrow \{b\} \times \mathcal{F}(\mathbb{F}^r)$$

To check compatibility of the charts, suppose (U_1, ϕ_1) and (U_2, ϕ_2) are charts on E with transition

$$g_{12}\colon U_1 \cap U_2 \longrightarrow GL(r, \mathbb{F}).$$

Then the charts $(U_1, \mathcal{F}(\phi_1))$ and $(U_2, \mathcal{F}(\phi_2))$ have transition map

$$\mathcal{F}(g_{12})\colon U_{12} \to GL(\mathcal{F}(\mathbb{F}^r))$$

given by either

$$\mathcal{F}(g_{12})(b) = \mathcal{F}(g_{12}(b))$$

in the covariant case, or

$$\mathcal{F}(g_{12})(b) = \mathcal{F}(g_{12}(b))^{-1}$$

in the contravariant case. We need this to be smooth for the charts to be compatible. But $\mathcal{F}$ defines a map

$$\mathrm{Hom}(\mathbb{F}^r, \mathbb{F}^r) \xrightarrow{\mathcal{F}} \mathrm{Hom}(\mathcal{F}(\mathbb{F}^r), \mathcal{F}(\mathbb{F}^r)),$$

which restricts to a map

$$\mathrm{GL}(r, \mathbb{F}) \xrightarrow{\mathcal{F}} \mathrm{GL}(\mathcal{F}(\mathbb{F}^r)), \tag{18}$$

which is a homomorphism if $\mathcal{F}$ is covariant and an antihomomorphism if $\mathcal{F}$ is contravariant.

Theorem 9.4. *Suppose that $\mathcal{F}$ has the property that (18) is smooth. Then $\mathcal{F}(E)$ is a smooth vector bundle over B, and if $\{g_{ij}\colon U_{ij} \to \mathrm{GL}(r, \mathbb{F}) \mid i, j \in I\}$ is a cocycle representing E, then a cocycle for $\mathcal{F}(E)$ is*

$$\{\mathcal{F}(g_{ij})\colon U_{ij} \to \mathrm{GL}(\mathcal{F}(\mathbb{F}^r))\}.$$

Example 9.5. If $\mathcal{F} = (-)^*$ is the dual functor, then we obtain the dual vector bundle. Similarly, we could let $\mathcal{F}$ be the tensor algebra, symmetric algebra, or alternating algebra over a vector space V, giving other bundles.

9.2 Alternating algebras

Definition 9.6. Let **Vect** be the category of finite-dimensional vector spaces and linear maps over the field $\mathbb{F} = \mathbb{R}, \mathbb{C}$. A covariant functor $\mathcal{F}$: **Vect** $\to$ **Vect** is **smooth** if for all objects V, W,

$$\mathrm{Hom}(V, W) \xrightarrow{\mathcal{F}} \mathrm{Hom}(F(V), F(W))$$

is smooth. Similarly for contravariant functors.

A smooth functor $\mathcal{F}$ can be applied to the fibers of a vector bundle $E \to B$ to yield a new vector bundle $\mathcal{F}(E) \to B$.

Example 9.7. Let $\mathcal{F}(V) = V^*$, $\mathcal{F}(f\colon V \to W) = (f^T\colon W^* \to V^*)$. The map

$$\mathrm{Hom}(V, W) \xrightarrow[\text{transpose}]{\mathcal{F}} \mathrm{Hom}(V^*, W^*)$$

is smooth because it's linear.

Let k be a commutative ring and V a k-module. For example, $k = \mathbb{R}$, $V = T_x M$, or $k = C^\infty(M)$, $V = \mathcal{T}(M)$.

Definition 9.8. The **alternating algebra** of V is the k-module $A(V)$ spanned by all symbols

$$u_1 \wedge u_2 \wedge \ldots \wedge u_n$$

for $n \in \mathbb{N}$, $u_1, \ldots, u_n \in V$. These are subject to all relations of the form:

$$u_1 \wedge \ldots \wedge (a u_i + a' u_{i'}) \wedge \ldots \wedge u_n = a(u_1 \wedge \ldots \wedge u_i \wedge \ldots \wedge u_n) + a'(u_1 \wedge \ldots \wedge u_{i'} \wedge \ldots \wedge u_n) \tag{19}$$

$$u_1 \wedge \ldots \wedge u_i \wedge \ldots \wedge u_j \wedge \ldots \wedge u_k = 0 \text{ if } u_i = u_j \text{ for } i \neq j. \tag{20}$$

for all $a, a' \in k$ and $u_i \in V$. If $n = 0$, we interpret $u_1 \wedge \ldots \wedge u_n$ as $1 \in k$.

$A(V)$ is an associative algebra over k with unit and multiplication $\wedge$ defined on generators by

$$(u_1 \wedge \ldots \wedge u_n) \wedge (v_1 \wedge \ldots \wedge v_m) = u_1 \wedge \ldots \wedge u_n \wedge v_1 \wedge \ldots \wedge v_m.$$

Axiom (19) is called **multilinarity** and axiom (20) is called **alternating.**

Example 9.9. $5 + 3u + v \wedge w \in A(v)$ for any $u, v, w \in V$.

Example 9.10. $u \wedge v = -(v \wedge u)$ for all $u, v \in V$. Why?

$$(u + v) \wedge (u + v) = 0$$

Then apply the multilinearity and alternating relations above.

Definition 9.11. Some variations on Definition 9.8.

(a) If we omit (20), the resulting algebra is the **tensor algebra** $T(V)$ with multiplication denoted $\otimes$.

(b) If we replace (20) by the following relation

$$u_1 u_2 \cdots u_i \cdots u_j \cdots u_n = u_1 u_2 \cdots u_j \cdots u_i \cdots u_n,$$

we obtain the **symmetric algebra** $S(V)$ of V with multiplication denoted by juxtaposition.

(c) If we replace (20) with the **Clifford Axiom**,

$$uu = q(u),$$

where $q\colon V \to k$ is a quadratic form, we get the **Clifford algebra** $\mathrm{Cl}(V, q)$ of the pair (V, q).

Definition 9.12. An element of $A(V)$ has **degree** n if it's a linear combination of generators $u_1 \wedge \ldots \wedge u_n$, with $u_i \in V$. The elements of degree n form a k-submodule, the n**-th alternating power** $A^n(V)$ of V.

We have that
$$A(V) = \bigoplus_{n=0}^{\infty} A^n(V)$$
where $A^0(V) = k$, $A^1(V) = V$. Therefore, $A(V)$ is a **graded algebra**, with
$$A^i(V) \wedge A^j(V) \subseteq A^{i+j}(V).$$

Remark 9.13. If $x \in A^i(V), y \in A^j(V)$, then $y \wedge x = (-1)^{ij}(x \wedge y)$.

Definition 9.14. The **graded commutator** is given on basis elements $x \in A^i(V)$, $y \in A^j(V)$ by
$$[x, y] = x \wedge y - (-1)^{ij}(y \wedge x)$$
and extended by linearity. This is also called the **Koszul sign rule.**

By Remark 9.13, $A(V)$ is **graded commutative:** $[x, y] = 0$ for all $x, y \in A(V)$.
The alternating algebra construction $V \mapsto A(V)$ defines a functor
$$A\colon \mathbf{Mod}_k \to \mathbf{Alg}_k,$$
where $\mathbf{Mod}_k$ is the category of k-modules and $\mathbf{Alg}_k$ is the category of k-algebras. If V, W are modules and $f\colon V \to W$ is k-linear, then define
$$A(f)\colon A(V) \to A(W)$$
on generators by
$$A(f)(u_1 \wedge \ldots \wedge u_n) = f(u_1) \wedge \ldots \wedge f(u_n),$$
and extend by k-lienarity.

This is then well-defined, k-linear and moreover multiplicative (which means that $A(f)(x \wedge y) = A(f)(x) \wedge A(f)(y)$ for all x, y). We also say that $A(f)$ is of **degree zero**, that is,
$$\deg(A(f)(x)) = \deg(x).$$
Therefore, $A(f)$ is a **homomorphism of graded algebras**.

Some questions we may ask are: what does a basis of $A(V)$ look like? What's the matrix of a linear transformation $A(f)$ relative to the matrix for f?

Let $V^* = \mathrm{Hom}_k(V, k)$ be the k-dual of V. Define a pairing
$$A^k(V^*) \times A^k(V) \to k$$

by

$$(\phi_1 \wedge \ldots \wedge \phi_k)(u_1 \wedge \ldots \wedge u_k) = \det \begin{pmatrix} \phi_1(u_1) & \ldots & \phi_1(u_k) \\ \vdots & \ddots & \vdots \\ \phi_k(u_k) & \ldots & \phi_k(u_k) \end{pmatrix}$$

for $\phi_i \in V^*$, $u_i \in V$. This is well-defined and k-bilinear.

Now assume that V is a **free** k-module of finite rank n. Choose a basis $b_1, \ldots, b_n$ of V, with dual basis $b_1^*, \ldots b_n^*$ such that $b_i^*(b_j) = \delta_{ij}$. Let $I = (i_1, \ldots, i_n)$ be a multi-index with $1 \leq i_\ell \leq r$. Write

$$b_I = b_{i_1} \wedge \ldots \wedge b_{i_r} \text{ and } b_I^* = b_{i_1}^* \wedge \ldots \wedge b_{i_r}^*.$$

We say that I is **increasing** if $1 \leq i_1 < \ldots < i_r \leq n$. Let $\mathcal{I}_r^n$ be the set of such multi-indices.

Lemma 9.15. *The elements* b_I *for* $I \in \mathcal{I}_r^n$ *span* $A^k(V)$. *Likewise, the elements* b_I^* *span* $A^k(V^*)$.

Proof. Let $u_1, \ldots, u_k \in V$. Then it's enough to show that $u_1 \wedge \ldots \wedge u_k$ is a linear combination of the b_I. Expressing

$$u_i = \sum c_{ij} b_i$$

with $c_{ij} \in k$ and using the properties of the wedge product, we see that

$$u_1 \wedge \ldots \wedge u_k = \sum_{j_1, j_2, \ldots, j_k = 1}^{k} c_{1j_1} c_{2j_2} \cdots c_{kj_k} b_{j_1} \wedge \ldots \wedge b_{j_k}.$$

Using the alternating rule, we may rearrange this to be in the form

$$u_1 \wedge \ldots \wedge u_k = \sum_{J \in \mathcal{I}_k^n} a_J b_J.$$

□

Lemma 9.16. $b_I^*(b_J) = \delta_{I,J}$ *for all* $I, J \in \mathcal{I}_k^n$.

Proof.

$$\begin{aligned} b_I^*(b_J) &= (b_{i_1}^* \wedge \ldots \wedge b_{i_k}^*)(b_{j_1} \wedge \ldots \wedge b_{j_k}) \\ &= \det \begin{pmatrix} b_{i_1}^*(b_{j_1}) & \cdots & b_{i_1}^*(b_{j_k}) \\ \vdots & \ddots & \vdots \\ b_{i_k}^*(b_{j_1}) & \cdots & b_{i_k}^*(b_{j_k}) \end{pmatrix} \\ &= \det(\delta_{i_p, j_q})_{1 \leq p,q \leq k} \\ &= \begin{cases} 1 & I = J \\ 0 & I \neq J \end{cases} \end{aligned}$$

□

Theorem 9.17. *Suppose that* V *is a free* k*-module of rank* n *with basis* $b_1, \dots, b_n$.

(i) The collections $\{b_I \mid I \in \mathcal{I}_k^n\}$ *and* $\{b_I^* \mid I \in \mathcal{I}_k^n\}$ *are bases of the* k*-modules* $A^k(V)$ *and* $A^k(V^*)$*, respectively. In particular, these modules are free of rank* $\#\mathcal{I}_k^n = \binom{n}{k}$*. Also,* $A^k(V) = A^k(V^*) = 0$ *for* $k > n$.

(ii) The pairing $A^k(V^*) \times A^k(V) \to k$ *is nondegenerate in the sense that the associated map* $A^k(V^*) \to A^k(V)^*$ *is an isomorphism.*

(iii) As a k*-module,* $A(V)$ *is free of rank* $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Now let V, W be free k-modules and $f\colon V \to W$ a linear map. Choose bases $b_1, \dots, b_n$ of V and $c_1, \dots, c_m$ of W. Let $(f_{ij}) \in M_{m\times n}(k)$ be the matrix of f, that is,

$$f_{ij} = c_i^*(f(b_j)) \in k.$$

Recall that $f\colon V \to W$ induces a k-linear map $A^k(f)\colon A^k(V) \to A^k(W)$ given by

$$A^k(f)(u_1 \wedge \dots \wedge u_k) = f(u_1) \wedge \dots \wedge f(u_k).$$

Theorem 9.18. *The matrix elements of* $A^k(f)$ *relative to the bases* $\{b_J \mid J \in \mathcal{I}_k^n\}$ *and* $\{c_I \mid I \in \mathcal{I}_k^m\}$ *are* $\det(f_{I,J})$*, where* $f_{I,J} \in M_{k\times k}(k)$ *is the* $k \times k$ *submatrix* $(f_{i_p,j_q})_{1\le p,q\le k}$ *with rows* $i_1, \dots, i_k$ *and columns* $j_1, \dots, j_k$.

Proof.

$$\begin{aligned} c_I^*(A^k(f)(b_J)) &= (c_{i_1}^* \wedge \dots \wedge c_{i_k}^*)(f(b_{j_1}) \wedge \dots \wedge f(b_{j_k})) \\ &= \det(c_{i_p}^*(f(b_{j_q})))_{1\le p,q\le k} \\ &= \det(f_{i_p,j_q})_{1\le p,q\le k} \\ &= \det(f_{I,J}) \end{aligned}$$

□

9.3 Differential Forms

Now set $k = \mathbb{R}$ and let $\mathbf{Vect}_{\mathbb{R}}$ be the category of finite-dimensional $\mathbb{R}$-vector spaces.

Corollary 9.19. *The functors* $A^k\colon \mathbf{Vect}_{\mathbb{R}} \to \mathbf{Vect}_{\mathbb{R}}$ *are smooth.*

Proof. For all V, W in $\mathbf{Vect}_{\mathbb{R}}$,

$$M_{m\times n}(\mathbb{R}) \cong \mathrm{Hom}_{\mathbb{R}}(V, W) \xrightarrow{A^k} \mathrm{Hom}_{\mathbb{R}}(A^k(V), A^k(W) \cong M_{\binom{m}{k}\times\binom{n}{k}}(\mathbb{R})$$

is smooth, and in fact multilinear. □

So for for every smooth vector bundle $\pi\colon E \to B$, with cocyles $\{g_{ij}\}$ we get a new vector bundles $A^k(E) \to B$ and $A^k(E^*) \to B$ with cocycles $\{A^k(g_{ij}\}$ and $\{A^k(g_{ij}^*)\}$, respectively.

We also have an alternating algebra bundle

$$A(E) = \bigoplus_{k=0}^{n} A^k(E),$$

and an isomorphism $A^k(E^*) \cong A^k(E)^*$.

Definition 9.20. In the special case that $E = T^*M$ for a manifold M, we say that $A(T^*M)$ is the bundle of **differential forms** on M, and use the notation

$$\Omega(M) = \bigoplus_{k=0}^{n} \Omega^k(M).$$

Definition 9.21. When $E = TM$ for a manifold M, we say that $A(TM)$ is the bundle of **multivector fields** on M and use the notation

$$\mathcal{X}(M) = \bigoplus_{k=0}^{n} \mathcal{X}^k(M).$$

Definition 9.22. Let $V^k = V \times V \times \ldots \times V$ and let W be any other k-module. A map $\mu\colon V^k \to W$ is **alternating multilinear** if

(a) $\mu(u_1, \ldots, u_i, \ldots, u_j, \ldots, u_k) = 0$ if $u_i = u_j$ for some $i \neq j$.

(b) $u_i \mapsto \mu(u_1, \ldots, u_i, \ldots, u_k)$ is linear for all i.

Then put

$$\mathrm{Alt}^k(V, W) = \{\mu\colon V^k \to W \mid \mu \text{ is alternating and multilinear}\},$$

and also $\mathrm{Alt}^k(V) = \mathrm{Alt}^k(V, k)$.

Example 9.23. If $V = k^n$, then $\mu = \det\colon V^n \to k$ is an element of $\mathrm{Alt}^n(V)$.

Furthermore, for any V, we may define $i\colon V^n \to A^n(V)$ by $i(u_1, \ldots, u_k) = u_1 \wedge \ldots \wedge u_k$. Then i is alternating multilinear.

Theorem 9.24 (The universal property of $A^k(V)$)**.** *For every alternating multilinear* $\mu\colon V^k \to W$*, there is a unique linear* $\overline{\mu}\colon A^k(V) \to W$ *satisfying* $\mu = \overline{\mu} \circ i$.

$$\begin{array}{ccc} V^k & \xrightarrow{\mu} & W \\ \downarrow i & \nearrow_{\exists!\overline{\mu}} & \\ A^k(V) & & \end{array}$$

Proof. Supposing that $\overline{\mu}$ exists, there's only one possible way to define it: we have

$$\mu(u_1, \ldots, u_k) = \overline{\mu}(i(u_1, \ldots, u_k)) = \overline{\mu}(u_1 \wedge \ldots \wedge u_k).$$

Since the products $u_1 \wedge \ldots \wedge u_k$ generate $A^k(V)$, this shows that $\overline{\mu}$ is unique.

Now define $\overline{\mu}$ on generators by $\overline{\mu}(u_1 \wedge \ldots \wedge u_k) = \mu(u_1, u_2, \ldots, u_k)$. This is well-defined because μ is alternating multilinear. So $\overline{\mu}$ is a k-linear map $A^k(V) \to W$ and $\mu = \overline{\mu} \circ i$. ☐

Corollary 9.25. $\mathrm{Alt}^k(V, W) \cong \mathrm{Hom}_k(A^k(V), W)$.

Proof. The isomorphism is given by $\overline{\mu} \mapsto \overline{\mu} \circ i$. ☐

Example 9.26. A special case of this is when $W = k$, when

$$\mathrm{Alt}^k(V) \cong A^k(V)^* \cong A^k(V^*)$$

Fact 9.27. *Let* $\pi \colon E \to B$ *be a vector bundle.*

(i) $\Gamma(E^*) \cong \Gamma(E)^*$

(ii) $\Gamma(A^k(E)) \cong A^k(\Gamma(E))$

Proof Sketch. Recall that $\Gamma(E)$ is the space of smooth sections $B \to E$ of E. $\Gamma(E)$ is a module over $C^\infty(B)$.

In (i), $E^* = \coprod_{b \in B} E_b^*$ is the dual vector bundle with fibers $E_b^* = \mathrm{Hom}_{\mathbb{R}}(E_b, \mathbb{R})$, while $\Gamma(E)^* = \mathrm{Hom}_{C^\infty(B)}(\Gamma(E), C^\infty(B))$ is the dual $C^\infty(B)$-module of $\Gamma(E)$. The rest of this will be on the homework.

In (ii), $A^k(E) = \coprod_{b \in B} A^k_{\mathbb{R}}(E_b)$ is the k-th alternating power of E, while $A^k(\Gamma(E)) = A^k_{C^\infty(B)}(\Gamma(E))$ is the k-th alternating power of the $C^\infty(B)$-module $\Gamma(E)$. ☐

We can put this previous fact together with the definition of the bundle of differential forms on M to see that

$$\begin{aligned} \Omega^k(M) &:= \Gamma(A^k(T^*M)) \cong \Gamma(A^k(TM)^*) \\ &\cong \Gamma(A^k(TM))^* && \text{Fact 9.27(i)} \\ &\cong A^k(\Gamma(TM))^* && \text{Fact 9.27(ii)} \\ &\cong A^k(\mathcal{T}(M))^* \\ &\cong \mathrm{Alt}^k(\mathcal{T}(M)) \end{aligned}$$

We have proved

Proposition 9.28. $\Omega^k(M) \cong \mathrm{Alt}^k(\mathcal{T}(M))$.

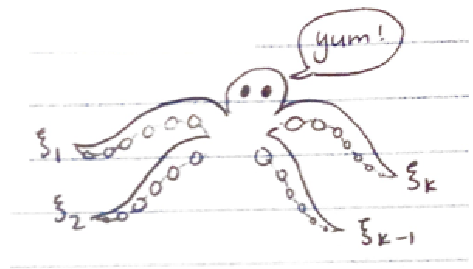


Figure 1: An octopus eating vector fields. (Illustration by Lila Greco)

Remark 9.29. A k-form on the manifold M is like an octopus that eats k-many vectors $\xi_1, \dots, \xi_k$ on M and spits out a real number. The expression of a k-form in a chart (U, ϕ) on M is as follows. Let $\alpha \in \Omega^k(M)$. Let ${}^{\partial}/_{\partial x_1}, \dots, {}^{\partial}/_{\partial x_n}$ be the frame on TU defined by ϕ, and $dx_1, \dots, dx_n$ the dual frame of T^*U. For $I \in \mathcal{I}_k^n$, put

$$dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_k}.$$

Then there exist unique functions $f_I \in C^\infty(U)$ such that

$$\alpha|_U = \sum_{I \in \mathcal{I}_k^n} f_I dx_I,$$

where $f_I = \alpha|_U({}^{\partial}/_{\partial x_{i_1}}, \dots, {}^{\partial}/_{\partial x_{i_k}})$.

9.4 The de Rahm complex

$$\Omega^0(M) = \Gamma * (A^0(T^*M)) = \Gamma(\text{trivial bundle } M \times \mathbb{R}) = C^\infty(M).$$

Let $f \in \Omega^0(M) = C^\infty(M)$. Recall that

$$df = pr_2 \circ Tf \colon TM \xrightarrow{Tf} T\mathbb{R} = \mathbb{R} \times \mathbb{R} \xrightarrow{pr_2} \mathbb{R}$$

For $\xi \in \mathcal{T}(M)$ we define $df(\xi) = \mathcal{L}_\xi(f)$ by

$$df(\xi)(x) = d_x f(\xi_x) \in \mathbb{R}$$

If $g \in C^\infty(M)$ we have

$$df(g\xi) = \mathcal{L}_{g\xi}(f) = g\mathcal{L}_\xi(f) = g df(\xi).$$

So df, considered as a map $\mathcal{T}(M) \to C^\infty(M)$ is $C^\infty(M)$-linear. So we may regard df as a section of T^*M,

$$df \in \Gamma(T^*M) = \Gamma(A^1(T^*M)) = \Omega^1(M).$$

This in turn tells us that d is a map

$$d\colon \Omega^0(M) \to \Omega^1(M).$$

Then d is $\mathbb{R}$-linear and satisfies the Leibniz rule:

$$d(fg) = (df)g + f(dg)$$

because

$$\mathcal{L}_\xi(fg) = \mathcal{L}_\xi(f)g + f\mathcal{L}_\xi(g)$$

for all $\xi \in \mathcal{T}(M)$.

Lemma 9.30. *Let* $M = U$ *open in* $\mathbb{R}^n$ *and* $f \in \mathcal{C}^\infty(M)$. *Then*

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$

Proof. We have that

$$df = \sum_{i=1}^{n} g_i \, dx_i$$

for some $g_i \in C^\infty(U)$, and these g_i are given by

$$df\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial f}{\partial x_i}.$$

□

Remark 9.31. The previous lemma tells us that in particular, $d(x_i) = dx_i$. This explains the notation dx_i for the 1-form dual to the vector field $\partial/_{\partial x_i}$.

We want to expand this map d into a sequence of maps

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \to 0$$

satisfying the graded Leibniz rule $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$ for $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^\ell(M)$.

Definition 9.32. For $\alpha \in \Omega^k(M)$, define $d\alpha \in \Omega^{k+1}(M)$ by

$$\begin{aligned} d\alpha(\xi_1, \ldots, \xi_{k+1}) = \sum_{\ell=1}^{k+1} (-1)^{\ell+1} \mathcal{L}_{\xi_\ell}(\alpha(\xi_1, \ldots, \widehat{\xi_\ell}, \ldots, \xi_{k+1})) \\ + \sum_{1 \le i < j \le n} (-1)^{i+j} \alpha([\xi_i, \xi_j], \xi_1, \ldots, \widehat{\xi_i}, \ldots, \widehat{\xi_j}, \ldots, \xi_\ell) \end{aligned}$$

for $\xi_1, \ldots, \xi_{k+1} \in \mathcal{T}(M)$.

Example 9.33. In the special case that $\alpha = f \in \Omega^0(M) = C^\infty(M)$, then

$$(df)(\xi) = \mathcal{L}_\xi(f).$$

This coincides with the definition of $d\colon \Omega^0 \to \Omega^1$.

Example 9.34. If $\alpha \in \Omega^1(M)$, then

$$(d\alpha)(\xi, \eta) = \mathcal{L}_\xi(\alpha(\eta)) - \mathcal{L}_\eta(\alpha(\xi)) - \alpha([\eta, \xi]) \tag{21}$$

Lemma 9.35. $d\alpha$ *is alternating multilinear over* $C^\infty(M)$ *and therefore defines a* $(k+1)$*-form. That is,* $d\colon \Omega^k \to \Omega^{k+1}$ *is well-defined.*

Proof. We only check the case when $k = 1$. For $k = 1$, $\alpha \in \Omega^1(M)$, so $d\alpha(\xi, \xi) = 0$ by (21). Moreover, (21) is additive with respect to η and ξ.

Now if $f \in C^\infty(M)$,

$$\begin{aligned} d\alpha(f\xi, \eta) &= \mathcal{L}_{f\xi}(\alpha(\eta)) - \mathcal{L}_\eta(\alpha(f\xi)) - \alpha([f\xi, \eta]) \\ &= f\mathcal{L}_\xi(\alpha(\eta)) - \mathcal{L}_\eta(f\alpha(\xi)) + \alpha([\eta, f\xi]) \\ &= f\mathcal{L}_\xi(\alpha(\eta)) - (\mathcal{L}_\eta(f)\alpha(\xi) + f\mathcal{L}_\eta(\alpha(\xi))) + \alpha(\mathcal{L}_\eta(f\xi)) \end{aligned} \tag{22}$$

Now

$$\begin{aligned} \alpha(\mathcal{L}_\eta(f\xi)) &= \alpha\left(\mathcal{L}_\eta(f)(\xi) + f\mathcal{L}_\eta(\xi)\right) \\ &= \mathcal{L}_\eta(f)\alpha(\xi) + f\alpha(\mathcal{L}_\eta(\xi)) \\ &= \mathcal{L}_\eta(f)\alpha(\xi) + f\alpha([\eta, \xi]) \\ &= \mathcal{L}_\eta(f)\alpha(\xi) - f\alpha([\eta, \xi]) \end{aligned}$$

Substittue this into (22) to see that

$$d\alpha(f\eta, \xi) = f\left(\mathcal{L}_\xi(\alpha(\eta)) - \mathcal{L}_\eta(\alpha(\xi)) - \alpha([\xi, \eta])\right).$$

□

Lemma 9.36. *Let* $M = U$ *be open in* $\mathbb{R}^n$, $\alpha \in \Omega^k(U)$,

$$\alpha = \sum_I f_I dx_I.$$

Then

$$d\alpha = \sum_I df_I \wedge dx_I = \sum_I \sum_{i=1}^n \frac{\partial f_I}{\partial x_i} dx_i \wedge dx_I.$$

Proof. Put $\partial_i = {}^{\partial}/_{\partial x_i}$, and $\partial_I = \partial_{i_1} \wedge \ldots \wedge \partial_{i_k}$. Recall that $f_I = \alpha(\partial_I)$. Similarly,

$$d\alpha = \sum_{J \in \mathcal{I}^n_{k+1}} g_J dx_J$$

with

$$\begin{aligned} g_J = d\alpha(\partial_J) &= \sum_{\ell=1}^{k+1} \partial_{j_\ell}(\alpha(\partial_{J\setminus\{j_\ell\}})) + 0 \\ &= \sum_{\ell} (-1)^{\ell+1} \partial_{j_\ell} f_{J\setminus\{j_\ell\}} \\ &= \sum_{\ell} (-1)^{\ell+1} \frac{\partial f_{J\setminus\{j_\ell\}}}{\partial x_\ell} \end{aligned}$$

So

$$d\alpha = \sum_J \sum_{\ell=1}^{k+1} (-1)^{\ell+1} \frac{\partial f_{J\setminus\{j_\ell\}}}{\partial x_\ell} dx_J = \sum_I \sum_{i=1}^{n} \frac{\partial f_I}{\partial x_i} dx_i \wedge dx_I.$$

Where we substitute $I = J \setminus \{j_\ell\}$ and $i = j_\ell$. □

Remark 9.37. The previous proof also shows that $dx_J = (-1)^{\ell+1} dx_i \wedge dx_I$. The graded Leibniz rule follows from this.

Theorem 9.38. $d^2 = 0$*, that is,* $d(d\alpha) = 0$ *for all* $\alpha \in \Omega^k(M)$.

Proof. Enough to show that $d^2\alpha|_U = 0$ for every coordinate neighborhood U on M. So without loss we may assume that $M = U \subseteq \mathbb{R}^n$ open, and

$$\alpha = \sum_I f_I dx_I.$$

Then

$$\begin{aligned} d(d\alpha) &= d\left(\sum_I \sum_i \frac{\partial f_I}{\partial x_i} dx_i \wedge dx_I\right) \\ &= \sum_I \sum_j \sum_i \frac{\partial^2 f_I}{\partial x_i \partial x_j} dx_i \wedge dx_j \wedge dx_I \\ &= \sum_I \sum_{1\le i<j\le n} \left(\frac{\partial^2 f_I}{\partial x_i \partial x_j} - \frac{\partial^2 f_I}{\partial x_j \partial x_i}\right) dx_i \wedge dx_j \wedge dx_I \\ &= 0 \end{aligned}$$

The mixed partial derivatives of f_I cancel because f is smooth. □

Corollary 9.39. $\Omega(M) = \bigoplus_{k=0}^{\infty} \Omega^k(M)$ *is a* ***commutative differential graded algebra****.*

9.5 De Rahm Cohomology

Definition 9.40. A k-form $\alpha \in \Omega^k(M)$ is **closed** if $d\alpha = 0$ (if α is in the kernel of d).

A k-form $\alpha \in \Omega^k(M)$ is **exact** if $\alpha = d\beta$ for some $\beta \in \Omega^{k-1}(M)$ (if α is in the image of d). Set

$$Z^k(M) = \{\alpha \in \Omega^k(M) \mid d\alpha = 0\} = \ker(d\colon \Omega^k(M) \to \Omega^{k+1}(M))$$

$$B^k(M) = \{d\beta \mid \beta \in \Omega^{k-1}(M)\} = \operatorname{im}(d\colon \Omega^{k-1}(M) \to \Omega^k(M))$$

Notice that $d^2 = 0$ implies that $B^k(M) \subseteq Z^k(M)$.

Definition 9.41. The k-th **de Rahm cohomology** of M is the $\mathbb{R}$-vector space

$$H^k_{DR}(M) = {Z^k(M)}\big/{B^k(M)}$$

Definition 9.42. The **cup product** of two classes $[\alpha] \in H^k_{DR}(M)$ and $[\beta] \in H^\ell_{DR}(M)$ is $[\alpha] \smile [\beta] = [\alpha \wedge \beta] \in H^{k+\ell}_{DR}(M)$.

The cup product is well-defined because of the following lemma.

Lemma 9.43.

(i) α, β closed $\implies$ $\alpha \wedge \beta$ closed.

(ii) α closed and β exact $\implies$ $\alpha \wedge \beta$ exact.

Proof.

(i) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{k\ell}\alpha \wedge d\beta = 0$

(ii) Let $\beta = d\gamma$. Then

$$d(\alpha \wedge \gamma) = d\alpha \wedge \gamma \pm \alpha \wedge d\gamma = 0 \wedge \gamma \pm \alpha \wedge \beta$$

Therefore, $\alpha \wedge \beta$ is exact. □

This also shows us that $Z(M) = \bigoplus_{k=0}^{\infty} Z^k(M)$ is a sub-algebra of $\Omega(M)$ that is also a commutative graded differential algebra. Similarly, $B(M)$ is a graded ideal of $Z(M)$.

Corollary 9.44. $H_{DR}(M) = \bigoplus_{k=0}^{\infty} H^k_{DR}(M)$ *is a commutative graded* $\mathbb{R}$*-algebra.*

Example 9.45.

$$\begin{aligned} H^0_{DR}(M) = Z^0(M) &= \ker(d\colon C^\infty(M) \to \Omega^1(M)) \\ &= \{f \in C^\infty(M) \mid df = 0\} \\ &= \{f \in C^\infty(M) \mid f \text{constant on connected components of } M\} \end{aligned}$$

Therefore, the zeroth de Rahm cohomology of M is $\mathbb{R}^{\pi_0(M)}$, where $\pi_0(M)$ is the set of connected components of M.

The de Rahm cohomology algebra of M satisfies several important properties. Namely, $M \mapsto H_{DR}(M)$ is functorial, homotopy invariant, and has the Mayer-Vietoris property.

9.6 Functoriality of H_{DR}

Let $F\colon M \to N$ be a smooth map. Recall that we have maps of smooth vector bundles

$$\begin{array}{ccccc} TM & & T^*M & & A(T^*M) \\ \downarrow & \overset{*}{\rightsquigarrow} & \downarrow & \overset{A}{\rightsquigarrow} & \downarrow \\ M & & M & & M. \end{array}$$

F induces a map on tangent spaces.

$$\begin{array}{ccc} TM & \xrightarrow{TF} & TN \\ \downarrow & & \downarrow \\ M & \xrightarrow{F} & N \end{array}$$

Let $\mathcal{F}$ be a contravariant functor. Then we get a pullback map on sections of $\mathcal{F}(TX)$

$$F^*\colon \Gamma(\mathcal{F}(TN)) \to \Gamma(\mathcal{F}(TM)).$$

In particular, when $\mathcal{F} = A$, we get

$$\mathcal{F}^*\colon \Omega(N) \to \Omega(M).$$

Example 9.46.

(a) For $k = 0$, $f \in \Omega^0(N) = C^\infty(N)$ and $F^*(f) = f \circ F$.

(b) For $f \in \Omega^0(N)$, let $\alpha = df$. Then

$$F^*(\alpha)_x(v) = \alpha_{F(x)}(T_xF(v)) = d_{F(x)}f(T_xF(v)) = dx(f \circ F)(v).$$

So $F^*(\alpha) = d(f \circ F) = dF^*(f)$, that is, $F^*(df) = dF^*(f)$.

Fact 9.47 (Other properties of pullback)**.**

(a) $F^*(d\alpha) = dF^*(\alpha)$ *for all* $\alpha \in \Omega^k(N)$.

(b) $F^*(\alpha \wedge \beta) = F^*(\alpha) \wedge F^*(\beta)$

(c) If $G\colon N \to P$ *is smooth as well, then* $(G \circ F)^* = F^* \circ G^*$*, and we get*

$$(G \circ F)^*\colon \Omega(P) \xrightarrow{G^*} \Omega(N) \xrightarrow{F^*} \Omega(M).$$

So Ω *is a contravariant functor from the category of Manifolds to the category of commutative differential graded algebras.*

We also get an induced map on cohomology. F^* induces an homomorphism of commutative graded algebras also denoted by F^*,

$$\begin{aligned} F^*\colon H_{DR}(N) &\longrightarrow H_{DR}(M) \\ [\alpha] &\longmapsto [F^*(\alpha)] \end{aligned}$$

9.7 Other properties of $H_{DR}(M)$

Definition 9.48. Smooth maps $F_0, F_1\colon M \to N$ are **homotopic** if there is a smooth map $F\colon M \times [0,1] \to N$ with $F(x,0) = F_0(x)$ and $F(x,1) = F_1(x)$. The notation is $F_t(x) = F(x,t)$.

Think of $M \times [0,1]$ as a manifold with boundary.

Theorem 9.49 (de Rahm)**.** *If two maps* $F_0, F_1\colon M \to N$ *are homotopic, then they induce the same map on cohomology, that is* $F_0^* = F_1^*\colon H_{DR}(N) \to H_{DR}(M)$.

In fact, de Rahm's theorem can say more: it says that the de Rahm cohomology of M is identical to the singular cohomology $H(M;\mathbb{R})$ of M as a topological space.

Example 9.50. M is **contractible** to a point $x_0 \in M$ if the maps $\mathrm{id}\colon M \to M$ and the constant map $f\colon x \mapsto x_0$ are homotopic. Then $\mathrm{id}^* = f^*\colon H_{DR}(M) \to H_{DR}(M)$. So

$$H_{DR}(M) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & k > 0. \end{cases}$$

Example 9.51. $H_{DR}(M) \cong H_{DR}(M \times [0,1])$

$H_{DR}(M)$ has a Mayer-Vietoris long exact sequence.

Lemma 9.52. *Suppose that* $U, V \subseteq M$ *are open, and* $U \cup V = M$. *Then there is a short-exact sequence*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega(M) & \xrightarrow{f} & \Omega(U) \oplus \Omega(V) & \xrightarrow{g} & \Omega(U \cap V) & \longrightarrow & 0 \\ & & \alpha & \longmapsto & (\alpha|_U, \alpha|_V) & & & & \\ & & & & (\beta, \gamma) & \longmapsto & \beta|_{U\cap V} - \gamma|_{U \cap V} & & \end{array}$$

Proof sketch. The fact that $g \circ f = 0$ is clear. If $g(\beta, \gamma) = 0$, then $(\beta, \gamma) = f(\alpha)$ for some α. So $\operatorname{im}(f) = \ker(g)$.

Injectivity of f is clear. For surjectivity of γ, use a partition of unity for the cover $\{U, V\}$ of M. □

Theorem 9.53 (Mayer-Vietoris)**.** *There is a long exact sequence of cohomology.*

$$\cdots \to H^k(U) \to H^k(V) \to H^k(M) \to H^k(U \cap V) \to H^{k+1}(U) \oplus H^{k+1}(V) \to \cdots$$

Example 9.54. We can use the previous theorem to calculate the de Rahm cohomology of S^n, for example.

$$H^k_{DR}(S^n) = \begin{cases} \mathbb{R} & k = 0, n \\ 0 & k \neq 0, n \end{cases}$$