Math 7510: Sheaves on Manifolds

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Contents

Contents by Lecture

Administrative

There is now a webpage with a list of things we want to understand by the end of the course, including papers that we'll hopefully have the background to read by the end of the course. Primarily we want to follow Kashiwara and Shapira's book *Sheaves on Manifolds*.

1 Noncommutative Algebra

Even though the course is geometry through and through, the initial motivation comes from noncommuative algebra.

Definition 1.1. If $\mathfrak{g}$ is a Lie algebra, we get a noncommutative associative algebra $U(\mathfrak{g})$ called the **universal enveloping algebra** that is defined as

$$U^{\hbar}(\mathfrak{g}) = {}^{T(\mathfrak{g})}\Big/_{\langle XY - YX - \hbar[X,Y]\rangle},$$

where

$$T\mathfrak{g} = \bigoplus_{n\in\mathbb{N}} \mathfrak{g}^{\otimes n}.$$

Theorem 1.2 (Poincaré-Birkhoff-Witt)**.** *This is flat in $\hbar$ if and only if these generators are a Gröbner basis if and only if*

$$\operatorname{gr} U(\mathfrak{g}) := \bigoplus_{n\in\mathbb{N}} \left({}^{U(\mathfrak{g})_{\deg\leqslant n}}\Big/_{U(\mathfrak{g})_{\deg\leqslant n-1}} \right) \cong \operatorname{Sym}\mathfrak{g}$$

Remark 1.3.

$$\begin{tikzcd} U(\mathfrak{g}) \arrow[d, "\operatorname{gr}"] & \supseteq & Z(U(\mathfrak{g})) \arrow[d] \arrow[r, equal] & (U(\mathfrak{g}))^{\mathfrak{g}\text{-invariants}} \cong \\ \operatorname{Sym}\mathfrak{g} & \supseteq & (\operatorname{Sym} g)^G & \end{tikzcd}$$

$(U\mathfrak{g})^{\mathfrak{g}\text{-invariants}} \cong (\operatorname{Sym}\mathfrak{g})^{\mathfrak{g}}$ if the action of $\mathfrak{g}$ on $U\mathfrak{g}$ is completely reducible.

$(\operatorname{Sym}\mathfrak{g})^{\mathfrak{g}} \cong (\operatorname{Sym}\mathfrak{g})^G$ if $g = \operatorname{Lie}(G)$ is connected.

$(\operatorname{Sym}\mathfrak{g})^G \cong (\operatorname{Sym}\mathfrak{l})^W$ where W is the Weyl group.

The linear term in $\hbar$ of the product on $U^{\hbar}\mathfrak{g}/\hbar^2$ gives a **Poisson (Lie) bracket** $\{-,-\}$ on $\operatorname{Sym}\mathfrak{g}$. A Poisson bracket is a Lie bracket such that

$$[f, gh] = \{f, g\}h + g\{f, h\}.$$

This one in particular satisfies

$$\{X, Y\} = [X, Y]$$

Definition 1.4. M is a **Poisson manifold** if the set of functions $\mathrm{Fun}(M)$ on M is equipped with a Poisson bracket.

This gives us (a unique) $\pi \in \Gamma(M; \bigwedge^2 TM)$, called an **alternating 2-tensor**. The Poisson bracket is related to π by

$$\{f, g\} = \langle \pi, df \wedge dg \rangle$$

We can't define the Poisson bracket this way from any arbitrary alternating 2-tensor, because we aren't guaranteed that the resulting bracket will satisfy the Jacobi identity. There needs to be an alternate definition.

π gives a map $\pi\cdot\colon T^*M \to TM$ given by $\alpha \mapsto \langle \pi, \alpha \wedge - \rangle$.

$$\langle \alpha, \pi \rangle = \sum \alpha(\vec{v}_i) \otimes \vec{w}_i.$$

Example 1.5. If $G = \mathrm{SO}(3, \mathbb{R})$ acts on $\mathrm{so}(3)^* = \mathbb{R}^3$ with the usual action of $\mathrm{so}(3)$.

???

Definition 1.6. M is **(Poisson) symplectic** if $\pi\cdot\colon T^*M \to TM$ is onto for all $m \in M$

Example 1.7. $M = \mathbb{R}^2$, $\pi = f(x, y)^d/_{dx} \wedge {}^d/_{dy}$ for some nowhere vanishing $f(x, y)$ (iff f is symplectic), π Poisson.

In this case, the inverse $\omega\colon TM \to T^*M$ exists, or $\omega \in \bigwedge^2 T^*M$ is the **symplectic form**.

Remember that we needed extra conditions so that an alternating 2-tensor π defines a Poisson bracket $\{f, g\} = \langle \pi, df \wedge dg \rangle$ that satisfies the Jacobi identity? Well, that condition turns out to be that ω is closed, that is, $d\omega = 0$.

Theorem 1.8. *If $\pi\cdot$ has constant rank near $m \in M$, then M near m has a foliation by submanifolds whose tangent spaces are the images of $\pi\cdot$, and are naturally symplectic.*

Example 1.9 (Bad example)**.** Let $\mathbb{R}$ act on $\mathbb{R}^4$ by

$$x \mapsto \begin{bmatrix} \cos x & \sin x & & \\ -\sin x & \cos x & & \\ & & \sqrt{2}\cos x & \sqrt{2}\sin x \\ & & -\sqrt{2}\sin x & \sqrt{2}\cos x \end{bmatrix}$$

Then take $G = \mathbb{R} \ltimes \mathbb{R}^4$. The orbits on $\mathfrak{g}^*$ are only locally closed. This is the irrational orbits on the torus issue.

Definition 1.10. Let M be a smooth manifold. Let $\mathrm{Vec}(M)$ be the sheaf of vector fields on M. This is a Lie algebra.

Definition 1.11. $\mathcal{D}_M := U(\mathrm{Vec}(M))$, the universal enveloping algebra of $\mathrm{Vec}(M)$. Recall that the universal enveloping algebra is a quotient of the tensor algebra. But we're not tensoring over $\mathbb{C}$, rather over $\mathcal{O}_M$, the set of functions on M.

There is an action of $\mathrm{Vec}M$ on $\mathcal{O}_M$, because derivatives act on functions. Therefore, $\mathcal{D}_M$ acts on $\mathcal{O}_M$ as differential operators (higher order derivatives).

Being a universal enveloping algebra, $\mathcal{D}(M)$ has a degeneration, via the associated graded algebra, to $\mathrm{Sym}(\mathrm{Vec}M)$.

So what is $\mathrm{Sym}(\mathrm{Vec}M)$? This is

$$\mathrm{Sym}(\Gamma(M;TM)) = \Gamma(M;\mathrm{Sym}\,TM) = p_*(\mathcal{O}_{T^*M})$$

Where $p\colon T^*M \to M$, and this is the pushforward of the sheaf on T^*M to the sheaf on M.

Then T^*M is Poisson, and even better, symplectic, and the symplectic 2-form is given as follows.

If $(m,f) \in T^*M$ for $m \in M$ and $f \in T^*_mM$, let $\vec{v},\vec{w} \in T_{(m,f)}(T^*M)$. We have

$$\omega(\vec{v},\vec{w}) = \text{exercise. There's only one possibility up to sign.}$$

Starting Point

Remark 1.12. Now let's put some of this stuff together. Let's say we're interested in representation theory. If we have G acting on some vector space V irreducibly, then we get an action of $\mathfrak{g}$ and $U(\mathfrak{g})$ on this vector space as well. Thus $Z(U(\mathfrak{g}))$ acts on V by scalars, by Schur's lemma. This gives a map $Z(U(\mathfrak{g})) \to \mathbb{R}$ defining this action.

Going backwards, we get a point in $\mathrm{Spec}(Z(U(\mathfrak{g})))$.

$$\begin{tikzcd}
G\text{-orbit closure} \arrow[r] \arrow[d] & \mathfrak{g}^* \arrow[d] \\
\mathrm{pt} \arrow[r] & \mathrm{Spec}\,Z(U(\mathfrak{g})) \cong \mathfrak{g}^*/\!/G
\end{tikzcd}$$

Example 1.13. G acts on $\mathbb{C}$, and the G-orbit closure is the fiber over 0 in the characteristic polynomial map. This is the so-called **nilpotent cone** N.

If instead $G = \mathrm{GL}_n(\mathbb{C})$, then N is the nilpotent matrices.

Definition 1.14. A $\mathcal{D}_M$-module is a sheaf over M with an action of $\mathrm{Vec}(M)$, or equivalently an action of $\mathcal{D}_M$.

Example 1.15. We already saw that $\mathcal{D}_M$ acts on $\mathcal{O}_M$.

Example 1.16. Let $M = \mathbb{C} = \mathrm{Spec}(\mathbb{C}[z])$. Then the global sections of $\mathcal{D}_M$ is the algebra

$$\mathbb{C}\left[\frac{d}{dz},\hat{z}\right] \Big/ \left\langle \left[\frac{d}{dz},\hat{z}\right] - 1 \right\rangle$$

The hat means that this isn't z, but rather multiplication by z, because it's an operator not a variable. $\mathcal{D}_M$ acts on $\mathbb{C}[z]$ by taking derivatives or multiplying by z.

Here are three $\mathcal{D}_M$-modules for this M. They are all cyclic.

$\mathcal{D}$-module	generator	linear ODE (relation)
Functions on $\mathbb{C}$	1	${}^{d}/_{dz}$
Distributions supported at 0	δ_0 (delta function)	$\widehat{z}$
Functions on $\mathbb{C}^\times$	z^{-1}	${}^{d}/_{dz}\widehat{z}$

We find the appropriate $\mathcal{D}$-module by quotienting by the right ideal generated by the linear ODE.

(Remark: The last is not finitely generated over $\mathcal{O}_M$, but it is over $\mathcal{D}_M$.)

What are the associated graded modules? Write $\operatorname{gr}\mathcal{D}_M = \mathbb{C}[\xi, z]/\langle [\xi, z] = 0\rangle$.

$\mathcal{D}$-module	$(\operatorname{gr}\mathcal{D}_M)$-module	$\operatorname{Spec} \subseteq T^*\mathbb{C} \cong C^2$
Functions on $\mathbb{C}$	$\xi = 0$	z-axis
Distributions supported at 0	$z = 0$	ξ-axis
Functions on $\mathbb{C}^\times$	$\xi z = 0$	both axes

We think of the picture as having a horizontal z-axis and a vertical ξ-axis.

Let's be concrete and actually prove some things this time. Let A be a noncommutative graded algebra, $A = \bigcup_{i\in\mathbb{N}} A_i$, with $A_i \leqslant A_{i+1}$, $A_iA_j \subseteq A_{i+j}$. Then

$$\operatorname{gr} A := \bigoplus {}^{A_i}/_{A_{i-1}}.$$

This is the **associated graded algebra.** We impose an extra assumption here, namely that $\operatorname{gr} A$ is commutative.

Now suppose that $a, b \in \operatorname{gr} A$ homogeneous with $a \in A_i/A_{i-1}$, $b \in A_j/A_{j-1}$, with lifts $\overline{a} \in A_i$, $\overline{b} \in A_j$.

Then define the poisson bracket of a and b by

$$\{a, b\} = (\overline{a}\overline{b} - \overline{b}\overline{a}) + A_{i+j-2} \in {}^{A_{i+j-1}}/_{A_{i+j-2}}$$

The commutator $\overline{a}\overline{b} - \overline{b}\overline{a}$ is an element of A_{i+j-1} because $\operatorname{gr} A$ is commutative, so the terms in A_{i+j} cancel.

Definition 1.17. Given a $\mathcal{D}_M$-module $\mathcal{F}$, a **good (increasing) filtration** $\mathcal{F}_i$ is

(1) compatible with $(\mathcal{D}_M)_j$. Therefore, $\mathcal{O}_{T*M} = \operatorname{gr}\mathcal{D}_M \circlearrowright \operatorname{gr}\mathcal{F}$.

(2) For all $\operatorname{gr}\mathcal{F}$ coherent over T^*M.

For $\mathcal{D}$-modules, you should picture distributions on a submanifold valued in a vector bundle with connection.

Remark 1.18 (Theorems to Come)**.**

(1) $\operatorname{supp}(\operatorname{gr}\mathcal{F}) \subseteq T^*M$ is **coisotropic** (at its smooth points). There are two ways to explain what coisotropic means. First, if C is smooth and contained in S symplectic, then $(T_cC)^\perp \leqslant T_cC$. The second version is that if $I = \operatorname{ann}(\operatorname{gr}\mathcal{F})$, then $\{I, I\} \subseteq I$, that is, I is closed under the Poisson bracket.

(2) The **characteristic cycle** (often denoted ss for **singular support**), defined by
$$\sum_{\text{top-dim components } C \text{ of support}} \operatorname{mult}_C[C]$$
is independent of the choice of filtration. (This lives inside formal $\mathbb{Z}$-linear combinations of subvarieties of fixed dimension).

Definition 1.19. Let S be a symplectic manifold. $L \subseteq S$ is **Lagrangian** if it is coisotropic and $\dim L = \frac{1}{2}\dim S$.

Definition 1.20. A $\mathcal{D}$-module $\mathcal{F}$ is **holonomic** if the singular support $\operatorname{ss}(\mathcal{F})$ is Lagrangian, and not just coisotropic.

Definition 1.21. If $L \subseteq T^*M$ is Lagrangian, then it is **conical** if invariant under scaling the fibers of T^*M.

Example 1.22. The singular support of a $\mathcal{D}$-module is necessarily conical.

In $T^*\mathbb{R}$, only get the z-axis or translates of the ξ-axis (where the axes are as before in the three examples of $\mathcal{D}$-modules).

1.1 Stuff that has nothing to do with $\mathcal{D}$-modules

Definition 1.23. If $Y \subseteq M$ is smooth and locally closed (for example a curve without endpoints), the **conormal bundle** is
$$CY := \{(m, \vec{v}) \in T^*M \mid m \in Y, \vec{v} \perp T_mY\}.$$

Example 1.24.

(1) The conormal bundle of M is just the zero section.

(2) The conormal bundle to a point y is T^*_yM.

Remark 1.25 (Fun Fact)**.** The conormal bundle is automatically conical and Lagrangian.

The locally closed condition on Y is irritating to work with, especially in algebraic geometry.

Definition 1.26. If Y is closed and irreducible and M smooth, with $Y \subseteq M$, then the **conormal variety** is

$$CY = \overline{CY_{\text{reg}}}.$$

This is conical, Lagrangian, and irreducible.

Example 1.27. Let M be a vector space and Y a subspace. Then $T^*M \cong M \times M^*$ and $CY = Y \times Y^\perp$.

Lemma 1.28 (Arnol'd)**.** *Let $X \subseteq T^*M$ be conical, closed, Lagrangian and irreducible.*

(1) $M \hookrightarrow T^*M$ as the zero section and $\pi \colon T^*M \to M$. Then $X \cap M = \pi(X)$. We know that $X \cap M$ is closed and $\pi(X)$ is irreducible, so that tells us that $Y = \pi(X) = X \cap M$ is both closed and irreducible.

(2) $X = CY$.

Proof.

(1) $\pi(X) \supseteq \pi(X \cap M) = X \cap M$.

Conversely, $y \in \pi(X)$ implies that there is some $\vec{v}$, $(y, \vec{v}) \in X$. This in turn implies that for all $z \in \mathbb{C}^\times$, $(y, z\vec{v}) \in X$ because X conical. Hence, as $z \to 0$, $(y, \vec{0}) \in X$ because X closed. Hence, $y \in X \cap M$.

(2) Since $Y_{\text{reg}} \subseteq Y$ is open and dense in Y, define

$$X^\circ = \pi^{-1}(Y_{\text{reg}}),$$

this is open and dense in X, because X is irreducible. Now X° is Lagrangian and therefore isotropic, so X° is contained inside the conormal bundle CY_{reg} to Y_{reg}. Again because X is Lagrangian, these have the same dimension. And these are both irreducible, so they therefore have the same closure, namely X. Hence, X is the conormal variety to Y. □

1.2 Application: Projective duality

Let $Y \subseteq V$ be closed and irreducible, where V is a vector space. Therefore,

$$CY \subseteq T^*V \cong V \times V^* \cong T^*(V^*).$$

We know that V^* is conical, and we want to apply Arnol'd's Lemma to $T^*(V^*)$, but we don't have all the assumptions. We need to assume that $Y \subseteq V$ is already conical, that is, Y is the cone over $\mathbb{P}Y \subseteq \mathbb{P}V$.

Given this, Arnol'd tells us that we can define the projective dual

$$Y^\perp := CY \cap (0 \times V^*)$$

where 0 is the zero section. Then $CY = C(Y^\perp)$.

Remark 1.29 (Warning!)**.** If $Y_1 \subseteq Y_2$, then this doesn't imply *anything* about their duals.

If Y is a vector subspace of V, then the projective dual is just the usual orthogonal compliment $Y^\perp$.

Theorem 1.30. *Let $G \circlearrowright V$ with finitely many orbits, V a $\mathbb{C}$-vector space and G connected. Then $G \circlearrowright V^*$ with finitely many orbits, and there is a canonical bijection by projective duality.*

Proof. First observe that the orbits are automatically conical because G acts linearly and Schur's Lemma and all the usual representation theory stuff; $\mathbb{C}^\times \circlearrowright V/G$ is the trivial action. Then take the projective dual of the orbit closures.

(Note that by Remark 1.29, this need not preserve the poset structures!) □

Example 1.31. If $V = M_{m\times n}$ with $m \times m$ lower triangular matrices B_-^m acting on the left and $n \times n$ upper triangular matrices B_+^n acting on the right. This means that we are acting by downward row operations on the left, and acting by rightward column operations on the right.

So the orbits correspond to the $m \times n$ **partial permutation matrices**, with at most a single 1 in each row and column.

What do the orbits look like on the dual? We are going to identify $(M_{m\times n})^*$ with $M_{m\times n}$ via the inner product defined by trace, followed by transpose.

$$(M_{m\times n})^* \overset{\text{tr}}{\cong} M_{m\times n} \overset{\text{transpose}}{\cong} M_{n\times m}$$

Then $B_+^n \circlearrowright M_{m\times n} \circlearrowleft B_-^n$.

Remark 1.32. "I didn't have time to print things this morning; let's see how it goes."

Remark 1.33 (Recall)**.** Here's the situation we have for the support cycle. M is a smooth variety, and $\mathcal{D}_M$ is it's sheaf of differential operators, filtered by order. Then

$$\operatorname{gr} \mathcal{D}_M \cong \pi_*(\mathcal{O}_{T^*M})$$

where $\pi\colon T^*M \to M$ is projection. $\mathcal{F}$ is a finitely generated $\mathcal{D}$-module.

Example 1.34. $M = \mathbb{A}^1_{\mathbb{C}}$,

$$\mathcal{D} = {}^{\mathbb{C}[\widehat{z}, {}^{d}/_{dz}]}\Big/_{\langle [\widehat{z}, {}^{d}/_{dz}] - 1 \rangle}.$$

We have three examples of $\mathcal{D}$-modules $\mathcal{F}$: functions on $\mathbb{C}$, functions on $\mathbb{C}^\times$, and distributions supported at zero.

1.3 Rees Algebra

Definition 1.35. Given an algebra A with a positive, increasing filtration $1 \in A_0 \subseteq A_1 \subseteq \ldots$, the **Rees algebra** $\widehat{A}$ is defined by

$$\widehat{A} := \bigoplus_{n \in \mathbb{N}} A_n t^n.$$

The Rees algebra comes with a map $k[t] \to \widehat{A}$, where k is some base ring, given by $t \mapsto 1 \cdot t^1$. Moreover, $\widehat{A} \subseteq A[t]$. More generally, we will later have that $\widehat{A} \subseteq A[t, t^{-1}]$.

The Rees ring is interesting because it interpolates between the algebra A and it's associated graded algebra.

$$\widehat{A}\Big/_{\langle t - 1 \rangle} \cong A$$

$$\widehat{A}\Big/_{\langle t - 0 \rangle} \cong \operatorname{gr} A$$

To filter a finitely generated A-module F, pick generators $m_1, \ldots, m_g$ and integers $d_1, \ldots, d_g$ and define

$$F_i := \sum_{j=1}^{g} A_{i-d_j} m_j \tag{1}$$

where $A_i := A_0$ for $i < 0$.

Definition 1.36. The **Rees module** $\widehat{F}$ is the $\widehat{A}$-module defined by

$$\widehat{F} := \bigoplus_{i \in \mathbb{N}} F_i t^i,$$

where F_i is as in (1).

If F is finitely generated over A and we use the filtration from (1), then $\widehat{F}$ is also finitely generated as an $\widehat{A}$-module.

Localizing, we get $\widehat{A}_t \cong A[t^{\pm 1}]$, which acts on $\widehat{F}_t \cong F[t^{\pm 1}]$.

Definition 1.37. An $\widehat{A}$**-lattice** E is an $\widehat{A}$-submodule of a $\widehat{A}_t$-module C, such that the natural map $E \otimes_{\widehat{A}} \widehat{A}_t \to C$ is an isomorphism. (Think $C = \bigcup_{n \in \mathbb{N}} t^{-n} E$.)

Definition 1.38. Given an algebra B, let $K^+(B)$ be the monoid of formal $\mathbb{N}$-linear combinations of isomorphism classes of finitely generated B-modules, modulo short exact sequences.

An element of $K^+(B)$ is an isomorphism class $[F]$ of a B-modules F, and if $0 \to F_1 \to F_2 \to F_3 \to 0$ is a short exact sequence of B-modules, then $[F_2] = [F_1] + [F_2]$.

Remark 1.39. Let L, L' be two lattices in $\widehat{F}_t$. For B commutative, get a map from $K^+(B)$ to effective cycles (an effective cycle is a linear combination of subvarieties).

Theorem 1.40. *Let F be a finitely generated A-module, so $\widehat{F}$ is a finitely generated $\widehat{A}$-module, where $\widehat{F}$ defined via the filtration* (1).

Let L, L' be two lattices in $\widehat{F}_t$. Then $[L/tL] = [L'/tL']$ in $K^+(\widehat{A}/\langle t \rangle)$. This then gives a homomorphism $K^+(\widehat{A}_t) \to K^+(\widehat{A}/\langle t \rangle)$.

Proof. Let's do a special case first. Call L and L' **adjacent** if

$$L \geqslant L' \geqslant tL \geqslant tL'.$$

We then get several short exact sequences:

$$0 \longrightarrow {}^{L'}\!/_{tL} \longrightarrow {}^{L}\!/_{tL} \longrightarrow {}^{L}\!/_{L'} \longrightarrow 0$$
$$0 \longrightarrow {}^{tL}\!/_{tL'} \longrightarrow {}^{L'}\!/_{tL'} \longrightarrow {}^{L'}\!/_{L'} \longrightarrow 0$$

Then in $K^+(\widehat{A}/\langle t \rangle)$, we have

$$[L/tL] = [L'/tL] + [L/L'] = [L'/tL] + [tL/tL'] = [L'/tL']$$

where $L/L' \cong tL/tL'$ because t acts invertibly on $\widehat{F}_t$. This concludes the proof of the special case.

For the general case, let $L^j = L + t^j L'$. Then for some $j \gg 0$, we get $L^j = L$, and for some $j \ll 0$, we get $t^j L'$. Claim that L^j is adjacent to L^{j+1} (exercise: this is not too hard to see). Then the special case finishes it. $\square$

The situation we want to apply this to is that F is a finitely generated A-module, so $\widehat{F}$ is a finitely generated $\widehat{A}$-module. Then $\widehat{F}$ is a lattice in $\widehat{F}_t \cong F[t^{\pm 1}]$. So by the theorem, we see that

$$[\operatorname{gr} F] = [\widehat{F}/t\widehat{F}] \in K^+(\widehat{A}/\langle t \rangle) = K^+(\operatorname{gr} A)$$

is well-defined.

1.4 Back to representation theory

Given $G \circlearrowright M$, we have by differentiating a map $\mathfrak{g} \to \mathrm{Vec}(M)$. Hence, we get a map $U(\mathfrak{g}) \to \Gamma(\mathcal{D}_M)$.

Example 1.41. $G \circlearrowright G/B$, such as $\mathrm{GL}(n)/B = \{\text{flags in } \mathbb{C}^n\}$. So we have $U(\mathfrak{g}) \to \Gamma(\mathcal{D}_{G/B})$.

Later, we'll prove the following theorem.

Theorem 1.42 (Beilinson-Bernstein)**.**

(1) $U(\mathfrak{g})_0 \xrightarrow{\sim} \Gamma(\mathcal{D}_{G/B})$*, where* $U(\mathfrak{g})_\lambda = U(\mathfrak{g})/I$*, where* I *is the central character* λ*,*

$$I = \ker(U(\mathfrak{g}) \to \mathrm{End}(V_\lambda)) \cap Z(U(\mathfrak{g}))$$

(2) $H^i(\mathcal{D}_{G/B}) = 0$ *for* $i > 0$*.*

(3) There is an equivalence of categories between $U(\mathfrak{g})_0$-**mod** *and* $\mathcal{D}_{G/B}$-**mod**.

Definition 1.43. The **central character** λ is generated by those elements of $U(\mathfrak{g})$ that act by scalars on V_λ, in the same way as $Z(U(\mathfrak{g}))$.

Example 1.44. For (2), the center $Z(U((2)))$ is generated by $H^2 + XY + YX$ possibly with a coefficient in front of H^2?

On the irrep V_n, this generator acts as $n^2 + n$.

A is a filtered algebra with increasing filtration $A_0 \subseteq A_1 \subseteq \ldots$ with the property that $\mathrm{gr}\, A$ is commutative. M is a filtered left A-module, and therefore $\mathrm{gr}\, M$ is a $\mathrm{gr}\, A$-module. We write $\overline{m}$ for the image of $m \in M$ inside $\mathrm{gr}\, M$, and similarly for the image of $a \in A$ inside $\mathrm{gr}\, A$.

$$\mathrm{ann}_{\mathrm{gr}\, A}(\overline{m}) = \{\overline{a} \in \mathrm{gr}\, A \mid \overline{am} = 0\} \twoheadleftarrow \{a \in A_j \mid am_i \in M_{i+j-1}\}$$

The thing on the right looks somewhat like the annihilator of m in A, but it's not quite.

Let $a \in A_j$, $b \in A_k$. We have that

(1) $am_i \in M_{i+j-1}$

(2) $bm_i \in M_{i+k-1}$

(3) $[a, b] \in A_{i+j-1}$

These three facts imply that $[a, b]m_i \in M_{i+j+k-1}$. This gives that $[\overline{a}, \overline{b}] \in \mathrm{ann}_{\mathrm{gr}\, A}(\overline{m})$.

Remark 1.45. Note that the ideal $\mathrm{ann}_{\mathrm{gr}\, A}(\overline{m})$ may not be radical itself!

2 CSM Classes

Remark 2.1. What got me into teaching this class is thinking about Chern-Schwartz-MacPherson classes via $\mathcal{D}$-modules. But before I start with that, I should probably start with Chern classes. To do that, we'll start with Euler classes.

Definition 2.2. If $\pi\colon V \to M$ is an oriented real vector bundle over a smooth manifold M, then the **Euler class** $e(V)$ is the Poincaré dual of $\sigma^{-1}(0)$, where $\sigma\colon M \to V$ is a generic section of π.

So what is $\sigma^{-1}(0)$? This set measures our inability to move M away from itself. You should think about it as a self-intersection of M inside V.

Note that $\sigma^{-1}(0)$ is **cooriented** inside V. The normal bundle of $\sigma^{-1}(0)$ inside M is $N_M(\sigma^{-1}(0)) \cong \sigma^*(V)$.

If M is oriented, then $\sigma^{-1}(0)$ is oriented, so the normal bundle is as well. If M is compact as well, then $\sigma^{-1}(0)$ defines an element of the homology of M, $[\sigma^{-1}(0)] \in H_{\dim M - d}(M)$, where d is the dimension of the fibers of π.

Hence, by Poincaré duality, the Euler class $e(V)$ lives in $H^d(M)$.

If M is not compact, we can use Borel-Moore homology to define $H_*(M)$ with locally finite chains. (When you take the Poincaré dual of Borel-Moore homology, you nevertheless end up with ordinary cohomology.)

If M is not oriented, then we don't get an element $[\sigma^{-1}(0)]$ of $H_*(M)$, but instead some wacky twisted homology. But Poincaré duality undoes this also.

So we don't need to care if M is oriented or compact or whatnot, the Euler class is still defined.

Proposition 2.3. *The Euler class is natural. Given the commutative diagram,*

$$\begin{tikzcd} f^*V \arrow[r] \arrow[d] & V \arrow[d] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

we have that

$$e(f^*V) = f^*(e(V))$$

So e is a map from isomorphism classes of oriented vector bundles on M to cohomology $H^*(M)$. Both taking isomorphism classes of vector bundles and $H^*(-)$ are functors from the category of smooth manifolds to the category of sets, so $e(-)$ defines a natural transformation between the two functors:

$$e\colon F \implies H^*(-)$$

where F is the functor taking a manifold M to the isomorphism classes of oriented vector bundles on M.

Definition 2.4. Let $\mathrm{EO}(n)$ be the set of real $n \times \mathbb{N}$-matrices of rank n. This is the **Stiefel manifold**. This is contained in $\mathbb{R}^\infty \setminus \{\text{infinite codimension}\}$.

Let $\mathrm{BO}(n)$ be $\mathrm{EO}(n)$ modulo the left action of $\mathrm{GL}_n(\mathbb{R})$. This is the same as $\mathrm{Gr}_n(\mathbb{R}^\infty)$.

Fact 2.5. *The functor F that takes M to isomorphism classes of vector bundles on M is represented by $BO(n)$. This means that $F \cong \mathrm{Map}_{\text{homotopy}}(M, BO(n))$.*

Let's do this with my favorite vector bundles instead! The best oriented vector bundles are complex vector bundles, classified by $\mathrm{Gr}_n \mathbb{C}^\infty$.

If $f \colon M \to \mathrm{Gr}_n(\mathbb{C}^\infty)$ is the **classifying map**, then get

$$f^* \colon H^*(\mathrm{Gr}_n \mathbb{C}^\infty) \to H^*(M).$$

Fortunately, $H^*(\mathrm{Gr}_n \mathbb{C}^\infty; \mathbb{Z})$ is much nicer than the corresponding thing over $\mathbb{R}$.

$$H^*(\mathrm{Gr}_n \mathbb{C}^\infty; \mathbb{Z}) \cong \mathbb{Z}[c_1^{(2)}, c_2^{(4)}, \ldots, c_n^{(2n)}].$$

What are these c_i^{2i}? (They're called **Chern classes**).

Definition 2.6. If $S^1 \circlearrowright M$, then let $ES^1 = \mathbb{C}^\infty \setminus \{0\}$. This has an action of $S^1 = \{e^{i\theta}\}$. This is homotopic to the unit sphere in $\mathbb{C}^\infty$.

The S^1**-equivariant cohomology** is

$$H^*_{S^1}(M) := H^*((M \times ES^1)/(S^1)_\Delta),$$

where $(M \times ES^1)/(S^1)_\Delta$ is the quotient of $M \times ES^1$ by the diagonal action of S^1.

What does the space $(M \times ES^1)/(S^1)_\Delta$ look like? If we forget the space M, we get $ES^1/S^1 \cong \mathbb{CP}^\infty$.

If instead $V \to M$ is a real oriented vector bundle with an action of S^1, then we can define the **equivariant Euler class** $e_{S^1}(V)$, as the Euler class of the vector bundle

$$(V \times ES^1)/(S^1)_\Delta \longrightarrow (M \times ES^1)/(S^1)_\Delta.$$

Remark 2.7. What does Euler have to do with this? He says that if you have a map in the plane, then $V - E + F = 2$. So he's computed the Euler class of the disk. Then people do it in the plane, and from there move onto surfaces. And then someone does it for the tangent bundle and someone else for arbitrary vector bundles. And now it's equivariant. So the moral of the story is that it's good to get in early on these things.

Example 2.8. Special case: $S^1 \circlearrowright M$ trivially. Then

$$H^*_{S^1}(M) = H^*(M \times (ES^1/S^1)) \cong H^*(M) \otimes H^*(\mathbb{CP}^\infty) = H^*(M) \otimes \mathbb{Z}[\hbar],$$

by the Kunneth theorem.

Now if $V \to M$ is a $\mathbb{C}$-vector bundle, then it's an S^1-equivariant vector bundle with respect to the trivial action on M. Then

$$e_{S^1}(V) \in H^{2\dim_{\mathbb{C}} V}(M)[\hbar^{(2)}] = \sum_{i=0}^{\dim_{\mathbb{C}}(V)} c_{\dim_{\mathbb{C}} V - i}(V)\hbar^i.$$

These are called the **Chern classes**. They're derived from Euler classes.

Definition 2.9. The **total Chern class** is defined as

$$\mathbf{c}(V) = \sum_i c_i(V)$$

Proposition 2.10 (Properties of Chern Classes)**.**

(a) $c_0 = 1$

(b) $c_{\dim_{\mathbb{C}} V}(V) = e(V)$

(c) $\mathbf{c}(V \oplus W) = \mathbf{c}(V)\mathbf{c}(W)$

(d) $c_i(V^*) = (-1)^i c_i(V)$

If V is not isomorphic to a direct sum of line bundles, then consider

$$\begin{tikzcd} \pi^*(V) \arrow[r] \arrow[d] & V \arrow[d] \\ F(M) \arrow[r, two heads, "\pi"] & M \end{tikzcd}$$

where $F(M)$ is the **frame bundle** of $V \to M$, $F(M) = \{(m, \text{basis of } V|_M)\}$. We have

$$H^*(M) \hookrightarrow H^*(F(M))$$

So how are we going to use this to study $\mathcal{D}$-modules? Let M be a complex manifold. Let $\mathcal{F}$ be a $\mathcal{D}_M$-module. Recall that we defined $\mathrm{ss}(F) \subseteq T^*M$. Then

$$[\mathrm{ss}(\mathcal{F})] \in H^*_{S^1}(T^*M) \cong H^*_{S^1}(M) \cong H^*(M)[\hbar]$$

Example 2.11. Let $i\colon K \hookrightarrow M$ be smooth and compact (and complex). Then $\mathcal{D}_M$ acts on "distributions on K." This $\mathcal{D}_M$-module is called $i_*(\mathcal{O}_K)$. Then the singular support of $i_*(\mathcal{O}_K)$ is the conormal bundle $C_M K$ to K inside T^*M.

$$\mathrm{ss}(i_*(\mathcal{O}_K)) = C_M K$$

Now consider $i^*(T^*M \to M)$. This fits inside the following diagram

$$\begin{tikzcd} i^*(T^*M \to M) \arrow[r, hook, "\bar{i}"] \arrow[d] & T^*M \arrow[d, "\pi"] \\ K \arrow[r, hook, "i"] & M \end{tikzcd}$$

We want $[C_M K] \in H^*_{S^1}(T^*M)$. We can consider this class in the cohomology of $i^*(T^*M \to M)$ instead.

$$\begin{array}{ccccc} [C_M K \subseteq T^*M] & \in & H^*_{S^1}(T^*M) & & \\ \uparrow & & \uparrow & & \\ [C_M K \subseteq i^*(T^*M \to M)] & \in & H^*_{S^1}(i^*(T^*M \to M)) & \xrightarrow{\cong} & H^*_{S^1}(K) \end{array}$$

There is a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & C_M K \arrow[r] \arrow[dr] & i^*(T^*M) \arrow[r] \arrow[d] & T^*K \arrow[r] \arrow[dl] & 0 \\ & & K & & \end{tikzcd}$$

Then we get

$$[C_M K \subseteq i^*(T^*M \to M)] = e_{S^1}(T^*K)$$

so therefore

$$[C_M K \subseteq T^*M] = i_* e_{S^1}(T^*K)$$

What does this look like in the dumb case $K = M$? There's no i_*, so we just get Chern classes of M.

2.1 The Deligne-Grothendieck Conjecture

Definition 2.12. A **constructible function** on X is a function $X \to \mathbb{C}$ taking finitely many values such that each level set is a finite disjoint union of locally closed subsets.

Example 2.13. The function $\mathbb{C} \to \mathbb{C}$ that is constantly 1 except on $\{z \mid \operatorname{im} z \neq 0\}$, where it's zero.

So every constructible $f\colon X \to \mathbb{C}$ looks like

$$\sum_i c_i 1_{Y_i}$$

nonuniquely, where $c_i \in \mathbb{C}$ and 1_{Y_i} is the characteristic of some locally closed $Y_i \subseteq X$.

Let $\mathbf{C}$ be the category of varieties over $\mathbb{C}$ with proper maps. There is a functor $H_*\colon \mathbf{C} \to \mathbf{Ab}$, and another functor const : $\mathbf{C} \to \mathbf{Ab}$, defined as follows.

Definition 2.14. The functor const takes a variety to it's group of constructible functions.

And if $f\colon X \to X'$ and $Y \hookrightarrow X$ is locally closed, then

$$\operatorname{const}(f)\colon 1_Y \mapsto \left(x' \mapsto \chi_c(Y \cap f^{-1}(x'))\right)$$

where χ_c is compactly-supported Euler characteristic.

Example 2.15 (Key Special Case). If X' is a point and $Y = X$, $Z \subseteq X$ closed, then $Z, X \backslash Z$ are locally closed. So for well-definedness, we need

$$\chi_c(X) = \chi_c(Z) + \chi_c(X \backslash Z)$$

But this is true! (Proof to come).

Theorem 2.16 (Deligne-Grothendieck Conjecture, MacPherson's Theorem)**.** *There is a unique natural transformation* csm: const $\to H_*$ *such that for a smooth manifold M,*

$$1_M \longmapsto \left(\sum_i c_i(TM)\right) \cup [M]$$

(This normalization condition is so that not everything maps to zero, so csm is nontrivial.)

Proof. The easier part is uniqueness, which we will do now. There are labor-saving several steps.

(1) It's enough to deal with 1_Y for Y locally closed, by the additivity.

(2) It's enough to deal with 1_Y for Y smooth, since varieties are stratified by smooth varieties.

So now we have $Y \hookrightarrow X$, and $Y \hookrightarrow \overline{Y} \hookrightarrow X$. However, $\overline{Y}$ may not be smooth, so we pick a resolution $\widetilde{Y}$ of $\overline{Y}$ – the Hironaka resolution of singularities.

$$\begin{tikzcd}
 & Y \arrow[r, hook] \arrow[dl, hook, "\text{strict}"'] \arrow[d, hook] & X \\
\widetilde{Y} \arrow[r] & \overline{Y} \arrow[ur, hook] &
\end{tikzcd}$$

$\widetilde{Y} \backslash Y$ is the normal crossings divisor. This is locally diffeomorphic to the space $\mathbb{C}^k \times \left(\mathbb{C}^n \backslash (\mathbb{C}^\times)^n\right)$.

Along all of these maps, 1_Y maps to 1_Y.

$$\begin{tikzcd}
 & 1_Y \arrow[r, mapsto] \arrow[dl, mapsto] \arrow[d, mapsto] & 1_Y \\
1_Y \arrow[r, mapsto] & 1_Y \arrow[ur, mapsto] &
\end{tikzcd}$$

This was a stupid diagram. But the point is that we get $1_Y \in \mathrm{const}(\widetilde{Y})$. Let

$$\widetilde{Y} \backslash Y = \bigcup_{i \in I} E_i,$$

where the E_i are normal crossing divisors. To avoid stupid cases like when the E_i self-intersect, we blow up again to get the simple normal crossing divisors. Now we get

$$1_Y = \sum_{S \subseteq I} (-1)^{|S|} 1_{\bigcap_S E_i}$$

where we just take $\widetilde{Y}$ if $S = \varnothing$.

Hence, on $\widetilde{Y}$,

$$\mathrm{csm}_{\widetilde{Y}}(1_Y) = \sum_{S \subseteq I} (-1)^{|S|} \, \mathrm{csm}_{\widetilde{Y}}(1_{\bigcap_S E_i})$$

We can rewrite this as

$$\begin{aligned} \mathrm{csm}_{\widetilde{Y}}(1_Y) &= \sum_{S \subseteq I} (-1)^{|S|} (i^{\widetilde{Y}}_{\bigcap_S E_i})_* \, \mathrm{csm}_{\bigcap E_i}(1_{\bigcap_S E_i}) \\ &= \sum_{S \subseteq I} (-1)^{|S|} (i^{\widetilde{Y}}_{\bigcap_S E_i})_* \left(\sum c_i (TM \cap \bigcap_S E_i) \cup [\bigcap_S E_i] \right) \end{aligned}$$

Later we'll see that this calculation works independently of our choice of resolution of singularities. □

2.2 Toric Varieties

Definition 2.17. If $P \subseteq \mathbb{R}^n$ is a convex polytope with $\mathbb{Z}^n$-vertices, then it's **toric variety** is

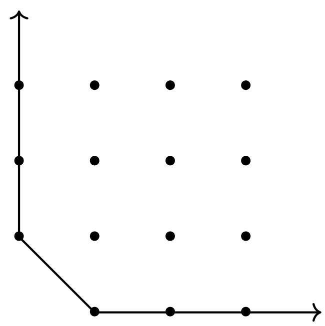
$$\mathrm{proj}\left(\mathbb{C}[\mathbb{Z}^{n+1} \cap \overline{\mathbb{R}_{\geqslant 0}(P \times \{1\})}] \right)$$

We take first $P \times \{1\} \subseteq \mathbb{R}^{n+1}$ if $P \subseteq \mathbb{R}^n$. We take the $\mathbb{R}_{\geqslant 0}$-linear combinations of this, and then the closure of that. Then intersecting it with $\mathbb{Z}^{n+1}$, we have a monoid M. Then take the monoid algebra $C[M]$ of this monoid, and then take proj of that.

Example 2.18. If $P = [0, 1]$, then $TV_P = \mathbb{CP}^1$. If P is a triangle in $\mathbb{R}^2$, then $TV_P = \mathbb{CP}^2$. If P is a square in $\mathbb{R}^2$ with vertices a, b, c, d, then

$$TV_P = \mathrm{proj}\left(\mathbb{C}[a, b, c, d] / \langle ad - bc \rangle \right) \cong \mathbb{CP}^1 \times \mathbb{CP}^1$$

Exercise 2.19. What do we get if P is the picture below?



To find this projective variety, first take the cone, which is all of the first quadrant. There are four generators, x at $(0, 1)$ and y at $(1, 0)$, and a and b the two vertices of the polytope. x and y are in degree zero, and a and b are in degree one, subject to the relation $ay - bx = 0$. So we get

$$\mathbb{C}[x, y, a, b] / \langle ay - bx \rangle.$$

2.3 CSM Classes on Toric Varieties

We still want the natural transformation csm: const $\to H_*$. We already saw uniqueness.

$$\begin{tikzcd} & A \arrow[r, hook, "\text{loc closed}"] \arrow[dl, hook, "\text{strict}"'] \arrow[d, hook, "\text{smooth}"] & X \\ \widetilde{A} \arrow[r, "\text{resolution}"'] & \overline{A} \arrow[ur, hook] & \end{tikzcd}$$

$$\widetilde{A} \backslash A = \bigcup_{i \in I} D_i$$

where D_i are simple normal crossing divisors. Then

$$\operatorname{csm}_{\widetilde{A}}(1_A) = \sum_{S \subseteq I} (-1)^S \operatorname{csm}\left(\bigcap_{i \in S} D_i\right)$$

These next two facts can be treated as black boxes, and in fact most algebraic geometers do so. They only hold over fields of characteristic zero.

Fact 2.20. *There is always such an $\widetilde{A}$ such that $\widetilde{A} \backslash A$ is a simple normal crossings divisor.*

Fact 2.21. *Given $\widetilde{A}_1, \widetilde{A}_2$, there is $\widetilde{A}_3 \twoheadrightarrow \widetilde{A}_1, \widetilde{A}_2$ such that we can build $\widetilde{A}_3$ from $\widetilde{A}_1$ (resp. $\widetilde{A}_2$) by successively blowing up along smooth "centers".*

Remark 2.22. We can associate to the simple normal crossing divisors a simplicial complex

$$\Delta(\widetilde{A}, \bigcup_I D_i),$$

called the **dual simplicial complex**, with vertex set I and $S \subseteq I$ is a face if and only if $\bigcap_S D_i \neq 0$.

Definition 2.23. The **log tangent bundle**

$$T(\widetilde{A} \cup D_i) \subseteq T\widetilde{A}.$$

is the vector fields tangent for all S to $\bigcap_S D_i$, on $\bigcap_S D_i$.

Example 2.24. If $\widetilde{A} = \mathbb{C}$, and $D_1 = \{0\}$, then

$$\Gamma(T\widetilde{A}) = \left\{ f(x)\frac{d}{dx} \right\} \qquad \mathcal{O}_{\widetilde{A}} \cdot \frac{d}{dx}$$

and

$$\Gamma(T(\widetilde{A}, D_1)) = \left\{ xf(x)\frac{d}{dx} \right\} \qquad \mathcal{O}_{\widetilde{A}} \cdot x\frac{d}{dx}$$

Definition 2.25. If $\widetilde{A} = \mathbb{C}^n$, $D_i = \{x_i = 0\}$, then $\Gamma(T(\widetilde{A}, \bigcup D_i))$ has an $\mathcal{O}_{\widetilde{A}}$-basis consisting of the $x_i{}^{d}/_{dx_i}$. Therefore this module is free, so it is the trivial vector bundle locally on general $\widetilde{A}$.

Now we have that

$$\mathrm{csm}_{\widetilde{A}}(1_A) = \sum_{S \subseteq I} (-1)^S \,\mathrm{csm}\left(\bigcap_{i \in S} D_i \right) = \sum c_i(T(\widetilde{A}, \bigcup D_i)) \frown [\widetilde{A}]$$

Now let's consider the case of toric varieties. Let $P \subseteq \mathbb{R}^n$ be a convex, compact polytope with vertices in $\mathbb{Z}^n$. We have an action of the torus $T = (\mathbb{C}^\times)^n$ on TV_P.

Remark 2.26. The orbits of this action correspond to faces of P. The way that we see this is that the orbit closures correspond to T-invariant subvarieties, which are then the faces of P.

Theorem 2.27 (Aluffi (maybe?))**.** *Let $T \cong (\mathbb{C}^\times)^n$ be the open torus orbit on TV_P. Then*

(a) $\mathrm{csm}_{TV_p}(1_T) = [TV_P] \in H_{2\dim P}(TV_p)$.

(b) $\mathrm{csm}_{TV_p}(1_{TV_p}) = \sum_{\text{faces } F \subseteq P} [TV_F \subseteq TV_P]$.

Proof of 2.27(a). The first case we will consider is $P = [0, \infty)$. To compute this toric variety, move the half-line up to level 1 and then take the cone, getting a quarter plane. This shape is generated by x in degree zero and a in degree 1, so

$$TV_P = \operatorname{proj} \mathbb{C}[x, a] \cong \mathbb{C}.$$

Therefore,

$$\operatorname{csm}_{\mathbb{C}}(1_{\mathbb{C}^\times}) = \operatorname{csm}(1_{\mathbb{C}}) - \operatorname{csm}(1_{\{0\}}) = ([\mathbb{C}] + [\{0\}]) - [\{0\}] = [\mathbb{C}] = [TV_P]$$

This lives inside the $\mathbb{C}^\times$-equivariant homology of the toric variety $H_*^{S^1}(TV_P)$ (see below).

Now let's consider the case of $(\mathbb{C}^\times)^n \hookrightarrow \mathbb{C}^n$. In this case, the CSM class is the total chern class of the log tangent bundle $T(\mathbb{C}^n, \mathbb{C}^n \backslash (\mathbb{C}^\times)^n)$. So

$$\begin{aligned}
\operatorname{csm}(1_{(\mathbb{C}^\times)^n}) &= \text{total Chern class}\left(T(\mathbb{C}^n, \mathbb{C}^n \backslash (\mathbb{C}^\times)^n\right) \\
&= \text{total Chern class}\left(\bigoplus_{i=1}^n T(\mathbb{C}, \mathbb{C} \backslash \mathbb{C}^\times)\right) \\
&= \sum_{S \subseteq [n]} (-1)^{|S|} \left(\text{total Chern class}(T\mathbb{C}^{[n]\backslash S}) \cup [\mathbb{C}^{[n]\backslash S}]\right) \\
&= \sum_{S \subseteq [n]} (-1)^{|S|} \left(\prod_{i \in S} \text{total Chern class}(T\mathbb{C}^i) \cup [\mathbb{C}^S]\right) \\
&= \sum_{S \subseteq [n]} (-1)^{|S|} \left(\prod_{i \in S} (1 + [0 \in \mathbb{C}^i]) \cup [\mathbb{C}^S]\right) \\
&= \sum_{S \subseteq [n]} (-1)^{|S|} \left(\sum_{R \subseteq S} [\mathbb{C}^R]\right) \\
&= \sum_{R \subseteq [n]} [\mathbb{C}^R] \sum_{S \supseteq R, S \subseteq [n]} (-1)^{|S|} \\
&= \sum_{R \subseteq [n]} [\mathbb{C}^R] (1-1)^{|[n]-R|} \\
&= [\mathbb{C}^n]
\end{aligned}$$

The next case is when TV_P is smooth. Then the previous case applies near each fixed point of the torus action. The fun thing is that the equivariant cohomology of this toric variety has an injective map

$$H^*_{(\mathbb{C}^\times)^n}(TV_p) \hookrightarrow H^*_{\mathbb{C}^n}\left(\coprod_{\text{corners of } P} \mathbb{C}^n \text{ nbhds}\right)$$

when P is compact.

So finally, what if the toric variety isn't smooth? Blow it up, and then apply what we have. This concludes the proof of Theorem 2.27(a). □

Remark 2.28 (Aside on equivariant homology)**.** What is the S^1 equivariant homology of a space M? Recall the Borel construction where we took $(M \times \mathbb{C}^\infty \backslash \{0\})/S^1$ and took the cohomology to get S^1-equivariant cohomology.

To get homology instead, consider $(M \times (\mathbb{C}^N \backslash \{0\}))/S^1$ inside $(M \times \mathbb{C}^\infty \backslash \{0\})/S^1$. Then we say that the S^1-equivariant homology is

$$H_*^{S^1}(M) := H_{*+2N}\left((M \times (\mathbb{C}^N \backslash \{0\}))/S^1\right) \text{ as } N \to \infty.$$

There's a theorem that says this is eventually stable, so well-defined.

In the case that M is smooth and compact of dimension n, then the homology and cohomology only exist in dimensions between 0 and n. The two are related by Poincaré duality. The equivariant cohomology goes up forever starting with dimension zero, and equivariant homology goes down forever starting with dimension n. Again, there is an action of (equivariant) cohomology on (equivariant) homology.

Example 2.29. For TV_P smooth, let's compute $\mathrm{csm}_{TV_P}(1_{TV_P})$. This is

$$\sum c_i(T(TV_P)) \cup [TV_P].$$

In degree zero, we get

$$[(TV_P)^T] = \sum_{\text{vertices of } P} [TV_v] = c_{\text{top}}(T(TV_P)) \cup [TV_P] = \chi(TV_P)$$

we also know that $c_{\text{top}}(T(TV_P))$ is $\dim_{\mathbb{C}}(T(TV_P)) = \dim_{\mathbb{R}} P$.

2.4 Independence for Deligne-Grothendieck

We still want the natural transformation $\mathrm{csm}\colon \text{const} \to H_*$. We already saw uniqueness in Section 2.1.

$$\begin{tikzcd}
 & A \arrow[r, hook, "\text{loc closed}"] \arrow[d, hook, "\text{smooth}"] \arrow[dl, hook, "\text{strict}"'] & X \\
\widetilde{A} \arrow[r, "\text{resolution}"'] & \overline{A} \arrow[ur, hook] &
\end{tikzcd}$$

$$\widetilde{A} \backslash A = \bigcup_{i \in I} D_i$$

where D_i are simple normal crossing divisors. Then

$$\operatorname{csm}_{\widetilde{A}}(1_A) = \sum_{S \subseteq I} (-1)^S \operatorname{csm}\left(\bigcap_{i \in S} D_i\right)$$

These next three facts can be treated as black boxes, and in fact most algebraic geometers do so. They only hold over fields of characteristic zero.

Fact 2.30 (Hironaka)**.** *There is always such an $\widetilde{A}$ such that $\widetilde{A} \backslash A$ is a simple normal crossings divisor.*

Fact 2.31 (Hironaka)**.** *Given $\widetilde{A}_1$, $\widetilde{A}_2$, there is $\widetilde{A}_3 \twoheadrightarrow \widetilde{A}_1, \widetilde{A}_2$ such that we can build $\widetilde{A}_3$ from $\widetilde{A}_1$ (resp. $\widetilde{A}_2$) by successively blowing up along smooth "centers".*

Fact 2.32 (Hironaka, "simultaneous resolution")**.** *If $B \subseteq A$ smooth, then there are simultaneous resolutions $\widetilde{B}$ and $\widetilde{A}$ of B and A, respectively, such that $\widetilde{B} \hookrightarrow \widetilde{A}$.*

Remark 2.33. Hironaka is a national treasure of Japan. Like buildings can be national monuments in the US, apparently people can be national treasures in Japan.

We now have enough to prove that the definition of csm is independent of the choice of $\widetilde{A}$.

Proof of independence for Theorem 2.16. It's enough to check that if $B \subseteq \widetilde{A} \backslash A$ is smooth and irreducible, and $\widetilde{\widetilde{A}}$ is the blowup of $\widetilde{A}$ along B, then $\widetilde{A}$, $\widetilde{\widetilde{A}}$ give the same csm_{1_A}.

Locally, we have that if $\widetilde{A} = \mathbb{C}^n$, and $A = (\mathbb{C}^\times)^n$, then B is contained in a coordinate hyperplane in $\mathbb{C}^n$ times some irrelevant $\mathbb{C}^k$.

The inclusion-exclusion of hyperplanes that don't contain B is the same in $\widetilde{A}$, $\widetilde{\widetilde{A}}$. So this allows us to reduce to the case that $A = \mathbb{C}^\times \times \mathbb{C}^{n-1}$.

So now, locally B is a point contained in $\widetilde{A} \backslash A = \mathbb{C}^m$. Then $\widetilde{A} = \mathbb{C}^{m+1}$ and $A = \mathbb{C}^\times \times \mathbb{C}^m$. This is just the toric case, where we know the answer, which is the sum of the classes of the faces not on $\widetilde{A} \backslash A$.

We still need to check the additivity of this recipe. We have $B \subseteq A \subseteq M$ all smooth. Then we want

$$\operatorname{csm}_M(1_A) = \operatorname{csm}_M(1_B) + \operatorname{csm}_M(1_{A \backslash B}).$$

To show this, we can use another Hiornaka fact on simultaneous resolution of singularities. So locally near a point of B, it looks like $\mathbb{C}^n \supseteq \mathbb{C}^k$. This again reduces to the toric case. □

This is the last part we needed for the proof of the Deligne-Grothendieck conjecture (Theorem 2.16).

2.5 Bott-Samelson Manifolds

Theorem 2.34 (Ginzburg 1986, to be proved later)**.** *If $i\colon A \hookrightarrow M$ is locally closed, A, M both smooth. We know that*

$$[\mathrm{ss}(i_*\mathcal{O}_A)] \in H^*_{S^1}(T^*M) = H^*_{S^1}(M) = H^*_{S^1}(M) = H^*(M)[\hbar],$$

but when we take $\hbar \mapsto -1$, we get

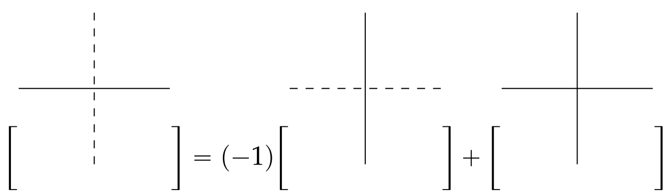
$$[\mathrm{ss}(i_*\mathcal{O}_A)]_{\hbar\mapsto -1} \cup [M] = (-1)^{\operatorname{codim} A}\,\mathrm{csm}_M(1_A)$$

Example 2.35. If $A = \mathbb{C}$ and $B = \{0\}$. Let $i\colon C^\times \to C$ and $j\colon \{0\} \to \mathbb{C}$. We have

$$\begin{aligned} \mathcal{O}_{\mathbb{C}} &= \langle 1\rangle = \mathcal{D}_A/\langle \tfrac{d}{dz}\rangle \\ i_*\mathcal{O}_{\mathbb{C}^\times} &= \langle z^{-1}\rangle = \mathcal{D}_A/\langle \tfrac{d}{dz}\hat{z}\rangle \\ j_*\mathcal{O}_{\{0\}} &= \langle \delta\rangle = \mathcal{D}_A/\langle \hat{z}\rangle \end{aligned}$$

Recall that, by taking associated graded rings, we pictured these $\mathcal{D}$-modules by looking at the axes in 2-d space with axes z and ξ. We have that

$$\mathrm{csm}(\mathbb{C}) = \mathrm{csm}(0) + \mathrm{csm}(\mathbb{C}^\times).$$



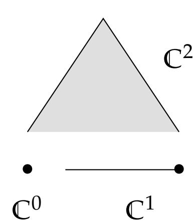
Conjecture 2.36. *If $X_0^W := BwP/P \subseteq G/P$, then*

$$\mathrm{csm}_{G/P}(1_{X_0^W}) = \sum_{v\in W/W_p} d_v[X^v].$$

Therefore $d_v \geqslant 0$.

Theorem 2.37 (Huh)**.** *Conjecture 2.36 holds on Grassmannians.*

Example 2.38. If $TV_P = \mathbb{CP}^2$, then the polytope P decomposes like



where $\mathbb{C}^0$ is the lower left vertex, $\mathbb{C}^1$ is the lower edge minus the lower left vertex, and $\mathbb{C}^2$ is the rest of the triangle.

Then

$$\begin{aligned}
\operatorname{csm}(\mathbb{C}^0) &= [\mathbb{CP}^0] \\
\operatorname{csm}(\mathbb{C}^1) &= [\mathbb{CP}^1] + [\mathbb{CP}^0] \\
\operatorname{csm}(\mathbb{C}^2) &= [\mathbb{CP}^2] + 2[\mathbb{CP}^1] + [\mathbb{CP}^0]
\end{aligned}$$

Example 2.39. If we ignore P, consider only $B \subseteq G$, then

$$\mathrm{GL}(n, \mathbb{C}) = \coprod_{w \in S_n} BwB$$

where the first B is upward row operations and the second is rightward column operations, w a permutation matrix.

To determine w in advance, given a matrix, look at the ranks.

Definition 2.40.

$$P \times^B Q = P \times Q / \sim$$

where $\sim$ is the equivalence relation $(p, q) \sim (pb^{-1}, bq)$ for all $b \in B$.

Definition 2.41. For G a lie group, $Q = i_1, \ldots, i_k$ a list of simple roots of G, the **Bott-Samelson** manifold is

$$\mathrm{BS}^Q = \left(P_{i_1} \times^B P_{i_2} \times^B \ldots \times^B P_{i_k}\right) \big/ B$$

The Bott-Samelson comes with a map to G/B.

$$\begin{tikzcd}
\mathbb{P}^1 \cong P_{i_k}/B \arrow[r] & \mathrm{BS}^Q \arrow[d] \\
& \mathrm{BS}^{Q \setminus \{i_k\}}
\end{tikzcd}$$

We can show that $\mathrm{BS}^{Q\setminus\{i_k\}}$ is smooth irreducible and proper by induction, so therefore BS^Q is as well.

Multiplication m is B-equivariant, so therefore $m(\mathrm{BS}^Q)$ is B-invariant, closed, and irreducible.

3 Derived Categories

3.1 General remarks on Localizations

Let **A** be an abelian category, for example $\mathbb{R}$-**mod**, or sheaves on a space X, or quasi-coherent sheaves on X, or coherent sheaves on X.

It's sometimes natural to consider the category of complexes on **A**, which we write as

$$\mathbf{Coh}(\mathbf{A}) = \{\cdots \to A^i \xrightarrow{d^i} A^{i+1} \to \cdots\}$$

We really care about cohomology of these complexes, not the complexes themselves.

We would like to pretend that any map of complexes $f\colon A^\bullet \to B^\bullet$ such that $f_*\colon H^\bullet(A) \xrightarrow{\sim} H^\bullet(B)$ is an isomorphism.

Definition 3.1. If $f\colon A^\bullet \to B^\bullet$ is such that f_* is an isomorphism on cohomology, then f is called a **quasi-isomorphism.**

We want to pretend that all quasi-isomorphisms in $\mathbf{Com}(\mathbf{A})$ are isomorphisms.

Definition 3.2. Suppose that **C** is a category and S a collection of morphisms in **C**. Then the **localization of C at** S is a category $\mathbf{C}[S^{-1}]$ with a functor $\gamma\colon \mathbf{C} \to \mathbf{C}[S^{-1}]$ such that all morphisms in S are sent by γ to an isomorphism in $\mathbf{C}[S^{-1}]$. Moreover, $\mathbf{C}[S^{-1}]$ must be **universal** among such categories: for any **D** and $\alpha\colon \mathbf{C} \to \mathbf{D}$ such that for $s \in S$, $\alpha(s)$ is an isomorphism in **D**, then

$$\begin{tikzcd} \mathbf{C} \arrow[rr, "\gamma"] \arrow[dr, "\alpha"'] & & \mathbf{C}[S^{-1}] \arrow[dl, dashed, "\exists!"] \\ & \mathbf{D} & \end{tikzcd}$$

Remark 3.3. Under mild assumptions, the localization always exits, and the objects of $\mathbf{C}[S^{-1}]$ are the objects of **C**, and the morphisms of $\mathbf{C}[S^{-1}]$ are "chains of roofs,"

$$\begin{tikzcd}[column sep=small] & Z_1 & & Z_2 & & \cdots & & Z_n & \\ X \arrow[ur, "f_1"] & & Y_1 \arrow[ul, "s_1"'] \arrow[ur, "f_2"] & & Y_2 \arrow[ul, "s_2"'] \arrow[ur, "f_2"] & & \cdots & & Y_n \arrow[ul, "s_n"'] \end{tikzcd}$$

with $s_i \in S$ and f_i any morphism in **C**.

Definition 3.4. If **A** is abelian, then $\mathbf{D}(\mathbf{A}) = \mathbf{Com}(\mathbf{A})[\mathrm{Qis}^{-1}]$ is the **derived category** of **A**. That is, in the category of complexes of **A**, we pretend that all quasi-isomorphisms are invertible.

The problem is that it's hard to say anything about these categories. So we will think about triangulated categories.

3.2 Triangulated Categories

The point of triangulated categories is to have localization in a much more manageable way.

Definition 3.5. An additive category $\mathbf{T}$ is **triangulated** if it has

(i) There is a functor $[1]\colon \mathbf{T} \to \mathbf{T}$ called the **degree-shift functor.** We often write the application of the functor $[1]$ as $X \mapsto X[1]$.

(ii) A class $\mathcal{E}$ of **distinguished triangles**, that is, diagrams
$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1].$$

Satisfying the following axioms (due to Verdier):

$\boxed{TR1}$ (a) $X \xrightarrow{\text{id}} X \to 0 \to X[1]$ is in $\mathcal{E}$

(b) $\mathcal{E}$ is closed under isomorphisms.

(c) For all $u\colon X \to Y$, there are v and w such that $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ is in $\mathcal{E}$.

$\boxed{TR2}$ If $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ is in $\mathcal{E}$, then $Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$ is in $\mathcal{E}$.

$\boxed{TR3}$ Given a diagram
$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, dashed, "\exists h"] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "v'"] & X'[1]
\end{tikzcd}$$

There is some h that fits in the diagram as shown. *Warning*! h may not be unique.

$\boxed{TR4}$ The **octahedral axiom.** It's annoying to state, very messy, and rarely used, so we will ignore it for now.

Proposition 3.6. *Let* $\mathbf{T}$ *be a triangulated category. For any* $U \in \mathbf{T}$*,* $\operatorname{Hom}(U,-)$ *applied to any distinguished triangle* $X \to Y \to Z \to X[1]$ *gives a long exact sequence of abelian groups.*

$$\cdots \to \operatorname{Hom}(U, Z[-1]) \to \operatorname{Hom}(U, X) \to \operatorname{Hom}(U, Y) \to \operatorname{Hom}(U, Z) \to \operatorname{Hom}(U, Z[1]) \to \cdots$$

Corollary 3.7 (The Five Lemma)**.** *If the maps* f, g *in the diagram below are isomorphisms, then* h *is an isomorphism as well.*

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "v'"] & X'[1]
\end{tikzcd}$$

Corollary 3.8. *For any $u\colon X \to Y$, the object Z completing the triangle $X \to Y \to Z \to X[1]$ from axiom $\boxed{\text{TR1}(c)}$ is unique up to isomorphism (but* not *unique isomorphism).*

Remark 3.9. We can define the "cone of the map $u\colon X \to Y$" to be the object Z in Corollary 3.8.

Corollary 3.10. *If $X \xrightarrow{u} Y \xrightarrow{v} \mathbb{Z} \xrightarrow{w} X[1]$ is in $\mathcal{E}$, then $vu = 0$, $wv = 0$, and $u[1]w = 0$.*

Remark 3.11. I'm really sorry that I'm not proving anything, but the proofs are not very revealing.

3.3 Homotopy Categories

Definition 3.12. Given an abelian category $\mathbf{A}$, consider $\mathbf{Com}(\mathbf{A})$. We say that $f, g\colon A^\bullet \to B^\bullet$ are **homotopic** if there is some $h\colon A^\bullet \to B^{\bullet-1}$ such that $f - g = d_B \circ h + h \circ d_A$.

Definition 3.13. The **homotopy category** of $\mathbf{A}$ is the category $\mathbf{H}(\mathbf{A})$ whose objects are complexes and morphisms between $A^\bullet$ and $B^\bullet$ are morphisms $A^\bullet \to B^\bullet$ in $\mathbf{Com}(\mathbf{A})$ modulo homotopy equivalence.

$$\mathrm{Hom}_{\mathbf{H}(\mathbf{A})}(A^\bullet, B^\bullet) = \mathrm{Hom}_{\mathbf{Com}(\mathbf{A})}(A^\bullet, B^\bullet)\big/\text{homotopy equivalence}.$$

Remark 3.14. We sometimes consider instead only those complexes bounded below, or which vanish in high positive degree, or which vanish in high negative degree. We denote these by $\mathbf{H}^b(\mathbf{A})$ or $\mathbf{H}^+(\mathbf{A})$ or $\mathbf{H}^-(\mathbf{A})$, respectively. If a fact holds for any of these cases, we refer to one of them generically by $\mathbf{H}^*(\mathbf{A})$.

Theorem 3.15. *$\mathbf{H}^*(\mathbf{A})$ is triangulated.*

Definition 3.16. Given $f\colon A^\bullet \to B^\bullet$, the **cone** of f is the complex with $\mathrm{cone}(f)^i = B^i \oplus A^{i+1}$ and differential

$$d_{\text{cone}} = \begin{bmatrix} d_B & f \\ 0 & -d_A \end{bmatrix}$$

Remark 3.17. For $f\colon A^\bullet \to B^\bullet$,

$$0 \to B^\bullet \hookrightarrow \mathrm{cone}(f) \twoheadrightarrow A^\bullet[1] \to 0$$

is exact.

Proof sketch of Theorem 3.15. Let $[1]\colon \mathbf{Com}(\mathbf{A}) \to \mathbf{Com}(\mathbf{A})$ be the usual degree shift functor on complexes, $A[1]^i = A^{i+1}$.

We say that the **standard distinguished triangles** of $\mathbf{H}^*(\mathbf{A})$ are of the form

$$A^\bullet \xrightarrow{f} B^\bullet \to \operatorname{cone}(f) \to A[1]$$

And then we say that the distinguished triangles of $\mathbf{H}^*(\mathbf{A})$ are the triangles isomorphic in $\mathbf{H}^*(\mathbf{A})$ to the standard ones.

Then we can check the axioms $\boxed{TR1}$ – $\boxed{TR4}$ via a long and annoying diagram chase. $\square$

Proposition 3.18. *A map $f\colon A^\bullet \to B^\bullet$ is a quasi-isomorphism if and only if* $\operatorname{cone}(f)$ *is* ***acyclic*** *(having zero cohomology).*

3.4 Verdier Quotients and Derived Categories

Suppose that $\mathbf{T}$ is a triangulated category and $\mathbf{N} \subset \mathbf{T}$ is a triangulated subcategory.

Lemma 3.19. *If $\mathbf{N} \subseteq \mathbf{T}$ is a subcategory that is both full and closed under isomorphisms, then $\mathbf{N}$ is a triangulated subcategory if and only if $\mathbf{N}$ is closed under $[1]$ and taking cones of morphisms in $\mathbf{N}$.*

Definition 3.20. The **Verdier Quotient $\mathbf{T}/\mathbf{N}$** is the category with objects the same as those in $\mathbf{T}$, and morphisms are roofs

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

such that $\operatorname{cone}(s) \in \mathbf{N}$ and modulo equivalence $\sim$, where we say that two roofs

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "f"] & & & X'' \arrow[dl] \arrow[dr] & \\ X & & Y & X & & Y \end{tikzcd}$$

and

are equivalent if there is a taller roof $X \leftarrow X''' \to Y$ that covers both. That is, there are arrows $X''' \to X''$ and $X''' \to X'$ such that

$$\begin{tikzcd} & X' \arrow[dl] \arrow[dr] & \\ X & X''' \arrow[l] \arrow[r] \arrow[u] \arrow[d] & Y \\ & X'' \arrow[ul] \arrow[ur] & \end{tikzcd}$$

commutes.

Proposition 3.21 (Universal Property of $\mathbf{T}/\mathbf{N}$)**.** $\mathbf{T}/\mathbf{N}$ *is universal among triangulated categories with* $Q\colon \mathbf{T} \to \mathbf{T}/\mathbf{N}$ *such that* Q *sends everything in* $\mathbf{N}$ *to zero.*

Fact 3.22. *Let* $S_{\mathbf{N}} = \{f\colon X \to Y \mid \operatorname{cone}(f) \in \mathbf{N}\}$. *Then* $\mathbf{T}/\mathbf{N} \cong \mathbf{T}[S_{\mathbf{N}}^{-1}]$.

Example 3.23. $\mathbf{H}^*(\mathbf{A}) \supset \operatorname{Acyclic}(\mathbf{A})$, which is the full subcategory of acyclic complexes.

Since $\operatorname{Acyclic}(\mathbf{A})$ is closed under shifts and taking cones, it is actually a triangulated subcategory of $\mathbf{H}^*(\mathbf{A})$.

Then

$$\begin{aligned} S_{\mathbf{N}} &= \{f\colon A^\bullet \to B^\bullet \mid \operatorname{cone}(f) \text{ is acyclic}\} \\ &= \{f\colon A^\bullet \to B^\bullet \mid f \text{ is quasi-iso}\} \end{aligned}$$

So Fact 3.22 implies that

$$\mathbf{H}^*(\mathbf{A})\big/_{\operatorname{Acyclic}(\mathbf{A})} \simeq \mathbf{H}^*(\mathbf{A})[\mathrm{Qis}^{-1}] \simeq \mathbf{Com}^*(\mathbf{A})[\mathrm{Qis}^{-1}] = \mathbf{D}^*(\mathbf{A})$$

So we have that $\mathbf{D}^*(\mathbf{A})$ is triangulated, and we have an explicit description of the shift, the cone, etc.

3.5 Derived Functors and $\mathbf{D}^b(\mathbf{Coh}(X))$

We do algebraic geometry, so we care about the derived category of bounded complexes on the category of coherent sheaves of X.

What are the functors we may want to consider on sheaves? Given $f\colon X \to Y$, there are functors f_*, f^*, and also there are functors $\mathbf{Hom}$, Γ, $\otimes$, etc.

If we have a functor $F\colon \mathbf{A} \to \mathbf{B}$, (e.g. $f^*\colon \mathbf{Coh}(Y) \to \mathbf{Coh}(X)$), when does this descend to a functor on derived categories?

$$\begin{tikzcd} \mathbf{Com}(\mathbf{A}) \arrow[r, "F"] \arrow[d, "\gamma"] & \mathbf{Com}(\mathbf{B}) \arrow[d, "\gamma"] \\ \mathbf{D}^b(\mathbf{A}) \arrow[r, dashed] & \mathbf{D}^b(\mathbf{B}) \end{tikzcd}$$

This almost never happens. We almost never have a functor that descends to the derived categories.

The solution to this is derived functors.

Definition 3.24. If F is right exact, then there is a functor $LF\colon \mathbf{D}^*(\mathbf{A}) \to \mathbf{D}^*(\mathbf{B})$, called the **left-derived functor**.

To compute $LF(A)$, for an object $A \in \mathbf{A}$, then we need to find a projective resolution $P^\bullet$ of A and compute $F(P^\bullet)$.

Definition 3.25. If F is left exact, then there is a functor $RF\colon \mathbf{D}^*(\mathbf{A}) \to \mathbf{D}^*(\mathbf{B})$, called the **right-derived functor**.

To compute $RF(A)$, we need to find an injective resolution $I^\bullet$ of $A \in \mathbf{A}$ and then compute $F(I^\bullet)$.

3.6 Derived Categories of Sheaves

Let's go back to the case of coherent/quasi-coherent sheaves. If $f\colon X \to Y$, then we may associate the pullback functor $f^*\colon \mathbf{Coh}(Y) \to \mathbf{Coh}(X)$. If f is proper, then we also have a pushforward $f_*\colon \mathbf{Coh}(X) \to \mathbf{Coh}(Y)$. Moreover, given a sheaf $\mathcal{F}$ on X, there is a functor $\mathcal{F} \otimes -\colon \mathbf{Coh}(X) \to \mathbf{Coh}(X)$. We may also have $\mathrm{Hom}(\mathcal{F}, -)\colon \mathbf{Coh}(X) \to \mathbf{Ab}$. We also have sheafy hom $\mathbf{Hom}(\mathcal{F}, -)\colon \mathbf{Coh}(X) \to \mathbf{Coh}(X)$.

Functor	Domain	Codomain	Exact on the ...
f^*	$\mathbf{Coh}(Y)$	$\mathbf{Coh}(X)$	right
f_*	$\mathbf{Coh}(X)$	$\mathbf{Coh}(Y)$	left
$\mathcal{F} \otimes -$	$\mathbf{Coh}(X)$	$\mathbf{Coh}(X)$	right
$\mathrm{Hom}(\mathcal{F}, -)$	$\mathbf{Coh}(X)$	$\mathbf{Ab}$	left
$\mathbf{Hom}(\mathcal{F}, -)$	$\mathbf{Coh}(X)$	$\mathbf{Coh}(X)$	left

Let's consider the case of $\mathcal{F} \otimes -$. We can construct the left-derived functor

$$\mathcal{F} \otimes^L -\colon \mathbf{D}^b(\mathbf{Coh}(X)) \to \mathbf{D}^b(\mathbf{Coh}(X))$$

by choosing for any other sheaf $\mathcal{G}$ a projective resolution $P^\bullet_{\mathcal{G}} \twoheadrightarrow \mathcal{G}$, and then

$$\mathcal{F} \otimes^L \mathcal{G} := \mathcal{F} \otimes P^\bullet_{\mathcal{G}}.$$

Then we can recover the classical derived functors via

$$H^i(\mathcal{F} \otimes^L \mathcal{G}) = \mathrm{Tor}^i(\mathcal{F}, \mathcal{G}).$$

We have to be careful in the case where $\mathbf{Coh}(X)$ doesn't have enough projectives. But we can use other sheaves to compute, for example, for $\otimes^L$ we can use locally free sheaves.

Example 3.26. What is the derived category of the space $X = \{\mathrm{pt}\}$? The only complexes on X are the ones where there is a single nonzero element, so this category is the one generated by complexes $\ldots \to 0 \to k \to 0 \to \ldots$, where k is the field.

?

Proposition 3.27 (Push-Pull). *Suppose that $f\colon X \to Y$ and $\mathcal{E} \in \mathbf{D}^b(\mathbf{Coh}(X))$ and $\mathcal{F} \in \mathbf{D}^b(\mathbf{Coh}(Y))$. Then*

$$Rf_*(Lf^*\mathcal{F} \otimes^L \mathcal{E}) \simeq Rf_*\mathcal{E} \otimes^L \mathcal{F}.$$

Proposition 3.28 (Flat Base Change). *Suppose we have the following diagram of spaces and maps.*

$$\begin{tikzcd} X \times Z \arrow[r, "v"] \arrow[d, tail, "g"] & X \arrow[d, "f"] \\ Z \arrow[r, "u"] & Y \end{tikzcd}$$

If u is flat, then

$$Rg_* \circ v^* = u^* \circ Rf_*.$$

The punchline to this is that derived categories allow us to package things nicely. Ordinarily these facts would need spectral sequences or something, but we don't need that here!

3.7 Bondal-Orlov Theorem

Remark 3.29. Given the category $\mathbf{D}^b(\mathbf{Coh}(X))$, how much can we say about X? Can we recover the scheme from its derived category of coherent sheaves?

Here's an example in the case of quasi-coherent sheaves where we can recover the scheme from the category.

Theorem 3.30 (Rosenberg). *Under very mild assumptions on X (maybe we need separated?), then $\mathbf{QCoh}(X)$ contains all the information needed to recover X.*

But we can do this in the case of derived categories.

Theorem 3.31 (Bondal-Orlov). *Suppose X is projective, smooth, and has ample (or anti-ample) ω_X, then $\mathbf{D}^b(\mathbf{Coh}(X)) \simeq \mathbf{D}^b(\mathbf{Coh}(Y)) \implies X \cong Y$.*

Conjecture 3.32. *If X is smooth and quasi-projective, then there are only finitely many X' such that $\mathbf{D}^b(\mathbf{Coh}(X)) \simeq \mathbf{D}^b(\mathbf{Coh}(X'))$ as triangulated categories.*

3.8 Fourier-Mukai Transform

Definition 3.33. Suppose that we have two schemes X and Y such that

$$\begin{tikzcd} & X \times Y \arrow[dl, "p_X"'] \arrow[dr, "p_Y"] & \\ X & & Y \end{tikzcd}$$

and $\mathcal{E}^\bullet \in \mathbf{D}^b(\mathbf{Coh}(X \times Y))$ and $\mathcal{F}$ a sheaf on X. Then the **Fourier-Mukai transform** is

$$\phi^{\mathcal{E}} \colon \mathbf{D}^b(\mathbf{Coh}(X)) \to \mathbf{D}^b(\mathbf{Coh}(Y))$$

given by

$$\mathcal{F} \longmapsto R_{p_Y *}(\mathcal{E}^\bullet \otimes^L (p_X^* \mathcal{F})).$$

So why is this called the Fourier-Mukai transform? If we take $X = Y = \mathbb{R}$, and $f \in C^\infty(\mathbb{R})$ standing in for the sheaf $\mathcal{F}$, then pushforward stands in for integration, and tensoring with $\mathcal{E}^\bullet$ is multiplying by e^x. Hence,

$$\phi(f) = \int_X f(x) e^{-xy}\, dx.$$

Theorem 3.34 (Orlov)**.** *If $F \colon \mathbf{D}^b(\mathbf{Coh}(X)) \to \mathbf{D}^b(\mathbf{Coh}(Y))$ is fully faithful, then there is $\mathcal{E} \in \mathbf{D}^b(\mathbf{Coh}(X \times Y))$ such that $F = \phi^{\mathcal{E}}$.*

Remark 3.35. If we work in the richer setting of **dg-categories** instead of triangulated categories, then we can state an even stronger result, due to Toën: Any functor between "dg-enhancements" is a Fourier-Mukai Transform.

3.9 Exceptional Collections

Recall the simple example Example 3.26. The point of exceptional collections is to use this example to deconstruct more complicated derived categories into simpler ones.

Definition 3.36. A sequence of objects $\langle A_0, \ldots, A_n \rangle$ in $\mathbf{D}^b(\mathbf{Coh}(X))$ is called a **strong exceptional collection** if

(a) $\mathrm{Ext}^i(A_p, A_q) = 0$ for all i and all $p > q$.

(b) $\mathrm{Ext}^i(A_p, A_p) = \begin{cases} k & i = 0 \\ 0 & \text{otherwise} \end{cases}$

We should think of this as almost an orthonormal basis for the derived category.

Example 3.37 (Beilinson)**.** Consider $X = \mathbb{P}^n$. Then $\langle \mathcal{O}(-n), \ldots, \mathcal{O}(-1), \mathcal{O} \rangle$ is a strong exceptional collection. For i, j such that $j > i$, we have that

$$\mathrm{Ext}^\bullet(\mathcal{O}(-i), \mathcal{O}(-j)) = H^\bullet(\mathcal{O}(i-j)) = 0$$

$$\mathrm{Ext}^\bullet(\mathcal{O}(-i), \mathcal{O}(-i)) = H^\bullet(\mathcal{O}(i) \otimes \mathcal{O}(-i)) = H^\bullet(0) = \begin{cases} k & \text{in degree zero} \\ 0 & \text{otherwise.} \end{cases}$$

Definition 3.38. A strong exceptional collection is called **full** if $\mathbf{D}^b(\mathbf{Coh}(X))$ is generated by the collection $\langle A_0, \dots, A_n \rangle$ as a category.

Theorem 3.39 (Beilinson 1971)**.** *$\langle \mathcal{O}(-n), \dots, \mathcal{O}(-1), \mathcal{O} \rangle$ is full for $\mathbf{D}^b(\mathbf{Coh}(X))$.*

Theorem 3.40 (Bondal)**.** *If $\mathbf{D} = \langle A_0, \dots, A_n \rangle$, and these form a strong exceptional collection. Let $\mathbf{D}' = \langle A_0, \dots, A_{n-1} \rangle$. Then there is a triangulated functor $P\colon \mathbf{D} \to \mathbf{D}'$ called the* ***projector****, where*

$$P(X) = \operatorname{cone}\left(R\operatorname{Hom}_{\mathbf{D}}(A_n, X) \otimes A_n \xrightarrow{ev} X \right)$$

Example 3.41. Consider $\mathbb{P}^1$. Then by Theorem 3.39, the exceptional collection is $\langle \mathcal{O}(-1), \mathcal{O} \rangle$. Let $X = \mathcal{O}(-2)$. The first step is to compute the cone

$$\operatorname{cone}\left(R\operatorname{Hom}(\mathcal{O}, X) \otimes \mathcal{O} \to X \right).$$

We have that

$$\begin{aligned} R\operatorname{Hom}(\mathcal{O}, \mathcal{O}(-2)) &\cong H^\bullet(R\mathbf{Hom}(\mathcal{O}, \mathcal{O}(-2))) \\ &\cong H^\bullet(\mathcal{O} \otimes \mathcal{O}(-2)) \\ &= H^\bullet(\mathcal{O}(-2)) = \begin{cases} k & \text{in degree 1} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

So to compute the cone, we have to compute

$$\operatorname{cone}\left(\begin{array}{ccccc} 0 & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}(-2) & \longrightarrow & 0 & \longrightarrow & 0 \end{array} \right) \cong \mathcal{O} \oplus \mathcal{O}(-2) \in \langle \mathcal{O}(-1) \rangle.$$

The coefficient here is $R\operatorname{Hom}(\mathcal{O}(-1), \mathcal{O} \oplus \mathcal{O}(-2))$, and

$$\begin{aligned} R\operatorname{Hom}(\mathcal{O}(-1), \mathcal{O} \oplus \mathcal{O}(-2)) &\simeq H^\bullet(\mathcal{O}(1) \otimes (\mathcal{O} \oplus \mathcal{O}(-2))) \\ &\simeq H^\bullet(\mathcal{O}(1)) \oplus \underbrace{H^\bullet(\mathcal{O}(-1))}_{0} \\ &\cong H^\bullet(\mathcal{O}(1)) \simeq \begin{cases} k^2 & \text{in degree zero} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Remark 3.42. This helps us determine the K-theory of these categories. The map $[-]\colon \mathbf{D}^b(\mathbf{Coh}(X)) \to K^0(X)$ given by

$$[\cdots \to A^i \to A^{i+1} \to \cdots] \longmapsto \sum_i (-1)^i [A^i]$$

sends the strong exceptional collection $\langle A_0, \dots, A_n \rangle$ for $\mathbf{D}^b(\mathbf{Coh}(X))$ to the generators of $K^0(X)$. Hence, $K^0(X) \cong \mathbb{Z}^{n+1}$ with generators $[A_i]$ for $i = 0, \dots, n-1$.

Theorem 3.43 (Orlov)**.** *If $X \to S$ is a fiber bundle and F_s is the fiber over s,*

$$\begin{tikzcd} F_s \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ \{s\} \arrow[r, hook] & S \end{tikzcd}$$

and $\mathcal{E}_0, \ldots, \mathcal{E}_n \in \mathbf{D}^b(\mathbf{Coh}(X))$ such that $\mathbf{E}_0|_{F_s}, \ldots, \mathbf{E}_n|_{F_s}$ is a full exceptional collection and $\mathcal{F}_0, \ldots, \mathcal{F}_m$ is a full exceptional collection on S, then

$$\mathbf{D}^b(\mathbf{Coh}(X)) = \langle \pi^*\mathcal{F}_0 \otimes^L \mathcal{E}_0, \ldots, \pi^*\mathcal{F}_m \otimes^L \mathcal{E}_0, \pi^*\mathcal{F}_0 \otimes^L \mathcal{E}_1, \ldots, \pi^*\mathcal{F}_m \otimes \mathcal{E}_n \rangle$$

4 Back to CSM Classes

4.1 Demazure Products

So far, if $T \circlearrowright TV_P$ for P a polytope in the weight lattice of T, then

$$\mathrm{csm}_{TV_P}(T) = [TV_P].$$

Therefore,

$$\mathrm{csm}(TV_P) = \sum_{\text{faces } F \subseteq P} [TV_F].$$

because

$$1_{TV_P} = \sum_F 1_{\text{corresponding } T\text{-orbit}}.$$

We want to compute the CSM class $\mathrm{csm}(X_o^w)$ where

$$X_o^w = BwB/B \subseteq G/B, \qquad G/B = \coprod_{w \in W} BwB/B.$$

By observing the diagram below, it is enough to compute $\mathrm{csm}(BS_o^w) \in H_*(BS^Q)$.

$$\begin{tikzcd} BS_o^Q \arrow[r, "\sim"] \arrow[d, hook] & X_O^w \arrow[r] \arrow[d] & G/B \\ BS^Q \arrow[r] & X^w := \overline{X_O^w} \arrow[ur] & \end{tikzcd}$$

For Q a word in the set of simple roots of G,

$$BS^Q = {}^{B \times^B P_{q_1} \times^B P_{q_2} \times^B \cdots \times^B P_q}\!\big/_{B} \xrightarrow{m} G/B.$$

The arrow here represents multiplication of all of the elements in the Bott-Samelson, and is B-equivariant. Recall also that $\times^B$ means we should divide by the diagonal action of B in each of the products: $b \cdot (g, h) = (gb^{-1}, bh)$.

If we forget the last letter of Q, then we get a fiber bundle

$$\mathbb{P}^1 \cong P_{q_{|Q|}}/B \to BS^Q \to BS^{Q\setminus\{q_{|Q|}\}}.$$

What do the fixed points of the torus action look like inside a Bott-Samelson? Elements of $(BS^Q)^T$ are tuples of elements in each of the parabolic subgroups corresponding to subwords R of Q, such that there is a 1 for $i \notin R$ and a simple reflection r_α for $i \in R$.

The image of m is closed, irreducible and B-invariant in G/B. Therefore, it is X^w for some w, which we will call the **Demazure product of** Q, $\mathrm{Dem}(Q)$.

We will consider $BS^R \subseteq BS^Q$ as submanifolds, for all subwords R of Q. Note also that $BS^{R_1 \cap R_2} = BS^{R_1} \cap BS^{R_2}$. Therefore, $m(BS^Q) \supseteq m(BS^R)$ for all R subwords of Q. Hence, $\mathrm{Dem}(Q) \geqslant \mathrm{Dem}(R)$.

Theorem 4.1. $\mathrm{Dem}(Q) = \max\{\prod R \in W \mid R \text{ subword of } Q\}$*, where* $\prod R$ *is the product of the simple reflections in* R*.*

Proof. We have that $m(BS^Q)^T = m((BS^Q)^T)$. The $\subseteq$ containment is easy, and the $\supseteq$ containment follows from Borel's theorem applied to the fiber over the T-fixed point. (Recall that Borel's theorem says that for X proper nonempty and S solvable, $X^S \neq \emptyset$.)

But from above, we know what $(BS^Q)^T$ looks like. It's tuples of 1's for $i \notin R$ and simple reflections for $i \in R$, as i runs over the subword R of Q. So multiplying these, we get

$$m((BS^Q)^T) = \{\textstyle\prod R \mid R \subseteq Q\}.$$

On the other hand,

$$m(BS^Q)^T = (X^w)^T = [1, w] \subseteq W,$$

where $w = \mathrm{Dem}(Q)$. Hence, the maximum of $\{\prod R \mid R \subseteq Q\}$. ☐

Theorem 4.2. *If* Q *is minimal length such that* $\mathrm{Dem}(Q) = w$*, then*

(1) $\prod Q = w$

(2) the map $BS^Q \to X^w$ *is birational and* $BS^Q_o \xrightarrow{\sim} X^w_o$ *is an isomorphism.*

Before we prove this theorem, we should say what exactly the open Bott-Samelson BS^Q_o is.

4.2 Variations on Bott-Samelsons

Definition 4.3. The **open Bott-Samelson** BS^Q_o is

$$BS^Q_o := {B \times^B (P_{q_1}\setminus B) \times^B (P_{q_2}\setminus B) \times^B \cdots \times^B (P_{q_{|Q|}}\setminus B)}\big/_{B}$$

There is still a B-equivariant multiplication map from BS^Q_o to G/B.

Proof of Theorem 4.2.

(1) There is $R \subseteq Q$ such that $\prod R = w$. Now we have that the T-fixed points of BS^R are mapped under m to $wB/B \subseteq (G/B)^T$. So by minimality, we have $|Q| = |R|$, and hence $Q = R$.

(2) By the previous part,

$$m^{-1}(wB/B)^T = m^{-1}(wB/B) \cap (BS^Q)^T = \{R \subseteq Q \mid \prod R = w\} = \{Q\}.$$

Hence, the fiber has just a single T-fixed point. Now apply 4.4 (below), so the fiber itself must be only one point, and therefore m is one-to-one over BwB/B.

In characteristic zero, if X is smooth, then $X \to Y$ has general fibers that are smooth. (This is "generic smoothness" if you look it up in Hartshorne). Hence, m is an isomorphism over $X_o^w = BwB/B$. □

Theorem 4.4 (Borel's Theorem, Upgraded)**.** *If T acts on X linearly, where X is projective (not just proper!), and X is not just a single point, then* $|X^T| > 1$.

Proof Sketch. Let's do this in the case that $T = \mathbb{C}^\times$ to get the idea. We have $\mathbb{C}^\times \circlearrowright \mathbb{CP}^n \supseteq X$ and X is not a point. Pick a point x not fixed by this action (else every point is fixed by T and we're done). Then the orbit looks like

$$\begin{aligned} \alpha\colon \mathbb{C}^\times &\longrightarrow \mathbb{CP}^n \\ z &\longmapsto z \cdot x \end{aligned}$$

Let Y be the closure of $\mathbb{C}^\times \cdot x$. Then this is isomorphic to $\mathbb{CP}^1$ under the identification $0 \sim \infty$ or isomorphic to $\mathbb{CP}^1$ with cusps at 0 and ∞. If the latter, we're done, so we want to rule out $\mathbb{CP}^1/(0 \sim \infty)$.

Let's call the north and south poles of $\mathbb{CP}^1$ n and s, respectively. We can decompose the action α as the composition

$$\alpha\colon \mathbb{C}^\times \hookrightarrow \mathbb{CP}^1 \longrightarrow \mathbb{CP}^n.$$

Let's look at the weight of $\alpha^*(\mathcal{O}(1)|_s)$. This is an integer, and

$$\mathrm{wt}(\alpha^*\mathcal{O}(1)|_s) = \mathrm{wt}(\alpha^*(\mathcal{O}(1)|_n) + \deg(Y)|\operatorname{stab}_{\mathbb{C}^\times}(x)|.$$

Notice that $\deg(Y)$ and the size of the stabilizer $\operatorname{stab}_{\mathbb{C}^\times}(x)$ are both positive integers (and not zero!), so it must be that

$$\mathrm{wt}(\alpha^*\mathcal{O}(1)|_S) \neq \mathrm{wt}(\alpha^*\mathcal{O}(1)|_n).$$

Therefore, n and s must be sent to different points by α, and we can rule out the case that Y is $\mathbb{CP}^1/(0 \sim \infty)$. □

Example 4.5. If $G = \mathrm{GL}(3)$ and $Q = 121$, then BS^{121} is the blowup of $\mathrm{GL}(3)/B$ along the Schubert variety given by the flag $\mathbb{C}^3 \to \mathbb{C}^2 \to L \to \mathbb{C}^0$.

So we described the Bott-Samelson manifold associated to a word Q as living inside $|Q|$-many copies of G/B.

$$\begin{array}{ccccc} BS^Q & \hookrightarrow & \prod^{|Q|} G/B & \xrightarrow{\mathrm{pr}_i} & G/B \\ \Cup & & & & \Cup \\ [p_1, \ldots, p_{|Q|}] & \longmapsto & & & \prod_{j=1}^{i} p_j B/B \end{array}$$

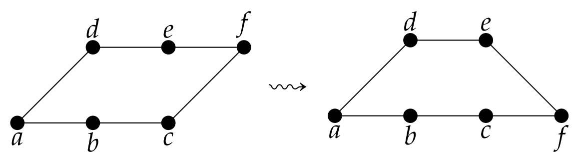
The next theorem is going to take us a while to prove. Probably the entirety of this lecture.

Theorem 4.6 (Bott-Samelson, Magyar, Grossberg-Karshon, Pasquier)**.** *BS^Q has a flat degeneration, topologically trivial, to a toric variety.*

$$\mathbb{C}^\times \circlearrowright \left(\vec{BS}^Q \to \mathbb{C} \right)$$

Note that, in the smooth category, $\vec{BS}^Q \cong BS^Q \times \mathbb{C}$.

Example 4.7. An example of such a family. Consider the toric varieties



On the left, the general fiber is $\mathbb{P}^1 \times \mathbb{P}^1 = F_0 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O})$, and on the right, we have the second Hirzebruch surface $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2)) = F_2$. This has a map $F_2 \to \mathbb{P}^1$.

If we label the vertices of the left-polytope as above, and label the vertices of the right polytope similarly, then the following equations hold in both of the coordinate rings of the toric varieties.

$$\begin{gathered} ac - b^2 \\ be - dc \\ ae - bd \end{gathered}$$

On the left polytope, we get the equations

$$\begin{gathered} af - be \\ df - e^2 \\ bf - ec \end{gathered}$$

and on the right polytope, we get the equations

$$af - bc$$
$$df - ec$$
$$bf - c^2.$$

Finally, we have the family over $\mathbb{C}[X, Y]$ given by

$$af - Xbe - Ybc$$
$$df - Xe^2 - Ybc$$
$$bf - Xec - Yc^2.$$

So how did Bott and Samelson think about Bott-Samelson manifolds? When they were around, algebraic groups weren't a thing and Lie groups were almost always compact. Instead of thinking of it as a product of minimal parabolics, they wrote one of these as

$$L \times^T L \times^T \cdots \times^T L/T$$

where T is the torus $T \cong U(1)^n$ contained in a compact group, such as $U(n)$, and L is the matrices that look like

$$\begin{bmatrix} * & & & & & \\ & * & & & & \\ & & \ddots & & & \\ & & & * & * & \\ & & & * & * & \\ & & & & & * \end{bmatrix}$$

$L \cong U(2) \times U(1)^{n-2}$. On $L \times^T L := (L \times L)/T_\Delta$, we still have an action of $(T \times T)/T_\Delta$. Therefore, we get an action

$$T^{|Q|} \circlearrowright BS^Q$$

but this is not algebraic. There is also a projection $T^|Q| \twoheadrightarrow U(1)^{|Q|}$, which acts on BS^Q faithfully. (T is n-dimensional, so $T^{|Q|}$ is is much larger than $U(1)^{|Q|}$).

The idea of Magyar is to not divide $P \times P$ by the action of B_Δ, but instead by $(N \times 1) \cdot T_\Delta$. Whereas B_Δ looks like pairs of upper triangular matrices, $(N \times 1) \cdot T_\Delta$ looks like pairs (X, Y) of an upper triangular matrix X and a diagonal matrix Y, sharing the same diagonal.

Let $B_M = (N \times 1) \cdot T_\Delta$. Then we have an action

$$T^{|Q|} \circlearrowright P_1 \times^{B_M} P_2 \times^{B_M} \cdots \times^{B_M} P_{|Q|}/B.$$

In this case, the action is algebraic, but it is not faithful. The only faithful portion comes from an action of $(\mathbb{C}^\times)^{|Q|}$.

What's the relation between B_M and B_Δ? If we define

$$\rho^\vee(t) = \begin{bmatrix} t & & & & \\ & t^2 & & & \\ & & t^3 & & \\ & & & \ddots & \\ & & & & t^n \end{bmatrix},$$

then this acts on $\mathfrak{b}$ with all negative weights outside . Then we get that

$$\lim_{t \to 0} (1, \rho^\vee(t)) \cdot B_\Delta = B_M.$$

(maybe we want $t \to \infty$ instead).

The idea of Pasquier is to consider

$$B^{|Q|} \circlearrowright (P_{q_1} \times P_{q_1} \times \cdots \times P_{q_{|Q|}}) \times \mathbb{C}$$

$$(b_1, \ldots, b_{|Q|}) \cdot (p_1, \ldots p_{|Q|}, t) = (p_1 b_1^{-1}, (\rho^\vee(t) \cdot b_1) p_2 b_2^{-1}, \ldots, t).$$

This quotient is a family over $\operatorname{Spec} \mathbb{C}[t]$. But there may be a serious problem with this: why can we divide by $B^{|Q|}$? There are bad examples (due to Nagata) of a non-reductive group (for example $B \circlearrowright R$ Noetherian such that R^B is not Noetherian).

The special cases that works are

(1) G/B.

(2) X/B, where $X \circlearrowleft G$ reductive. This is the space $(X \times G/B)/G$.

So to attempt to justify Magyar/Pasquier's approach, let's consider the diagram

$$\begin{tikzcd} P_{q_1} \times P_{q_1} \times \cdot \times P_{q_{|Q|}} \arrow[r, hook] \arrow[d, two heads] & G \times G \times \cdots \times G \arrow[d, two heads] \\ BS^Q \arrow[r, hook] & G \times^B G \times^B \cdots \times^B G/B \cong (G/B)^{|Q|} \end{tikzcd} \tag{2}$$

(This approach won't help us deal with the fiber over zero, so maybe it won't work…)

Definition 4.8. Over $\mathcal{B} = \{\text{borel subgroups of } G\}$, we have a bundle B/N of tori. Let $\mathbb{T} = \Gamma(\mathcal{B}, B/N)$. This is called the **abstract Cartan**.

Definition 4.9. If F is a flag in V (we're working in type A), then define

$$\operatorname{gr}_F V := F^1 \oplus {}^{F^2}\!/_{F^1} \oplus {}^{F^3}\!/_{F^2} \oplus \cdots \oplus {}^{F^n}\!/_{F^{n-1}}.$$

Now consider tuples $(F_1, \ldots, F_{|Q|})$ where F_1 is a flag in $\mathbb{C}^n$, F_2 is a flag in $\operatorname{gr}_{F_1} \mathbb{C}^n$, and so on, such that F_i is a flag in $\operatorname{gr}_{F_{i-1}}(F_{i-1}^n)$. Note that

$$\operatorname{gr}_{F_{i-1}} F_{i-1}^n = \operatorname{gr}_{F_{i-1}} \operatorname{gr}_{F_{i-2}} \cdots \operatorname{gr}_{F_1} \mathbb{C}^n.$$

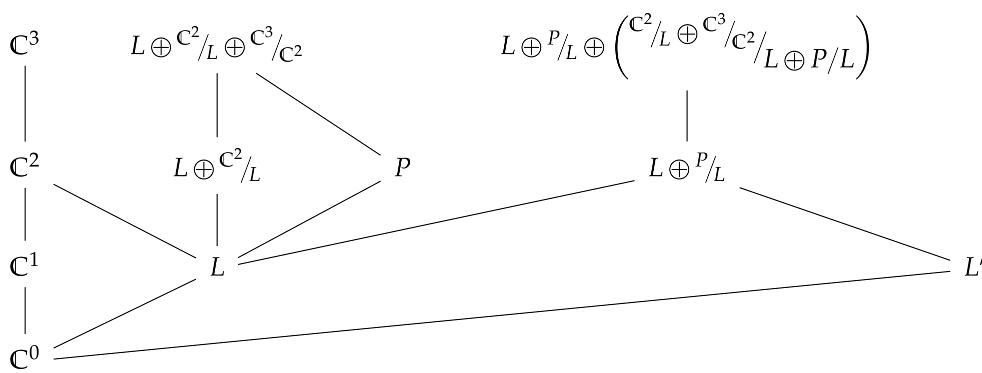
We have a torus $\mathbb{T} = \Gamma(\mathcal{B}, B/N)$ (the abstract Cartan) that acts on each $\operatorname{gr}_F V$, and therefore on the tuples

$$(F_1, \ldots, F_{|Q|})$$

with the condition above. This is a description of the lower right object in (2).

Note that F_i is the standard flag in $\operatorname{gr}_{F_{i-1}} F_{i-1}^n$ except in position q_i, which is the degenerate Bott-Samelson BS^Q.

Example 4.10. Let's go back to BS^{121}. Consider the flag $\mathbb{C}^3 \supseteq \mathbb{C}^2 \supseteq \mathbb{C}^1 \supseteq C^0$.



To summarize what we have so far, let's recall the several versions of the Bott-Samelson manifolds and their relations.

Demazure:	$BS^Q = P_{q_1} \times^B P_{q_2} \times^B \cdots \times^B P_{q_{\lvert Q\rvert}}/B$
Bott-Samelson:	$BS^Q_{\text{compact}} = L_{q_1} \times^{T_c} L_{q_2} \times^{T_c} \cdots \times^{T_c} L_{q_{\lvert Q\rvert}}/T_c$
Magyar:	$BS^Q_{\text{degen}} = P_{q_1} \times^{B_M} P_{q_2} \times^{B_M} \cdots \times^{B_M} P_{q_{\lvert Q\rvert}}/B$

where $B_M = T_\Delta \ltimes (N \times 1)$.

We have a series of diffeomorphisms (they're the same as real manifolds, but not as complex manifolds!)

$$BS^Q_{\text{degen}} \xrightarrow{\text{diffeo}} BS^Q_{\text{compact}} \xrightarrow{\text{diffeo}} BS^Q.$$

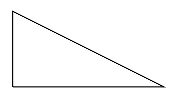
4.3 Abstract Toric Varieties

Definition 4.11. An **abstract toric variety** TV (as opposed to one embedded in projective space) is a normal scheme X with $T \circlearrowright X$ with open dense orbits.

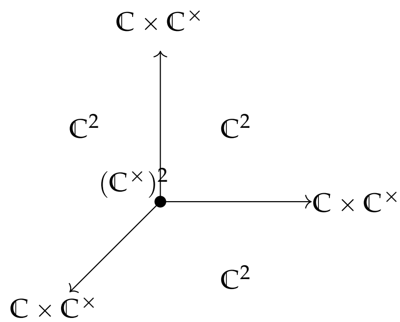
Form a polytope $P \subseteq \mathfrak{t}_c^* \supseteq T^* = \mathrm{Hom}(T, \mathbb{C})$, and associate a **fan of cones** $\subseteq \mathfrak{t}_c$, the dual cones around the faces of P.

This is enough information to reconstruct TV_P.

Example 4.12. If

$P =$ 

then the dual cones around the faces look like



Example 4.13. An example of a fan with no polytope.

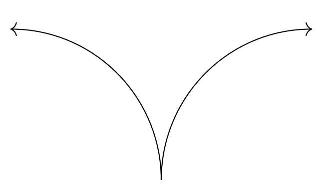
Start with an octohedron, and then split into the upper half (plus a little bit) union the lower half (plus a little bit). So the toric variety TV_P associated to the octohedron is the union of two open sets:

$$TV_P = (TV_P \backslash \{\mathrm{bot}\}) \cup (TV_P \backslash \{\mathrm{top}\}).$$

We can blow up each open set along the apex point that remains.

Then if we glue the blowup of the first open set with the bottom open set, then this gives us a fan with no polytope – the contradiction comes from considering the edge lengths of the middle square in the octohedron.

Example 4.14. Why is normal so important in the definition of a toric variety? Let's do an example of an abnormal toric variety. Consider $\mathbb{C}[x^2, x^3] \cong \mathbb{C}[y, z]/\langle y^2 - z^3 \rangle$. This lives inside $\mathbb{C}[x]$, but has a singularity. It looks like



4.4 Bott-Samelsons as Homology Classes

So now back to Bott-Samelsons. We have again the iterated $\mathbb{P}^1$ bundle

$$\begin{tikzcd} BS^Q_{\text{degen}} \arrow[d] & \mathbb{P}^1 \arrow[l] \\ BS^{Q\setminus\text{last}}_{\text{degen}} & \end{tikzcd}$$

Recall the big torus $\mathbb{T}$ from the discussion of the abstract Cartan. This acts on BS^Q_{degen} with $3^{|Q|}$-many orbits, acting on the front faces, back faces, or all of the faces.

Inside BS^Q_{degen}, we have some BS^R_{degen}. And under the action of $\mathbb{T}$, we have that BS^R_{degen} corresponds to

$$\begin{cases} \text{"all"} \in R \\ \text{"front"} \notin R. \end{cases}$$

Definition 4.15. $BS^Q_{R,\text{degen}}$ is the submanifold of BS^Q_{degen} that corresponds to

$$\begin{cases} \text{"all"} \notin R \text{"back"} \in R. \end{cases}$$

The classes of BS^R_{degen} form a basis for homology, and $BS^Q_{R,\text{degen}}$ is the dual basis for H_*.

Remark 4.16. Now recall that with respect to a flag F in V, we define

$$\mathrm{gr}_F(V) = F^1 \oplus {}^{F^2}\!/_{F_1} \oplus \ldots \oplus {}^{V}\!/_{F^{n-1}}.$$

Given a Hermitian metric on V, this is

$$\mathrm{gr}_F(V) F^1 \oplus (F^2 \cap (F^1)^\perp) \oplus \ldots \oplus (V \cap (F^{n-1})^\perp). \cong V$$

So we never need to worry about flags in the presence of a Hermitian metric.

So under this diffeomorphism $BS^Q_{\text{degen}} \to BS^Q$, let's find out where $BS^Q_{R,\text{degen}}$ goes. It's best to do this by example.

Example 4.17.

$$BS_{12}^{121} \longmapsto \left\{ \begin{tikzcd}[arrows=dash] \langle x,y,z\rangle \arrow[d] \arrow[dr] & & \\ \langle x,y\rangle \arrow[d] \arrow[dr] & P \arrow[d] \arrow[dr] & \\ \langle x\rangle \arrow[d] & L \arrow[dl] & L' \arrow[dll] \\ 0 & & \end{tikzcd} \;\middle|\; \langle x\rangle \perp L, P \perp \langle x,y\rangle \right\}$$

This forces $L = \langle y\rangle$ and $P = \langle y,z\rangle$. (Note: by demanding that two planes in $\mathbb{R}^3$ are perpindicular, we really mean "as perpindicular as possible," more concretely, we mean $P = (V^{k+1})^{\perp} \cap V^{k+2} \oplus V^k$, when V^i are the elements of the flag.)

$$BS^{12} \longmapsto \left\{ \begin{tikzcd}[arrows=dash] \langle x,y,z\rangle \arrow[d] \arrow[dr] & & \\ \langle x,y\rangle \arrow[d] \arrow[dr] & P \arrow[d] \arrow[dr] & \\ \langle x\rangle \arrow[d] & L \arrow[dl] & L' \arrow[dll] \\ 0 & & \end{tikzcd} \;\middle|\; L = L' \right\}$$

Denote the image of $BS^Q_{R,\text{degen}}$ under the diffeomorphism $BS^Q_{\text{degen}} \to BS^Q$ as BS^Q_R.

So now this diffeomorphism gives us the map m_* induced from

$$m\colon BS^Q \to G/B$$

on homology,

$$m_*\colon H_*(BS^Q) \to H_*(G/B),$$

where $H_*(BS^Q)$ has a $\mathbb{Z}$-basis consisting of classes $[BS^R]$, and $H_*(G/B)$ has a $\mathbb{Z}$-basis of classes $[X^w = \overline{BwB}/B]$. We can see that

$$m(BS^R) = X^{\mathrm{Dem}(R)},$$

so on homology,

$$m_*(BS^R) = \begin{cases} [X^{\Pi R}] & \text{if } R \text{ is a } \textbf{reduced word}\text{, that is, } |R| = \ell(\mathrm{Dem}(R)) \\ 0 & \text{otherwise.} \end{cases}$$

We can use this to understand the map on cohomology. We have a map

$$H^*(G/B) \longrightarrow H^*(BS^Q).$$

And $H^*(G/B)$ has a basis consisting of classes $[X_w = \overline{BwB}/B]$ and $H^*(BS^Q)$ has a basis consisting of classes $[BS^Q_R]$. This map is given by

$$[X_w] \longmapsto \sum_{\substack{R \subseteq Q \\ R \text{ reduced} \\ \prod R = w}} [BS^Q_R] = \sum_{\substack{R \subseteq Q \\ R \text{ reduced} \\ \prod R = w}} \prod_{r \in R} [BS^Q_r].$$

Remark 4.18. We've done all of this so far using homology and cohomology, but the story works the same way on T-equivariant cohomology.

4.5 The Anderson-Jantzen-Soergel/Billey Formula

Example 4.19 (Application)**.** Compute $[X_w]|_v \in H^*_T(vB/B)$. There is a map

$$(-)|_v \colon H^*_T(G/B) \longrightarrow H^*_T(vB/B).$$

This might be stupid if this was regular cohomology, but in equivariant cohomology the cohomology of a point isn't trivial. In fact, if we think about the direct sum over all of the points, we get an injective ring homomorphism

$$H^*_T(G/B) \hookrightarrow \bigoplus_{v \in W} H^*_T(vB/B).$$

So to do computations in $H^*_T(G/B)$, you can do computations in the big direct sum instead.

Now let Q be a reduced word for v. We have

$$\begin{array}{ccccc} BS^Q_Q = \{Q\} & \hookrightarrow & BS^Q & \longrightarrow & G/B \\ \sum_R \prod_{r \in R} [BS^Q_r]|_Q & \longleftarrow & \sum_R \prod_{r \in R} [BS^Q_r] & \longleftarrow & [X_w] \end{array}$$

And the leftmost thing lives inside $H^2_T(\mathrm{pt}) \cong T^*$, which is the weight lattice. Hence,

$$[BS^Q_r]|_Q = \left(\prod_{i < r} s_{\alpha_i} \right) \alpha_i.$$

This is due to Anderson-Jantzen-Soergel/Billey.

Recall the Anderson-Jantzen-Soergel/Billey formula from last time.

$$T \circlearrowright X_w = \overline{B_- w B}/B \subseteq G/B$$

$$[X_w] \in H_T^*(G/B) \longrightarrow H_T^*((G/B)^T) = \bigoplus_W H_T^* \cong \bigoplus_W \mathrm{Sym}(T^*)$$

$$[X_w]|_v = \sum_{\substack{R \subseteq Q \text{ reduced} \\ \prod R = w}} \prod_{r \in R} \left(\prod_{\substack{i<r \\ i \in R}} s_{q_i} \right) \cdot \alpha_r.$$

Theorem 4.20 (Kirwan)**.** *The map $H_T^*(G/B) \longrightarrow H_T^*((G/B)^T)$ above is injective.*

Example 4.21.

$$[X_{213}]^2 = \alpha[X_{213}] + [X_{312}]$$

where $\alpha \in H_T^2 = H_T^2(\mathrm{pt})$ is an equivariant correction term.

Proposition 4.22. *Let $\pi\colon G/B \to G/P_\alpha$, where P_α is a minimal parabolic. Then $\pi^{-1}(\pi(X_w)) \supseteq X_w$, with equality if and only if $w < wr_\alpha$.*

4.6 Deodhar decomposition of BS^Q

We have $BS^Q \hookrightarrow (G/B)^Q$. In terms of flags,

$$F_0 = B/B, F_1, F_2, \ldots, F_{|Q|}$$

$$(F_{i-1}, F_i) \in G \cdot ({}^B\!/_B r_\alpha {}^B\!/_B) \subseteq (G/B)^2 \iff \pi_i(F_{i-1}) = \pi_i(F_i)$$

Theorem 4.23 (Deodhar)**.** *Let $(F_0 = B/B, F_1, \ldots, F_{|Q|}) \in BS_O^Q$, (so $F_i \neq F_{i-1}$ because it's inside BS_O^Q). Suppose that under the map $BS^Q \hookrightarrow (G/B)^Q \to W^Q$, this flag maps to*

$$(1, w_1, w_2, \ldots, w_{|Q|}).$$

Then

(1) $w_i \in \{w_{i-1}, w_{i-1}r_{q_i}\}$ is encoded by $R \subseteq Q$.

*(2) If $w_{i-1}r_{q_i} < w_{i-1}$, then $w_i = w_{i-1}r_{q_i}$. In this case we say that the word R is **distinguished.***

(3) The stratum for a fixed distinguished $R \subseteq Q$ is isomorphic to $(\mathbb{A}^1)^a \times (\mathbb{G}_m)^b$, where a is the number of times $w_i = w_{i-1}r_{q_i}$, and b is the number of times $w_i = w_{i-1}$.

Proof.

(1) $\pi_i(F_i) = \pi_i(F_{i-1})$. So both map to $X_{w_i Y_{q_i}} \in W/W(P_{q_i})$, intersecting cells on G/P_{q_i}.

(2) If $w_{i-1}r_{q_i} < w_{i-1}$, then the q_i-plane in F_{i-1} is determinable from $\pi_i(F_{i-1})$. If $w_i = w_{i-1}$, both $\pi_i(F_{i-1}) = \pi_i(F_i)$ would extend to a flag in $X_{w_i} = X_{w_{i-1}}$ the same way. Therefore $F_i = F_{i-1}$, which is a contradiction, because we're inside the *open* Bott-Samelson BS_O^Q.

□

Example 4.24. Let's decompose $BS_O^{121} \subseteq GL(3)/B$.

4.7 CSM classes of Bott-Samelsons

Recall that, if Q is a word in the elements of the Weyl group, then

$$BS^Q = P_{q_1} \times^B \cdots \times^B P_{q_i} \times^B \cdots \times^B P_{q_{|Q|}}/B$$

and if R is a subword of Q, then we can realize BS^R inside BS^Q by replacing P_{q_i} with B for $q_i \notin R$.

$$BS^R = E_1 \times^B \cdots \times^B E_i \times^B \cdots \times^B E_{|Q|}/B, \qquad E_i = \begin{cases} P_{q_i} & q_i \in R \\ B & q_i \notin R. \end{cases}$$

Then last time, we defined the dual basis for $H_G^*(BS^Q)$ in terms of the non-algebraic (but smooth) submanifolds

$$BS^Q_{\perp R},$$

consisting of the flags in the Bott-Samelson where we demand that the new flags added are as perpindicular as possible to the old ones.

Now if $(F^i)_{i=1,\ldots,n}$ is a flag in a $\mathbb{C}$-vector space V, we obtain a degeneration of V to $\operatorname{gr} V$, given by the Rees module

$$\mathcal{V} = \operatorname{gr} V = \bigoplus_{i=0}^{\infty} F^i t^i.$$

We set $F^i = V$ for $i \geqslant n$, so this Rees module is

$$0 \oplus F^1 t \oplus F^2 t^2 \oplus \ldots \oplus t^n (V \otimes \mathbb{C}[t]) \oplus t^{n+1}(V \otimes \mathbb{C}[t]) \oplus \ldots$$

where $\mathcal{V}$ is a $\mathbb{C}[t]$-module. We have that

$$\mathcal{V}/_{(t-1)\mathcal{V}} \cong V, \qquad \mathcal{V}/_{t\mathcal{V}} \cong \operatorname{gr} V.$$

We can do this not over $\mathbb{C}$ but over $\mathrm{GL}(n)/B$, where V is the trivial $\mathbb{C}^n$-bundle, and (F^i) is the tautological flag. This is our plan.

$$BS^Q \subseteq (G/B)^{|Q|} = \left\{ (F_1, F_2, \dots) \,\middle|\, F_i \text{ flag in the fiber over } (F_1, \dots, F_i) \text{ of the trivial vector bundle} \right\}.$$

We can take the Rees degeneration interpolating between BS^Q and $\operatorname{gr} BS^Q$. This is a family of varieties $\mathcal{R}$ over $\mathbb{C}$, equivariant with respect to an action of $\mathbb{C}^\times$. The equivariance buys us that all fibers look the same except over zero. So we have that $\mathcal{R}|_1 = \mathcal{R}|_a \cong BS^Q$ for $a \neq 0$, and $\mathcal{R}|_0 = \operatorname{gr}(BS^Q)$.

$$\mathcal{R}|_1 \supseteq \bigcup_{q \in Q} BS^{Q \setminus q} \rightsquigarrow \bigcup_{q \in Q} \operatorname{gr} BS^{Q \setminus q} \subseteq \mathcal{R}|_0.$$

Definition 4.25. Define $\partial\mathcal{R}$ as the subfamily where the Rees construction is performed on $\bigcup_{q \in Q} BS^{Q \setminus q}$ instead.

Example 4.26. Consider $\mathbb{CP}^1$ as the set of lines in $\mathbb{C}^2$. Over this, we have the trivial vector bundle $\mathbb{C}^2 \to \mathbb{CP}^1$, but inside $\mathbb{C}^2$ there is a line L, so this is a trivial bundle over $\mathbb{CP}^1$ with the tautological bundle inside it. This degenerates through the Rees family to $\mathcal{O}(-1) \oplus \mathcal{O}(1)$.

Definition 4.27. The **log tangent bundle of** $\mathcal{R}$ is the sheaf of vector fields on $\mathcal{R}$ tangent to each fiber and, along each component of $\partial\mathcal{R}$, tangent to the component.

Denote the log tangent bundle of some scheme X by $\log(X)$

Remark 4.28 (Recall)**.** For $BS_0^Q \subseteq BS^Q$,

$$\operatorname{csm}(BS_0) = c(\text{log tangent bundle}),$$

where c denotes total Chern class in H^*.

We have a map

$$c(\log \mathcal{R}) \longmapsto c(\log BS^Q)$$

by naturality of Chern classes because $\mathcal{R}|_1 \cong BS^Q$. We can also look at

$$\operatorname{csm}(\operatorname{gr} BS_O^Q) = c(\log \operatorname{gr} BS^Q).$$

This is the one that we can compute due to Toric stuff we've done, and again there's a map

$$c(\log \mathcal{R}) \longmapsto c(\log \operatorname{gr} BS^Q).$$

However, we have that

$$\begin{array}{ccccc} H^*(BS^Q) & \xrightarrow{\cong} & H^*(\mathcal{R}) & \xrightarrow{\cong} & H^*(\operatorname{gr} BS^Q) \\ \Uparrow & & \Uparrow & & \Uparrow \\ c(\log BS^Q) & \longmapsto & c(\log \mathcal{R}) & \longmapsto & c(\log \operatorname{gr} BS^Q) \end{array}$$

Therefore, we can compute $\operatorname{csm}(BS^Q)$ by computing $\operatorname{csm}(\operatorname{gr} BS^Q)$. This gives us

Theorem 4.29.

$$\mathrm{csm}(\mathrm{gr}\, BS_0^Q) = \sum [\mathrm{gr}\, BS_R^Q] \longmapsto \sum [BS_{\perp R}^Q] = \mathrm{csm}(BS_0)$$

Corollary 4.30 (Knutson)**.** *Let Q be a reduced word with $\prod Q = v$. Inside BS^Q, there are both $BS^Q_{\perp R}$ and X^v_O. Considering the map $m\colon BS^Q \to G/B$,*

$$\mathrm{csm}(X_O^v) = m_* \left(\sum_{R \subseteq Q} [BS_{\perp R}^Q] \right).$$

Conjecture 4.31 (Aluffi-Michalcea)**.** *The CSM class of X_O^v is* ***Schubert positive****, that is,*

$$\mathrm{csm}(X_O^v) \in \sum_w \mathbb{N} \cdot [X^w].$$

Theorem 4.32 (Huh)**.** *The above conjecture is true on Grassmannians* $\mathrm{Gr}(k, n)$.

If we take Q a word in the simple reflections and the associated Bott-Samelson BS^Q. If $Q = (v_1, \dots, v_n)$, then

$$BS^Q = B \times^B \overline{Bv_1B} \times^B \overline{Bv_2B} \times^B \cdots / B.$$

In a special case, we have that

$$BS^{w_0^P} = \overline{Bw_o^PB}/B \cong P/B$$

Definition 4.33. Q is **reduced** if the sum of the lengths of the v_i is the length of the product of the v_i.

Remark 4.34. Q is reduced if and only if $BS^Q \to G/B$ is birational onto its image.

Example 4.35. Now if $Q = (r_\alpha, w)$ for a simple reflection r_α, then get

$$P_\alpha \circlearrowright \left(\begin{array}{ccc} X^w & \longrightarrow & BS^{(r_\alpha, w)} \\ & & \big\downarrow \\ & & BS^{r_\alpha} \cong \mathbb{P}^1 \end{array} \right)$$

So we get X^w living over $0 \in \mathbb{P}^1$ and $r_\alpha \cdot X^w$ living over $\infty \in \mathbb{P}^1$. We have $\alpha = [0] - [\infty] \in H_T^2(\mathbb{P}^1)$ on $\mathbb{P}^1$. So after tensoring with $\mathrm{frac}(H_T^*)$, we get

$$1 = \frac{[0] - [\infty]}{\alpha}.$$

Then apply π^*

$$1 = \frac{[X^w] - r_\alpha [X^w]}{\alpha} \in H_T^*(BS^{(r_\alpha, w)}).$$

Therefore,

$$m_*\left(\frac{[X^w]-r_\alpha[X^w]}{\alpha}\in H_T^*(G/B)\right)=\begin{cases}[X^{r_\alpha w}] & r_\alpha w>w\\ 0 & r_\alpha w<w.\end{cases}$$

Definition 4.36. The **divided difference operator** ∂_α is

$$\partial_\alpha=\frac{1}{\alpha}(1-r_\alpha).$$

Corollary 4.37 (Aluffi-Michalcea)**.** $\widetilde{r}_\alpha=r_\alpha+\hbar\partial_\alpha$

Corollary 4.38 (Lascoux)**.** $\widetilde{r}_\alpha^2=1$

Corollary 4.39. $\widetilde{r}_\alpha\,\mathrm{csm}(X^w)=\mathrm{csm}(X^{r_\alpha w})$

4.8 A few variations on Bott-Samelsons

Let $Q=(v_1,v_2,\dots,v_n)$ with $v_i\in W$. Then there is a variation on

$$BS^Q=\prod\overline{Bv_iB}/_{B^Q}$$

There is a map

$$BS^{(\dots,v_i,v_{i+1},\dots)}\twoheadrightarrow BS^{(\dots,v_i*v_{i+1},\dots,)}$$

where $*$ is the Demazure product on W. This comes from the multiplication map

$$\overline{Bv_iB}\times\overline{Bv_{i+1}B}\xrightarrow{m}\overline{Bv_i*v_{i+1}B}.$$

We can generalize further, replacing B by $P\geqslant B$ and replacing $W\cong B\backslash G/B$ by $W_P\backslash W\backslash W_P\cong P\backslash G/P$. Again, we can make $W_P\backslash W\backslash W_P$ into a monoid under the Demazure product, as we did with W in the previous paragraph.

There is a notion of height on $W_P\backslash W/W_P$, given by $\mathrm{ht}(W_pwW_p)=\min_{w\in W}\ell(W_pwW_P)$. This is equal to $\dim(PwP/P)$. The height is only subadditive under the Demazure product; in general we have $\ell(v*w)\leqslant\ell(v)+\ell(w)$.

Definition 4.40. If $(v_1,\dots,v_n)\in W_P\backslash W\backslash W_P$, we can define

$$BS^{(v_1,\dots,v_n)}=\overline{Pv_1P}\times^P\overline{Pv_2P}\times^P\cdots\times^P\overline{Pv_nP}/_P.$$

Remark 4.41. Notice that for the case when G/P is $\mathrm{Gr}(k,\mathbb{C}^n)$ with $k\leqslant n/2$,

$$W_P\backslash W\backslash W_P\cong P\backslash G/P\cong(G/P)^2/G=[0,k]$$

The critical special case of this is when H is an adjoint reductive group, for example $P\,\mathrm{GL}(n)$. In this case, we have $G=H_{\mathbb{C}((z))}$ (for example $P\,\mathrm{GL}(n,\mathbb{C}((z)))$), $P=H_{\mathbb{C}[[z]]}$, and the Levi of P is just $H_{\mathbb{C}}$.

$$W_G\cong W_H\ltimes\Lambda,$$

where Λ is the coweight lattice of H. Modding out both sides by W_H, we get that

$$W_H \backslash W_G / W_H$$

is the set Λ_+ of dominant coweights. Then the Demazure product on W becomes addition of coweights on $W_H \backslash W \backslash W_H$. Here the height function is the height of the coweights.

In Λ_+, every word is reduced.

5 Perverse Sheaves

5.1 $f_!$ and $f^!$

Let X, Y be topological spaces and $f \colon X \to Y$. Then there is a map f_* from sheaves on Y to sheaves on X, sending a sheaf $\mathcal{F}$ to $f_* \mathcal{F}$, which is given for $U \subseteq X$ open by

$$\Gamma(U; f_* \mathcal{F}) := \Gamma(f^{-1}(U), \mathcal{F})$$

Example 5.1. $Y \to \{\mathrm{pt}\}$, $\mathcal{F} = \mathbb{Q}_Y$ the constant sheaf.

Definition 5.2. Define a functor $f_! \colon \mathbf{Sh}(Y) \to \mathbf{Sh}(X)$ by

$$\Gamma(U; f_! \mathcal{F}) := \left\{ s \in \Gamma(f^{-1}U; \mathcal{F}) \mid f \colon \operatorname{supp}(s) \to X \text{ proper} \right\}$$

This gives a right-derived functor

$$Rf_! \colon \mathbf{D}^+(Y) \to \mathbf{D}^+(X).$$

We want to define $f^! \colon \mathbf{D}^+(X) \to \mathbf{D}^+(Y)$ right-adjoint to $Rf_!$. That is, if $\mathcal{F}$ is a sheaf on Y and $\mathcal{G}$ is a sheaf on X, then there is a natural isomorphism

$$\operatorname{Hom}_{\mathbf{D}^+(X)}(Rf_! \mathcal{F}, \mathcal{G}) \cong \operatorname{Hom}_{\mathbf{D}^+(Y)}(\mathcal{F}, f^! \mathbb{G}).$$

Example 5.3. If $X = \mathrm{pt}$, Y is a smooth and oriented manifold, and $G = \mathbb{Q}_{\mathrm{pt}}$ is the constant sheaf, then we want to have

$$f^!(\mathbb{Q}_{\mathrm{pt}}) = \mathbb{Q}_Y[\dim Y].$$

$$\operatorname{Hom}_{\mathbf{D}^+(\mathrm{pt})}\left(R\Gamma_c(Y; \mathbb{Q}_Y), \mathbb{Q}_{\mathrm{pt}}\right) \cong \operatorname{Hom}_{\mathbf{D}^+(Y)}(\mathbb{Q}_Y, \mathbb{Q}_Y[\dim Y])$$

To be continued.

6 Other stuff

Definition 6.1. Let X, Y be schemes with X a closed subscheme of Y. The **degeneration to the normal cone** is the blowup of $Y \times \mathbb{A}^1$ along $X \times \{0\}$.

$$\mathrm{Bl}_{X\times 0}(Y \times \mathbb{A}^1).$$

Example 6.2. If $Y = \mathrm{Spec}(R)$, $X = \mathrm{Spec}(R/I)$, then $Y \times \mathbb{A}^1 = \mathrm{Spec}(R[t])$, and

$$X \times 0 = \mathrm{Spec}(R[t])/\langle I, t\rangle.$$

$$\mathrm{Bl}_{X\times 0}(Y \times \mathbb{A}^1) = \mathrm{proj}\left(\bigoplus_n \langle I, t\rangle^n z^n\right) \subseteq R[t, z],$$

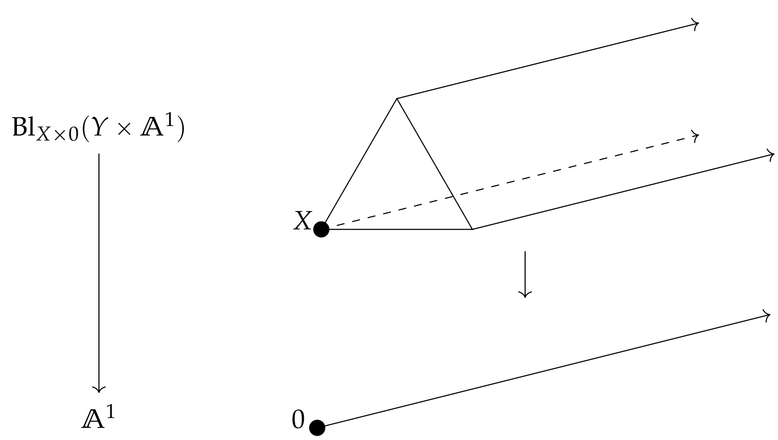
where $R[t, z]$ has the grading with t in degree zero, and z in degree 1.

In this case, the normal cone is

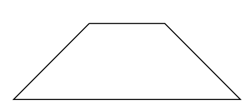
$$\mathrm{Spec}\left(\bigoplus_n I^n/I^{n+1}\right) = \mathrm{gr}_I R.$$

?

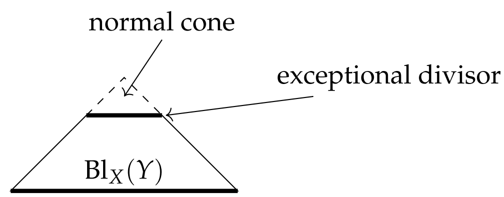
Example 6.3. If X is a point inside $Y = \mathbb{P}^2$. We know how to draw $\mathbb{P}^2$; it's the toric variety with moment polytope a triangle. And $\mathbb{A}^1$ has moment polytope a half-line, so $Y \times \mathbb{A}^1$ is a semi-infinite Toblerone bar.



To blowup at $X \times \{0\}$, we chop off the corner. There is a map $Y \times \mathbb{A}^1$ to $\mathbb{A}^1$, where most fibers are copies of $\mathbb{P}^2$, but at 0, the fiber is this toric variety with moment polytope



The various parts of this can be labelled.



6.1 Brick Manifolds

There is a **brick variety** inside the Bott-Samelson such that the following commutes.

$$\begin{tikzcd} BS^Q \arrow[r, two heads] & X^{\mathrm{Dem}(Q)} \\ \mathrm{Brick}^Q \arrow[u] \arrow[r, hook] & \mathrm{Dem}(Q)B/B \end{tikzcd}$$

Fact 6.4.

(a) *$BS^Q \supseteq BS^R$ for $R \geqslant Q$, and*

$$BS^R = \bigcap_{r \notin R} BS^{Q \setminus r},$$

so we see that $\mathrm{Brick}^Q \supseteq \mathrm{Brick}^R$ *for $R \subseteq Q$ such that* $\mathrm{Dem}(R) = \mathrm{Dem}(Q)$.

(b) *Moreover,* $\mathrm{Brick}^R = \bigcap_{r \notin R} \mathrm{Brick}^{Q \setminus r}$.

(c) $\bigcup_{q \in Q} \mathrm{Brick}^{Q \setminus q}$ *is a simple normal crossings divisor.*

Definition 6.5. For $M \supseteq D$ a simple normal crossings divisor, define the **dual complex** $\Delta(M, D)$ with vertex set the components $\mathrm{comps}(D)$ of D, and $F \subseteq \mathrm{comps}(D)$ a face if and only if $\bigcap_{C \in F} C \neq \varnothing$.

Example 6.6. For BS^Q, the vertex set are the letters of the word Q and the faces are all possible faces (it's a simplex!) because the condition always holds for BS^Q.

Example 6.7. If $M = TV_p$ is smooth and compact, then

$$\mathrm{comps}(D) = \{TV_s \mid S \text{ facet of } P\}.$$

$$\Delta(M, D) = \text{ dual to } \partial P \simeq S^{\dim P - 1} \simeq \text{sphere} \cap \mathrm{fan}(P).$$

Theorem 6.8 (Knutson-Miller 2003)**.** *Let*

$$D = \bigcup_{\substack{q \in Q \\ \mathrm{Dem}(Q\backslash q) = \mathrm{Dem}(Q)}} \mathrm{Brick}^{Q\backslash q}.$$

Then $\Delta\left(\mathrm{Brick}^Q, D\right)$ *is homeomorphic to a sphere.*

Definition 6.9. For $v \leqslant \mathrm{Dem}(Q)$, the preimage of X_v^O under the B-equivariant map $m\colon BS^Q \to G/B$ is defined as

$$\mathrm{Brick}_v^Q := m^{-1}(X_v^O)$$

Again, this is a smooth manifold, and

$$\bigcup_{\substack{q \in Q \\ \mathrm{Dem}(Q\backslash q) \geqslant v}} \mathrm{Brick}_v^Q$$

is a simple normal crossings divisor.

Definition 6.10.

$$\partial \mathrm{Brick}_v^Q := \bigcup_{\substack{q \in Q \\ \mathrm{Dem}(Q\backslash q) \geqslant v}} \mathrm{Brick}_v^{Q\backslash q}$$

Definition 6.11. The **subword complex** is the complex with vertex set Q and faces $F \subseteq Q$ if and only if $\mathrm{Dem}(Q\backslash F) \geqslant v$.

Theorem 6.12 (Knutson-Miller)**.** $\Delta(Q, v)$ *is homeomorphic to either a ball or a sphere, and*

$$\Delta(Q, v) \supseteq \bigcup_{q \in Q} \Delta(Q\backslash q, v) = \partial\Delta(Q, v).$$

6.2 Gross-Hacking-Keel

Let $M = \overline{M}\backslash\partial\overline{M}$. Assume $\overline{M}$ is smooth and compact, and that $\partial\overline{M}$ is an **anticanonical** simple normal crossings divisor.

Definition 6.13. An **anticanonical** divisor is $\sigma^{-1}(0)$, for some nonzero $\sigma \in \Gamma\left(\overline{M}, \bigwedge^{\dim M} T\overline{M}\right)$.

Assume further that the stratification coming from $\partial\overline{M}$ includes a zero-dimensional stratum.

Example 6.14 (Non-examples)**.** Elliptic curves in $\mathbb{CP}^2$, or curves with 1 node, because although they are normal crossing divisors but not simple normal crossing divisors.

Gross-Hacking-Keel make a ring R with basis the lattice points in the cone complex C of $(\overline{M}, \partial\overline{M})$. This cone complex is some piecewise-linear object with lattice points. If M is a torus and $\overline{M}$ is a toric variety, then this cone complex is actually a fan.

Conjecture 6.15. *This cone complex (and therefore the ring structure) depends only on M, not $\overline{M}$.*

To define this ring, the zero element $\vec{0} \in C$ corresponds to the identity of R, and having a basis, we get $\mathrm{tr}\colon R \to \mathbb{C}$ sending $r \in R$ to the coefficient of 1 in r.

Conjecture 6.16. *For $r, s \in R$, $\langle r, s\rangle = \mathrm{tr}(rs)$ is nondegenerate.*

Definition 6.17. Define $\langle r_1, r_2, \ldots, r_k\rangle$ as follows. Each r_i is a lattice point in a cone in the cone complex C, each of which corresponds to a list of divisors with coefficients in $\mathbb{N}$. Hence, each r_i corresponds to a map $\mathrm{comps}(\partial\overline{M}) \to \mathbb{N}$. So we can associate to the list $r_1, \ldots, r_k$ a sum of coefficient vectors from each r_i.

Then $\langle r_1, \ldots, r_k\rangle$ is the number of rational curves $\mathbb{P}^1 \to \overline{M}$ meeting each $D \subseteq \partial\overline{M}$ in the correct multiplicity with certain homology class $H_2(\overline{M})$.

The ring R is defined using the quantum cohomology on $\overline{M}$.

Remark 6.18. This is what quantum cohomology is all about. It's about counting curves where you're allowed some quantum tunneling between some points.

6.3 An application of Brick manifolds

We have a resolution of singularities given by Bott-Samelson manifolds.

$$BS^Q \xrightarrow{\text{birational}} X^w \hookrightarrow G/B$$

Definition 6.19. The (closed) **Richardson Varieties** inside G/B are $X^w \cap X_v$.

To get resolve the Richardson varieties, consider the maps $BS^Q \to X^w$ and $BS^R \to w_0 \cdot X_v = X^{w_0 v}$, where

$$w_0 = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}$$

is the long element of the Weyl group. Define

$$w_0 \cdot BS^R = P^-_{r_1} \times^{B_-} P^-_{r_2} \times^{B_-} \cdots \times^{B_-} P^-_{r_{|R|}} w_0 B/B.$$

We have that BS^Q and $w_0 \cdot BS^R$ are transverse by Kleiman 1973. Also, the Brick manifold resolves the Richardson variety.

$$
\begin{tikzcd}
 & & \mathrm{Brick}^{Q\overleftarrow{R}} \arrow[dll] \arrow[dd] \arrow[drr] & & \\
BS^Q \arrow[dd, two heads] & & & & w_0 \cdot BS^R \arrow[dd] \\
 & & X^w \cap X_v \arrow[dll] \arrow[drr] & & \\
X^w \arrow[drr] & & & & X_v \arrow[dll] \\
 & & G/B & &
\end{tikzcd}
$$

Assume that $X^w \cap X_v \neq \varnothing$, which happens when $w \geqslant v$.

Example 6.20 (Escobar)**.** A very fun example of a brick manifold.

$$Q = 1234\ 1234\ 123\ 12\ 1$$

This is w_0 for $\mathrm{GL}(5)$. The Coxeter element is $\chi = 1234$, and the rest of the word is called the χ-sorted word for w_0.

$$
\begin{aligned}
\dim \mathrm{Brick}^Q &= \dim BS^Q - \dim X^{\mathrm{Dem}(Q)} \\
&= |Q| - \ell(\mathrm{Dem}(Q)) \\
&= \ell(\chi) + \ell(w_0) - \ell(w_0) = \ell(\chi) = \mathrm{rank}(G/Z(G)).
\end{aligned}
$$

There is an action $T \circlearrowright \mathrm{Brick}^Q$; both T and Brick^Q have the same dimension, so you may worry that the action isn't faithful, but it is. Therefore, Brick^Q is a smooth projective toric variety, so it comes from some polytope.

Fact 6.21. *The polytope of the Brick manifold is the associahedron, whose faces correspond to subdivisions of the* $(n+2)$*-gon.*

6.4 Duistermaat-Heckman Theorem

This is an application of csm classes. Assume that $T \circlearrowright M$, where M is a compact oriented manifold. We get a map $\int : H_T^*(M) \to H_T^*$. How do we compute it? Well, look at the fixed points.

$$
\begin{tikzcd}
H_T^*(M) \arrow[r] \arrow[dr] & H_T^* \\
 & H_T^*(M^T) \arrow[u, dashed]
\end{tikzcd}
$$

There is no dashed map in the diagram above that makes it commute. But once we tensor everything with the fraction field of H_T^*, the diagonal map above is an isomorphism.

Fact 6.22. $H_T^*(M) \otimes \mathit{frac}(H_T^*) \cong H_T^*(M^T) \otimes \mathit{frac}(H_T^*)$

On M we have some classes

$$\sum_{f \in M^T} \alpha_f [f]$$

with $\alpha_f \in H_T^*$ and $[f] \in H_T^{\dim M}(M)$. These are easy to integrate:

$$\int \sum_{f \in M^T} \alpha_f [f] = \sum_{f \in M^T} \alpha_f.$$

Now assume that $T \circlearrowright M$ has isolated fixed points, so $|M^T| < \infty$.

If we're given a class $c \in H_T^*(M)$, how do we figure out what the coefficients α_f are, under the isomorphism $H_T^*(M) \otimes \mathrm{frac}(H_T^*) \cong H_T^*(M^T) \otimes \mathrm{frac}(H_T^*)$?

If c is of this form, then

$$c|_g = \alpha_g [g]|_g.$$

Therefore,

$$\int_M c = \sum_{f \in M^T} \frac{c|_{[g]}}{[g]|_g}$$

If $U \ni g$ is a T-equivariant neighborhood inside M, then

$$[g]|_g = \prod_{\text{weights } \lambda \text{ in } T_g M} \lambda.$$

We also have that g is the transverse intersection of T-invariant hyperplanes.

Remark 6.23. For a reference for this stuff, see *The moment map and equivariant cohomology*, Atiyah-Bott 1984.

Theorem 6.24 (Atiyah-Bott, Berline-Vergne)**.**

$$\int_M c = \sum_{f \in M^T} \frac{c|_{[g]}}{[g]|_g}$$

holds for any $c \in H_T^*(M)$.

Proof of Theorem 6.24. M a compact smooth manifold, $\alpha \in H_T^*(M)$. Let's not assume isolated fixed points for now. Let's compute what α looks like.

$$H_T^*(M)_{\mathrm{loc}} \cong \bigoplus_{C \text{ component of } M^T} H_T^*(C)_{\mathrm{loc}}$$

$$\alpha = \sum_C \frac{\alpha|_C}{e(N_C M)}$$

$$N_C M = \bigoplus_{\lambda \in T^*} (N_C M)_\lambda$$

$$e(N_C M) = \prod_\lambda e((N_C M)_\lambda) = \prod_\lambda \prod_{\text{Chern roots}} (\lambda + r_i)$$

Note that none of these λ in the product are zero, because they are in directions transverse to the fixed-points component C. So the Euler class is nonzero, and we may divide by it. So we have

$$\int \alpha = \sum_C \int \frac{\alpha|_C}{e(N_C M)} = \sum_C \frac{\int \alpha|_C}{e(N_C M)}.$$

If each component is a point, then $\int \alpha|_C = \alpha|_C$, so

$$\int \alpha = \sum_C \frac{\alpha|_C}{e(N_C M)}.$$

□

6.5 The Cartan model of $H_T^*(M)$

If you wanted ordinary cohomology of M, you'd look at the de Rahm complex. Under the Cartan model, however, you look at forms taking values in $\mathrm{Sym}(\mathfrak{t})^*$. This makes a complex

$$\Omega^\bullet(M; \mathrm{Sym}(\mathfrak{t})^*)^T = (\Omega^\bullet(M) \otimes \mathrm{Sym}(\mathfrak{t})^*)^T$$

with differential

$$\widetilde{d} = d \otimes 1 + \sum_i \iota_{X_i} \otimes X^i$$

where $\{X_i\}$ is a basis for $\mathfrak{t}$, and $\{X^i\}$ is a basis for $\mathfrak{t}^*$; each X^i is given degree 2 in $\mathrm{Sym}(\mathfrak{t})^*$.

Now assume that (M, ω) is symplectic; so $d\omega = 0$, and therefore ω defines a class in $H^2(M)$. However, $\widetilde{d}(\omega \otimes 1) \neq 0$. Let

$$\widetilde{\omega} = \omega \otimes 1 - 1 \otimes \Phi,$$

where Φ is designed to make $\widetilde{d}\widetilde{\omega} = 0$. (It turns out Φ is the moment map).

6.6 Duistermaat-Heckman Measures

Theorem 6.25 (Duistermaat-Heckman)**.**

$$\int e^{\widetilde{\omega}} = \sum_{f \in M^T} \frac{e^{\widetilde{\omega}}|_f}{\prod\{\textit{weights in } T_f M\}} = \sum_{f \in M^T} \frac{e^{-\Phi(f)}}{\prod\{\textit{weights of } T_f M\}} \in \overline{H_T^*} = \overline{\mathrm{Sym}(\mathfrak{t})^*}$$

Next, we can Fourier transform this thing. This should map to a sum of products of integration operators with delta functions at the points $\Phi(f)$.

Given $\Phi\colon M \to \mathfrak{t}^*$, consider the symplectic volume $\omega^{\wedge \frac{1}{2} \dim M}$. We can push forward this measure to $\mathfrak{t}^*$, called the **Duistermaat-Heckman measure** on $\mathfrak{t}^*$. This is the Fourier transform of $\int e^{\widetilde{\omega}}$.

To Fourier transform this sum, let's do it piece by piece. First, the Fourier transform of $e^{-\Phi(f)}$ is

$$\delta_{\Phi(f)},$$

which is a distribution on $\mathfrak{t}^*$.

Now choose $X \in \mathfrak{t}$ such that for any weight λ of $T_f M$,

$$\langle X, \lambda \rangle \neq 0.$$

This holds if and only if X is a vector field on M with zeros only at M^T. We have that

$$\Lambda = \Lambda_+ \coprod \Lambda_-,$$

where $\Lambda_\pm$ is the set of weights μ such that $\langle X, \mu \rangle$ is positive or negative, respectively.

Definition 6.26.

$$\text{Fourier Transform}\left(\frac{e^{-s}}{\prod_{\substack{\lambda \in \Lambda \\ \langle X, \lambda \rangle \neq 0}} \lambda}\right) = (-1)^{|\Lambda_-|}\left(\text{integrate } \delta_S \text{ in directions } \Lambda_+ \coprod -\Lambda_-\right)$$

Example 6.27. Consider $M = \mathbb{CP}^2 \circlearrowleft T^2$. Then $[\widetilde{\omega}] = c_1(\mathcal{O}(1))$, and the Fourier transform of $\int e^{\widetilde{\omega}}$ is Lebesgue measure supported only inside the moment polytope, and zero outside.

Fact 6.28. *The composition*

$$M \xrightarrow{\Phi} \mathfrak{t}^* \xrightarrow{\cdot X} \mathbb{R}$$

is a Morse function. The eigenvalues of the Hessian are the $\langle X, \lambda \rangle$ for $\lambda \in \mathfrak{t}^*$.

This gives us a Morse decomposition of M,

$$M = \coprod_{f \in M^T} M_f.$$

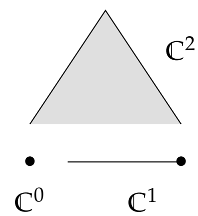
Hence,

$$\mathrm{csm}(1_M) = \sum \mathrm{csm}(M_f).$$

Example 6.29. Continuing the previous example, if $M = \mathbb{CP}^2 \circlearrowleft T^2$, then the Morse function is given by

$$[x, y, z] \longmapsto \frac{(|x|^2, |y|^2, |z|^2)}{|x|^2 + |y|^2 + |z|^2} \longmapsto \frac{|x|^2 - |z|^2}{|x|^2 + |y|^2 + |z|^2}.$$

The Morse decomposition is



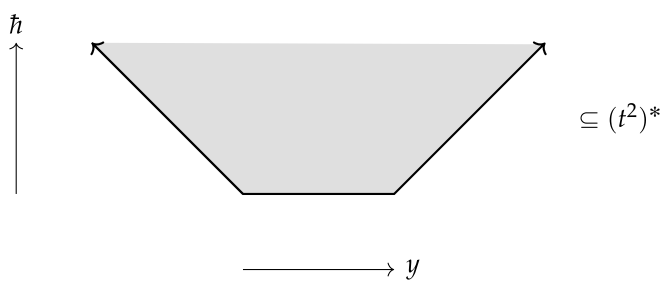
Definition 6.30. If $S = \sum m_i [S_i]$ is a T-invariant cycle on M, then define

$$\int_S \alpha := \int_M \alpha S.$$

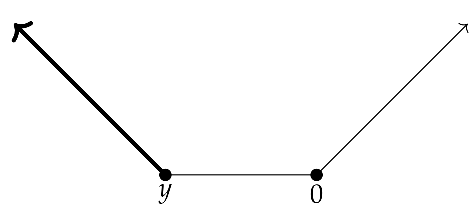
Example 6.31. If $S \hookrightarrow M$ is a submanifold, then we write

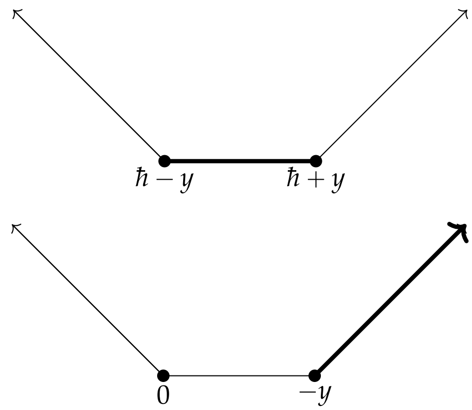
$$\int_S \alpha = \int_S \alpha|_S$$

Example 6.32. Consider $M = \mathbb{CP}^1 = \{0\} \sqcup \mathbb{C}^\times \sqcup \{\infty\}$ with an action of $\mathbb{C}^\times$ fixing 0 and ∞. Then we get $(\mathbb{C}^\times)^2 \circlearrowright T^*M \cong \mathcal{O}(-2)$. This has the polytope



The weights in the normal bundle to the cotangent space at zero is the characteristic cycle of distrubutions supported at zero. So we get





Setting $\hbar = 0$ (flattening the picture) we get from this Lebesgue measure on the half-line, interval, and half line again.

Consider $\pi\colon T^*M \to M$. This gives a form on T^*M by $\pi^*(\widetilde{\omega})$. Let $\zeta\colon M \to T^*M$ be the zero section. Then $[\zeta]$ is the class of $M \subseteq T^*M$, which is just $\operatorname{csm}(1_M)$.

$$\begin{aligned}\int_M e^{\widetilde{\omega}} &= \int_{T^*M} [\zeta] e^{\pi^*(\widetilde{\omega})} \\ &= \int_{T^*M} \operatorname{csm}(1_M) e^{\pi^*\widetilde{\omega}} \\ &= \int_{T^*M} \sum_{f \in M^T} \operatorname{csm}(M_f) e^{\pi^*\widetilde{\omega}} \\ &= \sum_f \int_{\operatorname{cc}(M_f)} e^{\widetilde{\omega}} \qquad (3)\end{aligned}$$

If we Fourier transform both sides of Eq. (3), on the left hand side we get the Duistermaat-Heckman measure on M, and on the right hand side we get the sum of Duistermaat-Heckman measures of the components of the Morse decomposition $M = \coprod_f M_f$.

$$DH(M) = \sum_f DH(M_f)$$

Lemma 6.33. *If $f \notin A \subseteq M$, where A is locally closed and M is smooth and compact, both with an action of T, then $\hbar$ divides $\operatorname{csm}(1_A)|_f$.*

$\hbar$ is the dilation equivariant parameter: $H^*_{\mathbb{C}^\times}(\mathrm{pt}) = \mathbb{Z}[\hbar]$.

Proof. The dumb case is if $\dim A = 0$, then A is not only locally closed but closed. So f is far away. So assume that $\dim A > 0$.

The proof proceeds by decreasingly special cases.

In the first case, let $M = TV_P$ be a toric variety and $A = T$. Then we get $\mathrm{csm}(A) = [M] \in H_*(M)$ from $(-\hbar)^{\dim A} \in H^*_{T\times\mathbb{C}^\times}(T^*M)$. So $\hbar$ divides this CSM class.

The second, slightly more general case, is $M = \mathbb{C}^n$, $A = \mathbb{C}^k \times T^{n-k}$, and $f = \vec{0} \in \mathbb{C}^n$. Therefore,

$$A = \coprod_{S\in[k]} T^S \times T^{n-k},$$

$$\mathrm{csm}(A) = \sum_{S\in[k]} \mathrm{csm}(T^S \times T^{n-k}) \in H^*_{T\times\mathbb{C}^\times}(\mathbb{C}^k \times T^{n-k})$$

Now restrict to $\vec{0} \in M$ to see that $\hbar^{n-k}$ divides $\mathrm{csm}(A)|_{\vec{0}}$.

The third case is when $M\backslash A$ is a simple normal crossings divisor containing the point f. Nearby f, we can reduce to the second case.

For the general case, consider the resolution

$$\begin{tikzcd}
& A \arrow[rr, hook] \arrow[dl, hook] \arrow[dr, hook] & & M \\
\widetilde{A} \arrow[rr] \arrow[urrr, bend right=60, "\pi"'] & & \overline{A} \arrow[ur, hook] &
\end{tikzcd}$$

Then $\widetilde{A}\backslash A$ is a simple normal crossings divisor, so we apply case 3 to get the lemma on $\widetilde{A}$.

Assume that $f \hookrightarrow \overline{A}$. Note that the map $\widetilde{A} \to \overline{A}$ is both proper and T-equivariant. Then by Borel's fixed point theorem, there is at least one fixed point of the torus action, so there is a map $b\colon [f] \to \widetilde{A}$. Now

$$\mathrm{csm}(A) = \pi_*(c(\text{log tangent bundle of } (\widetilde{A}, \widetilde{A}\backslash A)))$$

$$b^*\pi^*\pi_*(c(\log)) = b^*(c(\log)\pi^*\pi_* 1)$$

$$\pi_*(c^\hbar(\text{log tangent bundle})) = \pi_*\left(\sum_{\text{components } D \text{ of } (\widetilde{A})^T} \frac{c^\hbar|_D}{e(N_D\widetilde{A})}|_f\right) = \sum_D \pi_*\left(\frac{c^\hbar|_D}{e(N_D\widetilde{A})}\right) \in H^*_{T\times\mathbb{C}^\times}$$

By case 3, $\hbar$ divides $c^\hbar$. Also, $\hbar$ does not divide $e(N_D\widetilde{A}) \in H^*_T(D)[\hbar]$. Therefore, $\hbar$ divides the CSM class of A. $\square$

$$DH(M) = \sum_f \text{Fourier Transform}\left(\int_{cc(M_f)} e^{\widetilde{\omega}}\right)\Big|_{\hbar\to 0}$$

$$\int_{cc(M_g)} e^{\widetilde{\omega}}\Big|_{\hbar\to 0} = \sum_{f\in (T^*M)^{T\times\mathbb{C}^\times}} \left(\frac{e^{\widetilde{\omega}}|_f [cc(M_g)]|_f}{\prod_\lambda \lambda(\hbar-\lambda)} \right)\Big|_{\hbar\to 0}$$

where λ runs over all weights of $T_f M$. Sending $\hbar \to 0$ kills each $[cc(M_g)]|_f$ unless $f = g$. Then $[cc(M_g)]|_g$ is the conormal bundle to M_g near g. Therefore,

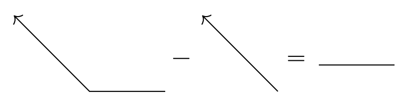
$$\int_{cc(M_g)} e^{\widetilde{\omega}}\Big|_{\hbar\to 0} = \frac{e^{\widetilde{\omega}}|_g [cc(M_g)]|_g}{\prod_\lambda -\lambda^2}.$$

$[cc(M_g)]|_g$ is all the weights in $T_g(T^*M)$ that are not in the conormal bundle of M_g, so this cancels with some stuff in the denominator, and we get

$$\int_{cc(M_g)} e^{\widetilde{\omega}}\Big|_{\hbar\to 0} = \frac{e^{-\Phi(g)}}{\prod \text{weights in } T_g(CM_g)} = (-1)^{\operatorname{codim} M_g} \frac{e^{-\Phi(g)}}{\prod \text{weights in } T_g M}.$$

This proves the Duistermaat-Heckman theorem.

Example 6.34. $\mathbb{CP}^1 = \{0\} \sqcup \mathbb{C}^{-1}$. The characteristic cycle looks like



6.7 Spherical actions

Say $G \circlearrowright M$ manifold. Therefore, have

$$\begin{array}{rcl} G \circlearrowright T^*M & \xrightarrow{\Phi_G} & \mathfrak{g}^* \\ (m, \vec{v}) & \longmapsto & (x \mapsto \langle X|_m, \vec{v}\rangle) \end{array}$$

$$\phi^{-1}(0)/G \sim T^*(M/G)$$

Proposition 6.35. *$\phi_G^{-1}(0)$ is the union of the conormal bundles to G-orbits in M.*

Example 6.36. $\mathbb{C}^\times \circlearrowright \mathbb{C}$ via $\Phi_{\mathbb{C}^\times}(m, \vec{v}) = m\vec{v}$; $m, \vec{v} \in \mathbb{C}$.

The interesting case is $G \circlearrowright M$ with finitely many orbits. We can perhaps think of B acting on G/B.

Definition 6.37. $G \circlearrowright M$ is **spherical** if $B_G \circlearrowright M$ has an open orbit. This is equivalent to the fact that $B_G \circlearrowright M$ has finitely many orbits.

Theorem 6.38. *Assume that $M = \operatorname{proj}(R)$ with $R = \bigoplus_n R_n$. Assume further that $G \circlearrowright R$ homogeneously and R is a domain. Then M is spherical if and only if each R_n is a multiplicity-free G-representation.*

Half of a proof. ($\Longrightarrow$). $V_\lambda \subseteq R_n$ if and only if $\Gamma(M; \mathcal{O}(n)) = R_n$ for large n. It follows that the multiplicity of V_λ in R_n is equal to

$$\dim (R_n)^\lambda = \dim (\Gamma(M; \mathcal{O}(n)))^\lambda ;$$

R_n^λ is all of the B-weight vectors of weight λ inside R_n.

Now $B \circlearrowright M$ has a dense orbit, so the right-hand-side is at most 1-dimensional. $\square$

Example 6.39. Say that $M = G/P$. Then

$$\begin{tikzcd}
T^*(G/P) \arrow[r, "\Phi_G"] \arrow[rr, bend left, "\Phi_B"] & \mathfrak{g}^* \arrow[r, two heads] & \mathfrak{b}^* \\
\Phi_B^{-1}(0) \arrow[u, hook] \arrow[r] & \mathfrak{b}^\perp \cap \Phi_G(T^*(G/P)) \arrow[u] &
\end{tikzcd}$$

$M/B \cong W/W_P \cong$ components of $\Phi_B^{-1}(0)$.

$\Phi_G(T^*(G/P))$ is G-invariant, conical, and in fact a nilpotent orbit closure.

Definition 6.40. $\mathfrak{b}^\perp \cap \Phi_G(T^*(G/P))$ is called the **orbital scheme**, and the components are called **orbital varieties**.

Remark 6.41. Actually, nobody other than Allen calls it the "orbital scheme," but they should. People often call both the whole thing and it's components "orbital varieties," but that can be confusing and "variety" should be reserved for things that are reduced. Maybe they're scared of the word "scheme," in which case they should get over it.

Theorem 6.42 (Spaltenstein (1977?))**.** $G = \mathrm{GL}(n)$.

$\overline{\mathcal{O}}_\lambda$ = *nilpotent matrices with Jordan canonical form corresponding to a partition* $\lambda \vdash n$.

$$T^*(G/P_\lambda) \twoheadrightarrow \overline{\mathcal{O}}_\lambda$$

P_λ *is block upper triangular matrices with blocks corresponding again to the partition* $\lambda \vdash n$. *Use* tr *to identify* $\mathfrak{g}$ *with* $\mathfrak{g}^*$.

$$\begin{array}{rcl} \overline{\mathcal{O}}_\lambda \cap \mathfrak{n} & \longrightarrow & SYT \\ X & \longmapsto & (J_1, \ldots, J_n) \end{array}$$

where J_i *is the Jordan Canonical form of the upper-left* $i \times i$*-block of the matrix* X. *Then the theorem is that the components of* $\overline{\mathcal{O}}_\lambda \cap \mathfrak{n}$ *correspond bijectively to standard Young tableaux of shape* λ.

Going back to the diagram in Example 6.39, we have a map between $\Phi_B^{-1}(0)$ and the orbital scheme $\mathfrak{b}^\perp \cap \Phi_G(T^*(G/P_\lambda))$. The former has components corresponding to W/W_{P_λ}, and the latter has components corresponding to SYT_λ. So certainly

$$\Phi_B^{-1}(0) \twoheadrightarrow \mathfrak{b}^\perp \cap \Phi_G(T^*(G/P))$$

is a surjection. What's the relation between these spaces of components, that is the relation between W/W_{P_λ} and SYT_λ.

Example 6.43. Suppose that $G/P = \mathrm{Gr}(n, \mathbb{C}^{2n})$ and

$$\lambda = \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}$$

Then $\overline{\mathcal{O}}_\lambda = \{M \in M_{n\times n} \mid M^2 = 0,\ \mathrm{tr}(M) = 0\}$. Then $W = S_{2n}$ and $W_{P_\lambda} = S_n \times S_n$.

W/W_{P_λ} is paths from $(0,0)$ to (n,n) in $\mathbb{Z}^2$, an element recording a sequence steps up and steps to the right.

SYT_λ is paths from $(0,0)$ to (n,n) entirely above the diagonal.

Take some partition $\lambda \vdash n$.

$$CX^\lambda = \overline{CX_o^\lambda} \subseteq T^* \mathrm{Gr}(n, \mathbb{C}^{2n})$$

$$T^* \mathrm{Gr}(n, \mathbb{C}^{2n}) = \{(V, M) \in \mathrm{Gr}(n, \mathbb{C}^{2n}) \times M_{2n\times 2n} \cong \mathfrak{gl}_{2n}^* \mid \ker M \geqslant V \geqslant \mathrm{im}\, M\}$$
$$CX_o^\lambda = \{(V, M) \in X_o^\lambda \times \mathfrak{n} \mid \ker M \geqslant V \geqslant \mathrm{im}\, M\}$$

Definition 6.44. For a matrix A, define $A_<$ to be the same matrix but with the lower triangle (including the diagonal) zeroed out.

Theorem 6.45 (Melnikov (2003?))**.** *Each B-orbit on $\mathfrak{n} \cap \{M^2 = 0\}$ contains a unique $\pi_<$, where $\pi \in S_{2n}$ such that $\pi^2 = 1$ (think of it as a permutation matrix).*

6.8 D-modules of twisted differential operators

$$\begin{tikzcd} BS^{Q\setminus\mathrm{first}} \arrow[r] & BS^Q \arrow[r, two heads, "m"] \arrow[d] & X^W & \subseteq & G/B \\ & \mathbb{P}^1 & & & \end{tikzcd}$$

$$[0] - [\infty] = \alpha_{\mathrm{first}} \in H_T^2(\mathbb{P}^1 = P_{\mathrm{first}}/B)$$

$$\frac{[BS^{Q\setminus\text{first}}] - r_\alpha[BS^{Q\setminus\text{first}}]}{\alpha_{\text{first}}} = [BS^Q] \xmapsto{m} [X^w] = \underbrace{\frac{1}{\alpha}(1-r_\alpha)}_{\partial_\alpha}[X^{r_\alpha w}]$$

This is due to Bernstein-Gelfand-Gelfand in 1973. Notice that $\partial_\alpha^2 = 0$.

Let's look at $r_\alpha + \hbar\partial_\alpha \circlearrowright H_T^*(G/B)[\hbar] \cong H^*_{T\times\mathbf{C}^\times}(T^*(G/B))$. We have that

$$(r_\alpha + \hbar\partial_\alpha)^2 = 1.$$

Recall that

$$\operatorname{csm}(X_o^w) = \sum_{R\subseteq Q} m_*([BS_R]).$$

This is (combinatorially) equivalent to

$$(r_\alpha + \hbar\partial_\alpha)\operatorname{csm}(X_o^w) = \operatorname{csm}(X_o^{r_\alpha w}).$$

You can deduce one from the other through some not-so-interesting combinatorics, due to Aluffi-Mihalcea (although they have set $\hbar \mapsto -1$).

G acts on G/B from the left, but nothing acts on G/B on the right. But G/B is homotopic to G/T, which has a map from $B/T \cong N$.

$$\begin{tikzcd} G/B & \simeq & G/T \arrow[d, hook, "\substack{\text{general}\\ \text{orbit}}"] & B/T \arrow[l] & \cong & N \\ & & \mathfrak{g}^* & & & \end{tikzcd}$$

However, G/B is projective and G/T is affine, so they are not equivalent except in a topological sense.

However, G/T has an action of W on the right, which freely and transitively permutes $N(T)/T \hookrightarrow G/T$. Also $BN(T)/T = [BwT/T]$. Each of these is closed, if and only if $BwT \subseteq G$ is closed, if and only if $B\backslash BwT \subseteq B\backslash G$ is closed. However, $B\backslash BwT$ is a T-fixed point, so $[BwT/T]$ is indeed closed.

Claim that there is a degeneration of G/T to $T^*(G/B)$. It's easy to see that there is a degeneration from G/T to the nilpotent cone $\mathcal{N}$, given by

$$G\cdot\lambda \longmapsto \lim_{z\to 0} z(G\cdot\lambda) = \lim_{z\to 0} G\cdot(z\lambda)$$

for $\lambda \in \mathfrak{t}^*_{\text{reg}} \subseteq \mathfrak{g}^*$. The family of these comes from

$$\operatorname{Spec}(\operatorname{Fun}(\mathfrak{g}) \hookleftarrow \operatorname{Fun}(\mathfrak{g})^G = \operatorname{Fun}(\mathfrak{t})^W)$$

Recall that

$$T^*(G/B) = \{(F, X) \mid F \text{ flag}, X \text{ nilpotent preserving } F\}$$

So what we want (in type A, at least) is

$$\left\{(F,X,\lambda)\in G/B\times\mathfrak{g}\times\mathfrak{t}\;\middle|\; X\cdot F\leqslant F, X|_{F_i/F_{i-1}}=\lambda_i\right\}\xrightarrow{\lambda}\mathfrak{t}$$

Remark 6.46. Where we're heading is

$$G/T \rightsquigarrow T^*(G/B)$$

while

$$BwT/T \rightsquigarrow \mathrm{ss}(X_o^w)$$

Consider the following diagram of sheaves of filtered algebras on G/B (not all commutative algebras, but we insist that they are **almost commutative:** the associated graded is commutative). Let $\lambda\in\mathfrak{t}^*$ regular.

$$\begin{tikzcd} \mathcal{D}^\lambda_{G/B} \arrow[r, rightsquigarrow] \arrow[d, dashed, "?"] & \mathcal{D}_{G/B} \arrow[d, rightsquigarrow] \\ \mathcal{O}_{G\cdot\lambda} \arrow[r, rightsquigarrow] & \mathcal{O}_{T^*(G/B)} \end{tikzcd}$$

Definition 6.47. If M is a manifold, equal to the quotient of $\widetilde{M}$ by a free action of T

$$M=\widetilde{M}/T,$$

and $\lambda\in\mathfrak{t}^*$, then the **λ-twisted differential operators** on M are $\mathcal{D}^\lambda_M := (\mathcal{D}_{\widetilde{M}})^T/\langle\lambda\rangle$, where $\langle\lambda\rangle$ is the ideal coming from the map

$$\begin{tikzcd} \mathrm{Sym}(\mathfrak{t})=U(\mathfrak{t}) \arrow[r] \arrow[d, "\lambda"] & (\mathcal{D}_{\widetilde{M}})^T \\ \mathbb{C} & \end{tikzcd}$$

which lands in the center of $(\mathcal{D}_{\widetilde{M}})$.

Example 6.48. $M=G/B$ and $\widetilde{M}=G/N$. Then $G/B=(G/N)/T$. For example, $\lambda\in T^*$ if and only if there is $\mathcal{L}_\lambda\to G/B$ a line bundle,

$$\begin{tikzcd} \mathcal{L} \arrow[r, equal] \arrow[d] & G\times^B\mathbb{C}_\lambda \\ G/B \arrow[r, phantom, "="] & G/B\times^T\mathbb{C}_\lambda \end{tikzcd}$$

Then D^λ_M is differential operators on $\mathcal{L}_\lambda$.

Example 6.49. We have $\mathrm{SL}(2)/N \cong \mathbb{C}^2 \backslash \{0\}$, and so $\mathcal{D}_{\mathrm{SL}(2)/N} \cong \mathcal{D}_{\mathbb{C}^2\backslash\{0\}}$. This is generated by $\widehat{x}, \widehat{y}, {}^{d}/_{dx}, {}^{d}/_{dy}$.

$$T = \left\{ \begin{bmatrix} z & \\ & z^{-1} \end{bmatrix} \right\}$$

$$(\mathcal{D}_{\mathrm{SL}(2)/N})^T = \left\langle \begin{bmatrix} \widehat{x}\widehat{y} & \widehat{x}\,{}^{d}/_{dx} \\ \widehat{y}\,{}^{d}/_{dy} & {}^{d}/_{dx}\,{}^{d}/_{dy} \end{bmatrix} \right\rangle$$

Once you work out the commutation relations among these four operators, this turns out to be

$$(\mathcal{D}_{\mathrm{SL}(2)/N})^T \cong U(\mathfrak{gl}(2)).$$

Theorem 6.50 (Beilinson-Bernstein)**.** $\Gamma(\mathcal{D}^{\lambda}_{G/B}) \cong U(\mathfrak{g})/\langle \lambda \rangle$.

The only irreps on which the center of $U(\mathfrak{g})$ acts in the same way as it does on V_λ have high weights $W \cdot (\lambda + \rho) - \rho$.

The easy representations of $U(\mathfrak{g})/\langle\lambda\rangle$ are the Verma modules $L(w) := L(w(\lambda + \rho) - \rho)$. If $L(w) \geqslant L(v)$, then $w(\lambda+\rho) - \rho - (v(\lambda+\rho) - \rho)$ is a sum of positive roots. Therefore, $w \cdot \lambda - v \cdot \lambda$ is in the root lattice.

But if λ is general, then $w \cdot \lambda - v \cdot \lambda$ being in the root lattice is impossible. Hence, all $L(w)$ are irreducible.

6.9 A bit of silliness

Let's say we want to sum some function f over some range $[0, b]$. Say

$$F(b) = \sum_{n=0}^{b} f(b).$$

What's the inverse of summing? Differences!

$$\sum = \frac{1}{\Delta}$$

$$(\Delta f)(a) = f(a+1) - f(a) = (e^D - 1)(f)$$

where D is the differential operator, which we have exponentiated. The last equality above by Talyor series. Therefore,

$$\frac{1}{\Delta} = \frac{1}{e^D - 1}$$

Then as a power series in D, this thing has a pole at $D = 0$, whatever that means. So

$$\frac{1}{\Delta} = \frac{1}{D}\left(\frac{D}{e^D - 1}\right) = \frac{1}{D}\,(1 + \text{Bernoulli numbers})$$

But what's the inverse of the differentiation operation? Integration! So

$$\sum = \frac{1}{\Delta} = \int + \text{error term}$$

If you work it all out (maybe for polynomials), you get the Euler summation formula. (Derivation due to Legendre.)

6.10 Calabi-Yau, Hirzebruch-Riemann-Roch

If M is a real manifold, then smooth sections $\Gamma(M; \bigwedge^k T^*M)$ and the exterior derivative d give rise to the de Rahm complex of M, which gives $H^*(M; \mathbb{R})$.

If instead M is a compact complex manifold, there is higher sheaf cohomology $H^{p,q}(M; \mathbb{C}) := H^q(M; \bigwedge^p T^*M)$. This is Dolbeault cohomology; $H^{p,q}(M)$ are also called **Hodge groups**.

$$H^k(M) \cong \bigoplus_{p+q=k} H^{p,q}(M)$$

This isomorphisms, however, is not natural.

Instead of just a line of cohomology, we now have a diamond, called the **Hodge diamond**. It's left-right symmetric $H^{p,q}(M) \cong H^{q,p}(M)$, which is one analogue of Poincaré duality. We also have a top-bottom symmetry

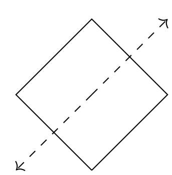
$$H^{p,q} \cong (H^{n-p,n-q})^*$$

This is Poincaré duality, from Serre duality.

$$\begin{tikzcd}
& H^{n,n} & \\
H^{n,0} \arrow[ur] & & H^{0,n} \arrow[ul] \\
& H^{0,0} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Example 6.51. If $M = \coprod C^k$, then $H^{p,q} = 0$ for $p \neq q$.

When might



also be a symmetry of the Hodge diamond? We must have that $H^{0,n}(M) \cong H^{0,0}(M) = \mathbb{C}$

So $\mathbb{C} \cong H^{0,n}(M) = \Gamma(M; \bigwedge^n T^*M)$

Definition 6.52. This condition is called **Calabi-Yau**.

Example 6.53. If $M = \Sigma^g$, then the Hodge diamond has dimensions

$$\begin{tikzcd}
& 1 & \\
g \arrow[ur] & & g \arrow[ul] \\
& 1 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

So the only Riemann surfaces that are Calabi-Yau are genus 1 (elliptic curves).

Definition 6.54. The **Hodge-Poincaré** Polynomial of M is

$$HP(x, y) = \sum_{p,q} x^p y^q \dim H^{p,q}(M)$$

The Euler characteristic is $\chi = HP(-1, -1)$.

$$\chi_y := HP(-1, y)$$

is the **Hirzebruch χ_y-genus**.

How do we compute χ_Y?

$$\begin{aligned}
\chi_Y(M) &= \sum_{p,q} (-1)^p y^q \dim H^p(M; \textstyle\bigwedge^q T^*M) \\
&= \sum_q y^q \left(\sum_p (-1)^p \dim H^p(M; \textstyle\bigwedge^q T^*M) \right) \\
&= \sum_q y^q \chi(M; \textstyle\bigwedge^q T^*M)
\end{aligned}$$

This is the "K-theory version of integrating the class $[\bigwedge^q T^*M]$".

Definition 6.55. For a line bundle $\mathcal{L}$ on M, the **Todd class** of $\mathcal{L}$ is

$$\mathrm{Td}(\mathcal{L}) = \frac{c_1(\mathcal{L})}{1 - e^{-c_1(\mathcal{L})}} \in H^*(M).$$

Definition 6.56. For $\mathcal{V} \to M$ a complex vector bundle, with Chern roots $(\mathcal{L}_i)$, then

$$\mathrm{Td}(\mathcal{V}) = \prod \mathrm{Td}(\mathcal{L}_i).$$

Theorem 6.57 (Hirzebruch-Riemann-Roch). *If $\mathcal{L} \to M$ is a complex line bundle over a compact complex smooth manifold M,*

$$\sum_i (-1)^i H^i(M; \mathcal{L}) = \int_M e^{c_1(\mathcal{L})} \mathrm{Td}(TM)$$

Example 6.58. $M = \mathbb{CP}^1$ and $\mathcal{L} = \mathcal{O}(k)$ with $k \geqslant 0$.

$$k + 1 = \int_{\mathbb{CP}^1} (1 + k[\mathrm{pt}])(1 + [\mathrm{pt}])$$

Remark 6.59. There's a version of Hirzebruch-Riemann-Roch that works for general vector bundles $\mathcal{V} \to M$, not just line bundles. You have to replace $e^{c_1(\mathcal{L})}$ by the Chern character, which is a map $\mathrm{Ch}\colon K(M) \to H^*(M)$.

We can use Hirzebruch-Riemann-Roch to compute $\chi_Y(M)$. Let TM have Chern roots $\mathcal{L}_i$, with $c_1(\mathcal{L}_i) = r_i$.

Let e_q be the q-th elementary symmetric polynomial.

Theorem 6.60 (Hirzebruch signature theorem)**.** *Let L_i be the Chern roots of TM, with first Chern classes r_i. Then*

$$\begin{aligned}
\chi_y(M) &= \sum_q y^q \int_M Td(M)\, e_q(\{\exp(c_1(L_i^*))\}) \\
&= \int_M Td(M) \sum_q y^q\, e_q(\{\exp(-r_i)\}) \\
&= \int_M Td(M) \sum_q\, e_q(\{y \exp(-r_i)\}) \\
&= \int_M \prod_i Td(L_i) \prod_i (1 + y \exp(-r_i)) \\
&= \int_M \prod_i r_i \frac{1 + y \exp(-r_i)}{1 - \exp(-r_i)}
\end{aligned}$$

At $y = -1$ the integrand is just $\prod_i r_i = e(TM)$, so this does indeed recover the Euler characteristic.

Fact 6.61. *χ_Y extends to an additive function $Var(\mathbb{C}) \to \mathbb{C}[y]$, where $Var(\mathbb{C})$ is the Grothendieck group of varieties over $\mathbb{C}$, with operation disjoint union. Then setting $y = -1$ gives usual Euler characteristic.*

$$\begin{tikzcd}
Var(\mathbb{C}) \arrow[r, "\chi_y"] \arrow[dr, "\chi"'] & \mathbb{C}[y] \arrow[d, "y=-1"] \\
& \mathbb{C}
\end{tikzcd}$$

Remark 6.62. There is a theory of CSM classes using this! (See arχiv 1303.4454). Say $T \hookrightarrow M = TV_p$ is smooth. Then the χ_y-csm class is

$$T_{y*}(T) = (1 + y)^n\, \mathrm{Td}(\textstyle\bigwedge^n T^*M)$$