# MATH 6530: K-THEORY AND CHARACTERISTIC CLASSES

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Cornell University
Fall 2017

Last updated November 8, 2018.
The latest version is online here.

# Contents

## Contents by Lecture

## Administrative

- There is a course webpage here.
- Office hours are Monday 1-2pm and Friday 2-3pm, but subject to change.
- There will be approximately four homework sets and a small final project for those who really want or need a grade. Homework must be typed.
- Inna's notes are on the class webpage, and are more complete than these.

## 1 Vector bundles

What are we studying in this class? Mostly, we'll talk about **vector bundles**. These seem like geometric objects, but really they're topological objects. They come up most naturally when we talk about geometry of manifolds – tangent lines to curves on a manifold don't ever intersect, and indeed they know nothing about one another. The fact that they might look like it is an illusion of the fact that we choose coordinates and go into $\mathbb{R}^n$.

We should get away from coordinates then, and look at manifolds as intrinsic objects. Using this, we can define tangent spaces at a point.

**Definition 1.1.** A **vector bundle** on a **base space** $B$ is is a topological space $E$ (the **total space**) together with a map $p\colon E \to B$ such that for all $b \in B$,

- $p^{-1}(b)$ has the structure of a vector space, and
- for all $b \in B$, there is a neighborhood $U$ of $b$ and an integer $k$ (the **rank**) with a homeomorphism $\phi_b\colon U \times \mathbb{R}^k \to p^{-1}(U)$ such that the following diagram commutes,

$$\begin{array}{ccc} U \times \mathbb{R}^k & \xrightarrow{\phi_b} & p^{-1}(U) \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

  and the restriction of $\phi_b$ to each fiber is a linear homomorphism.

**Example 1.2.** The **trivial bundle** $pr_1\colon B \times \mathbb{R}^k \to B$ is a vector bundle, called the **trivial bundle of rank** $k$ over $B$.

**Remark 1.3.** There are really two structures contained in the definition of a vector bundle. One is that of a fiber bundle, where the fibers are allowed to be anything: tori, spheres, or other things without vector space structure.

The second thing is the linear structure on the fibers.

**Example 1.4.** The **tangent bundle** to a manifold $M$ embedded in $\mathbb{R}^N$ is

$$E = \{(x, v) \mid x \in M, v \text{ tangent to } M \text{ at } x\}$$

this is a subspace of $M \times \mathbb{R}^N$, and the projection onto the first factor is the map $p\colon E \to M$.

But we want an intrinsic definition of tangent bundles that doesn't depend on the embedding.

**Example 1.5.** Define the tangent bundle to an $n$-manifold $M$ by

$$TM = \coprod_{x \in M} T_x M$$

as a set, with $p\colon TM \to M$ defined in the obvious way. We can induce a topology on this set by the map $\coprod_{x \in U} T_x M \to \mathbb{R}^n \times \mathbb{R}^n$ given by choosing coordinates around $x$ in the first coordinate, and using the appropriate tangent vector in the second.

**Example 1.6.** The **normal bundle** to an embedded $n$-manifold $M \hookrightarrow \mathbb{R}^N$ is

$$\nu = \{(x, v) \mid x \in M, v \text{ normal to } M\} \subseteq M \times \mathbb{R}^N.$$

This has rank $k = N - n$.

**Example 1.7.** How do we construct vector bundles in general? If $p\colon E \to B$ is a vector bundle, with homeomorphisms

$$\phi_\alpha\colon p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$$
$$\phi_\beta\colon p^{-1}(U_\beta) \to U_\beta \times \mathbb{R}^k$$

Then we have the composite homeomorphism

$$(U_\alpha \cap U_\beta) \times \mathbb{R}^k \xrightarrow{\phi_\alpha^{-1}} p^{-1}(U_\alpha \cap U_\beta) \xrightarrow{\phi_\beta} (U_\alpha \cap U_\beta) \times \mathbb{R}^k$$

Restricting to the second coordinate, this gives an element $GL_k(\mathbb{R})$ above every point. This gives a smooth map

$$g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to GL_k(\mathbb{R}).$$

These maps are called the **transition functions,** and they satisfy three things

(a) $g_{\alpha\alpha} = \mathrm{id}$

(b) $g_{\alpha\beta}(x) = g_{\beta\alpha}(x)^{-1}$

(c) $g_{\alpha\beta}(x)g_{\beta\gamma}(x)g_{\gamma\alpha}(x) = \mathrm{id}$ for $x \in U_\alpha \cap U_\beta \cap U_\gamma$.

**Proposition 1.8.** *Given an atlas* $\{U_\alpha\}$ *of* $X$ *and functions* $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}_k(\mathbb{R})$ *satisfying*

*(a)* $g_{\alpha\alpha} = \mathrm{id}$

*(b)* $g_{\alpha\beta}(x) = g_{\beta\alpha}(x)^{-1}$

*(c)* $g_{\alpha\beta}(x)g_{\beta\gamma}(x)g_{\gamma\alpha}(x) = \mathrm{id}$ *for* $x \in U_\alpha \cap U_\beta \cap U_\gamma$*,*

*you can construct a vector bundle*

$$E = \coprod_\alpha U_\alpha \times \mathbb{R}^k \Big/ (x, v) \sim (x, g_{\alpha\beta}(x) \cdot v)$$

**Remark 1.9.** We've been working with the assumption that the rank $k$ is the same everywhere, but this is not required by the definition. We may have different rank in different connected components, but they are the same on the same connected components. We don't usually care about bundles with different ranks on different components though, so we'll almost always assume the rank is uniform.

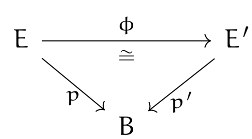
**Remark 1.10.** We haven't used any properties of $\mathbb{R}$, but we may use any other field (such as $\mathbb{C}$).

So far we haven't conclusively demonstrated that any vector bundles are nontrivial. Let's do that now.

**Definition 1.11.** An **isomorphism of vector bundles** over $B$ is a map $\phi\colon E \to E'$ such that

(a) $\phi$ is an isomorphism, and

(b) the diagram below commutes




and on each fiber it is a linear isomorphism.

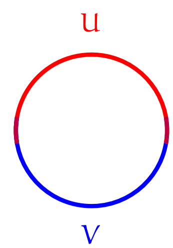
**Definition 1.12.** $\mathrm{Vect}_n(B)$ is the set of vector bundles of rank $n$ over $B$.

Question: What is $\mathrm{Vect}_n(B)$?

**Example 1.13.** If $B = *$, then $\mathrm{Vect}_n(B) = \{* \times \mathbb{R}^n\}$.

**Example 1.14.** If $B = S^1$, then there are several bundles just of rank 1: the tangent bundle $TS^1$, the trivial bundle $S^1 \times \mathbb{R}$, and there's also the Möbius bundle.

We can construct the Möbius bundle as follows. Take two open sets $U$ and $V$ on $S^1$




Then the Möbius bundle is $(U \times \mathbb{R}) \coprod (V \times \mathbb{R})$, glued on the left hand side by 1 and the right hand side by $-1$.

**Definition 1.15.** A **section** of $p \colon E \to B$ is a map $s \colon B \to E$ such that $ps = \mathrm{id}_B$.

**Example 1.16.** The **zero section** $s_0 \colon B \to E$, $b \mapsto (b, 0)$ sends a point in $B$ to the zero vector in the corresponding fiber.

**Example 1.17.** If $E \cong B \times \mathbb{R}^k$ is trivial, then there is an everywhere nonzero section: choose any nonzero $x \in \mathbb{R}^k$, and then the section is $b \mapsto (b, x)$.

**Lemma 1.18.** *The tangent bundle on $S^1$ is trivial.*

*Proof sketch.* We can imagine both of these bundles as a circle with a line on each point, either tangent to the circle ($TS^1$) or normal to the circle ($S^1 \times \mathbb{R}$). In either case, we can arrange the lines to make a cylinder by either rotating by $\pi/2$ in the plane of the circle or rotating by $\pi/2$ around a tangent vector. So these bundles look the same. □

**Lemma 1.19.** *The Möbius bundle is not trivial.*

*Proof.* Vector bundle isomorphisms preserve zero sections. So if $E \cong E'$ then so are $E \setminus s_0(B) \cong E' \setminus s_0'(B)$. Now let $E = TS^1 \cong S^1 \times \mathbb{R}$ and let $E'$ be the Möbius bundle. $E \setminus s_0(S^1)$ is $S^1 \times \mathbb{R} \setminus \{0\}$, which is disconnected, but $E' \setminus s_0(S^1)$ is connected (as we know from slicing a Möbius strip along the middle circle). □

**Definition 1.20.** If a vector bundle has rank 1, we call it a **line bundle**.

**Example 1.21.** Let $B = \mathbb{RP}^n$. Let $\gamma_{1n} \subseteq \mathbb{RP}^n \times \mathbb{R}^{n+1}$ be the bundle

$$\gamma_{1n} = \{(\ell, v) \mid v \in \ell\}.$$

This is the **tautological line bundle** over $\mathbb{RP}^n$.

**Lemma 1.22.** $\gamma_{1n}$ *has no everywhere nonzero sections.*

*Proof.* Notice that $\mathbb{RP}^n = S^n/\{\pm 1\}$. A section $s\colon \mathbb{RP}^n \to \gamma_{1n}$ is of the form $s(\pm x) = (\pm x, t(x)x)$, where we write $\pm x$ for the image of the point $x \in S^n$ in $\mathbb{RP}^n = S^n/\{\pm 1\}$. We must have that $t(x) = -t(-x)$, and $t\colon S^n \to \mathbb{R}$ is an odd function, so $t$ must hit zero somewhere. Hence, the section $s$ cannot be everywhere nonzero. □

**Lemma 1.23.** *Let $L$ be a line bundle over a base $B$. If $L$ has an everywhere nonzero section, then it is trivial.*

*Proof.* Assume that $s\colon B \to L$ is everywhere nonzero. Then define a map $f\colon L \to B \times \mathbb{R}$ by $(b, v) \mapsto (b, c)$, where $v = c \cdot s(b)$ for a unique $c$. This is an isomorphism. □

**Lemma 1.24.** *Let $f\colon E \to E'$ be a map of vector bundles. Then $f$ is an isomorphism if and only if it is a linear isomorphism on each fiber.*

*Proof.* Hatcher Lemma 1.1 □

**Proposition 1.25.** $p\colon E \to B$ *is a trivial vector bundle if and only if there are* $n$ *sections* $s_1, \ldots, s_n$ *that are linearly independent at each point of* $b$.

*Proof.* First, if $E \cong B \times \mathbb{R}^n$, then set $s_i(b) = (b, e_i)$.

Conversely, define $f\colon E \to B \times \mathbb{R}^n$ by

$$(b, v) \mapsto (b, (c_1, \ldots, c_n)),$$

where we write $v$ uniquely as

$$v = \sum_{i=1}^{n} c_i s_i(b)$$

in the basis $s_1(b), \ldots, s_n(b)$. □

**Example 1.26.**

$$TS^1 = \left\{ \big((\cos\theta, \sin\theta), (-t\sin\theta, t\cos\theta)\big) \;\middle|\; \theta \in S^1, t \in \mathbb{R} \right\}$$

Define a map $S^1 \to TS^1$ by

$$\theta \mapsto \big((\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)\big)$$

This gives an everywhere nonzero section, which defines a basis of each fiber. Hence, $TS^1$ is trivial.

**Theorem 1.27.** *If* $E \to B$ *is a fiber bundle with fiber* $F$*, then there is a long exact sequence of homotopy groups*

$$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \cdots \to \pi_1(F) \to \pi_1(E) \to \pi_1(B)$$

**Corollary 1.28.** *For a vector bundle,* $\pi_n(E) \cong \pi_n(B)$ *since the fibers are all contractible.*

## 1.1 Grassmannians

**Definition 1.29.** The **Grassmannian** $\mathrm{Gr}_n(\mathbb{R}^k)$ of $n$-planes in $\mathbb{R}^k$ is the space of $n$-dimensional linear subspaces of $\mathbb{R}^k$.

**Definition 1.30.** The **Stiefel manifold** $V_n(\mathbb{R}^k)$ is the set of all orthogonal $n$-frames in $\mathbb{R}^k$. This is a subspace of $(S^{k-1})^n$, and inherits its topology and manifold structure from that space.

**Fact 1.31.** $V_n(\mathbb{R}^k)$ *and* $\mathrm{Gr}_n(\mathbb{R}^k)$ *are compact.*

*Proof.* Notice that $V_n(\mathbb{R}^k)$ is a closed subspace of a compact space, and therefore it is compact. There is an action of the orthogonal group $O(n)$ on $V_n(\mathbb{R}^k)$, and the quotient of $V_n(\mathbb{R}^k)$ by this action is $\mathrm{Gr}_n(\mathbb{R}^k)$. Hence, $\mathrm{Gr}_n(\mathbb{R}^k)$ is compact. □

**Lemma 1.32.** $\mathrm{Gr}_n(\mathbb{R}^k)$ *is Hausdorff.*

*Proof.* It suffices to show that for any two $n$-planes $\omega_1, \omega_2$ in $\mathrm{Gr}_n(\mathbb{R}^k)$, there is a function to $R$ which has different values on $\omega_1$ and $\omega_2$. For any point $p \in \mathbb{R}^k$, let $f_p(\omega)$ be the Euclidean distance from $\omega$ to $p$. For any $(v_1, \dots, v_n) \in V_n(\mathbb{R}^k)$ representing $\omega$,

$$f_p(\omega) = \sqrt{p \cdot p - (p \cdot v_1)^2 - \ldots - (p \cdot v_n)^2}.$$

So $f_p$ is clearly continuous and well-defined as a function $\mathrm{Gr}_n(\mathbb{R}^k) \to \mathbb{R}$.

If $p \in \omega_1 \setminus \omega_2$, then this gives the required function. □

**Theorem 1.33.** $\mathrm{Gr}_n(\mathbb{R}^k)$ *is a manifold.*

*Proof.* If $\omega$ is an $n$-plane, let $\omega^\perp$ be the orthogonal $(k-n)$-plane in $\mathbb{R}^k$. Then let

$$U = \left\{ n\text{-planes which do not meet } \omega^\perp \right\}$$

This is homeomorphic to the set of graphs of linear maps $\omega \to \omega^\perp$, which is the space of $n \times (k-n)$ matrices. This is homemorphic to $\mathbb{R}^{n(k-n)}$.

This gives an atlas on $\mathrm{Gr}_n(\mathbb{R}^k)$. □

The previous theorem also shows that $\dim \mathrm{Gr}_n(\mathbb{R}^k) = n(k-n)$.

Since $\mathrm{Gr}_n(\mathbb{R}^k)$ is Hausdorff, we can try to construct a CW-structure on it. This is relatively simple once we figure out the correct cells to look at. Let $p_i\colon \mathbb{R}^k \to \mathbb{R}^i$ be the projection onto the first $i$ coordinates, so $p_k$ is the identity and $p_0$ is constant. As $i$ goes from $k$ to $0$ the dimension of an $n$-plane $\omega$ drops from $n \to 0$. Let $\sigma_i$ be the smallest integer $j$ such that $\dim p_j(\omega) = j$. The sequence $\sigma = (\sigma_1, \ldots, \sigma_n)$ is called a **Schubert symbol**.

If we let $e(\sigma)$ be the subset of $\mathrm{Gr}_n(\mathbb{R}^k)$ having $\sigma$ as their Schubert symbol, we notice these are spaces whose $n$-planes have matrices with columns $\sigma_1, \ldots, \sigma_n$ holding pivots after row reduction. These are the **Schubert cells** of $\mathrm{Gr}_n(\mathbb{R}^k)$.

**Remark 1.34.** Note that this doesn't rely on any properties of $\mathbb{R}$ other than that $\mathbb{R}^m$ is homeomorphic to an open cell of dimension $m$. Thus we could have done the exact same analysis for $\mathrm{Gr}_n(\mathbb{C}^k)$.

We want a Grassmannian of *all* $n$-planes, not just those in a particular dimension. We have inclusions

$$\mathrm{Gr}_n(\mathbb{R}^k) \subseteq \mathrm{Gr}_n(\mathbb{R}^{k+1}) \subseteq \mathrm{Gr}_n(\mathbb{R}^{k+2}) \subseteq \cdots \mathrm{Gr}_n(\mathbb{R}^\infty),$$

where $\mathbb{R}^\infty = \bigoplus_{i=1}^\infty \mathbb{R}$. Each of these inclusions respects the CW-structure on the Grassmannian $\mathrm{Gr}_n(\mathbb{R}^k)$, so we get a CW-structure on $\mathrm{Gr}_n := \mathrm{Gr}_n(\mathbb{R}^\infty)$.

**Definition 1.35.** $\mathrm{Gr}_n := \mathrm{Gr}_n(\mathbb{R}^\infty)$.

**Definition 1.36.** The **universal bundle** of $n$-planes is

$$\gamma_n := \gamma_{n,\infty} = \{(\omega, \nu) \mid \omega \in \mathrm{Gr}_n, \nu \in \omega\}.$$

This may look like we just made things harder! $\mathrm{Gr}_n$ and $\gamma_n$ are larger than their counterparts in $\mathbb{R}^k$. But for algebraic topologists, $\mathrm{Gr}_n$ and $\gamma_n$ are much more natural. They have nice topological structure.

**Definition 1.37.** Let $G$ be a topological group. Then $EG$ is any weakly contractible space with a continuous free $G$-action and $BG$ is the quotient of $EG$ by this action. $BG$ is called the **classifying space** of $G$.

**Example 1.38.** When $G = \mathbb{Z}$, $B\mathbb{Z} = S^1$ and $E\mathbb{Z} = \mathbb{R}$.

When $G = \mathbb{Z}/2$, $B(\mathbb{Z}/2) = \mathbb{RP}^\infty$ and $E(\mathbb{Z}/2) = S^\infty$.

In general, for $G$ discrete, $BG$ is a $K(G,1)$ space, which means

$$\pi_i BG = \begin{cases} 1 & i = 0 \\ G & i = 1 \\ 0 & i > 1 \end{cases}$$

**Remark 1.39.** $B$ is one of the most mysterious functors in all of algebraic topology. It's evil insofar as it hides a lot of information, but at the same time it has lots of nice properties.

The next theorem illustrates how in the case of $O(n)$, $BG$ has both nice homotopy-theoretic properties and a nice combinatorial description via $\mathrm{Gr}_n$.

**Theorem 1.40.** $\mathrm{Gr}_n \simeq BO(n)$.

*Proof.* First, we claim that $EO(n) = V_n(\mathbb{R}^\infty)$. To show this will suffice to prove the theorem, because we know that $V_n(\mathbb{R}^\infty)$ has a free action of $O(n)$, and $\mathrm{Gr}_n$ is the quotient of $V_n(\mathbb{R}^\infty)$ by this action. But what we don't know is that $V_n(\mathbb{R}^\infty)$ is weakly contractible.

The map $V_n(\mathbb{R}^k) \to S^{k-1}$ given by projecting an $n$-frame onto its last vector is a fiber bundle with fiber $V_{n-1}(\mathbb{R}^{k-1})$, considering $\mathbb{R}^{k-1}$ as the hyperplane orthogonal to the last vector. Thus there is a long exact sequence in homotopy

$$\cdots \to \pi_{m+1}S^{k-1} \to \pi_m V_{n-1}(\mathbb{R}^{k-1}) \to \pi_m V_n(\mathbb{R}^k) \to \pi_m S^{k-1} \to \cdots$$

Since $\pi_m S^{k-1} = 0$ for $m < k-2$, $\pi_m V_{n-1}(\mathbb{R}^{k-1}) \cong \pi_m V_n(\mathbb{R}^k)$ for $m < k-2$. By iterating this and takin $k$ large enough, we note that

$$\pi_m V_n(\mathbb{R}^k) \cong \pi_m V_1(\mathbb{R}^{k-n+1}) = \pi_m(S^{k-n}).$$

Thus for $k$ large enough we can show that $\pi_m V_n(\mathbb{R}^k) = 0$ for $m < k-n$.

Now consider $\pi_m V_n(\mathbb{R}^\infty)$. An element in this group is a homotopy class of maps $S^m \to V_n(\mathbb{R}^\infty) = \bigcup_{k=n}^\infty V_n(\mathbb{R}^k)$. Since $S^m$ is compact, this map factors through the inclusion $V_n(\mathbb{R}^k) \to V_n$ for some $k$. Assuming $k$ is sufficiently large, we this map factors through $V_n(\mathbb{R}^k)$.

$$\begin{array}{ccc} S^m & \longrightarrow & V_n(\mathbb{R}^\infty) \\ & \searrow \quad \nearrow & \\ & V_n(\mathbb{R}^k) & \end{array}$$

And since the first map is nullhomotopic, then the composite must be as well. Thus, $\pi_m V_n(\mathbb{R}^\infty) = 0$. $\square$

**Remark 1.41.** A similar proof shows that $\mathrm{Gr}_n(\mathbb{C}^\infty) \simeq BU(n)$.

**Remark 1.42.** In general, we cannot always factor a map from $S^m$ to a colimit through a finite stage, but it works if each map is given by a closed inclusion of Hausdorff spaces.

## 1.2 Classification of Vector bundles

How do we classify vector bundles? We will manage to classify them, but the result will be computationally useless for our purposes. We'll also spend a great deal of time figuring out how to make this result useful.

**Theorem 1.43.** *There is a bijection of sets* $\mathrm{Vect}_n(B) \cong [B, \mathrm{Gr}_n]$.

This result is magical! It gives a geometric classification through homotopical data! We can ignore geometric structure and instead use topological information.

We need a few ingredients to prove Theorem 1.43.

**Definition 1.44.** A space is **paracompact** if every open cover has a locally finite subcover.

**Lemma 1.45.** *Given any open cover* $\{U_\alpha\}$ *of a paracompact space* $X$*, there is a countable open cover* $\{V_i\}$ *such that:*

*(a) for all* $i$, $V_i$ *is a disjoint union of spaces* $U'_\alpha$ *with* $U'_\alpha \subseteq U_\alpha$.

*(b) there is a partition of unity* $\{\phi_i\}$ *subordinate to* $V_i$.

**Definition 1.46.** If $f\colon B' \to B$ is a continuous map and $p\colon E \to B$ is a vector bundle, then the **pullback bundle** of $p\colon E \to B$ along $f$ is the categorical pullback

$$\begin{array}{ccc} f^*(E) & \longrightarrow & E \\ \downarrow & \ulcorner & \downarrow p \\ B' & \xrightarrow{f} & B. \end{array}$$

Explicitly, this is the set

$$f^*(E) = \{(b', e) \mid b' \in B', e \in E, f(b') = p(e)\}.$$

What is the map $[B, \mathrm{Gr}_n] \to \mathrm{Vect}_n(B)$ in the theorem? It is given by sending the class of $f\colon B \to \mathrm{Gr}_n$ to $f^*\gamma_n$.

$$\begin{array}{ccc} [B, \mathrm{Gr}_n] & \xrightarrow{\cong} & \mathrm{Vect}_n(B) \\ [f] & \longmapsto & [f^*\gamma_n] \end{array}$$

We should check that this map is well-defined. That is essentially the content of the next lemma.

**Lemma 1.47.** *If* $f, g\colon X \to Y$ *are homotopic and* $p\colon E \text{ to} Y$ *is any vector bundle, then* $f^*E \cong g^*E$.

*Proof.* Let $H\colon X \times I \to Y$ be a homotopy from $f$ to $g$. In particular, $H|_{X\times\{0\}} = f$ and $H|_{X\times\{1\}} = g$. Notice that $H^*E$ is a bundle over $X \times I$, and the restriction of this bundle to $X \times \{0\}$ is $f^*E$ and the restriction to $X \times \{1\}$ is $g^*E$.

So it suffices to show that given any bundle $E'$ over $X \times I$, the restrictions over $X \times \{0\}$ and $X \times \{1\}$ are isomorphic.

*Case 1:* First, assume that $E'$ is trivial.

$$\begin{array}{ccc} E' & \xrightarrow{\cong} & X \times I \times \mathbb{R}^k \\ & \searrow \quad \swarrow & \\ & X \times I & \end{array}$$

Let $E'_0$ be the restriction of $E'$ to $X \times \{0\}$, and let $E'_1$ be the restriction of $E'$ to $X \times \{1\}$. The required isomorphism between $E'_0$ and $E'_1$ is evident from the following diagram.

$$\begin{array}{ccc} E'_0 & \xrightarrow{\cong} & X \times \{0\} \times \mathbb{R}^k \\ & \searrow \quad \swarrow & \| \\ & X \times \{0\} & \| \\ & \| & \| \\ E'_1 & \xrightarrow{\cong} & X \times \{1\} \times \mathbb{R}^k \\ & \searrow \quad \swarrow & \\ & X \times \{1\} & \end{array}$$

Now for any map $\rho\colon X \to [0, 1]$, let $\Gamma_\rho \subseteq X \times I$ be the be the graph of $\rho$. This same proof as above shows that the restriction of $E'$ over $\Gamma_\rho$ is isomorphic to $E'_0$, for any $\rho$. Let $E'_\rho$ be the restriction of $E'$ to $\Gamma_\rho$.

*Case 2:* Dispose of the assumption that $E'$ is trivial, and instead assume that $E'$ is trivial over $U \times I$ for $U \subseteq X$ open. Let $\rho\colon X \to I$ be any function with support contained in $U$. Then $E'_\rho \cong E'_0$; indeed, outside $U$, $E'_\rho = E'_0$, and inside $U$, we can use the isomorphism from the previous case.

*Case 3:* Assume only that $E'$ is a vector bundle over $X \times I$; no triviality assumptions. By Lemma 1.45, there is a countable open cover $\{U_i\}$ of $X$ such that $E'$ is trivial over $U_i \times I$. Let $\{\phi_i\}$ be the subordinate partition of unity. Let

$$\psi_i = \sum_{j=1}^{i} \phi_j,$$

with $\psi_0 \equiv 0$, and $\psi_\infty \equiv 1$.

Claim that $E'_{\psi_i} \cong E'_{\psi_{j-1}}$. This follows from case 2, because the support of $\psi_i - \psi_{i-1}$ is contained within $U_i$. ☐

**Lemma 1.48.** *For any vector bundle $p\colon E \to B$ of rank $n$, the data of a map $f\colon B \to \mathrm{Gr}_n$ such that $E \cong f^*\gamma_n$ is equivalent to the data of a map $g\colon E \to \mathbb{R}^\infty$ which is a linear injection on each fiber.*

*Proof.* Assume we are given a map $f\colon B \to \mathrm{Gr}_n$ and an isomorphism $E \cong f^*\gamma_n$. We have the data of this diagram:

$$\begin{array}{ccccccc}
E & \xrightarrow{\cong} & f^*\gamma_n & \longrightarrow & \gamma_n & \longrightarrow & \mathbb{R}^\infty \\
& \searrow & \downarrow & \lrcorner & \downarrow & & \\
& & B & \xrightarrow{f} & \mathrm{Gr}_n & &
\end{array}$$

The map $g$ is the composite along the top row of this diagram.

Conversely, given a map $g\colon E \to \mathbb{R}^\infty$ that is a linear isomorphism of each fiber, define

$$f(b) := g(p^{-1}(b)) \in \mathrm{Gr}_n\,.$$

Note that $p^{-1}(b)$ is an $n$-dimensional vector space, as a fiber of $p\colon E \to B$. Then applying $g$, we get an $n$-dimensional subspace of $\mathbb{R}^\infty$.

The vector bundle isomorphism $E \to f^*\gamma_n$ is as follows:

$$\begin{aligned} E &\longrightarrow f^*\gamma_n \\ e &\longmapsto (p(e), g(e)) \end{aligned}$$

Note that $g(e) \in g(p^{-1}(p(e)))$ since $e \in p^{-1}(p(e))$. This is an isomorphism because $g$ is a linear injection on the fibers; we can recover $e$ uniquely from $p(e)$ and $g(e)$. ☐

*Proof of Theorem 1.43.* We will prove both injectivity and surjectivity.

*Injectivity.* Suppose that $E \cong f^*\gamma_n \cong (f')^*\gamma_n$. We want to show that $f \simeq f'$. By Lemma 1.48, take $g, g'\colon E \to \mathbb{R}^\infty$ corresponding to these maps. Claim that it suffices to show that $g \simeq g'$.

Why? Suppose that $G\colon E \times I \to \mathbb{R}^\infty$ is a homotopy from $g$ to $g'$ such that $G|_{E\times\{t\}}$ is a linear injection on fibers for all $t \in [0,1]$. Then we may define a homotopy $F\colon B \times I \to \mathrm{Gr}_n$ between $f$ and $f'$ by

$$F(b,t) = G(p^{-1}(b), t) \subseteq \mathrm{Gr}_n\,.$$

It's tempting to define

$$G(e,t) = g(e)t + g'(e)(1-t), \tag{1.1}$$

but this doesn't necessarily work! This may pass through $0$, but if $g(p^{-1}(b))$ and $g'(p^{-1}(b))$ only intersect at $0$ for all $b$, then we're fine. Since we're working in $\mathbb{R}^\infty$, we have lots of space, so we can homotope things around.

Define homotopies

$$L^o((x_1, x_2, \ldots), t) = t(x_1, \ldots) + (1-t)(x_1, 0, x_2, 0)$$
$$L^e((x_1, x_2, \ldots), t) = t(x_1, \ldots) + (1-t)(0, x_1, 0, x_2, 0, \ldots)$$

Then we may construct a homotopy from $g$ to $g'$ via the following procedure:

(1) Homotope $g$ to be in all odd coordinates.

(2) Homotope that to $g'$ living in even coordinates using (1.1).

(3) Homotope from even coordinates back to all coordinates.

This shows that the map $[B, \mathrm{Gr}_n] \to \mathrm{Vect}_n(B)$ is injective.

*Surjectivitiy.* Suppose $E$ is trivial, say $E \cong B \times \mathbb{R}^k$. Then take by Lemma 1.45 a countable cover $\{U_i\}$ with subordinate partition of unity $\{\phi_i\}$. Then define $g_i \colon E \to \mathbb{R}^n$ as follows:

- above $U$, take the composite

$$\begin{array}{rcccl} E|_U & \longrightarrow & U \times \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & & (b, v) & \longmapsto & \phi_i v \end{array}$$

- outside of $U$, send everything to zero.

Then we can define $g \colon E \to \mathbb{R}^\infty \cong (\mathbb{R}^n)^\infty$ by

$$e \mapsto (g_1(e), g_2(e), \ldots)$$

By Lemma 1.48, this corresponds to the required $f \colon B \to \mathrm{Gr}_n$. $\square$

**Definition 1.49.** The **classifying map** $f \colon B \to \mathrm{Gr}_n$ of an $n$-dimensional vector bundle $p \colon E \to B$ is the preimage of $[p] \in \mathrm{Vect}_n(B)$.

**Remark 1.50.** If instead we want to classify principal $G$-bundles for some group $G$ over a base $X$, then the same proof gives a bijection $\mathrm{Vect}_n(X) \cong [X, BG]$.

**Definition 1.51.** The **Whitney Sum** of two vector bundles $E \to B$ and $E' \to B$ is a new vector bundle $E \oplus E' \to B$, which is the direct sum on fibers.

Here are two descriptions of the Whitney sum:

(1) as a pullback

$$\begin{array}{ccc} E \times_B E' & \longrightarrow & E' \\ \downarrow & & \downarrow \\ E & \longrightarrow & B \end{array}$$

(2) Using Theorem 1.43, we have an isomorphism between vector bundles over $B$ and homotopy classes of maps $B \to \mathrm{Gr}_n$. If $E$ corresponds to $f\colon B \to \mathrm{Gr}_m$ and $E'$ corresponds to $\mathrm{Gr}_n$, what does $E \oplus E'$ correspond to?

$$f \oplus f'\colon B \xrightarrow{\Delta} B \times B \xrightarrow{f \times f'} \mathrm{Gr}_m \times \mathrm{Gr}_n \xrightarrow{\oplus} \mathrm{Gr}_{m+n}$$

The map $\oplus\colon \mathrm{Gr}_m \times \mathrm{Gr}_n \to \mathrm{Gr}_{m+n}$ comes from interleaving two copies of $\mathbb{R}^\infty$ and sending $(\omega, \xi) \in \mathrm{Gr}_m \times \mathrm{Gr}_n$ to the image under the interleaving.

# 2 Cohomology and Characteristic Classes

**Definition 2.1.** The **loop space** of a pointed space $X$ is the space $\Omega X$ consisting of all loops $S^1 \to X$ with the weak topology.

**Definition 2.2.** Let $Z$ be a space. We write $Z_+$ for the space $Z$ with a disjoint basepoint added, $Z_+ := Z \sqcup \{*\}$.

**Remark 2.3.** Adding a basepoint is often a stupid operation when we're trying to look at maps into a space. For example, pointed maps $S^n \to Z_+$ are stuck at the basepoint, so the homotopy groups are all trivial.

On the other hand, maps out of $Z_+$ are perfectly fine to think about.

**Definition 2.4.** Let **CW** denote the category of CW-complexes.

**Remark 2.5.** We will not think about pairs of spaces $(X, A)$; instead, we will look at the space $X/A$, and declare that the image of $A$ under projection $X \to X/A$ is the basepoint.

**Definition 2.6.** The **mapping cone** of an inclusion $\alpha\colon A \hookrightarrow X$ is

$$C\alpha = X \coprod CA \big/ (x, 1) \sim \alpha(x)$$

**Definition 2.7.** A **generalized cohomology theory** is a sequence of functors $h^n\colon \mathbf{CW}_*{}^{\mathrm{op}} \to \mathbf{Ab}$ together with natural **suspension isomorphisms**

$$\sigma_i\colon h^{i+1}(\Sigma X) \to h^i(X)$$

such that the following axioms hold.

(a) **homotopy invariance:** If $f_1, f_2 \colon X \to Y$ are homotopic, then $h^i(f_1) = h^i(f_2)$.

(b) **exactness:** If $\alpha \colon A \hookrightarrow X$ is an inclusion $\beta \colon X \to C\alpha$, then the sequence

$$h^i(C\alpha) \to h^i(X) \to h^i(A)$$

is exact.

(c) **additivity:** If $\{X_j\}$ is a collection of pointed spaces, then

$$h^i\left(\bigvee_j X_j\right) \cong \prod_j h^i(X_j)$$

**Remark 2.8.** There is an additional axiom, known as the **dimension axiom.**

(d) **dimension:** $h^i(S^0) \cong \begin{cases} \mathbb{Z} & i = 0 \\ 0 & \text{otherwise} \end{cases}$

But we don't include it because there is exactly one cohomology theory that satisfies all axioms (a)-(d): ordinary (singular) cohomology.

If you've seen cohomology before, you are probably wondering where the long exact sequence comes from. The axioms only have three-term sequences which are exact at the middle. But that's enough to reconstruct the long exact sequence.

Consider the sequence of maps

$$A \xrightarrow{\alpha} X \xrightarrow{\beta} C\alpha \xrightarrow{\gamma} C\beta \longrightarrow \cdots$$

Notice that $C\alpha \simeq X/A$, and $C\beta \simeq \Sigma A$, and $C\gamma \simeq \Sigma X$. Then using the suspension isomorphisms, we have the long exact sequence of cohomology.

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & h^i(\Sigma X) & \longrightarrow & h^i(\Sigma A) & \longrightarrow & h^i(X/A) & \longrightarrow & h^i(X) & \longrightarrow & h^i(A) & \longrightarrow \cdots \\ & \cong \downarrow \sigma_{i-1} & & \cong \downarrow \sigma_{i-1} & & & & & & & \\ & h^{i-1}(X) & & h^{i-1}(A) & & & & & & & \end{array}$$

**Theorem 2.9.** *Suppose that we are given a sequence $X_0, X_1, \ldots$ of pointed spaces and weak equivalences $X_i \xrightarrow{\sim} \Omega X_{i+1}$ for all $i$. Then the sequence of functors $h^n \colon \mathbf{CW_*}^{op} \to \mathbf{Ab}$ defined by*

$$h^n(Y) = \begin{cases} [Y, X_n] & n \geq 0 \\ [Y, \Omega^{-n} X_0] & n < 0 \end{cases}$$

*is a generalized cohomology theory.*

**Remark 2.10.** The converse of this theorem actually holds as well; it's called the **Brown Representability Theorem**.

**Example 2.11.** Let $X_i = K(\mathbb{Z}, i)$ be an Eilenberg-MacLane space. Then

$$\widetilde{H}^i(Y) = [Y, K(\mathbb{Z}, i)]$$

$$\widetilde{H}^i(S^0) = [S^0, K(\mathbb{Z}, i)] = \pi_0 K(\mathbb{Z}, i) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & \text{otherwise} \end{cases}$$

More generally, we can replace $\mathbb{Z}$ by any discrete group $G$ to get singular cohomology with $G$-coefficients.

*Proof of Theorem 2.9.* First, let's define the suspension isomorphism. We want to show that $h^{n+1}(\Sigma Y) \cong h^n(Y)$. We have that

$$h^{n+1}(\Sigma Y) = [\Sigma Y, X_{n+1}] \cong [Y, \Omega X_{n+1}] \cong [Y, X_n] = h^n(Y)$$

We must also check the three axioms of a generalized cohomology theory.

Homotopy invariance is clear, because we are only dealing with homotopy classes of maps.

Additivity follows from the universal property of the product.

It remains to check exactness. Consider

$$Y \xhookrightarrow{\alpha} Z \longrightarrow C\alpha,$$

where

$$C\alpha := Z \cup_\alpha CY = \left. Z \sqcup Y \times I \middle/ \begin{matrix} (y,0) \sim (y',0) \\ (y,1) \sim \alpha(y). \end{matrix} \right.$$

We want to show that

$$[C\alpha, X_n] \to [Z, X_n] \to [Y, X_n]$$

is exact at the middle.

To show that the composite is zero, suppose that we are given $f\colon C\alpha \to X_n$. Then $f|_Y$ is null-homotopic, with a null-homotopy $f|_{CY}\colon Y \times I \to X_n$, which is constant on $Y \times \{0\}$.

Conversely, let $f\colon Z \to X_n$ be such that $f|_Y$ is null-homotopic. Then there exists $h\colon Y \times I \to X_n$ such that $h|_{Y\times\{0\}}$ is constant and $h|_{Y\times\{1\}} = f|_Y$.

Define $g\colon C\alpha \to X_n$ by

$$\begin{cases} g(z) = f(z) & z \in Z \\ g(y,t) = h(y,t) & (y,t) \in Y \times I. \end{cases}$$

Then $g$ is a map whose image under $[C\alpha, X_n] \to [Z, X_n]$ is $f$. ☐

**Remark 2.12.** Our goal is to use this to understand $[B, Gr_n]$. As remarked before, however, this is hopeless. But, we understand $[B, K(\mathbb{Z}, i)]$ and $[Gr_n, K(\mathbb{Z}, i)]$. Then given $[f] \in [B, Gr_n]$, we get a map $f^* \colon [Gr_n, K(\mathbb{Z}, i)] \to [B, Gr_n]$.

$$\begin{array}{ccc} [Gr_n, K(\mathbb{Z}, i)] & \xrightarrow{f^*} & [B, K(\mathbb{Z}, i)] \\ \downarrow \cong & & \downarrow \cong \\ H^i(Gr_n) & \longrightarrow & H^i(B) \end{array}$$

Although we have no hope of computing homotopy classes of maps $[B, Gr_n]$ even for spheres, we do know the cohomology of spheres! The moral is that these maps give invariants of vector bundles, which we *can* compute. These are called **characteristic classes**.

The strategy is

(a) compute $\widetilde{H}^*(Gr_n, \mathbb{Z}/2)$,

(b) compute $\operatorname{im} f^*$ as an invariant of $E \to B$,

(c) hope that this retains useful information.

## 2.1 Cohomology of Grassmannians

**Example 2.13.** Let's first consider the case $n = 1$, where $Gr_1 = \mathbb{RP}^\infty$. This has another useful description of $\mathbb{RP}^\infty$ as a quotient of $S^\infty$ by a $\mathbb{Z}/2$ action. $S^\infty$ has a cells structure with two zero-cells, two one-cells, two two-cells, etc. The action of $\mathbb{Z}/2 = \{\pm 1\}$ switches the cells in each dimension. Depending on whether or not the dimension is even, the action of $\mathbb{Z}/2$ switches the orientation when it swaps the cells. Hence, the dimension of the boundaries is 2 in even dimensions and 0 in odd dimensions. But with $\mathbb{Z}/2$ coefficients, the dimension of the boundaries is always zero. Hence,

$$H^*(\mathbb{RP}^\infty, {}^{\mathbb{Z}}/_{2}) = {}^{\mathbb{Z}}/_{2}$$

in each dimension, and as a ring,

$$H^*(\mathbb{RP}^\infty, {}^{\mathbb{Z}}/_{2}) \cong {}^{\mathbb{Z}}/_{2}[x],$$

with $x$ in degree 1.

We could theoretically do a similar thing using the Schubert cell structure on Grassmannians, but that uses a lot of (really cool) combinatorics that we don't have time for. So we'll do something harder.

**Remark 2.14.** The next theorem is one of those theorems whose proof is less useful than its consequences.

> I used to think that proofs were important and theorems were just made up, but now I think that theorems are important and proofs are just made up.
>
> – Mike Hopkins (paraphrased)

**Definition 2.15.** Let $B$ be a paracompact space, and $p\colon E \to B$ a vector bundle over $B$. Let $D(E)$ be the unit disk bundle, and $S(E)$ the sphere bundle (boundary of the disk bundle). Then define the **Thom space** of $E$

$$\mathrm{Th}(E) := {}^{D(E)}\!/_{S(E)}$$

For general $B$, we define $\mathrm{Th}(E)$ as the one-point compactification of $E$.

**Example 2.16.** Let $E \cong B \times \mathbb{R}^n$. Then $\mathrm{Th}(E) = B_+ \wedge S^n$.

**Theorem 2.17** (Thom Isomorphism Theorem)**.** *Let $p\colon E \to B$ be an $n$-dimensional fiber bundle. There exists a natural class $c \in \widetilde{H}^n(\mathrm{Th}(E), \mathbb{Z}/2)$ such that the restriction of $c$ to any fiber $F$ is a generator of $\widetilde{H}^n(S^n)$ and the map*

$$\begin{aligned} \widetilde{H}^i(B_+, \mathbb{Z}/2) &\xrightarrow{\ \Phi\ } \widetilde{H}^{i+n}(\mathrm{Th}(E), \mathbb{Z}/2) \\ b &\longmapsto p^*(b) \smile c \end{aligned}$$

*is an isomorphism for all $i$. (This is not a ring map!)*

**Definition 2.18.** The class $c$ in the Thom Isomorphism Theorem is called the **Thom class**.

**Remark 2.19.** The Thom class $c$ of a bundle $p\colon E \to B$ is natural in the following sense: given $f\colon B' \to B$, the Thom class of $f^*(E)$ is $f^*(c)$, where $f^*\colon \widetilde{H}^n(\mathrm{Th}(E); \mathbb{Z}/2) \to \widetilde{H}^n(\mathrm{Th}(E); \mathbb{Z}/2)$.

**Theorem 2.20.** *As a ring,*

$$\widetilde{H}^*(\mathrm{Gr}_{n+}, \mathbb{Z}/2) \cong {}^{\mathbb{Z}}\!/_{2}[w_1, \dots, w_n]$$

*with $w_i$ in degree $i$.*

**Remark 2.21.** If we have an oriented bundle, then the Thom isomorphism theorem holds with $\mathbb{Z}$ coefficients. The problem with unoriented bundles is that we struggle choose generators.

**Remark 2.22.** In algebra, we often add together elements of different degrees. But in topology, the different degrees in a cohomology ring come from different dimension. So adding elements of different degrees is weird and doesn't quite make sense. Yet we do it anyway when we define total Chern classes and total Whitney classes. Allen Knutson calls these "abominations."

If we're topologists, we try to avoid working with elements of mixed degree. So we won't talk about total Chern classes or total Whitney classes here.

**Definition 2.23.** Let $p\colon E \to B$ be a vector bundle and let $c$ be the Thom class of $E$. Let $j\colon D(E) \to \mathrm{Th}(E)$ be the projection from the disk bundle onto the Thom bundle.

Define $e := (p^*)^{-1}(j^*c) \in \widetilde{H}^n(B; \mathbb{Z}/2)$. We will call this the $\mathbb{Z}/2$**-Euler class** (this is not standard terminology!).

**Remark 2.24.** Like the Thom class, the $\mathbb{Z}/2$-Euler class is natural in the sense that given a vector bundle $p\colon E \to B$ with $\mathbb{Z}/2$-Euler class $e$ and a function $f\colon B' \to B$, the $\mathbb{Z}/2$-Euler class of $f^*(E)$ is $f^*(e)$.

**Remark 2.25.** Here is a geometric description of the Euler class. Let $p\colon E \to B$ be a vector bundle where $B$ is a manifold. Let $s\colon B \to E$ be a generic section. Let $\zeta$ be the zero section. Then $s(B) \cap \zeta(B)$ is the Poincaré dual of $e$. Therefore, a nonzero Euler class implies that there do not exist everywhere-nonzero sections, and hence the bundle is nontrivial.

**Lemma 2.26.** *For $\gamma_n \to \mathrm{Gr}_n$, the $\mathbb{Z}/2$-Euler class is nonzero.*

*Proof.* By Remark 2.24, it suffices to find any bundle with nonzero $\mathbb{Z}/2$-Euler class. If we find such a bundle, it will be a pullback of the universal bundle by the classification of vector bundles (Theorem 1.43). Therefore, a bundle with a nonzero $\mathbb{Z}/2$-Euler class shows that the $\mathbb{Z}/2$-Euler class of the universal bundle is nonzero.

We will show that the universal bundle

$$\gamma_{n,n+1} \to \mathrm{Gr}_n(\mathbb{R}^{n+1})$$

is nontrivial. Notice that $\mathrm{Gr}_n(\mathbb{R}^{n+1}) \cong \mathrm{Gr}_1(\mathbb{R}^{n+1}) = \mathbb{RP}^n$ by taking the orthogonal compliment of any $n$-plane.

Now consider the isomorphism $\mathbb{RP}^n \cong S^n/\{\pm 1\}$. We may think of $\gamma_{n,n+1}$ as the quotient of a bundle over $S^n$ by $\pm 1$. In particular, the identification is $(x, v) \sim (-x, v)$.

We must show that this bundle over $\mathbb{RP}^n$ has a nontrivial $\mathbb{Z}/2$-Euler class. By Remark 2.25, it is enough to find a section of this bundle that intersects the zero section transversely. Then the Poincaré dual of that will be the $\mathbb{Z}/2$-Euler class.

The section we choose is $s\colon \mathbb{RP}^n \to \gamma_{n,n+1}$ that takes a point $x$ to the projection of $(1, 0, \ldots, 0)$ onto $x^\perp$. This is zero only at $(1, 0, \ldots, 0)$ and at $(-1, 0, \ldots, 0)$ in $\mathbb{RP}^n$, so intersects the zero section transversely.

Hence, it defines a nonzero element of $H_0(\mathbb{RP}^n; \mathbb{Z}/2) \cong H_0(\mathrm{Gr}_n(\mathbb{R}^{n+1}); \mathbb{Z}/2)$, and its Poincaré dual is a nonzero element of $H^n(\mathrm{Gr}_n(\mathbb{R}^{n+1}))$, and therefore the $\mathbb{Z}/2$-Euler class of $\gamma_{n,n+1}$ is nonzero. □

**Definition 2.27.** We have $\mathrm{Th}(E) = D(E)/S(E)$. This yields a long exact sequence in cohomology

$$\cdots \longrightarrow \widetilde{H}^i(\mathrm{Th}(E);\mathbb{Z}/2) \xrightarrow{j^*} \widetilde{H}^i(D(E);\mathbb{Z}/2) \longrightarrow \widetilde{H}^i(S(E);\mathbb{Z}/2) \longrightarrow \widetilde{H}^{i+1}(\mathrm{Th}(E);\mathbb{Z}/2) \longrightarrow \cdots$$

Now apply the Thom Isomorphism Theorem (Theorem 2.17), and notice moreover that $\widetilde{H}^i(B) \cong \widetilde{H}^i(D(E))$ since $B \simeq D(E)$.

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \widetilde{H}^i(\mathrm{Th}(E);\mathbb{Z}/2) & \xrightarrow{j^*} & \widetilde{H}^i(D(E);\mathbb{Z}/2) & \longrightarrow & \widetilde{H}^i(S(E);\mathbb{Z}/2) & \longrightarrow & \widetilde{H}^{i+1}(\mathrm{Th}(E);\mathbb{Z}/2) & \longrightarrow \cdots \\
 & \Phi\uparrow\cong & & \cong\uparrow & & \| & & \uparrow & \\
\cdots \longrightarrow & \widetilde{H}^{i-n}(B_+;\mathbb{Z}/2) & \xrightarrow{\smile e} & \widetilde{H}^i(B;\mathbb{Z}/2) & \longrightarrow & \widetilde{H}^i(S(E);\mathbb{Z}/2) & \longrightarrow & \widetilde{H}^{i-n+1}(B;\mathbb{Z}/2) & \longrightarrow \cdots
\end{array}$$

The bottom row here is called the **Gysin sequence**.

*Proof of Theorem 2.20.* Proof by induction on $n$, using the Gysin sequence for the universal bundle $\gamma_n \to \mathrm{Gr}_n$.

If $n = 0$, $\mathrm{Gr}_0$ is a point, and $H^*(\mathrm{Gr}_0;\mathbb{Z}/2)$ is a polynomial ring on zero generators.

If $n > 0$, assume that $H^*(\mathrm{Gr}_{n-1};\mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, \ldots, w_{n-1}]$. The sphere bundle on the universal bundle $\gamma_n$ is

$$S(\gamma_n) = \{(\omega, v) \mid v \in \omega,\ \|v\| = 1\}$$

There is a natural projection $p' \colon S(\gamma_n) \to \mathrm{Gr}_{n-1}$ given by

$$\begin{array}{ccc} S(\gamma_n) & \xrightarrow{p'} & \mathrm{Gr}_{n-1} \\ (\omega, v) & \longmapsto & \omega \cap v^\perp \end{array}$$

This defines a fiber bundle with fiber $S^\infty$ consisting of all unit vectors orthogonal to $\omega \cap v^\perp$. Since $S^\infty$ is contractible, $p'$ induces isomorphisms on all homotopy groups by Theorem 1.27 and therefore also on cohomology rings:

$$H^*(S(\gamma_n);\mathbb{Z}/2) \cong H^*(\mathrm{Gr}_{n-1};\mathbb{Z}/2).$$

The diagram

$$\mathrm{Gr}_n \xleftarrow{p} S(\gamma_n) \xrightarrow{\sim} \mathrm{Gr}_{n-1}$$

Gives a ring homomorphism

$$\eta \colon \widetilde{H}^*(\mathrm{Gr}_n;\mathbb{Z}/2) \to \widetilde{H}^*(S(\gamma_n);\mathbb{Z}/2) \cong \widetilde{H}^*(\mathrm{Gr}_{n-1};\mathbb{Z}/2) \tag{2.1}$$

So we may replace the terms $H^i(S(\gamma_n);\mathbb{Z}/2)$ in the Gysin sequence for the universal bundle $\gamma_n \to \mathrm{Gr}_n$, to get the following sequence.

$$\cdots \longrightarrow \widetilde{H}^{i-n}(\mathrm{Gr}_{n+};\mathbb{Z}/2) \xrightarrow{\smile e} \widetilde{H}^i(\mathrm{Gr}_n;\mathbb{Z}/2) \xrightarrow{\eta} \widetilde{H}^i(\mathrm{Gr}_{n-1};\mathbb{Z}/2) \longrightarrow \widetilde{H}^{i-n+1}(\mathrm{Gr}_{n+};\mathbb{Z}/2) \longrightarrow \cdots$$

For $i < n-1$, the ring map $\eta$ is an isomorphism because the groups $\widetilde{H}^{i-n+1}(Gr_{n+};\mathbb{Z}/2)$ and $\widetilde{H}^{i-n}(Gr_{n+};\mathbb{Z}/2)$ vanish. This moreover means that for each generator $w_j \in H^*(Gr_{n-1};\mathbb{Z}/2)$, there is a unique $w_j' \in \widetilde{H}^*(Gr_n;\mathbb{Z}/2)$ such that $\eta(w_j') = w_j$ for $j < n-1$.

For $i = n-1$, the map $H^0(Gr_n;\mathbb{Z}/2) \xrightarrow{\smile e} H^n(Gr_n;\mathbb{Z}/2)$ is injective by Lemma 2.26. Therefore, we have the following diagram.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{H}^{n-1}(Gr_n) & \longrightarrow & \widetilde{H}^{n-1}(Gr_{n-1}) & \xrightarrow{0} & \widetilde{H}^0(Gr_{n+}) & \xrightarrow{\smile e} & \widetilde{H}^n(Gr_n) \\
 & & & & & & \wr\| & & \\
 & & & & & & \mathbb{Z}/2 & & \\
 & & & & & & \cup & & \\
 & & & & & & 1 & \longmapsto & e \neq 0
\end{array}
$$

This shows that $\eta$ is an isomorphism in degree $n-1$. This means that there must be some $w_{n-1}' \in \widetilde{H}^*(Gr_n;\mathbb{Z}/2)$ such that $\eta(w_{n-1}') = w_{n-1} \in H^*(Gr_{n-1};\mathbb{Z}/2)$.

Now because $H^*(Gr_{n-1};\mathbb{Z}/2)$ is generated by $w_1,\ldots,w_{n-1}$ as a ring and $\eta$ is a ring homomorphism, it must be surjective in each degree. Hence, the Gysin sequence splits into short exact sequences for all $i$:

$$0 \longrightarrow \widetilde{H}^{i-n}(Gr_{n+}) \xrightarrow{\smile e} \widetilde{H}^i(Gr_n) \xrightarrow{\eta} \widetilde{H}^i(Gr_{n-1}) \longrightarrow 0.$$

So define $w_n' = e \in \widetilde{H}^n(Gr_n)$.

Claim that $w_1',\ldots,w_n'$ are generators for $H^*(Gr_n;\mathbb{Z}/2)$ as a polynomial ring. To show this, it suffices to show that for all $i$, every element of $H^i(Gr_n;\mathbb{Z}/2)$ can be uniquely written as a polynomial in $w_1',\ldots,w_n'$. For $i < n$, this follows because $\eta$ is an isomorphism $H^i(Gr_{n-1};\mathbb{Z}/2) \cong H^i(Gr_n;\mathbb{Z}/2)$. For $i \geq n$, we proceed by induction. Let $x \in H^i(Gr_n;\mathbb{Z}/2)$. Then $\eta(x) \in H^i(Gr_{n-1};\mathbb{Z}/2)$ is polynomial in $w_1,\ldots,w_{n-1}$. Since $\ker(\eta) = \operatorname{im}(\smile e)$, we may write

$$x = p(w_1',\ldots,w_{n-1}') + w_n' \cdot y$$

for $y \in H^{i-n}(Gr_n;\mathbb{Z}/2)$. By induction, $y$ is polynomial in $w_1',\ldots,w_n'$, and therefore $x$ is as well. So any element of $H^*(Gr_n;\mathbb{Z}/2)$ may be written as a polynomial in $w_1',\ldots,w_n'$. □

## 2.2 Characteristic Classes

**Definition 2.28.** A characteristic class for $n$-dimensional real vector bundles is a function $\xi$ assigning to each vector bundle $E \xrightarrow{p} B$ an element $\xi(E) \in H^i(B;\mathbb{Z}/2)$ for some $i$ such that:

(a) $\xi(E)$ depends only on the isomorphism class of $E$;

(b) for any $f\colon B' \to B$, $\xi(f^*(E)) = f^*(\xi(E))$.

**Lemma 2.29.** *Characteristic classes for $n$-dimensional real vector bundles correspond to (homogeneous) elements of $H^i(\mathrm{Gr}_n;\mathbb{Z}/2)$.*

*Proof.* First, given a characteristic class $\xi$, we get an element $\xi(\gamma_n)$ of $H^i(\mathrm{Gr}_n;\mathbb{Z}/2)$.

On the other hand, given a cohomology class $c \in H^i(\mathrm{Gr}_n;\mathbb{Z}/2)$, we get a characteristic class $\xi$ defined by $\xi(E) := f^*(c)$, where $f\colon B \to \mathrm{Gr}_n$ is the classifying map for $E \to B$.

One can check that this is a bijection. □

Here is another proof of Lemma 2.29.

*Proof.* If we identify $n$-dimensional real vector bundles with their classifying maps, then we may think of such bundles as the collection of morphisms represented by the functor $[-, \mathrm{Gr}_n]$. In this situation, a characteristic class is a natural transformation

$$[-;\mathrm{Gr}_n] \implies H^*(-;\mathbb{Z}/2),$$

which corresponds by the Yoneda lemma to an element of $H^*(\mathrm{Gr}_n;\mathbb{Z}/2)$. □

Recall that $H^*(\mathrm{Gr}_n;\mathbb{Z}/2) = \mathbb{Z}/2[w_1,\ldots,w_n]$ with $w_i$ in degree $i$. In the course of the proof, we constructed in equation (2.1) a map

$$\eta\colon H^*(\mathrm{Gr}_n;\mathbb{Z}/2) \to H^*(\mathrm{Gr}_{n-1};\mathbb{Z}/2)$$

that is an isomorphism in degrees less than $n$. This is the map of polynomial rings that evaluates the last generator $w_n$ at $0$.

$$\begin{array}{ccc} \mathbb{Z}/2[w_1,\ldots,w_n] & \xrightarrow{\eta} & \mathbb{Z}/2[w_1,\ldots,w_{n-1}] \\ w_n & \longmapsto & 0 \end{array}$$

We also canonically defined $w_n = e$ when demonstrating Theorem 2.20 by induction.

**Definition 2.30.** The **Stiefel-Whitney classes** are the ones associated via Lemma 2.29 to the generators $w_i$ of $H^*(\mathrm{Gr}_n;\mathbb{Z}/2)$. They are written $w_i(E)$ for a vector bundle $E \to B$.

**Remark 2.31.** Since all elements of $H^*(\mathrm{Gr}_n;\mathbb{Z}/2)$ are polynomial in the $w_i$, it follows from Lemma 2.29 that we can learn everything about characteristic classes by studying the Stiefel-Whitney classes.

Usually, when characteristic classes are introduced, they are given with four axioms. Here, we will prove these axioms.

**Lemma 2.32.** *Given* $f\colon B' \to B$, $w_i(f^*E) = f^*(w_i(E))$.

**Lemma 2.33.** *For any vector bundle* $E \to B$, $w_i(E \oplus \varepsilon^k) = w_i(E)$, *where* $\varepsilon^k$ *is a trivial bundle of rank* $k$.

*Proof.* It suffices to show this for $k = 1$. Let $f\colon B \to \mathrm{Gr}_n$ be the classifying map for $E \to B$. Consider

$$\begin{array}{ccc} E \oplus \varepsilon^1 & \longrightarrow & \gamma_n \oplus \varepsilon^1 \\ \downarrow & \ulcorner & \downarrow \\ B & \xrightarrow{f} & \mathrm{Gr}_n \end{array}$$

By Lemma 2.29 and the pullback property of characteristic classes, we have that

$$w_i(E \oplus \varepsilon^1) = w_i(f^*(\gamma_n \oplus \varepsilon^1)) = f^*(w_i(\gamma_n \oplus \varepsilon^1))$$

On the other hand,

$$w_i(E) = w_i(f^*(\gamma_n)) = f^*(w_i(\gamma_n)).$$

So it suffices to show that $w_i(\gamma_n \oplus \varepsilon^1) = w_i(\gamma_n)$.

To compute $w_i(\gamma_n \oplus \varepsilon^1)$, we first need to find its classifying map $g\colon \mathrm{Gr}_n \to \mathrm{Gr}_{n+1}$. Then

$$w_i(\gamma_n \oplus \varepsilon^1) := g^*(w_i) \in H^*(\mathrm{Gr}_n; \mathbb{Z}/2)$$
$$w_i(\gamma_n) := w_i \in H^*(\mathrm{Gr}_n; \mathbb{Z}/2)$$

Note that in the first line, $w_i$ is the polynomial generator of $H^*(\mathrm{Gr}_{n+1}; \mathbb{Z}/2)$, and on the second line, $w_i$ is the polynomial generator of $H^*(\mathrm{Gr}_n; \mathbb{Z}/2)$.

If $g^* = \eta\colon H^*(\mathrm{Gr}_{n+1}; \mathbb{Z}/2) \to H^*(\mathrm{Gr}_n; \mathbb{Z}/2)$, then we're done. This is what we claim.

To that end, recall that $\eta$ constructed as follows. The map

$$\begin{array}{ccc} S(\gamma_{n+1}) & \xrightarrow{\sim} & \mathrm{Gr}_n \\ (\omega, \nu) & \longmapsto & \omega \cap \nu^{\perp} \end{array}$$

defines a fiber bundle with fiber $S^\infty$; since $S^\infty$ is contractible, the long exact sequence of homotopy gives an isomorphism

$$H^*(S(\gamma_{n+1}); \mathbb{Z}/2) \cong H^*(\mathrm{Gr}_n; \mathbb{Z}/2).$$

Then $\eta$ is the composite of this map with the map induced on cohomology by

$$S(\gamma_{n+1}) \hookrightarrow D(\gamma_{n+1}) \xrightarrow{\sim} \mathrm{Gr}_{n+1}.$$

Altogether in one diagram, the map on cohomology induced by the following determines $\eta$.

$$\begin{array}{ccccccc} \mathrm{Gr}_{n+1} & \xleftarrow{\sim} & D(\gamma_{n+1}) & \xhookleftarrow{i} & S(\gamma_{n+1}) & \xrightarrow{\sim} & \mathrm{Gr}_n \\ & & & & (\omega, \nu) & \longmapsto & \omega \cap \nu^\perp \end{array}$$

To see that $g^* = \eta$, let's verify that the pullback of $\gamma_{n+1} \to \mathrm{Gr}_{n+1}$ along $g$ gives $\gamma_n \oplus \varepsilon^1$. This involves two pullbacks – we first pull back $\gamma_{n+1} \to \mathrm{Gr}_{n+1}$ along the map $S(\gamma_{n+1}) \to \mathrm{Gr}_{n+1}$ and show that it splits as $t \oplus t^\perp$ with $t^\perp$ trivial, and then show that $t^\perp$ is isomorphic to the pullback of $\gamma_n \to \mathrm{Gr}_n$ along the weak equivalence $S(\gamma_{n+1}) \xrightarrow{\sim} \mathrm{Gr}_n$.

First, pull the universal bundle $\gamma_{n+1} \to \mathrm{Gr}_{n+1}$ back to a bundle $\widetilde{E}$ over $S(\gamma_{n+1})$. An element of $\widetilde{E}$ looks like $(\omega, \nu, u)$ with $\nu, u \in \omega$ and $\nu$ a unit vector.

$$\begin{array}{ccccc} \gamma_{n+1} & \longleftarrow & & & \widetilde{E} \\ \downarrow & & & & \downarrow \uparrow t \\ \mathrm{Gr}_{n+1} & \xleftarrow{\sim} & D(\gamma_{n+1}) & \xhookleftarrow{i} & S(\gamma_{n+1}) \end{array}$$

This has a section $t\colon S(\gamma_{n+1}) \to \widetilde{E}$ given by

$$(\omega, \nu) \mapsto (\omega, \nu, \nu)$$

Because $\nu$ is a unit vector, this is an everywhere nonzero section. Therefore, $\widetilde{E}$ contains a trivial bundle given by the image of $t$. So we decompose $\widetilde{E} = t \oplus t^\perp$. Note that elements of $t^\perp$ look like $(\omega, \nu, u)$ with $u \in \nu^\perp$ and $\nu$ a unit vector.

Now remains to see what the pullback of the universal bundle $\gamma_n \to \mathrm{Gr}_n$ along $S(\gamma_{n+1}) \to \mathrm{Gr}_n$ looks like. We hope it looks like $t^\perp$.

$$\begin{array}{ccc} t^\perp & \dashrightarrow & \gamma_n \\ \downarrow & & \downarrow \\ S(\gamma_{n+1}) & \xrightarrow{\sim} & \mathrm{Gr}_n \end{array}$$

Given $(\lambda, w) \in \gamma_n$, the pullback along the map $S(\gamma_{n+1}) \to \mathrm{Gr}_n$ looks like $(\lambda, \nu, w)$ with $w \in \nu^\perp \cap \lambda$. This is exactly $t^\perp$.

Now, the pullback of $\gamma_n$ all the way along $g\colon \mathrm{Gr}_n \to \mathrm{Gr}_{n+1}$ is $\gamma_n \oplus \varepsilon^1$. This shows that $g^* = \eta$. We now know that

$$w_i(g^*(\gamma_{n+1})) = \eta w_i(\gamma_{n+1}).$$

Therefore,

$$\begin{aligned} \eta\, w_i(\gamma_{n+1}) &= \begin{cases} 0 & \text{if } i \geq n+1 \\ w_i & \text{if } i \leq n \end{cases} \\ &= w_i(\gamma_n) \end{aligned}$$

□

We will write the Künneth Theorem down here, because we will need it.

**Theorem 2.34** (Künneth)**.** *For cohomology with coefficients in a field* $k$*,*

$$H^*(X \times Y; k) \cong H^*(X; k) \otimes_k H^*(Y; k).$$

## 2.3 Axioms for Stiefel-Whitney classes

**Theorem 2.35** (Whitney Sum Formula)**.**

$$w_i(E \oplus E') = \sum_{j+k=i} w_j(E) \smile w_k(E')$$

*Proof.* We will first prove this for Grassmannians, as usual.

Consider the bundle $\gamma_m \times \gamma_n \to \mathrm{Gr}_m \times \mathrm{Gr}_n$. The classifying map of this bundle is

$$\oplus\colon \mathrm{Gr}_m \times \mathrm{Gr}_n \to \mathrm{Gr}_{m+n}$$

We want to compute $\oplus^*(w_i)$, which is by Lemma 2.29 the $i$-th Whitney class of the sum $\gamma_m \oplus \gamma_n$.

Proof by induction on $m+n$. The base case is trivial.

For $m+n > 0$, let $g_n\colon \mathrm{Gr}_{n-1} \to \mathrm{Gr}_n$ be the map that induces

$$\eta\colon H^*(\mathrm{Gr}_n; \mathbb{Z}/2) \to H^*(\mathrm{Gr}_n - 1; \mathbb{Z}/2),$$

where $\eta(w_i) = w_i$ for $i < n$ and $\eta(w_n) = 0$.

We know that

$$H^*(\mathrm{Gr}_n \times \mathrm{Gr}_m; \mathbb{Z}/2) \cong \mathbb{Z}/2[w_{n1}, \ldots, w_{nn}] \otimes \mathbb{Z}/2[w_{m1}, \ldots, w_{mm}]$$

Moreover, $\oplus^* w_i$ is some polynomial in the $w_{ij}$, say

$$\oplus^*(w_i) = q_i(w_{m1}, \ldots, w_{nn}).$$

Now consider

$$g_m \times 1\colon \mathrm{Gr}_{m-1} \times \mathrm{Gr}_n \to \mathrm{Gr}_m \times \mathrm{Gr}_n\,.$$

Evaluating on $\oplus^* w_i$, we have

$$(g_m \times 1)^* \oplus^* w_i = q_i(w_{m-1,1}, \ldots w_{m-1,m-1}, 0, w_{n1}, \ldots, w_{nn}) \tag{2.2}$$

On the other hand, $g_m$ is the classifying map of the bundle $\gamma_{m-1} \oplus \varepsilon^1$. Therefore, by Lemma 2.29, we have

$$(g_m \times 1)^* \oplus^* w_i = w_i(\gamma_{m-1} \oplus \varepsilon^1 \oplus \gamma_n)$$

Now by Lemma 2.33, this is

$$w_i(\gamma_{m-1} \oplus \gamma_n)$$

And then by induction, we have

$$w_i(\gamma_{m-1} \oplus \gamma_n) = \sum_{j+k=i} w_j(\gamma_{m-1}) \smile w_k(\gamma_n) \tag{2.3}$$

Equating (2.2) and (2.3), we have

$$q_i(w_{m1}, \dots, w_{nn}) \equiv \sum_{j+k=i} w_{mj} \smile w_{nk} \pmod{w_{mm}}.$$

Analogously,

$$q_i(w_{m1}, \dots, w_{nn}) \equiv \sum_{j+k=i} w_{mj} \smile w_{nk} \pmod{w_{nn}}.$$

So by the Chinese Remainder Theorem,

$$q_i(w_{m1}, \dots, w_{nn}) = \sum_{j+k=i} w_{mj} \smile w_{nk} \pmod{w_{mm}w_{nn}}.$$

If $i < m+n$, this congruence must be equality because $w_{mm}w_{nn}$ has grading $m+n$. If $i > m+n$, $w_i = 0$, so its pullback must also be zero, and the formula on the right is zero as well.

The only case that remains to check is when $i = m+n$. That is, we must check that

$$\oplus^*(w_{m+n}) = w_{mm}w_{nn}.$$

Notice that $w_{m+n}$ is the $\mathbb{Z}/2$-Euler class of $\gamma_{m+n}$, $w_{mm}$ is the $\mathbb{Z}/2$-Euler class of $\gamma_m$, $w_{nn}$ is the $\mathbb{Z}/2$-Euler class of $\gamma_n$.

This equality is true for Thom classes by the Künneth theorem, since

$$\widetilde{H}^*\left( {}^{D(E\times E')}/_{S(E\times E')}; {}^{\mathbb{Z}}/_{2} \right) \cong \widetilde{H}^*\left( {}^{D(E)}/_{S(E)}; {}^{\mathbb{Z}}/_{2} \right) \otimes \widetilde{H}\left( {}^{D(E')}/_{S(E')}; {}^{\mathbb{Z}}/_{2} \right)$$

This implies that it is also true for $\mathbb{Z}/2$-Euler classes, since the $\mathbb{Z}/2$-Euler class is a pullback of the Thom class.

Hence, we have shown that

$$w_i(\gamma_m \times \gamma_n) = \sum_{j+k=i} w_{mj} \smile w_{nk}.$$

Now let $E, E'$ be any two bundles over $B$, of dimensions $m$ and $n$. Assume that $E$ and $E'$ are classified by maps $f, f'$. Now consider the pullback diagram

$$\begin{array}{ccccccc}
E \oplus E' & & \longrightarrow & & & & \gamma_{m+n} \\
\downarrow & & & & & & \downarrow \\
B & \xrightarrow{\Delta} & B \times B & \xrightarrow{f \times f'} & \mathrm{Gr}_m \times \mathrm{Gr}_n & \xrightarrow{\oplus} & \mathrm{Gr}_{m+n}
\end{array}$$

Then

$$\begin{aligned} w_i(E \oplus E') &= \Delta^*(f \times f')^* \oplus^* w_i \\ &= \Delta^*(f \times f')^* \left( \sum_{j+k=i} w_{mj} \smile w_{nk} \right) \\ &= \Delta^* \left( \sum_{j+k=i} w_j(E) \smile w_k(E') \right) \in H^*(B \times B; \mathbb{Z}/2) \\ &= \sum_{j+k=i} w_i(E) \smile w_k(E') \in H^*(B; \mathbb{Z}/2) \end{aligned}$$

□

**Remark 2.36.** Another way to see the that the product of Euler classes is again an Euler class, at least for manifolds, is to use the fact that the Euler class is Poincaré dual to the intersection of a generic section with the zero section. Given sections $\psi\colon \mathrm{Gr}_m \to \gamma_m$ of $\gamma_m$, and $\phi\colon \mathrm{Gr}_n \to \gamma_n$ of $\gamma_n$, this gives a section $\psi \times \phi$ of $\gamma_m \times \gamma_n$. The product of the intersection of $\psi$ with the zero section and the intersection of $\phi$ with the zero section is equal to the intersection of $\psi \times \phi$ with the zero section of $\gamma_m \times \gamma_n$.

**Remark 2.37.** There is another easier proof of the Whitney sum formula that uses the **splitting principle** to reduce the proof to the case of line bundles. From there, the only ingredient is the Thom isomorphism theorem.

Let's summarize what we know about Stiefel-Whitney classes.

**Theorem 2.38** ("Axioms" for Stiefel-Whitney Classes)**.**

*(1) For every* $j \geq 0$*, there is a Stiefel-Whitney class* $w_j(E) \in H^j(B; \mathbb{Z}/2)$*, with* $w_0(E) = 1$ *and* $w_j(E) = 0$ *if* $j$ *is larger than the rank of* $E$*.*

*(2) Given any map* $f\colon B' \to B$*,* $w_j(f^*(E)) = f^*(w_j(E))$

*(3)* $w_i(E \oplus E') = \sum_{j+k=i} w_j(E) \smile w_k(E')$

*(4) For* $\gamma_n \to \mathrm{Gr}_n$*,* $w_n(\gamma_n) \neq 0$*.*

These are often taken as the axioms for Stiefel-Whitney classes, and in fact characterize them uniquely. We will prove this later once we've discussed the splitting principle.

To see the utility of these axioms, let's prove Lemma 2.33 using them.

**Lemma 2.39** (Lemma 2.33, repeated)**.** *For any vector bundle* $E \to B$*,* $w_i(E) = w_i(E \oplus \varepsilon^k)$*, where* $\varepsilon^k$ *is a trivial bundle of rank* $k$*.*

*Proof.* Claim that $w_i(\varepsilon^k) = 0$ for $i > 0$ or $w_i(\varepsilon^k) = 1$ for $i = 0$ over any base. Given a trivial bundle $\varepsilon^k \to B$, the classifying map factors through a point

$$\begin{array}{ccccc} \varepsilon^k & & \longrightarrow & & \gamma_k \\ \downarrow & & & & \downarrow \\ B & \xrightarrow{r} & * & \longrightarrow & \mathrm{Gr}_k \end{array}$$

Therefore,

$$w_i(\varepsilon^k) = r^*(w_i(\varepsilon^k \to 0)) = \begin{cases} 1 & i = 0 \\ 0 & \text{else.} \end{cases}$$

Now by the Whitney Sum Formula,

$$w_i(E \oplus \varepsilon^n) = \sum_{j+k=i} w_j(E) \smile w_k(\varepsilon^n) = w_i(E).$$

□

## 2.4 Some computations

In this section, we will only use the four axioms of Stiefel-Whitney classes that we proved previously in Theorem 2.38.

**Proposition 2.40.** *If* $E \cong E'$*, then* $w_i(E) = w_i(E')$ *for all* $i$.

*Proof.* If $E \cong E'$, then their classifying maps are homotopic. □

**Proposition 2.41.** *For all* $i > 0$ *and any base* $B$*,* $w_i(\varepsilon^k) = 0$.

*Proof.* Consider the pullback diagram.

$$\begin{array}{ccc} \varepsilon^k & \longrightarrow & \varepsilon^k \\ \downarrow & \ulcorner & \downarrow \\ B & \xrightarrow{\mathrm{pr}} & \{*\} \end{array}$$

This shows that $w_i(\varepsilon^k \to B) = \mathrm{pr}^* w_i(\varepsilon^k \to \{*\})$. And $w_i(\varepsilon^k \to \{*\}) \in H^i(\{*\}) = 0$. □

**Proposition 2.42.** *If* $E$ *is a rank* $n$ *bundle over a paracompact base* $B$ *with an everywhere nonzero section, then* $w_n(E) = 0$*. Moreover, if* $E$ *has* $k$ *everywhere independent sections, then*

$$w_n(E) = w_{n-1}(E) = \cdots = w_{n-k+1}(E) = 0.$$

*Proof.* Let $s_1, \dots, s_k$ be everywhere independent sections of $E$. Let $E'$ be the span of $s_1, \dots, s_k$; it is a trivial bundle of rank $k$: $E' \cong \varepsilon^k$. Let $E''$ be the orthogonal compliment of $E'$, so $E'' = (E')^\perp$ (this is where we use paracompactness). Then $E \cong E' \oplus E'' = \varepsilon^k \oplus E''$.

Now we may use the Whitney sum formula

$$w_i(E) = \sum_{j+k=i} w_j(\varepsilon^k) \smile w_k(E'') = w_i(E'').$$

Note that $E''$ has rank $n-k$. Therefore, $w_i(E) = 0$ if $i > n-k$. $\square$

This gives a bound on the number of possible linearly independent sections of $E$.

**Proposition 2.43.** *For every* $k$*, there exists a unique polynomial* $q_i$ *such that whenever* $E \oplus E' \cong \varepsilon^n$*,*

$$w_i(E') = q_i(w_1(E), w_2(E), \dots, w_i(E)).$$

*Proof.* By induction on $i$. When $i = 0$, $w_0(E') = 1$.

When $i = 1$,

$$w_1(\varepsilon^n) = w_0(E) \smile w_1(E') + w_1(E) \smile w_0(E')$$

But $w_1(\varepsilon^n) = 0$ by Proposition 2.41, and $w_0(E) = w_0(E') = 1$. Hence, we have

$$w_1(E') = -w_1(E) = w_1(E),$$

where the last equality holds because we work mod 2.

Now suppose that $q_0, \dots, q_{i-1}$ exist. Then

$$\begin{aligned} 0 = w_i(\varepsilon^n) &= \sum_{k+j=i} w_k(E) \smile w_j(E') \\ &= w_i(E') + \sum_{\substack{k+j=i \\ j<i}} w_k(E) \smile w_j(E') \\ &= w_i(E') + \sum_{\substack{k+j=k \\ j<i}} w_k(E) \smile q_j(w_1(E), \dots, w_j(E)) \end{aligned}$$

Then by rearranging, we have

$$w_i(E') = q_i(w_1(E), \dots, w_k(E)) := \sum_{\substack{k+j=i \\ j<i}} w_k(E) \smile q_j(w_1(E), \dots, w_j(E)) \quad \square$$

**Definition 2.44.** We write $\overline{w}_i(E)$ for $q_i(w_1(E), \dots, w_i(E))$. These are called many things, among them **dual/orthogonal/normal Stiefel-Whitney classes.**

**Definition 2.45.** The **total Stiefel-Whitney class** of a bundle is

$$w(E) := w_0(E) + w_1(E) + w_2(E) + \ldots \in H^*(\mathrm{Gr}_n; \mathbb{Z}/2).$$

This is well-defined because for $j > \mathrm{rank}(E)$, $w_j(E) = 0$.

**Remark 2.46.** We call total Stiefel-Whitney classes an abomination because they sum elements of mixed degree, and therefore doesn't have a clear geometric interpretation.

Using the total Stiefel-Whitney class, we can rewrite the Whitney sum formula as

$$w(E)w(E') = w(E \oplus E').$$

In the case of Proposition 2.43, the dual Stiefel-Whitney classes of $E$ are the coefficients of the inverse power series of $w(E)$, so

$$w(E)\overline{w}(E) = 1.$$

Sometimes, it is convenient to use an abomination.

The following is a consequence of Proposition 2.43.

**Lemma 2.47** (Whitney Duality Theorem)**.** *Let $TM$ be the tangent bundle to $M \hookrightarrow \mathbb{R}^N$, and let $\nu$ be the normal bundle. Then $w_i(\nu) = \overline{w}_i(TM)$.*

**Remark 2.48.** Note that $\overline{w}_i(TM)$ is independent of the embedding! Hence, the class of a normal bundle is independent of the embedding of $M$ into $\mathbb{R}^N$ for some $N$. In fact, we need an embedding $TM \hookrightarrow \mathbb{R}^N$, but instead only an immersion; the tangent bundle doesn't notice if two places far apart map to the same place in $\mathbb{R}^N$, only that the tangent space is locally nicely included in $\mathbb{R}^N$. This can give us bounds on the dimension $N$ into which we can immerse a manifold.

**Proposition 2.49.** *Let $E \to B$ be a bundle, and assume that $B$ is compact. Then there is some $E'$ such that $E \oplus E' \cong \varepsilon^N$ for some $N$.*

*Proof.* As in Lemma 1.48, it suffices to construct $g \colon E \to \mathbb{R}^N$ that is a linear injection on fibers.

For each $x \in B$ there is some $U_x$ such that $p^{-1}(U_x) \cong U_x \times \mathbb{R}^n$. By Urysohn's Lemma there is a map $\phi_x \colon B \to [0,1]$ which is $0$ outside of $U_x$ and nonzero at $x$.

Then $\{\phi_x^{-1}(0,1]\}$ is an open cover of $B$. Since $B$ is compact, there is a finite subcover $U_1, \ldots, U_m$ defined by $\phi_1, \ldots, \phi_m$. These define maps $g_1, \ldots, g_m \colon U_i \to \mathbb{R}^n$ defined by

$$g_i \colon (x, v) \longmapsto \phi_i(x)v.$$

Then paste these together to get

$$\begin{aligned} g\colon E &\longrightarrow \mathbb{R}^{nm} \\ e &\longmapsto (g_1(e), \dots, g_m(e)) \end{aligned} \qquad \square$$

The following example shows that we definitely need the compactness assumption in Proposition 2.49.

**Claim 2.50.** There is no bundle $E \to \mathbb{RP}^\infty$ such that $E \oplus \gamma_1 \cong \varepsilon^n$.

*Proof.* $\gamma_1$ has two nonzero Stiefel-Whitney class: $w_0(\gamma_1) = 1$ and $w_1(\gamma_1)$, with

$$w_1(\gamma_1) = x \in H^*(\mathbb{RP}^\infty) = \mathbb{Z}/2[x].$$

By the Whitney sum formula,

$$w_i(\varepsilon^n) = \sum_{j+k=i} w_j(\gamma_1) w_k(E)$$

But $w_i(\varepsilon^n) = 0$, and the only nonzero classes $w_j(\gamma_1)$ are $w_1(\gamma_1)$ and $w_0(\gamma_1)$. So

$$0 = w_i(E) + w_1(\gamma_1) w_{i-1}(E) \implies w_i(E) = x w_{i-1}(E).$$

This inductively shows that $w_i(E) = x^i$.

In particular, these are never zero in $H^*(\mathbb{RP}^\infty) = \mathbb{Z}/2[x]$, so the hypothetical bundle $E$ must have infinite dimension (else $w_j(E) = 0$ for $j > \operatorname{rank}(E)$). But $E \oplus \gamma_1 = \varepsilon^n$ has finite rank, so no such $E$ exists. $\square$

*Abominable proof of Claim 2.50.* This proof uses total Stiefel-Whitney classes. We have $w(\varepsilon^n) = 1$ and $w(\gamma_1) = 1 + x$. Then

$$\overline{w}(\gamma_1) = 1 + x + x^2 + \dots$$

Hence, if $E$ exists, $w_i(E) = x^i$ for all $i$. So $E$ must be infinite dimensional. But $E \oplus \gamma_1 = \varepsilon^n$ has finite rank, so no such $E$ exists. $\square$

**Example 2.51.** Consider $TS^n$ for the $n$-sphere $S^n \subseteq \mathbb{R}^{n+1}$. There is a normal bundle $\nu \to S^n$ that is trivial. We have $TS^n \oplus \nu = \varepsilon^{n+1}$, and since $\nu$ is trivial, this means that $TS^n$ and $\varepsilon^{n+1}$ have the same Stiefel-Whitney classes. So

$$w_i(TS^n) = 0$$

for all $i > 0$. Stiefel-Whitney classes cannot detect that $TS^n$ is nontrivial. Later we will see that $w_n(TS^n) = 2e$, where $e$ is the Euler class. Hence, $w_n(TS^n) = 0$ since we work mod 2.

**Example 2.52.** Consider the bundle $T\mathbb{RP}^n$. We can't compute this directly, but we can compute $w_i(T\mathbb{RP}^n \oplus \varepsilon^1)$. We will use the description of $T\mathbb{RP}^n$ as

$$T\mathbb{RP}^n = TS^n/\{\pm 1\}.$$

A point in $T\mathbb{RP}^n$ will be written as a pair $((x, v), (-x, -v))$ with $x \perp v$. This determines a linear map

$$\begin{aligned} \ell \colon \mathbb{R}x &\longrightarrow (\mathbb{R}x)^\perp \\ x &\longmapsto v \end{aligned}$$

and vice versa. This canonically identifies the fiber above $x$ with the vector space $\mathrm{Hom}(\mathbb{R}x, \mathbb{R}x^\perp)$, and thereby identifies $T\mathbb{RP}^n$ with $\mathrm{Hom}(\gamma_{1n}, \gamma_{1n}^\perp)$.

$$\begin{aligned} T\mathbb{RP}^n \oplus \varepsilon^1 &\cong \mathrm{Hom}(\gamma_{1n}, \gamma_{1n}^\perp) \oplus \mathrm{Hom}(\gamma_{1n}, \gamma_{1n}) \\ &\cong \mathrm{Hom}(\gamma_{1n}, \gamma_{1n}^\perp \oplus \gamma_{1n}) \\ &\cong \mathrm{Hom}(\gamma_{1n}, \varepsilon^{n+1}) \\ &\cong \mathrm{Hom}(\gamma_{1n}, \varepsilon^1)^{\oplus(n+1)} \end{aligned}$$

Notice that all of these bundles are self dual. Hence,

$$\mathrm{Hom}(\gamma_{1n}, \varepsilon^1)^{\oplus(n+1)} \cong \mathrm{Hom}((\varepsilon^1)^\vee, \gamma_{1n}^\vee) \cong (\gamma_{1n}^\vee)^{\oplus(n+1)} \cong \gamma_{1n}^{\oplus(n+1)}$$

Therefore, in $H^*(\mathbb{RP}^n; \mathbb{Z}/2) = {}^{\mathbb{Z}/2[x]}/_{\langle x^{n+1}\rangle}$,

$$w(T\mathbb{RP}^n) = w(T\mathbb{RP}^n \oplus \varepsilon^1) = w(\gamma_{1n}^{\oplus(n+1)}) = (1+x)^{n+1}$$

Separating the individual classes from the total Stiefel-Whitney class, this shows that

$$w_i(T\mathbb{RP}^n) = \binom{n+1}{i} x^i \pmod 2$$

**Definition 2.53.** A manifold is **parallelizable** if its tangent bundle is trivial.

**Remark 2.54** (Notation)**.** If $M$ is a manifold, we write $w_i(M)$ for the Stiefel-Whitney class $w_i(TM)$ of its tangent bundle.

**Lemma 2.55.** *$\mathbb{RP}^n$ is parallelizable only if $n = 2^k - 1$ for some $k$.*

*Proof.* From Example 2.52, we know that $w_i(T\mathbb{RP}^n) = \binom{n+1}{i} x^i \pmod 2$ for all $i > 0$. When $n = 2^k - 1$, this is zero (look at Pascal's triangle mod 2). □

**Theorem 2.56** (Stiefel)**.** *Suppose that there exists a bilinear operation $P \colon \mathbb{R}^n \times \mathbb{R}^n$ without zerodivisors. Then $\mathbb{RP}^{n-1}$ is parallelizable.*

**Remark 2.57.**

(a) These can exist only when $n = 2^k$

(b) These exist for $n = 1, 2, 4, 8$: the real numbers, complex numbers, quaternions, and octonions.

*Proof of Theorem 2.56.* We're going to use $P$ to construct $n-1$ everywhere independent sections on $T\mathbb{RP}^{n-1}$. To do this, we will make use of the isomorphism

$$T\mathbb{RP}^{n-1} \cong \text{Hom}(\gamma_{1,n-1}, \gamma_{1,n-1}^{\perp}).$$

Given any $T \colon \mathbb{R}^n \to \mathbb{R}^n$ linear, we can construct for any line $\ell$ through the origin a map

$$\overline{T} \colon \ell \to \ell^{\perp}$$

that for any $x \in \ell$ assigns the projection of $T(x)$ onto $\ell^{\perp}$. This defines

$$\overline{T} \colon \gamma_{1,n-1} \to \gamma_{1,n-1}^{\perp}.$$

Now suppose that we have $T_1, \ldots, T_n$ linear such that $T_1(x), \ldots, T_n(x)$ are linearly independent for all $x$, and $T_1(x) = x$. Then for all $x$, $\overline{T}_2(x), \ldots, \overline{T}_n(x)$ are linearly independent sections of $T\mathbb{RP}^{n-1}$.

It remains to construct $T_1, \ldots, T_n$. Let $e_1, \ldots, e_n$ be the standard basis for $\mathbb{R}^n$, and define $S = P(-, e_1) \colon \mathbb{R}^n \to \mathbb{R}^n$. This is a linear map, and since there are no zerodivisors, it has trivial kernel. Therefore, $S$ is an isomorphism. Now define

$$T_i(x) = P(S^{-1}(x), e_i)$$

Notice that $T_1 = \text{id}$.

Moreover, we claim that these $T_i$ are linearly independent. For $x \neq 0$, suppose

$$c_1 T_1(x) + \ldots + c_n T_n(x) = 0$$

for nonzero real numbers $c_1, \ldots, c_n$. Then

$$P(S^{-1}(x), c_1 e_1 + \ldots + c_n e_n) = 0$$

yet neither $S^{-1}(x)$ nor $\sum_i c_i e_i$ are zero. This contradicts that $P$ has no zerodivisors. Hence, all the $c_i$ must be zero. □

**Corollary 2.58.** *$S^n$ is an H-space only if $n = 2^k - 1$.*

**Corollary 2.59.** *$\mathbb{RP}^n$ is parallelizable if and only if $n = 0, 1, 3, 7$.*

*Proof.* Combine Theorem 2.56 and Lemma 2.55. □

**Theorem 2.60** (Whitney Immersion)**.** *Any $n$-manifold $M$ has an immersion into $\mathbb{R}^{2n-1}$.*

Now suppose that an $n$-manifold $M$ has an immersion onto $\mathbb{R}^{n+k}$. How small can we make $k$? Well, we have that

$$TM \oplus \nu(M) \cong \varepsilon^{n+k}$$

Then $w_i(\nu) = \overline{w}_i(TM)$.

Recall that if $i$ is larger than the rank of a bundle $E$, then $w_i(E) = 0$. Therefore, if $w_i(E) \neq 0$, then $i < \text{rank}(E)$. Since the tangent bundle of $TM$ has rank $n$, then $\nu(M)$ has rank $k$.

Recall from Example 2.52 that $w_i(T\mathbb{RP}^n) = \binom{n+i}{i}x^i$.

**Example 2.61.** Let $M = \mathbb{RP}^9$. Then $w_i(TM) = x^i$ if $i \in \{2, 8\}$. So we may compute that $\overline{w}_i(TM) = x^i$ for $i \in \{2, 4, 6\}$. Hence, $\mathbb{RP}^9$ cannot be immersed in anything of dimension 15 or less.

**Example 2.62.** If $M = \mathbb{RP}^{2^k}$, then $w_i(TM) = x^i$ for $i \in \{0, 1, 2^k\}$. We can compute that $\overline{w}_i(TM) = x^i$ for $i \in \{1, 2, \ldots, 2^k - 1\}$. Hence, if $\mathbb{RP}^{2^k}$ is immersed in $\mathbb{R}^{2^k+\ell}$, then $\ell \geq 2^k - 1$.

So the bound in the Whitney Immersion Theorem is sharp.

# 3 Cobordism

## 3.1 Stiefel-Whitney Numbers

Stiefel-Whitney numbers are a much coarser invariant than Stiefel-Whitney classes, but they are still surprisingly powerful. Stiefel-Whitney classes allow us to compare vector bundles on manifolds, while Stiefel-Whitney numbers allow us to compare things between manifolds, which gives some interesting results.

**Definition 3.1.** Let $M$ be an $n$-manifold, and let $[M] \in H_n(M; \mathbb{Z}/2)$ be the fundamental class. Let $r_1, \ldots, r_n \geq 0$ such that

$$r_1 + 2r_2 + 3r_3 + \ldots + nr_n = n.$$

Then the $(r_1, \ldots, r_n)$**-th Stiefel-Whitney number** is

$$(w_1(TM)^{r_1} \smile w_2(TM)^{r_2} \smile \cdots \smile w_n(TM)^{r_n}) \frown [M] \in \mathbb{Z}/2.$$

For shorthand, we write

$$w_1^{r_1} w_2^{r_2} \cdots w_n^{r_n}[M].$$

**Lemma 3.2.** *Let* $M, N$ *be* $n$*-manifolds. Then*

$$w_1^{r_1} w_2^{r_2} \cdots w_n^{r_n}[M \sqcup N] = w_1^{r_1} w_2^{r_2} \cdots w_n^{r_n}[M] + w_1^{r_1} w_2^{r_2} \cdots w_n^{r_n}[N]$$

*Proof.* We have that

$$H_n(M \sqcup N; \mathbb{Z}/2) \cong H_n(M; \mathbb{Z}/2) \times H_n(N; \mathbb{Z}/2)$$

and

$$H^n(M \sqcup N; \mathbb{Z}/2) \cong H^n(M; \mathbb{Z}/2) \times H^n(N; \mathbb{Z}/2).$$

The pullback of the tangent bundle of $M \sqcup N$ along the inclusion $M \hookrightarrow M \sqcup N$ is the tangent bundle of $M$, and likewise for the tangent bundle of $N$.

The result now follows from the fact that Stiefel-Whitney classes commute with pullbacks of vector bundles. □

**Example 3.3.** Consider $\mathbb{RP}^n$. Recall that

$$w_i(T\mathbb{RP}^n) = \binom{n+1}{i} x^i \pmod 2$$

Therefore,

$$w_1^{r_1} w_2^{r_2} \cdots w_n^{r_n}[M] \equiv \underbrace{\binom{n+1}{1}^{r_1}}_{r_1 \neq 0} \underbrace{\binom{n+1}{2}^{r_2}}_{r_2 \neq 0} \cdots \underbrace{\binom{n+1}{n}^{r_n}}_{r_n \neq 0} \pmod 2.$$

We only include the term $\binom{n+1}{j}^{r_j}$ when $r_j$ is not even to avoid defining $0^0$ (don't forget that we're working mod 2!).

Notice that this isn't always zero. In particular, for $n$ even, we have

$$w_n(T\mathbb{RP}^n) = (1+n)x^n \equiv x^2 \pmod 2$$
$$w_1(T\mathbb{RP}^n) = x$$

And therefore, when $r_1 = n$ or $r_n = 1$, the corresponding Stiefel-Whitney number of $\mathbb{RP}^n$ is 1.

For $n$ odd, say $n = 2k - 1$. Then

$$(1+x)^{2k} \equiv (1+x^2)^k \pmod 2.$$

Hence,

$$\binom{2k}{2i} \equiv \binom{k}{i} \pmod 2$$
$$\binom{2k}{2i+1} \equiv 0 \pmod 2.$$

This shows in particular that all Stiefel-Whitney classes vanish when $n$ is odd. Any sequence of $(r_1, \ldots, r_n)$ such that

$$r_1 + 2r_2 + 3r_3 + \ldots + nr_n = n$$

must have odd $j$ such that $r_j \neq 0$, so inside the product there is at least one zero. Hence, all Stiefel-Whitney numbers of $\mathbb{RP}^{2k-1}$ are zero:

$$w_1^{r_1} \cdots w_n^{r_n}[\mathbb{RP}^{2k-1}] = 0.$$

**Theorem 3.4** (Pontrjagin). *If $B$ is a compact smooth $(n+1)$-manifold with boundary $M$, then all the Stiefel-Whitney numbers of $M$ are zero.*

*Proof.* Consider $i\colon M \hookrightarrow B$. Then there is a long exact sequence in homology

$$\begin{array}{ccccccccc} H_{n+1}(M;\mathbb{Z}/2) & \xrightarrow{i_*} & H_{n+1}(B;\mathbb{Z}/2) & \longrightarrow & H_{n+1}(B,M;\mathbb{Z}/2) & \xrightarrow{\partial} & H_n(M;\mathbb{Z}/2) & \xrightarrow{i_*} & H_n(B;\mathbb{Z}/2) \\ & & & & [B,M] & \longmapsto & [M] & & \end{array}$$

So for all $\nu \in H^n(M;\mathbb{Z}/2)$,

$$\nu[M] = \nu(\partial[B,M]) = (\delta\nu)[B,M]$$

where $\delta\colon H^n(M;\mathbb{Z}/2) \to H^{n+1}(B,M;\mathbb{Z}/2)$ is the map on cohomology corresponding to $\partial$.

$TB|_M = \varepsilon^1 \oplus TM$, where $\varepsilon^1$ is the trivial bundle on $M$ orthogonal to $B$. Since adding a trivial bundle doesn't change Stiefel-Whitney classes,

$$i^* w_i(TB|_M) = w_i(i^* TB|_M) = w_i(TM).$$

Therefore,

$$\begin{aligned} w_1^{r_1} w_2^{r_2} \cdots w_n^{r_n}[M] &= (\delta(w_1^{r_1} w_2^{r_2} \cdots w_n^{r_n}))[B,M] \\ &= (\delta i^*(w_1^{r_1} w_2^{r_2} \cdots w_n^{r_n}))[B,M] \\ &= 0 \end{aligned}$$

since $\delta \circ i^* = 0$ in the long exact sequence of cohomology. □

## 3.2 Cobordism Groups

**Definition 3.5.** Let $M$ and $N$ be two $n$-manifolds. We say that $M$ and $N$ are **cobordant** if there is an $(n+1)$-manifold such that $\partial W = M \sqcup N$. Then $W$ is a **cobordism** between $M$ and $N$.

**Remark 3.6.** The "co-" in "cobordant" is *not* the same as the "co-" in cohomology. It means "together," and saying that $M$ and $N$ are cobordant means that together, they form the boundary of another manifold $W$.

**Example 3.7.** $S^1$ is cobordant to $S^1 \sqcup S^1$, and to $S^1 \sqcup S^1 \sqcup S^1$.

**Example 3.8.** Cobordisms are not unique. $S^1 \times I$ is a cobordism between $S^1$ and $S^1$, but so is the torus with two ends chopped off.

**Definition 3.9.** The **unoriented cobordism group** $\mathfrak{N}_n$ is the abelian of manifolds up to cobordism.

**Lemma 3.10.** $\mathfrak{N}$ *is an abelian group.*

*Proof.* The addition is defined as $[M] + [N] = [M \sqcup N]$, with unit $0 = [\varnothing]$. Thus, a manifold is in the class of the identity if it is the boundary of an $(n+1)$-manifold.

The inverse of a class $[M]$ is itself, because there is a cobordism $M \times I$ with boundary $[M \sqcup M]$, so $[M] + [M] = 0$.

This group is abelian because $[M \sqcup N] \cong [N \sqcup M]$. □

By Theorem 3.4, we can determine when a manifold is the boundary of another. This is useful for the purposes of cobordism.

**Corollary 3.11.** *If* $[M] = [N]$ *in* $\mathfrak{N}$*, then their Stiefel-Whitney numbers are equal.*

*Proof.* Combine Theorem 3.4 and Lemma 3.2. □

**Question 3.12.** Is the converse of Theorem 3.4 true? If $M$ and $N$ have equal Stiefel-Whitney numbers, are they cobordant?

We will spend the next few lectures investigating the answer to this question. Spoiler alert: Thom proves that the answer is yes, and moreover he identifies the structure of the cobordism groups. For this, we need to understand Thom spaces better.

## 3.3 Geometry of Thom Spaces

Recall that for a vector bundle $p\colon E \to B$, the Thom space of $E$ is

$$\mathrm{Th}(E) = {}^{D(E)}\!/_{S(E)}.$$

This is alternatively described as the one-point compactification of $E$, but that's not always quite true (although it is for the cases we care about).

**Example 3.13.** The Thom space of a trivial bundle $\varepsilon^k$ over a point is $S^k$, because $S^k$ is the quotient of the unit ball in $\mathbb{R}^k$ by its boundary.

**Definition 3.14.** The **smash product** of two pointed spaces $(X, x_0)$ and $(Y, y_0)$ is

$$X \wedge Y := {}^{X \times Y}\!/_{(X \times \{y_0\}) \cup (\{x_0\} \times Y)}.$$

This works a lot like a tensor product does: a tensor $a \otimes b$ is zero if either $a$ or $b$ are zero. Similarly, a point $(x, y) \in X \wedge Y$ is the basepoint if either $x$ or $y$ is the basepoint of $X$ or $Y$.

The smash product is useful because it's easy to state the Künneth theorem for the smash product of pointed spaces with coefficients in a field.

$$\widetilde{H}^*(X \wedge Y; k) \cong \widetilde{H}^*(X; k) \otimes \widetilde{H}^*(Y; k)$$

**Example 3.15.** $S^n \wedge S^m \cong S^{n+m}$. Why? Write $S^i = I^i/\partial I^i$. Therefore,

$$S^n \wedge S^m = {I^n}/{\partial I^n} \wedge {I^m}/{\partial I^m} = {I^n \times I^m}/{(\partial I^n \times I^m) \cup (I^n \times \partial I^m)}$$

$$= {I^n \times I^m}/{\partial(I^n \times I^m)} = S^{n+m}$$

**Example 3.16.** For any space $X$, $S^1 \wedge X$ is the reduced suspension $\Sigma X$ of $X$.

$$S^1 \wedge X = {I^1}/{\partial I^1} \wedge X = {I \times X}/{(\partial I \times X) \cup (I \times \{x_0\})} = \Sigma X$$

**Lemma 3.17.** *For any two bundles* $p\colon E \to B$ *and* $p'\colon E' \to B'$,

$$\mathrm{Th}(E \times E') \cong \mathrm{Th}(E) \wedge \mathrm{Th}(E')$$

*Proof.* $\mathrm{Th}(E \times E')$ is the one-point compactification of $E \times E'$. Therefore, $\mathrm{Th}(E) \times \mathrm{Th}(E')$ is the product of the one-point compactifications, so it is compact and contains $E \times E'$. We have a map

$$g\colon \mathrm{Th}(E) \times \mathrm{Th}(E') \longrightarrow \mathrm{Th}(E \times E')$$

that is the identity on $E \times E' \subseteq \mathrm{Th}(E) \times \mathrm{Th}(E')$. Then $g$ takes everything to the extra point in $\mathrm{Th}(E \times E')$. In particular,

$$(E \cup \{*\}) \times (\{*'\} \cup E') = (E \times \{*'\}) \cup (\{*\} \times E') \mapsto \{*\}$$

So $g$ factors through $\mathrm{Th}(E) \wedge \mathrm{Th}(E')$, and it isn't difficult to check that this is a bijection on points. □

**Lemma 3.18.**

$$\mathrm{Th}(E \oplus \varepsilon^k) \cong S^k \wedge \mathrm{Th}(E).$$

*Proof.* Note that $E \oplus \varepsilon^k \cong E \times \varepsilon^k$, when we think of $\varepsilon^k$ as a trivial bundle over a point. Then use Lemma 3.17.

$$\mathrm{Th}(E \oplus \varepsilon^k) = \mathrm{Th}(E \times \varepsilon^k) = \mathrm{Th}(E) \wedge \mathrm{Th}(\varepsilon^k) = \mathrm{Th}(E) \wedge S^k$$

The last equality comes from Example 3.13. □

**Definition 3.19.** A space $X$ is $n$**-connected** if $\pi_i(X) = 0$ for all $i \le n$.

**Definition 3.20.** Given a space $X$, the **suspension homomorphism** $\pi_i(X) \to \pi_{i+1}(\Sigma X)$ is given as follows. A class $[f] \in \pi_i(X)$, is represented by $f\colon S^i \to X$, and the corresponding class in $\pi_{i+1}(\Sigma X)$ is $\Sigma f\colon \Sigma S^i \to \Sigma X$, since $\Sigma S^i \cong S^{i+1}$.

**Exercise 3.21.** Prove that the suspension homomorphism is well-defined: if $f$ and $g$ represent the same class in $\pi_i(X)$, then $\Sigma f$ and $\Sigma g$ represent the same class in $\pi_{i+1}(\Sigma X)$.

**Theorem 3.22** (Freudenthal Suspension Theorem)**.** *If $X$ is an $n$-connected CW complex, then the suspension homomorphism*

$$\pi_i(X) \to \pi_{i+1}(\Sigma X)$$

*is an isomorphism for $i < 2n$ and a surjection for $i = 2n$.*

**Theorem 3.23** (Hurewicz)**.** *For any space $X$ and any positive integer $i$, there is a homomorphism*

$$\begin{aligned} \pi_i(X) &\xrightarrow{h} H_i(X) \\ f &\longmapsto f_*[S^i] \end{aligned}$$

*When $X$ is $(n-1)$-connected, this is an isomorphism for $i \leq n$ and surjective for $i = n+1$.*

**Definition 3.24.** The map $h$ in Theorem 3.23 is called the **Hurewicz homomorphism**.

**Lemma 3.25.** *For $k > n$, the group $\pi_{n+k}(\mathrm{Th}(\gamma_k))$ is independent of $k$.*

**Remark 3.26.** $\mathrm{Th}(\gamma_k)$ is often referred to as $MO(k)$ in other sources. For complex vector bundles, it is called $MU(k)$, and for oriented bundles, it is called $MSO(k)$.

The following proposition from Thom's original paper is quite horrible to prove, so we present it without proof.

**Proposition 3.27.** *Let $X$ and $Y$ be simply connected CW complexes and let $f\colon X \to Y$. Suppose that for all primes $p$, the induced map $f^*\colon H^i(Y;\mathbb{Z}/p) \to H^i(X;\mathbb{Z}/p)$ is an isomorphism for all $i < k$ and injective for $i = k$.*

*Then there is some $g\colon Y^{(k)} \to X^{(k)}$ such that $f \circ g|_{Y^{k-1}} \simeq 1_{Y^{k-1}}$ and $g \circ f|_{X^{k-1}} \simeq 1_{X^{(k-1)}}$.*

What this is really saying is that, if $f\colon X \to Y$ induces isomorphisms on all mod $p$-cohomology for $i < k$ and is injective for $i = k$, then $X$ and $Y$ have the same homotopy $k$-type.

*Proof of Lemma 3.25.* Consider the diagram that induces $\eta$ on cohomology.

$$\begin{array}{ccccccc} \mathrm{Gr}_k & \xleftarrow{\sim} & S(\gamma_{k+1}) & \hookrightarrow & D(\gamma_{k+1}) & \xrightarrow{\sim} & \mathrm{Gr}_{k+1} \\ \nu^\perp \cap \omega & \longleftarrow & (\omega, \nu) & & & & \\ & & \nu \in \omega & & & & \end{array}$$

Since $\mathrm{Gr}_k$ and $S(\gamma_{k+1})$ are both CW complexes, and the map $S(\gamma_{k+1}) \to \mathrm{Gr}_k$ is a weak equivalence, there is a homotopy inverse $\mathrm{Gr}_k \to S(\gamma_{k+1})$. Hence, we have some map

$$i\colon \mathrm{Gr}_k \to \mathrm{Gr}_{k+1}$$

that induces $\eta$ on cohomology.

This gives a pullback diagram.

$$\begin{array}{ccc} \gamma_k \oplus \varepsilon^1 & \dashrightarrow & \gamma_{k+1} \\ \downarrow & & \downarrow \\ \mathrm{Gr}_k & \xrightarrow{i} & \mathrm{Gr}_{k+1} \end{array}$$

This induces a map on Thom spaces

$$\Sigma\, \mathrm{Th}(\gamma_k) \cong \mathrm{Th}(\gamma_k \oplus \varepsilon^1) \to \mathrm{Th}(\gamma_{k+1}).$$

Altogether, we have a map

$$\pi_{n+k}\, \mathrm{Th}(\gamma_k) \xrightarrow{f} \pi_{n+k+1}(\Sigma\, \mathrm{Th}(\gamma_k)) \cong \pi_{n+k+1}(\mathrm{Th}(\gamma_k \oplus \varepsilon^1)) \xrightarrow{i_*} \pi_{n+k+1}\, \mathrm{Th}(\gamma_{k+1})$$

To show that this map is an isomorphism, we will show that both $f$ and $i_*$ are isomorphisms.

(1) $f$ is an isomorphism by the Freudenthal suspension theorem if we can show that $\mathrm{Th}(\gamma_k)$ is $k$-connected; the Freudenthal suspension theorem applies because $n < k$.

$\widetilde{H}^i(\mathrm{Th}(\gamma_k)) \cong \widetilde{H}^{i-k}(\mathrm{Gr}_{k+1})$ if $i < k$. Therefore, $H_i(\mathrm{Th}(\gamma_k)) = 0$ for $i < k$. If we know that $\mathrm{Th}(\gamma_k)$ is simply connected, then we can apply the Hurewicz theorem to conclude $\pi_i(\mathrm{Th}(\gamma_k)) = 0$ for $i < k$.

So it suffices to show that $\pi_1(\mathrm{Th}(\gamma_k)) = 0$. Notice that $\mathrm{Th}(\gamma_k) = D(\gamma_k)/S(\gamma_k)$, so $\pi_1(\mathrm{Th}(\gamma_k))$ is a quotient of $\pi_1(D(\gamma_k))$.

We have a long exact sequence of homotopy groups coming from the fiber sequence

$$O(k) \to EO(k) \to BO(k).$$

This gives isomorphisms $\pi_i(BO(k)) \cong \pi_{i-1}(O(k))$ because $EO(k)$ is contractible. In particular, $\pi_1(\mathrm{Gr}_k) \cong \pi_0(O(k)) \cong \mathbb{Z}/2$. Hence,

$$\pi_1(D(\gamma_k)) \cong \pi_1(\mathrm{Gr}_k) \cong \pi_1(BO(k)) \cong \pi_0(O(k)) \cong \mathbb{Z}/2.$$

Since $\mathrm{Th}(\gamma_k) = D(\gamma_k)/S(\gamma_k)$, we need to know that the quotient by $S(\gamma_k)$ collapses the $\mathbb{Z}/2$: we know

$$\pi_1\, \mathrm{Th}(\gamma_k) = {}^{\pi_1(\mathrm{Gr}_k)}\!\big/_{\mathrm{im}\, \pi_1(\mathrm{Gr}_{k-1})}$$

To show that $\pi_1(\mathrm{Th}(\gamma_k)) = 0$, it now suffices to show that $i_* \colon \pi_1(\mathrm{Gr}_{k-1}) \to \pi_1(\mathrm{Gr}_k)$ is surjective, so the image of $\pi_1(\mathrm{Gr}_{k-1})$ inside $\pi_1(\mathrm{Gr}_k)$ is all of $\mathbb{Z}/2$. But this is induced b the map $O(k-1) \to O(k)$ given by

$$\begin{aligned} O(k-1) &\longrightarrow O(k) \\ A &\longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

This is surjective on connected components, which implies that the induced map $\pi_1 BO(k-1) \to \pi_1 BO(k)$ is surjective. Finally, we know that $\mathrm{Gr}_i \cong BO(i)$, so $\pi_1(\mathrm{Gr}_{k-1}) \to \pi_1(\mathrm{Gr}_k)$ is surjective. Hence,

$$\pi_1(\mathrm{Th}(\gamma_k)) = 0,$$

so the Hurewicz theorem applies and we may conclude that $\pi_i(\mathrm{Th}(\gamma_k)) = 0$ for $i < k$.

(2) Now we need to show that $i_*$ is an isomorphism. On cohomology, $i^*$ is an isomorphism $H^i(\mathrm{Gr}_{k+1}) \to H^*(\mathrm{Gr}_k)$ up to degree $k+1$. Hence, for cohomology of the Thom bundles, $i^*$ is an isomorphism for $j < 2k+2$:

$$i^* \colon H^j(\mathrm{Th}(\gamma_{k+1})) \cong H^j(\mathrm{Th}(\gamma_k \oplus \varepsilon^1)).$$

Hence, Proposition 3.27 applies and therefore $i_*$ is an isomorphism up to dimension $2k$. In particular $i_*$ is an isomorphism on homotopy groups $\pi_{n+k+1}(-)$ for $n < k$. □

**Theorem 3.28** (Thom)**.** *For* $k > n+2$,

$$\mathfrak{N}_n \cong \pi_{n+k}(\mathrm{Th}(\gamma_k))$$

Notice that the right-hand-side of this isomorphism is well-defined by Lemma 3.25 for $k > n$.

## 3.4 L-equivalence and Transversality

To prove Theorem 3.28, we need a lot of results about smooth manifolds. Since the point of this class isn't to learn about smooth manifolds, we will cite a lot of these things without proof. Most of it comes out of Thom's original paper.

**Remark 3.29.** We will abuse notation and abbreviate $\mathrm{Gr}_k := \mathrm{Gr}_k(\mathbb{R}^N)$ for $N \geq 2k+5$. In the cases we care about in the lemmas below, we need a compact manifold; $\mathrm{Gr}_k(\mathbb{R}^N)$ is compact. Moreover, maps here are well-defined and independent of $N$ when $N$ is sufficiently large. Likewise, write $\gamma_k := \gamma_{kN}$.

**Definition 3.30.** Let $f\colon X^n \to M^p$ be a $C^n$ map from an $n$-manifold to a $p$-manifold. Let $N^{p-q} \subseteq M$ be a submanifold of $M$ of codimension $q$. For $y \in N$, $T_yM \supseteq T_yN$. Let $x \in f^{-1}(y)$. We say that $f$ is **transverse to** $N$ **at** $y$ if

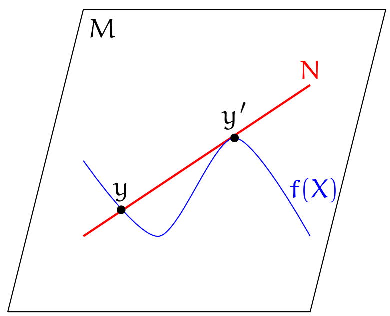
$$df_x\colon T_xX \to T_yM \to {}^{T_yM}/_{T_yN}$$

is an epimorphism.

$f$ is **transverse to** $N$ if this holds for all $x, y$.

Notice that if $f^{-1}(y) = \emptyset$, transversality automatically holds.

**Example 3.31.** Let $X = \mathbb{R}$, $M = \mathbb{R}^2$, and $N = \mathbb{R}$.




At $y$, $T_xX \twoheadrightarrow {}^{T_yM}/_{T_yN}$ is transverse.

At $y'$, $T_xX \xrightarrow{0} {}^{T_{y'}M}/_{T_{y'}N}$ is not transverse.

**Definition 3.32.** A homotopy $X \times [0,1] \to Y$ is an **isotopy** if for all $t \in [0,1]$, the map $X \times \{t\} \to Y$ is smooth.

**Definition 3.33.** Let $N$ be a submanifold of a manifold $M$ of codimension $q$. A **tubular neighborhood** of $N$ in $M$ is an embedding of a $q$-disk bundle on $N$ into $M$ such that $N$ is the zero section.

**Theorem 3.34.** *Assume that*

- $X$ *is a smooth* $n$*-manifold;*
- $M$ *is a* $p$*-manifold;*
- $N \subseteq M$ *is a paracompact submanifold of* $M$ *of codimension* $q$*;*
- $T$ *is a tubular neighborhood of* $N$ *in* $M$*;*
- $f\colon X \to M$ *is a* $C^n$ *map;*
- $y \in T_yM$ *and* $x \in f^{-1}(Y)$.

*Then we may conclude the following.*

(a) *If* $f\colon X \to M$ *is transverse to* $N$*, then* $f^{-1}(N)$ *is a* $C^n$ *submanifold of* $X$ *of codimension* $q$*.*

(b) *There is a homeomorphism* $A$ *of* $T$ *arbitrarily close to the identity and equal to the identity on* $\partial T$*, such that* $A \circ f$ *is transverse to* $N$*.*

(c) *If* $f\colon X \to M$ *is transverse to* $N$*, then* $N$ *is compact. Then for any* $A$ *(as in (b)) sufficiently close to the identity,* $A \circ f$ *is transverse to* $N$ *and* $f^{-1}(N)$*,* $(A \circ f)^{-1}(N)$ *are isotopic in* $X$*.*

**Remark 3.35.** Theorem 3.34(b) says that we may always wiggle the tubular neighborhood a little bit so that, after with composing with $f$, it is transverse to $N$. Theorem 3.34(c) implies that $f^{-1}(N)$ and $(A \circ f)^{-1}(N)$ are isomorphic. Similar results hold for manifolds $M$ with boundary.

Now let's relate this to cobordism.

**Theorem 3.36.** *Let* $f, g\colon X \to M$ *be* $C^m$ *maps where* $m \geq n$*, both transverse to a submanifold* $N$ *of codimension* $q$*. If* $f$ *and* $g$ *are homotopic, then* $f^{-1}(N)$ *is cobordant to* $g^{-1}(N)$*.*

*Proof.* We may assume that this homotopy is smooth. So consider $F\colon X \times I \to M$. By Theorem 3.34(b) there is some $A$ such that $A \circ F$ is transverse to $N$. By Theorem 3.34(c), $f^{-1}(N)$ is isotopic to $(A \circ F|_{X\times\{0\}})^{-1}(N)$. In particular, this implies that $f^{-1}(N)$ is *cobordant* to $(A \circ F|_{X\times\{0\}})^{-1}(N)$. Likewise, $g^{-1}(N)$ is cobordant to $(A \circ F|_{X\times\{1\}})^{-1}(N)$.

Now by Theorem 3.34(a), $(A \circ F)^{-1}(N)$ is a submanifold of $X \times I$ with boundary

$$(A \circ F|_{X\times\{0,1\}})^{-1}(N) = (A \circ F|_{X\times\{0\}})^{-1}(N) \cup (A \circ F|_{X\times\{0\}})^{-1}(N)$$

Composing with the cobordisms in the first paragraph, we obtain a cobordism between $f^{-1}(N)$ and $g^{-1}(N)$. □

This motivates the following definition.

**Definition 3.37.** Let $W_0, W_1$ be two $k$-submanifolds of an $n$-manifold $X$. Then we say that $W_0$ and $W_1$ are **L-equivalent in** $X$ if there exists a manifold $Y$ with boundary $W_0 \sqcup W_1$ and an embedding $f\colon Y \to X \times I$ such that

$$f^{-1}(X \times \{0\}) = W_0 \text{ and } f^{-1}(X \times \{1\}) = W_1.$$

We write $L_k(X)$ for the set of L-equivalence classes of $k$-submanifolds.

**Example 3.38.** If $W_0 = S^1 \sqcup S^1$ and $W_1 = S^1$ inside the plane $X$, but $W_0 \cap W_1 \neq \emptyset$, then there's no embedded cobordism between them. But there is an embedded pair of pants linking them in $X \times I$.

**Lemma 3.39.** *If $n > 2k+2$, then $L_k(S^n)$ is an abelian group. The map $\phi\colon L_k(S^n) \to \mathfrak{N}_k$ taking the L-equivalence class of $W$ to the cobordism class of $W$ is an isomorphism.*

*Proof.* For $n > 2k+2$, any two embedded k-submanifolds can be homotoped (and indeed, isotoped) to be disjoint. Thus, disjoint union is a well-defined operation on $L_k(S^n)$.

We say that $[\emptyset]$ is the identity in $L_k(S^n)$.

$L_k(S^n)$ has inverses given by the horseshoe L-equivalence: Therefore, $2[W] = 0$, so $[W] = -[W]$. Hence $L_k(S^n)$ is a group.

Now to show that the map $\phi\colon L_k(S^n) \to \mathfrak{N}_k$ is an isomorphism, it suffices to check that this is a bijection since these have the same group structure.

To check surjectivity, assume $[W] \in \mathfrak{N}_k$. Then there is an embedding

$$W \hookrightarrow \mathbb{R}^{2k+2} \hookrightarrow S^n$$

(recall that we are assuming that $n \geq 2k+2$ Remark 3.29). So $[W]$ is a class in $L_k(S^n)$.

To check injectivity, consider an embedded submanifold $W \hookrightarrow S^n$ such that $[W] = 0$ in $\mathfrak{N}_k$. Write $W = \partial B$ for a $(k+1)$-manifold $B$. Embed $B$ into $S^n$ via

$$f\colon B \hookrightarrow \mathbb{R}^{2(k+1)} \hookrightarrow S^n.$$

Use Urysohn's Lemma to pick a function $\phi\colon B \to I$. Then $\phi^{-1}(0) = W$, and $\phi^{-1}(1) = \emptyset$. Then $(f, \phi)\colon B \to S^n \times I$ witnesses an L-equivalence between $W$ and $\emptyset$. Hence, $[W] = 0$ in $L_k(S^n)$. □

**Construction 3.40.** For $X$ an $n$-manifold, we define a map $J\colon L_{n-k}(X) \to [X, \mathrm{Th}(\gamma_k)]$ by first choosing an embedding $X \hookrightarrow \mathbb{R}^N$. Then for each $w \in W$, we have a normal bundle at $w$ inside $X$:

$$N_w W := (T_w W)^\perp \cap T_w X.$$

Then $N_w W$ is a k-plane in $\mathbb{R}^N$, so an element of $\mathrm{Gr}_k = \mathrm{Gr}_k(\mathbb{R}^N)$ (see Remark 3.29). This gives a map $f\colon W \to \mathrm{Gr}_k$.

Now let $N$ be a tubular neighborhood of $W$ in $X$; think of it as a pullback of the disk bundle of $\gamma_k$: $N = f^*(D(\gamma_k))$. Then $f$ induces a map $\widetilde{f}\colon \mathrm{Th}(f^*(\gamma_k)) \to \mathrm{Th}(\gamma_k)$. So define

$$f'\colon X \to \mathrm{Th}(\gamma_k)$$

by

$$f'(x) = \begin{cases} * & \text{if } x \notin N, \\ \widetilde{f}(x) & \text{if } x \in N. \end{cases}$$

The image of $W$ under J is this map $f' \colon X \to \mathrm{Th}(\gamma_k)$.

Now, $\mathrm{Gr}_k$ embeds into $\mathrm{Th}(\gamma_k)$ as the zero section. So

$$(f')^{-1}(\mathrm{Gr}_k) = W.$$

Why do we find this construction useful? Let $X = S^{n+k}$. Then we have a map

$$L_n(S^{n+k}) \to \pi_{n+k}\,\mathrm{Th}(\gamma_k).$$

To prove Theorem 3.28, we want to show that this is an isomorphism of groups. Then we may apply Lemma 3.39 to conclude that

$$\mathfrak{N}_k \cong L_n(S^{n+k}) \cong \pi_{n+k}\,\mathrm{Th}(\gamma_k)$$

**Remark 3.41.** When might we expect $[X, Y]$ to be a group?

If we have $f, g \colon X \to Y$ with $X$ **cogrouplike**, meaning that it has a nice map $p \colon X \to X \vee X$, then we might define the product of $f$ and $g$ by

$$X \xrightarrow{p} X \vee X \xrightarrow{f \vee g} Y \vee Y \xrightarrow{\text{fold}} Y.$$

For example, the pinch map $S^n \to S^n \vee S^n$ satisfies this property.

Alternatively, if we had a retraction map $r \colon Y \times Y \to Y \vee Y$, then we might define the product of $f$ and $g$ by

$$X \xrightarrow{\text{diag}} X \times X \xrightarrow{f \times g} Y \times Y \xrightarrow{r} Y \vee Y \xrightarrow{\text{fold}} Y.$$

Classically, the conditions for this second approach to work were answered in **cohomotopy theory**, which studies homotopy classes of maps into spheres instead of out of spheres. This theory is now pretty much defunct.

**Lemma 3.42.** *J is independent of the choice of* $X \hookrightarrow \mathbb{R}^N$.

*Proof.* If $i_0, i_1 \colon X \hookrightarrow \mathbb{R}^N$, we may assume for large enough $N$ that $i_0(X) \cap i_1(X) = \emptyset$. Moreover, we may assume that there is an embedding $X \times I \to \mathbb{R}^N$ that is $i_0$ on $X \times \{0\}$ and $i_1$ on $X \times \{1\}$. Finally, we may assume that $X \times I$ is embedded orthogonally to its boundary.

Let $W$ be some $k$-submanifold of $X$. The embedding above restricts to an embedding of $W \times I \to \mathbb{R}^N$. A tubular neighborhood $N$ of $W \times I$ under this embedding is orthogonal to the boundary of $X \times I$ by our assumption; thus $N \cap X \times \{0\}$ is a tubular neighborhood of $i_0(W)$ and $N \cap X \times \{1\}$ is a tubular neighborhood of $i_1(W)$. We can then apply the construction of J to this $N$ and the embedding of $W \times I$ to produce a map $X \times I \to \mathrm{Th}(\gamma_k)$. This restricts to the maps constructed for $W$ under $i_0$ and $i_1$, respectively. Thus, the two maps are homotopic. □

**Lemma 3.43.** $L$*-equivalent submanifolds give homotopic maps under* $J$*.*

*Proof sketch.* Let $W_0$, $W_1$ be L-equivalent. So there is some submanifold $B \subseteq S^{n+k} \times I$ with $B \cap (S^{n+k} \times \{i\}) = W_i$. Let $T$ be a tubular neighborhood of $B$. Then $T \cap (S^{n+k} \times \{i\})$ is a tubular neighborhood of $W_i$. We get a map

$$S^{n+k} \times I \to \mathrm{Th}(\gamma_k)$$

that is a homotopy. □

Now claim that $J$ is a group homomorphism.

Given $W, W' \subseteq S^{n+k}$, and tubular neighborhoods $N, N'$ of $W$ and $W'$ inside $S^{n+k}$, $N \sqcup N'$ is a tubular neighborhood of $W \sqcup W'$. Applying $J$ to $W$ and $W'$, we get two maps $f\colon W \to \mathrm{Gr}_k$ and $f'\colon W' \to \mathrm{Gr}_k$. The group operation on $L_n(S^{n+k})$ is given by disjoint union, so if we collapse everything outside of a tubular neighborhood of $W \sqcup W'$ inside $S^{n+k}$, we can realize the disjoint union of $f$ with $f'$ as

$$S^{n+k} \to S^{n+k} \vee S^{n+k} \xrightarrow{f \vee f'} \mathrm{Th}(\gamma_k) \vee \mathrm{Th}(\gamma_k) \xrightarrow{\nabla} \mathrm{Th}(\gamma_k).$$

This is exactly the same as the group operation on $[S^{n+k}, \mathrm{Th}(\gamma_k)]$. Hence, $J$ is a group homomorphism.

This next lemma shows that $J$ is injective.

**Lemma 3.44.** *Let* $f, f'\colon X \to \mathrm{Th}(\gamma_k)$*. If* $f$ *is homotopic to* $f'$ *and both are transverse to* $\mathrm{Gr}_k$*, then* $f^{-1}(\mathrm{Gr}_k)$ *is* $L$*-equivalent to* $(f')^{-1}(\mathrm{Gr}_k)$*.*

*Proof sketch.* Let $F\colon X \times I \to \mathrm{Th}(\gamma_k)$ be a homotopy. Then $F^{-1}(\mathrm{Gr}_k)$ is a submanifold, and gives the desired L-equivalence. □

This lemma actually gives us something more: an inverse to $J$ sending $f\colon S^{n+k} \to \mathrm{Th}(\gamma_k)$ to $f^{-1}(\mathrm{Gr}_k)$. Hence, we have shown the following.

**Lemma 3.45.** $L_n(S^{n+k}) \cong \pi_{n+k}(\mathrm{Th}(\gamma_k))$.

Modulo checking some details, this in fact shows Theorem 3.28.

## 3.5 Characteristic Numbers and Boundaries

**Corollary 3.46** (Corollary to Theorem 3.28)**.** *If* $M$ *is an* $n$*-manifold all of whose characteristic numbers are zero, then* $M$ *is the boundary of an* $(n+1)$*-manifold.*

*Proof.* Suppose that we have an $n$-manifold $M$ and an embedding $M \hookrightarrow S^{n+k}$ (recall $k > n+2$, so such an embedding exists). Under the isomorphism $L_n(S^{n+k}) \cong \pi_{n+k}(\mathrm{Th}(\gamma_k))$, we have a map $f\colon S^{n+k} \to \mathrm{Th}(\gamma_k)$ with $M = f^{-1}(\mathrm{Gr}_k)$. Thus, $f$ restricts to a map $\widetilde{f}\colon M \to \mathrm{Gr}_k$.

$f$ sends $M$ to the zero section of $\gamma_k$, and $\widetilde{f}^*(\gamma_k) = \nu_M$ is the normal bundle to $M$ inside $S^{n+k}$. So $\widetilde{f}$ is the classifying map of $\nu_M$. Since $\nu_M \oplus TM$ is a trivial bundle, the characteristic classes of $TM$ are uniquely determined by the characteristic classes of $TM$.

Recall that $M$ is a boundary if and only if $f$ is null-homotopic, by Lemma 3.45. So it suffices to show that the characteristic numbers of $\nu_M$ are zero implies that $f$ is null-homotopic.

If there is some $\alpha \in H^n(Gr_k; \mathbb{Z}/2)$ such that $\widetilde{f}^*(\alpha) \in H^n(M; \mathbb{Z}/2)$ is nonzero, then $\widetilde{f}^*(\alpha) \frown [M] \neq 0$. If $\alpha$ is additionally monomial in the generators of $H^n(Gr_k; \mathbb{Z}/2)$, then it follows that tehre is a characteristic number of $\nu_M$ (and thus a characteristic number of $M$) which is nonzero. So it suffices to show that $f$ is null-homotopic when all elements $\alpha \in H^n(Gr_k; \mathbb{Z}/2)$ pull back to zero along $\widetilde{f}$.

Consider the following diagram, where $N$ is a tubular neighborhood of $M$, (and can be considered as a disk bundle of $M$)

$$\begin{array}{ccccc} N & \hookrightarrow & S^{n+k} & \xrightarrow{f} & Th(\gamma_k) \\ \uparrow & & & & \uparrow \\ M & & \xrightarrow{\quad\widetilde{f}\quad} & & Gr_k \end{array}$$

From this, we get the following diagram

$$\begin{array}{ccccc} H^{n+k}(N; \mathbb{Z}/2) & \xleftarrow[g]{} & H^{n+k}(S^{n+k}; \mathbb{Z}/2) & \xleftarrow[f^*]{} & H^{n+k}(Th(\gamma_k); \mathbb{Z}/2) \\ \cong\uparrow & & & & \cong\uparrow \\ H^n(M; \mathbb{Z}/2) & & \xleftarrow[\quad\widetilde{f}^*\quad]{} & & H^n(Gr_k; \mathbb{Z}/2) \end{array}$$

where the vertical maps are Thom isomorphisms. Notice that the map $g$ is nonzero, since $N$ is a disk bundle over $M$. Since the vertical maps are isomorphisms, we now know that $\widetilde{f}^* = 0 \iff f^* = 0$.

So finally, we want to say that $\widetilde{f}^* = 0$ (iff $f^* = 0$) implies that $f$ is nullhomotopic. We leave this last statement without proof in order to avoid a detour into the Steenrod algebra. □

This theorem is really cool because it classifies cobordism in terms of characteristic numbers! This comes from a classification of cobordism in terms of homotopy.

# 4 K-Theory

## 4.1 Bott Periodicity

Now that we've calculated the cohomology groups of Grassmannians (and therefore the homology), let's turn our attention to their homotopy groups. In general, these are not known. However, we can recover some of their homotopy groups using Theorem 1.40. Recall that this gives a weak homotopy equivalence $\mathrm{Gr}_n \simeq BO(n)$. So to study the homotopy groups of Grassmannians, we will study the orthogonal groups $O(n)$.

**Lemma 4.1.** $\pi_i O(n-1) \cong \pi_i O(n)$ *for* $i < n-2$.

*Proof.* Consider the action of $O(n)$ on $\mathbb{R}^n$. This induces an action of $O(n)$ on $S^{n-1}$, and the stabilizer of a point is $O(n-1)$ – rotations of the sphere fixing the axis through that point and the origin. Hence, $S^{n-1} \cong O(n)/O(n-1)$ by the orbit-stabilizer theorem, and there is a fiber sequence

$$O(n-1) \to O(n) \to S^{n-1}.$$

Thus, there is a long exact sequence in homotopy

$$\cdots \to \pi_{i+1} S^{n-1} \to \pi_i O(n-1) \to \pi_i O(n) \to \pi_i S^{n-1} \to \cdots$$

When $i < n-1$, $\pi_i S^{n-1} = 0$. Hence, for $i < n-2$, $\pi_i O(n-1) \cong \pi_i O(n)$. □

This lemma says that the homotopy groups of the orthogonal group are **stable**. Moreover, we may put together all of the orthogonal groups to get the **infinite orthogonal group** $O$. We will study this instead of the individual orthogonal groups.

**Definition 4.2.** The **infinite orthogonal group** is the colimit of the finite orthogonal groups over the inclusions $O(n) \hookrightarrow O(n+1)$ as the upper-left-corner of $O(n+1)$.

We may likewise define similar constructions for other Lie groups.

**Definition 4.3.** The **symplectic group** $\mathrm{Sp}(n)$ is the group of $2n \times 2n$ matrices that preserve the inner product defined by the block matrix

$$\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$$

**Definition 4.4.** Sp is the **infinite symplectic group**, $SO$ is the **infinite special orthogonal group**, and $U$ is the **infinite unitary group**. Each are colimits of the finite versions along inclusions $G(n) \hookrightarrow G(n+1)$.

**Theorem 4.5** (Bott Periodicity)**.**

*(a)* $\pi_i U \cong \pi_{i+2} U$

*(b)* $\pi_i O \cong \pi_{i+8} O$

The proof of Bott Periodicity relies entirely on the following lemma.

**Lemma 4.6.** *There are weak equivalences*

$$\Phi\colon BU \to \Omega U$$

*and*

$$\begin{array}{ll} \Phi_1\colon B\,Sp \to \Omega(U/Sp), & \Phi_2\colon BO \to \Omega(U/O), \\ \Phi_3\colon U/Sp \to \Omega(SO/U), & \Phi_4\colon U/O \to \Omega(Sp/U), \\ \Phi_5\colon SO/U \to \Omega SO, & \Phi_6\colon Sp/U \to \Omega\,Sp, \end{array}$$

*called* ***Bott maps****.*

Before we prove this lemma, we will illustrate how it proves Theorem 4.5, together with the lemma below.

**Lemma 4.7.** *For* $i > 2$*,*

*(a)* $\pi_i BX \cong \pi_{i-1} X$

*(b)* $\pi_i X \cong \pi_{i-1} \Omega X$

*For a topological group* $G$*, there is a weak equivalence* $\Omega BG \simeq G$*.*

*Proof.*

(a) $BX \cong EX/X$, and $EX$ is contractible. Hence, have a fiber sequence

$$X \to EX \to BX.$$

Passing to the long exact sequence of homotopy, we obtain the desired result.

(b) For all spaces $Y$, there is a fibration sequence

$$\Omega Y \to PY \to Y$$

where $PY$ is the paths in $Y$ starting at the basepoint and ending who knows where. Passing to the long exact sequence of homotopy, we obtain the desired result.

Finally, when $X$ is a topological group, $\pi_1$ is abelian and $\pi_0$ is a group, so these equivalences hold for all $i$. □

*Proof of Bott Periodicity (Theorem 4.5).*

(a) To show that $\pi_i U \cong \pi_{i+2} U$, observe that there is a weak equivalence by Lemma 4.6.
$$U \simeq \Omega BU \xrightarrow{\Phi} \Omega^2 U.$$

(b) We will show $\pi_i O \cong \pi_{i+8} O$ in two steps. First, we show $\pi_i \operatorname{Sp} \cong \pi_{i+4} O$:
$$\operatorname{Sp} \simeq \Omega B \operatorname{Sp} \xrightarrow{\Phi_1} \Omega^2(U/\operatorname{Sp}) \xrightarrow{\Omega\Phi_3} \Omega^3(SO/U) \xrightarrow{\Omega^3\Phi_5} \Omega^4 SO \simeq \Omega^4 O$$
Second, $\pi_i O \cong \pi_{i+4} \operatorname{Sp}$ because
$$O \simeq \Omega BO \xrightarrow{\Omega\Phi_2} \Omega^2(U/O) \xrightarrow{\Omega^2\Phi_4} \Omega^3(\operatorname{Sp}/U) \xrightarrow{\Omega^3\Phi_6} \Omega^4 \operatorname{Sp}.$$
Putting these two together, we obtain the desired result. □

**Remark 4.8** (Note to reader.)**.** It's probably best if you read Inna Zakharevich's notes on this part. I didn't quite follow the proof of Bott periodicity, and I'm certain some stuff in here is wrong (or at least misleading!). On the other hand, if you want to read it and send me all the errors, I would really appreciate that.

So to complete the proof of Bott periodicity, we must prove Lemma 4.6. We will construct only the map $\Phi\colon BU \to \Omega U$. To do so, we need the following theorem about H-spaces.

**Theorem 4.9.** *If $f\colon X \to Y$ is an H-map of connected H-spaces that induces an isomorphism on homology, then $f$ is a weak equivalence.*

We will also need the following calculation of the (co)homology of $U$.

**Proposition 4.10.** $H^*(BU) \cong \mathbb{Z}[c_i \mid i \in \mathbb{N}]$ *with $c_i$ in degree $2i$.*

*Proof.* $H^*(BU(n)) \cong H^*(\mathrm{Gr}_n(\mathbb{C}^\infty)) \cong \mathbb{Z}[c_1, \ldots, c_n]$ with $c_i$ in degree $2i$. Then $H^*(BU) \cong \lim H^*(BU(n)) \cong \mathbb{Z}[c_i \mid i \in \mathbb{N}]$. □

**Fact 4.11.** $H_*U$ *is an exterior algebra generated by elements $x_{2i-1}$ for $i \geq 1$.*

The final ingredient is the following theorem.

**Theorem 4.12.** *Let $X$ be an H-space such that $H_*X$ is a transgressively generated exterior algebra. Then $H_*\Omega X$ is a polynomial algebra generated by the adjoints.*

This theorem uses the Serre spectral sequence applied to the fibration sequence $\Omega X \to PX \to X$ under some assumptions for $X$.

The product structure on homology might come from an H-space structure $\mu\colon X \times X \to X$, then
$$H_i X \times H_j X \to H_{i+j}(X \times X) \xrightarrow{\mu_*} H_{i+j} X$$
gives a graded ring structure on homology. This structure is called the **Pontrjagin ring**.

**Theorem 4.13.** *The Pontryjagin ring of* $U(n)$ *is given by the exterior algebra*

$$H_*U(n) = \bigwedge \mathbb{Z}[e_1, e_3, \dots, e_{2n-1}].$$

*with* $e_i$ *in degree* $i$. *The map* $U(n) \to U(n+1)$ *takes* $e_i$ *to* $e_i$.

*Proof sketch.* Recall that $U(n) \cong S^1 \times SU(n)$. So by the Künneth theorem,

$$H_*U(n) \cong H_*S^1 \otimes H_*SU(n)$$

and notice that $H_*S^1 \cong \bigwedge_{\mathbb{Z}}[e_1]$, so

$$H_*U(n) \cong \bigwedge \mathbb{Z}[e_1] \otimes H_*SU(n)$$

So we just need to know what $H_*SU(n)$ looks like.

Put coordinates on points $(\theta, x) \in \Sigma\mathbb{CP}^{n-1} = S^1 \wedge \mathbb{CP}^{n-1}$ with $\theta \in [-\pi/2, \pi/2]$ and $x \in \mathbb{CP}^{n-1}$ a unit vector $(x_1, x_2, \dots, x_n) \in \mathbb{C}^n$. Then define

$$f_n \colon \Sigma\mathbb{CP}^{n-1} \to SU(n)$$

by

$$f_n(\theta, x) = \left( I_n - 2e^{i\theta}\cos(\theta) \begin{bmatrix} |x_1|^2 & x_1\overline{x}_2 & \cdots x_1\overline{x}_n & \\ \vdots & \vdots & \ddots & \vdots \\ x_n\overline{x}_1 & x_n\overline{x}_2 & \cdots & |x_n|^2 \end{bmatrix} \right) \begin{bmatrix} e^{-2i\theta} & \\ & I_{n-1} \end{bmatrix}$$

If we think of $\Sigma\mathbb{CP}^{k-1}$ as sitting inside $\Sigma\mathbb{CP}^{n-1}$ as the first $k$ coordinates, then we have a map which factors through $SU(k) \to SU(n)$.

For $n \geq k_1 > k_2 > \cdots > k_j \geq 2$, we have a map

$$f_{k_1,\dots,k_j} \colon \Sigma\mathbb{CP}^{k_1-1} \times \cdots \times \Sigma\mathbb{CP}^{k_j-1} \to SU(n)$$

which is the product of $f_{k_1}, f_{k_2}, \dots, f_{k_j}$. This gives one cell of $SU(n)$, and all of these together give a cellular structure on $SU(n)$.

From this, we can read off the structure of the Pontrjagin ring. □

**Remark 4.14.** When $X = U$, the Pontrjagin ring structure on $H_*U$ is homotopy commutative, because given two elements $A$ and $B$ of $U$, they live in some finite stage of the colimit $U = \mathrm{colim}_n\, U(n)$, say $A, B \in U(N)$. Then we may represent these in $U(2N)$ by block matrices

$$\begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \qquad \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}$$

Then we may homotope $B$ to live in the odd-indexed rows and columns, and $A$ to live in the even-indexed rows and columns, so the multiplication is commutative up to homotopy in $U$. Hence, $H_*U$ is a commutative ring.

*Proof of Lemma 4.6.* We have two colimits:

$$BU = \operatorname*{colim}_n BU(n)$$

$$BU(n) = Gr_n(\mathbb{C}^\infty) = \operatorname*{colim}_N Gr_n(\mathbb{C}^N)$$

By properties of colimits, we can therefore rewrite $BU$ as

$$BU = \operatorname*{colim}_n Gr_n(\mathbb{C}^{2n}).$$

Recall that $Gr_n(\mathbb{C}^{2n}) \cong V_n(\mathbb{C}^{2n})/U(n)$. This is how we proved that this Grassmannian is the classifying space of $U$.

We can consider $V_n(\mathbb{C}^{2n})$ as $U(2n)/U(n)$, where $U(n)$ acts on the last $n$ column vectors. Hence,

$$Gr_n(\mathbb{C}^{2n}) \cong {}^{U(2n)/U(n)}\big/_{U(n)};$$

the second quotient acts on the first $n$ column vectors in the Stiefel manifold. Hence,

$$Gr_n(\mathbb{C}^{2n}) \cong {}^{U(2n)}\big/_{U(n) \times U(n)}.$$

Now define

$$\Phi_{k,n} \colon {}^{U(k+n)}\big/_{U(k) \times U(n)} \to \Omega U(k+n)$$

by sending a matrix $T$ to

$$T \mapsto \left(\theta \mapsto T\alpha_\theta T^{-1}\alpha_\theta^{-1}\right)$$

where $\alpha_\theta(\vec{x}, \vec{y}) = (e^{i\theta}\vec{x}, e^{-i\theta}\vec{y})$ for $(\vec{x}, \vec{y}) \in \mathbb{C}^k \times \mathbb{C}^n$.

In particular, we have

$$\Phi_{n,n} \colon {}^{U(2n)}\big/_{U(n) \times U(n)} \to \Omega U(2n).$$

Now define

$$\Phi = \operatorname*{colim}_n \Phi_{n,n}.$$

Notice that $\alpha$ is natural in the sense that

$$\begin{array}{ccc} \mathbb{C}^k \times \mathbb{C}^n & \xrightarrow{\alpha_\theta} & \mathbb{C}^{k+n} \\ \big\downarrow & & \big\downarrow \\ \mathbb{C}^{k'} \times \mathbb{C}^{n'} & \xrightarrow{\alpha_\theta} & \mathbb{C}^{k'+n'} \end{array}$$

This implies that the following diagram commutes, which shows that $\Phi_{k,n}$ is natural in $k$ and $n$.

$$\begin{array}{ccc} {U(k+n)}/{U(k)\times U(n)} & \xrightarrow{\Phi_{k,n}} & \Omega U(k+n) \\ \downarrow & & \downarrow \\ {U(k'+n')}/{U(k')\times U(n')} & \xrightarrow{\Phi_{k',n'}} & \Omega U(k'+n') \end{array}$$

Now let $n = 1$ and $k' = n' = n$. Then the diagram above becomes

$$\begin{array}{ccc} {U(1+n)}/{U(1)\times U(n)} & \xrightarrow{\Phi_{1,n}} & \Omega U(1+n) \\ \downarrow & & \downarrow \\ {U(2n)}/{U(n)\times U(n)} & \xrightarrow{\Phi_{n,n}} & \Omega U(2n) \end{array}$$

What is $U(1+n)/U(1)\times U(n)$? It's the Grassmannian of 1-planes in $\mathbb{C}^{n+1}$, or $\mathbb{CP}^n$. Hence, when we take the colimit along $n$ in the above diagram, we get

$$\begin{array}{ccc} \mathbb{CP}^\infty & \longrightarrow & \Omega U \\ \downarrow{\scriptstyle \bar{J}} & & \downarrow{\scriptstyle \Omega J} \\ BU & \xrightarrow{\Phi} & \Omega U \end{array} \tag{4.1}$$

Where $J\colon U(1+n) \to U(2n)$ is the inclusion of the first $1+n$ coordinates, and $\bar{J}$ is the map induced on the quotient.

Recall that our goal is to show that $\Phi$ an equivalence. To this end, we will use Theorem 4.9 and show that $BU$, $\Omega U$ are connected H-spaces.

$BU$ is an H-space because there are maps $U(n) \times U(m) \to U(n+m)$ given by block diagonals. The functor $B$ preserves products, because $B = N|-|$, and the nerve functor $N$ is a right adjoint, and the geometric realization $|-|$ preserves products by its construction. Hence, $B(U(n)\times U(m)) = BU(n) \times BU(m)$ and there are maps $BU(n)\times BU(m) \to BU(n+m)$. Taking the colimit in both $n$ and $m$, we obtain $BU \times BU \to BU$ and $BU$ is then an H-space.

$\Omega U$ is an H-space by composition of loops; alternatively, there is another description of the H-space structure because we may pointwise multiply loops using the group structure. But to apply Theorem 4.9, we need to know that $BU$ and $\Omega U$ are connected. But $\Omega U$ may very well not be connected!

Nevertheless, recall that $\Phi_{k,n}$ is defined via a commutator $T\alpha_\theta T^{-1}\alpha_\theta^{-1}$, and the determinant of a commutator like that is always 1. Hence, $\Phi_{k,n}$ lands in

$\Omega SU$ instead of $\Omega U$; $\Omega SU$ is connected. Hence, we really care about the diagram

$$\begin{array}{ccc} \mathbb{CP}^\infty & \longrightarrow & \Omega SU \\ \Big\downarrow \bar{J} & & \Big\downarrow \Omega J \\ BU & \xrightarrow{\Phi} & \Omega SU \end{array} \tag{4.2}$$

To show that $\Phi$ is an H-map of H-spaces, consider the following diagram

$$\begin{array}{ccc} {U(k+n)}/{U(k)\times U(n)} \times {U(k'+n')}/{U(k')\times U(n')} & \xrightarrow{\Phi_{k,n}\times\Phi_{k',n'}} & \Omega U(k+n)\times \Omega U(k'+n') \\ \Big\downarrow & & \Big\downarrow \\ U(k+k'+n+n') & \xrightarrow{\Phi_{k+k',n+n'}} & \Omega U(k+k'+n+n') \end{array}$$

Taking the colimit over all $n, k, n', k'$ shows that $\Phi$ is an H-map of H-spaces.

Now, Theorem 4.9 applies and we only need to check that $\Phi_*$ is an isomorphism on homology.

To show that $\Phi$ is an isomorphism on Homology, we will prove that the other three maps in the diagram (4.2) are isomorphisms on homology, and therefore $\Phi$ is as well.

For the first map, $\Omega J$, we will show that $J\colon U(n+1) \to U(2n)$ is an isomorphism on Homology up to degree $2n+2$. Notice that $J\colon SU(n+1) \to SU(n+k)$ is an inclusion of cells which is an isomorphism on homology up to degree $2n+1$. Hence, $J$ is an isomorphism on homology in the colimit as $n \to \infty$.

Then by Theorem 4.12, $H_*\Omega SU$ is generated by the adjoints of the cells $f_k\colon \Sigma\mathbb{CP}^{k-1} \to SU$, which are explicitly the maps

$$\widetilde{f}_k\colon \mathbb{CP}^{k-1} \to \Omega SU.$$

Now, we know that $H_{2i}\mathbb{CP}^\infty = \mathbb{Z}\{b_{2i}\}$; then $b_{2i}$ mapsto a polynomial generator of $H_*\Omega SU$. So it suffices to show that $H_*BU \cong \mathbb{Z}[z_{2i} \mid i \geq 1]$ and $b_{2i} \mapsto z_{2i}$.

On cohomology, we know that $H^*(BU) = \mathbb{Z}[c_{2i} \mid i \geq 1]$ and $H^*(\mathbb{CP}^\infty) \cong \mathbb{Z}[x]$, with $x$ in degree 2. The map induced

$$BU(1) \cong \mathbb{CP}^\infty \to BU$$

is given by $c_2 \mapsto x$ and $c_{2i} \mapsto 0$, for $i > 1$. Then (skipping some steps) on homology, we have that $b_{2i} \mapsto z_{2i}$, where $z_{2i}$ is the dual of $b_{2i}$.

Hence, $\Phi$ is an isomorphism on homology. Thus, $\pi_*(BU) \cong \pi_*\Omega SU$. But we wanted to know that $\pi_i U \cong \pi_{i+2}(U)$. But we know

$$\pi_i U \cong \pi_i(S^1 \times SU) \cong \begin{cases} \mathbb{Z} & i = 1, \\ \pi_i SU & i > 1. \end{cases}$$

Therefore,

$$\pi_i \Omega U \cong \begin{cases} \mathbb{Z} & i = 0, \\ \pi_i \Omega SU & i > 0. \end{cases}$$

So we have shown that $\pi_i(BU \times \mathbb{Z}) \cong \pi_i \Omega U$ for all $i$. □

Using Bott periodicity, to know all of the homotopy groups of $U$, we only need to know $\pi_0 U$ and $\pi_1 U$. Soon, we will compute $\pi_1 U$.

## 4.2 The K-theory spectrum

**Definition 4.15.** A **spectrum** is a sequence of spaces $X = X_0, X_1, X_2, \ldots$ together with maps $\Sigma X_i \to X_{i+1}$, or equivalently, maps $X_i \to \Omega X_{i+1}$.

**Definition 4.16.** The K**-theory spectrum** $KU$ is the spectrum

$$KU_i = \begin{cases} \mathbb{Z} \times BU & i \text{ even} \\ U & i \text{ odd.} \end{cases}$$

By Bott periodicity, the maps $KU_i \to \Omega KU_{i+1}$ are weak equivalences.

**Question 4.17.** What cohomology does this spectrum give? By Remark 2.10, there is an associated cohomology theory.

**Theorem 4.18.** *For any compact connected space $X$, the cohomology theory defined by the spectrum $K$ has $0$-th space isomorphic to the free abelian group generated by vector bundles over $X$, subject to the relation that $[E \oplus E'] = [E] + [E']$ for vector bundles $E$ and $E'$ over $X$.*

*Proof.* Let $A$ be the free abelian group generated by vector bundles, subject to the relation $[E \oplus E'] = [E] + [E']$.

By definition, $\widetilde{K}^0(X_+) = [X_+, \mathbb{Z} \times BU]$. A map $f \colon X_+ \to \mathbb{Z} \times BU$ has image in some $\{i\} \times BU$, since $X_+$ is connected. So we may just consider $f \colon X_+ \to BU$.

Now write $BU = \operatorname{colim} BU(n)$. Since $X$ is compact, $f$ factors through some $BU(n)$ for some $n$. Hence, this gives a rank $n$ vector bundle $E$ on $X$. However, this $n$ is not necessarily well-defined. Composition with the inclusion $BU(n) \to BU(n+1)$ gives a classifying map for $E \oplus \varepsilon^1$. So define a map

$$\begin{aligned} [X_+, \mathbb{Z} \times BU] &\longrightarrow A \\ [f] &\longmapsto [E] - [\varepsilon^{\dim E - i}] \end{aligned}$$

This is well-defined, because

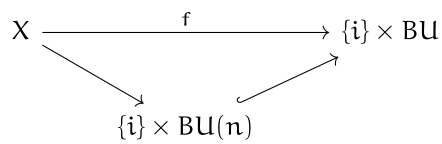
$$[E] - [\varepsilon^{\dim E - i}] = [E \oplus \varepsilon^1] - [\varepsilon^{\dim E + 1 - i}].$$

Moreover, we say that $[\varepsilon^{-n}] = -[\varepsilon^n]$.

Why is this a group homomorphism? Well $\mathbb{Z} \times BU \simeq \Omega U$, and the group operation on $BU$ is given by $\oplus\colon BU \times BU \to BU$ that takes the alternating-rows-and-columns block sum of matrices. For classifying maps, this corresponds to the Whitney sum of vector bundles.

Now we must check that this is both injective and surjective. Surjectivity first. For any $a_1[E_1] + \ldots + a_n[E_n]$ in $A$ for integers $a_i$, we may write this as $[E] - [E']$. Since $X$ is compact, there is a vector bundle $F$ such that $E' \oplus F \cong \varepsilon^n$. Therefore, $[E] - [E'] = [E \oplus F] - [\varepsilon^n]$, so let $f$ be the classifying map of $E \oplus F$ and let $i = \dim(E \oplus F) - n$. Then the image of $[f]$ is $\sum_i a_i[E_i]$.

To check injectivity, suppose that $[f] \mapsto 0$. We know that $f$ factors through $\{i\} \times BU(n)$ for some $n$.




If $E$ has dimension $n$ with $[f] \mapsto [E]$, then $[E] - [\varepsilon^{\dim E - i}] = 0$ implies that $i = 0$ (the bundles must be the same dimension to cancel). Hence,

$$[E] - [\varepsilon^{\dim E}] = 0.$$

Thus, there is some vector bundle $F$ such that $E \oplus F \cong \varepsilon^{\dim E} \oplus F$. Moreover, there is $F'$ such that $F \oplus F' \cong \varepsilon^m$, and therefore $E \oplus \varepsilon^m \cong \varepsilon^{\dim E + m}$. The classifying map for $E \oplus \varepsilon^m$ is

$$X \to BU(\dim E) \hookrightarrow BU(\dim E + m).$$

Yet this is homotopic to the classifying map of $\varepsilon^{\dim E + m}$, so this is null-homotopic. Then the composite

$$X \to BU(\dim E) \hookrightarrow BU(\dim E + m) \hookrightarrow BU$$

is null-homotopic. Hence, $X \to BU$ is null-homotopic, so $[f] = 0$. □

**Proposition 4.19.** *For* $i \geq 0$, $\widetilde{K}^i(X_+) = K^0(\Sigma^i X)$.

*Proof.* In general,

$$\begin{aligned} \widetilde{K}^i(X_+) &= [X_+, K_i] \\ &= [X_+, \Omega^i(\mathbb{Z} \times BU)] \\ &= [\Sigma^i X_+, \mathbb{Z} \times BU] \\ &= \widetilde{K}^0(\Sigma^i X_+) = K^0(\Sigma^i X) \end{aligned}$$

□

**Remark 4.20.** Topological $K^0$ is the group completion of the monoid of vector bundles under Whitney sums. But these group completions are not always so nice – just in this case, we can say nice things about them.

For example, the additive monoid $\mathbb{R} \cup \{\infty\}$ with $a + \infty = \infty$, $\infty + b = \infty$, and $\infty + \infty = \infty$ has trivial group completion. Usually, the best we can say about a group completion is "I don't know."

**Remark 4.21.** When we don't want a disjoint basepoint, if $X$ is a pointed compact connected space, then any map $f\colon X \to \mathbb{Z} \times BU$ always lands in the component $\{0\} \times BU$ of $\mathbb{Z} \times BU$. In this case, we get sums of classes of bundles of total dimension zero. Sometimes the notation $\widetilde{K}^0(X)$ is used for this group – this is entirely consistent with what we have defined.

## 4.3 Some properties of K-theory

**Proposition 4.22.** *If $X$ is pointed, with basepoint $x_0$, then*

$$\widetilde{K}^0(X) = \ker(\widetilde{K}^0(X_+) \xrightarrow{i_*} \widetilde{K}^0(S^0)),$$

*where $X_+$ is $X$ with a disjoint basepoint and the map $i_*$ is induced by $i\colon S^0 \to X_+$, $0 \mapsto x_0$, $1 \mapsto +$.*

*Proof.* This follows from the long exact sequence for homotopy:

$$\widetilde{K}^{-1}(X^0) \to \widetilde{K}^0(X) \cong \widetilde{K}^0(X_+/S^0) \to \widetilde{K}^0(X_+) \to \widetilde{K}^0(S^0) \to \widetilde{K}^1(X_+/S^0) \cong \widetilde{K}^1(X) \to \cdots$$

Here, $\widetilde{K}^0(X) \cong [S^0, U] = 0$ since $U$ is connected. Therefore, $\widetilde{K}^0(X)$ is the kernel of $\widetilde{K}^0(X_+) \to \widetilde{K}^0(S^0)$. □

A consequence of this is that all trivial bundles are zero in $\widetilde{K}^0(X)$, because the map

$$\widetilde{K}^0(X_+) \to \widetilde{K}^0(S^0) \cong \mathbb{Z}$$

is given by $[E] \mapsto \dim E$. Therefore, we get the following:

**Proposition 4.23.**

1. *If $[E] = [E']$ in $\widetilde{K}^0(X)$, then there are trivial bundles $\varepsilon^k$ and $\varepsilon^{k'}$ such that $E \oplus \varepsilon^k \cong E' \oplus \varepsilon^{k'}$. In particular, if $[E] = 0$, then $E \oplus \varepsilon^k \cong \varepsilon^{k'}$.*
2. *If $[E] \in \widetilde{K}^0(X)$, and $E'$ satisfies $E \oplus E' \cong \varepsilon^k$, then $[E] + [E'] = 0$ in $\widetilde{K}^0(X)$.*

**Definition 4.24.** We call a bundle $E$ such that $E \oplus \varepsilon^k \cong \varepsilon^{k'}$ **stably trivial**.

We have already used the following proposition, but we may as well write it down.

**Proposition 4.25.** *For a cofiber sequence $A \hookrightarrow X \to X/A$, we get a long exact sequence*

$$\cdots \to \widetilde{K}^i(X/A) \to \widetilde{K}^i(X) \to \widetilde{K}^i(A) \to \widetilde{K}^{i+1}(X/A) \to \cdots$$

Since $\Omega^2 K_i \cong K_i$, then we have

$$\widetilde{K}^i(X) = [X, K_i] \cong [X, \Omega^2 K_i] = [\Sigma^2 X, K_i] \cong \widetilde{K}^{i-2}(X)$$

Therefore, we may rewrite the long exact sequence for K-theory as a periodic exact sequence

$$\begin{array}{ccccc} \widetilde{K}^i(X/A) & \longrightarrow & \widetilde{K}^i(X) & \longrightarrow & \widetilde{K}^i(A) \\ \uparrow & & & & \downarrow \\ \widetilde{K}^{i+1}(A) & \longleftarrow & \widetilde{K}^{i+1}(X) & \longleftarrow & \widetilde{K}^{i+1}(X/A) \end{array}$$

There is a ring structure on K-theory that ultimately comes from a ring structure on the spectrum K, but we won't talk about that. Instead, we'll just define this on $K^0$ and work with that.

**Definition 4.26.** Put a ring structure on $K^0(X)$ by $[E] \cdot [E'] = [E \otimes E']$.

This works on reduced K-theory $\widetilde{K}^0(X)$ as well.

## 4.4 An example: K-theory of $S^2$

**Example 4.27.** What is the ring structure on $K^0(S^2)$?

We can compute the reduced K-theory of the 2-sphere:

$$\widetilde{K}^0(S^2) = [S^2, \mathbb{Z} \times BU] = [S^2, BU] = [S^0, \Omega^2 BU] = [S^0, \mathbb{Z} \times BU] = [S^0, BU] \cong \mathbb{Z}.$$

Then we know that $\widetilde{K}^0(S^2) = \ker(K^0(S^2) \to K^0(S^0)) \cong \mathbb{Z}$, given by dimension. With some algebra, we can learn that

$$K^0(S^2) \cong \mathbb{Z}^2$$

as a group. What is the ring structure on this?

There are two generators: $[\varepsilon^1]$ and the tautological line bundle $\gamma_{11}$ on $\mathbb{CP}^1 \cong S^2$. Let H be the class of the tautological line bundle.

How do we know that $H \neq [\varepsilon^1]$? If $\gamma_{1,1} \oplus \varepsilon^k \cong \varepsilon^{k+1}$, then the characteristic classes of $\gamma_{1,1}$ would be all zero. However, we know that the first Chern class of this bundle is nonzero $c_1(\gamma_{1,1}) \neq 0$. Therefore, $K^0(S^2) \cong \mathbb{Z}^2$ generated by $[\varepsilon^1]$ and H.

Now claim that $\gamma_{1,1} \oplus \gamma_{1,1} \cong (\gamma_{1,1} \otimes \gamma_{1,1}) \oplus \varepsilon^1$. We will return to this in a second.

Consider an $n$-dimensional bundle $E$ over $S^2$. This has a classifying map $S^2 \to BU(n)$, which comes from $\Sigma S^1 \to BU(n)$. Under the adjunction $\Sigma \dashv \Omega$, this corresponds to a map

$$S^1 \to \Omega BU(n) \simeq U(n).$$

Such a function is called a **clutching function**. (Notice that nothing about this depends on using $S^2$ – we could use any sphere.)

Now, if we write $S^2 = D^2 \cup_{S^1} D^2$, we can take the two hemispheres as open sets of an atlas for $S^2$. To define a vector bundle over $S^2$, we only need specify one transition function from the southern hemisphere to the northern hemisphere. This is what the clutching function does.

Clutching functions play nicely with tensor products and Whitney sums of bundles.

**Proposition 4.28.**

*(a) The clutching function of a Whitney sum $E \oplus F$ is the block diagonal of the clutching functions for $E$ and $F$.*

*(b) The clutching function of a tensor product $E \otimes F$ is the tensor product of the clutching functions for $E$ and $F$.*

**Example 4.29** (Example 4.27, continued)**.** What is the clutching function of $\gamma_{1,1}$? There are two open sets of $\mathbb{CP}^1$ that we have to worry about, $U_0$ and $U_1$ with

$$U_0 = \{[z_0 \colon z_1] \mid z_0 = 1, |z_1| \leq 1\},$$
$$U_1 = \{[z_0 \colon z_1] \mid z_1 = 1, |z_0| \leq 1\}.$$

The transition function between these two charts is given by multiplication by $z_1$: $z_0 = 1/z_1$. Hence, the clutching function of $\gamma_{1,1}$ is $f \colon S^1 \to U(1)$, $f(z) = z$.

Therefore, the clutching function of $\gamma_{1,1} \oplus \gamma_{1,1}$ is

$$\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix}$$

and the clutching function of $(\gamma_{1,1} \otimes \gamma_{1,1}) \oplus \varepsilon^1$ is

$$\begin{bmatrix} z^2 & 0 \\ 0 & 1 \end{bmatrix}$$

These are clearly homotopic. Hence, in $K^0(S^2)$, $H + H = H^2 + 1$. So we may conclude that, as a ring,

$$K^0(S^2) \cong \mathbb{Z}[H]/\langle (H-1)^2 \rangle$$

while $\widetilde{K}^0(S^2) = \mathbb{Z}\{(H-1)\}$ as a group.

**Remark 4.30.** When most people prove Bott periodicity, they're proving the statement that $\widetilde{K}^i(X) \cong \widetilde{K}^{i+2}(X)$. This is done by considering an external product

$$K^0(X) \times K^0(Y) \to K^0(X \times Y)$$

and then noticing that when $Y = S^2$, this is an isomorphism. Hence, there is an isomorphism on reduced K-theory

$$\widetilde{K}^0(X) \otimes \widetilde{K}^0(S^2) \cong \widetilde{K}^0(X \wedge S^2).$$

Now notice that $\widetilde{K}^0(S^2) \cong \mathbb{Z}$, so

$$\widetilde{K}^0(X) \cong \widetilde{K}^0(\Sigma^2(X)) = \widetilde{K}^2(X).$$

But our approach reveals more about where this is really coming from, and resembles Bott's approach (although he proved the same statement that we did using Morse theory).

## 4.5 Power Operations

**Definition 4.31.** A commutative ring $R$ is a **pre-$\lambda$-ring** if there exist functions (*not* necessarily ring homomorphisms!) $\lambda^n \colon R \to R$ satisfying

(L1) $\lambda^0(r) = 1$ for all $r \in R$,

(L2) $\lambda^1 = \mathrm{id}_R$,

(L3) $\lambda^n(r+s) = \sum_{i=0}^{n} \lambda^i(r)\lambda^{n-i}(s)$.

**Remark 4.32.** Sometimes, pre-$\lambda$-rings are called $\lambda$-rings, and in that case the things that we call $\lambda$-rings are called **special $\lambda$-rings**.

**Example 4.33.** On $K^0(X)$, define $\lambda^n[E] = [\bigwedge^n E]$, where $\bigwedge^n E$ is the $n$-th exterior power. Then notice that $\lambda^n[E] = 0$ if $\dim(E) < n$.

For a genuine $\lambda$-ring, you should think of the operations $\lambda^n$ as analogous to the elementary symmetric polynomials.

**Definition 4.34.** For a pre-$\lambda$-ring $R$, let $\Lambda(R)$ be the set of power series $f(t) \in R[[t]]$ with constant term 1, considered as an abelian group under multiplication of power series.

Define a homomorphism $R \to \Lambda(R)$ of abelian groups by

$$r \longmapsto \lambda^0(r) + \lambda^1(r)t + \lambda^2(r)t^2 + \dots$$

This is an abelian group homomorphism by Definition 4.31(L3).

**Construction 4.35.** Define multiplication on $\Lambda(R)$ as follows. If

$$\alpha(t) = 1 + a_1 t + a_2 t^2 + \dots$$
$$\beta(t) = 1 + b_1 t + b_2 t^2 + \dots$$

then suppose that (completely formally)

$$\alpha(t) = \prod_{n \geq 1} (1 + \xi_n t)$$
$$\beta(t) = \prod_{n \geq 1} (1 + \eta_n t)$$

Then write

$$\prod_{n,m} (1 + \xi_m \eta_n t) = 1 + P_1 t + P_2 t^2 + \dots$$

where $P_i$ are symmetric polynomials in $\xi$ and $\eta$. So each $P_n$ may be written in terms of the $a_i$ and the $b_i$. Each $P_n$ depends on $a_1, \dots, a_n$ and $b_1, \dots, b_n$. Notice that even if each previous step was not well-defined, we end up with $P_i$ that depend only on the $a_i$ and $b_i$, not on $\xi_i$ and $\eta_i$.

Then define $\alpha(t) * \beta(t)$ by

$$\alpha(t) * \beta(t) := 1 + P_1(a_1, b_1)t + P_2(a_1, a_2, b_1, b_2)t^2 + \dots$$

**Example 4.36.**

$$\begin{aligned}
P_1 &= \sum_{m,n} \xi_m \eta_n = a_1 b_1 \\
P_2 &= \sum_{(m_1,n_1) \neq (m_2,n_2)} \xi_{m_1} \xi_{m_2} \eta_{n_1} \eta_{n_2} \\
&= \sum_{\substack{m_1 \neq m_2 \\ n_1 \neq n_2}} \xi_{m_1} \xi_{m_2} \eta_{n_1} \eta_{n_2} + \sum_{m, n_1 \neq n_2} \xi_m^2 \eta_{n_1} \eta_{n_2} + \sum_{n, m_1 \neq m_2} \xi_{m_1} \xi_{m_2} \eta_n^2 \\
&= a_2 b_2 + b_2(a_1^2 - 2a_2) + a_2(b_1^2 - 2b_2) \\
&= b_2 a_1^2 + a_2 b_1^2 - 3a_2 b_2
\end{aligned}$$

**Proposition 4.37.** *With this definition of multiplication* $(*)$*, and with the "addition"* $(\times)$ *given by normal multiplication of power series,* $\Lambda(R)$ *becomes a ring with unit* $1 + t$.

**Remark 4.38.** As sets, $\Lambda(R) = 1 + tR[[t]]$, but these are *not* the same as rings.

**Definition 4.39.** A **$\lambda$-ring** $R$ is a pre-$\lambda$-ring such that the operations $\lambda^n$ satisfy the additional rules

(L4) $\lambda^n(1) = 0$ for $n > 1$,

(L5) $\lambda^n(rs) = P_n(\lambda^1(r), \dots, \lambda^n(r), \lambda^1(s), \dots, \lambda^n(s))$,

(L6) $\lambda^n(\lambda^m(r)) = L_{n,m}(\lambda^1(r), \dots, \lambda^{nm}(r))$.

**Construction 4.40.** We may make $\Lambda(R)$ into the universal $\lambda$-ring as follows. If

$$\alpha(t) = 1 + a_1 t + a_2 t + \dots,$$

suppose that we know

$$\alpha(t) = \prod_{n \geq 1} (1 + \xi_n t).$$

Then write

$$\prod_{i_1 < \dots < i_n} (1 + \xi_{i_1} \xi_{i_2} \cdots \xi_{i_n} t) = 1 + L_{n,1} t + L_{n,2} t^2 + \dots$$

for some $L_{n,i}$, such that $L_{n,i}$ depends only on $a_1, \dots, a_{in}$. Then define the $\lambda$ operation on $\Lambda(R)$

$$\lambda^n \alpha(t) = 1 + L_{n,1} t + L_{n,2} t^2 + \dots \tag{4.3}$$

**Example 4.41.** $L_{2,1} = \sum_{i<j} \xi_i \xi_j = a_2$

**Fact 4.42.** *$\Lambda(R)$ is a $\lambda$-ring, with $\lambda$-operations given by* (4.3).

**Example 4.43.** $K^0(X)$ is a $\lambda$-ring, with $\lambda^n[E] = [\bigwedge^n E]$.

Let $s_k(y_1, \dots, y_k)$ be the polynomial in $y_1, \dots, y_k$ such that

$$s_k(\sigma_1, \dots, \sigma_k) = \sum x_i^k$$

where the $\sigma_i$ are the elementary symmetric polynomials. (Look up **Newton's identities**).

**Definition 4.44.** Let $R$ be a $\lambda$-ring, and define the $k$-th **Adams operation**

$$\psi^k(r) = s_k(\lambda^1(r), \lambda^2(r), \dots, \lambda^k(r))$$

**Theorem 4.45.** *The Adams operations $\psi^k$ are ring homomorphisms $K^0(X) \to K^0(X)$ such that*

*(1)* $\psi^k f^* = f^* \psi^k$ *for all* $f\colon X \to Y$*,*

*(2)* $\psi^k[L] = [L^k]$ *when* $L$ *is a line bundle over* $X$*,*

*(3)* $\psi^k \circ \psi^\ell = \psi^{k\ell}$*,*

*(4)* $\psi^p \alpha = \alpha^p \pmod p$ *when* $p$ *is prime, in the sense that for each* $\alpha$*, there exists some* $\beta \in K^0(X)$ *such that* $\psi^p \alpha = \alpha^p + p\beta$

*(5) when* $X \cong S^n$*, then* $\psi^k(\alpha) = k^n a$*.*

**Remark 4.46.** The Adams operations descend to ring homomorphisms on reduced K-theory $\psi^k\colon \widetilde{K}^0(X) \to \widetilde{K}^0(X)$.

## 4.6 When is the Hopf Invariant one?

The attaching map of a $2n$-cell to an $n$-cell is $f\colon S^{2n-1} \to S^n$.

$$H^i(S^n \cup_f D^{2n}) = \begin{cases} \mathbb{Z} & i \in \{0, n, 2n\}, \\ 0 & \text{otherwise.} \end{cases}$$

If $n > 1$, there are no boundary maps between these cohomology groups. If $\alpha \in H^n(S^n \cup_f D^{2n}) = \mathbb{Z}$ generates this cohomology group, $\alpha \smile \alpha = h\beta$ in $H^{2n}(S^n \cup_f D^{2n}) = \mathbb{Z}$ for some $h \in \mathbb{Z}$.

**Definition 4.47.** This $h$ is the **Hopf invariant** of $f$.

Say we have a division algebra structure on $\mathbb{R}^n$ given by $g\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. We may use this to construct a map

$$\widehat{g}\colon S^{2n-1} \to S^n.$$

Write

$$S^{2n-1} \cong \partial(D^n \times D^n) \cong (\partial D^n \times D^n) \cup (D^n \times \partial D^n),$$

and similarly write $S^n$ as a union of hemispheres:

$$S^n \cong D^n_+ \cup_{S^{n-1}} D^n_-.$$

So define $\widehat{g}$ by

$$\widehat{g}(x,y) = \begin{cases} \frac{|y|}{|g(x,y)|} g(x,y) \in D^n_+ & \text{if } (x,y) \in \partial D^n \times D^n, y \neq 0 \\ \frac{|x|}{|g(x,y)|} g(x,y) \in D^n_- & \text{if } (x,y) \in D^n \times \partial D^n, x \neq 0 \\ \cdots & x = 0 \text{ or } y = 0 \end{cases}$$

**Claim 4.48.** $\widehat{g}$ has Hopf invariant 1.

The Hopf invariant also appears in K-theory. Let $X = S^n \cup_f D^{2n}$. Then there is a cofiber sequence

$$S^n \hookrightarrow X = (S^n \cup_f D^{2n}) \to {}^{S^n \cup_f D^{2n})}\!/_{S^n} \cong S^{2n}$$

that induces a short exact sequence in K-theory

$$0 \to \widetilde{K}^0(S^{2n}) \to \widetilde{K}^0(X) \to \widetilde{K}^0(S^n) \to 0$$

but $\widetilde{K}^0(S^{2n}) \cong \mathbb{Z}\{\alpha'\}$ and $\widetilde{K}^0(S^n) \cong \mathbb{Z}\{\beta'\}$. Let $\alpha$ be the image of $\alpha'$ in $\widetilde{K}^0(X)$ and let $\beta$ be any preimage of $\beta'$ in $\widetilde{K}^0(X)$. Since $\widetilde{K}^0(S^n)$ has trivial multiplication, we have $(\beta')^2 = 0$. By exactness of this sequence, we must have $\beta^2 = h\alpha$ for some $h \in \mathbb{Z}$.

**Fact 4.49.** *This integer* $h$ *is the Hopf invariant.*

This fact follows from calculations using the Atiyah-Hirzebruch spectral sequence, which we won't discuss here.

**Theorem 4.50.** $h = \pm 1$ *only if* $n = 2, 4, 8$.

*Proof.* If $n$ is odd, then the Hopf invariant must be zero: it is the integer $h$ such that $\alpha \smile \alpha = h\beta$ for $\alpha \in H^n(S^n \cup_f D^{2n})$. In particular, $\alpha$ is in odd cohomological degree, so $\alpha \smile \alpha = (-1)^{|\alpha||\alpha|}\alpha \smile \alpha$, hence $\alpha \smile \alpha = 0$. So when $n$ is odd, the Hopf invariant is zero.

So let $n = 2m$ for some integer $m$. It suffices to show that $\beta^2 \equiv 0 \pmod 2$ unless $m = 1, 2, 4$. To do this, we will use Theorem 4.45.

By property (4), $\beta^2 \equiv 0 \pmod 2 \iff \psi^2\beta \equiv 0 \pmod 2$. By property (5), $\psi^2\beta = 2^m\beta + k\alpha$ and $\psi^3\beta = 3^m\beta + \ell\alpha$.

By property (3), $\psi^2\psi^3 = \psi^3\psi^2$, so

$$3^m(2^m\beta + k\alpha) + \ell\alpha = 2^m(3^m\beta + \ell\alpha) + k\alpha$$

Rearranging, we see that

$$3^m(3^m - 1)k = 2^m(2^m - 1)\ell.$$

It suffices to show that $k \equiv 0 \pmod 2$. $2^m$ does not divide $3^m$, so to show $k \equiv 0 \pmod 2$, we must demonstrate that $2^m$ does not divide $3^m - 1$ unless $m = 1, 2, 4$. What is the largest power of 2 dividing $3^m - 1$? Call this $\nu(m)$.

If $m$ is odd, then $3^m - 1 \equiv 3 - 1 \equiv 2 \pmod 8$, so $\nu(m) = 1$. Also note that $3^m + 1 \equiv 4 \pmod 8$.

If $m$ is even, then $3^m + 1 \equiv 2 \pmod 8$. Write $m = 2^r j$ with $j$ odd. Then (repeatedly factoring a difference of squares)

$$\begin{aligned} 3^{2^r j} - 1 &= (3^{2^{r-1}j} + 1)(3^{2^{r-1}j} - 1) \\ &= (3^{2^{r-1}j} + 1)(3^{2^{r-2}j} + 1)(3^{2^{r-2}j} - 1) \\ &= \cdots = \prod_{\ell=0}^{r-1} (3^{2^L j} + 1)(3^j - 1). \end{aligned}$$

Modulo 8, the first term is 2 except when $L = 0$, in which case it's 4. The second term is likewise 2. Hence, $\nu(2^L j) = L + 2$. So $2^m$ divides $3^m - 1$ when $nu(2^L j) \leq L + 2$, with $m = 2^L j$. This inequality holds when $L = 0, 1, 2$ and $j = 1$, so we must have $m = 2^L j \in \{1, 2, 4\}$. Hence, $n = 2m \in \{2, 4, 8\}$, as desired. □

## 4.7 The Splitting Principle

**Theorem 4.51** (Splitting Principle)**.** *Let $X$ be a compact Hausdorff space. For any bundle $p\colon E \to X$, there is a compact Hausdorff space $X'$ and a map $f\colon X' \to X$ such that $f^*\colon K^0(X) \to K^0(X')$ is injective and $f^*(E)$ splits as a sum of line bundles.*

To illustrate how this principle is useful, we can use these to prove the properties of the Adams operations. We will only prove properties (1) - (4). Property (5) isn't hard, but it takes time (and it's in Hatcher).

**Lemma 4.52.** *The pullback of a sum is the sum of the pullbacks.*

*Proof of Theorem 4.45.* First, notice that $\psi^k([L_i]) = [L_i]^k = [L_i^{\otimes k}]$, and since $\psi^k$ is a group homomorphisms, we know that $\psi^k$ applied to the sum of line bundles is the sum of $\psi^k$ applied to these line bundles. Hence,

$$\psi^k([L_1 \oplus \ldots \oplus L_n]) = [L_1^{\otimes k} \oplus \ldots \oplus L_n^{\otimes k}].$$

To check (1), it suffices to check it for the $\lambda^n$'s by definition. Consider two bundles $E, E'$, and $\psi^k(E \oplus E')$. By the splitting principle, let $f\colon X' \to X$ split $E$, so

$$f^*(E \oplus E') \cong L_1 \oplus \ldots \oplus L_m \oplus f^*E'$$

Let $f'\colon X'' \to X'$ split $f^*E'$. Then

$$(f'f)^* = L_1 \oplus \ldots \oplus L_m \oplus L_1' \oplus \ldots \oplus L_n'.$$

We have the following commutative diagram

$$\begin{array}{ccc} K^0(X) & \xhookrightarrow{(f'f)^*} & K^0(X'') \\ \downarrow{\psi^k} & & \downarrow{\psi^k} \\ K^0(X) & \xhookrightarrow{(f'f)^*} & K^0(X'') \end{array}$$

This diagram shows that $\psi^k$ commutes with pullbacks, using injectivity properties of the splitting maps.

We should check that the Adams operations are also ring homomorphisms. So consider $E \otimes E'$. Again choose a splitting map $f\colon X' \to X$ for $E$, so that

$$f^*(E \otimes E') \cong (L_1 \oplus \ldots \oplus L_m) \otimes f^*(E')$$

Then choose a splitting map $f'\colon X'' \to X'$ for $f^*(E')$, so that

$$(ff')^*(E \otimes E') \cong (f')^*(L_1) \oplus \ldots (f')^*(L_m) \otimes (L_1' \oplus \ldots \oplus L_n') \cong \bigoplus (f')^* L_i \otimes L_j'.$$

Hence, we have split $E \otimes E'$ as a sum of (products of) line bundles. Then, we know that for line bundles $L_1$ and $L_2$, $\psi^k(L_1 \otimes L_2) = \psi^k(L_1)\psi^k(L_2)$.

Now we can combine the diagram above and the splitting principle to conclude property (2).

Property (3) follows from $(L^{\otimes k})^{\otimes \ell} \cong L^{\otimes (k\ell)}$ for line bundles.

To show property (4), write any element $\alpha \in K^0(X)$ as $\alpha = [E] - [E']$. Then for $p$ prime,

$$\begin{aligned} f^*(\psi^p(E)) &= \psi^p(L_1 \oplus \ldots \oplus L_m) = L_1^p \oplus \ldots \oplus L_m^p \\ &\equiv (L_1 \oplus \ldots \oplus L_m)^p = f^*(E) \pmod{p}. \end{aligned}$$

□

**Definition 4.53.** For any bundle $p\colon E \to X$, define the **flag bundle** $g\colon F(E) \to X$ with total space $n$-tuples of orthogonal lines in the same fiber of $E$. The fibers of this bundle are Stiefel manifolds $V_n(\mathbb{C}^n)$.

Note that $F(E)$ is compact.

**Claim 4.54.** $g^*(E)$ splits as a sum of line bundles.

To prove this claim, we need a few statements that we won't prove.

**Proposition 4.55.** *As a ring* $K^0(\mathbb{CP}^n) \cong \mathbb{Z}[L]/\langle (L-1)^{n+1}\rangle$ *where* $L$ *is the canonical line bundle on* $\mathbb{CP}^n$.

**Theorem 4.56** (Liray-Hirsch). *Let* $p\colon E \to B$ *be a fiber bundle with* $E, B$ *compact Hausdorff, and with fiber* $F$ *such that* $K^*(F)$ *is free. Suppose that there are* $c_1, \ldots, c_k \in K^*(E)$ *such that they restrict to a basis for* $K^*(F)$ *for all fibers* $F$. *If* $F$ *is a finite cell complex with cells only in even dimensions, then* $K^*(E)$ *is a free module over* $K^*(B)$ *with basis* $c_1, \ldots, c_k$.

**Remark 4.57.** The Leray-Hirsch theorem holds for generalized cohomology theories, not only K-theory. It is usually stated for singular cohomology $H^*$. It can be used to prove the Thom isomorphism theorem.

The assumption that $F$ is a finite cell complex with cells only in even dimension can be replaced by a different assumption on the base $B$ instead.

*Proof of Theorem 4.51.* Let $P(E)$ be the projective bundle of $E$, with fibers $\mathbb{CP}^{n-1}$ if $E$ has rank $n$. There is a canonical line bundle $L \to P(E)$, and classes

$$[1], [L], [L^2], \ldots, [L^{n-1}] \in K^*(P(E)).$$

Notice that for $\iota\colon F \to P(E)$ the inclusion of a fiber, $\iota^*[L^k] = L^k$, so these $L^i$ form a basis for $K^*(P(E))$.

These are exactly the conditions for the Liray-Hirsch theorem, so $K^*(P(E))$ is free over $K^*(B)$ with basis $[1], [L], [L^2], \ldots, [L^{n-1}]$. Therefore, $\widetilde{K}^0(B) \hookrightarrow \widetilde{K}^0(P(E))$. $g^*(E)$ contains $L$ as a subbundle, so $g^*E = L \oplus E'$, where $E'$ has rank $n-1$.

Then recursively repeat this process on $E'$ to get a sum of line bundles for $E$, giving a point in the flag bundle $F(E)$. □

# 5 Where do we go from here?

If you took an algebraic geometer from the 1950's and took them to a conference today, they wouldn't understand everything, but they would understand what the problems are and why people want to understand them.

If you took a combinatorialist from the 1950's and took them to a conference today, they would mostly understand what's going on.

But with the possible exception of Peter May, if you took an algebraic topologist from the 1950's and took them to a conference today, they wouldn't recognize it as the same field.

So far, we've done algebraic topology from the 1960's, but the point of a graduate class is to introduce you to the stuff that's going on in algebraic topology today. So let's take a while to talk about how these things appear in modern algebraic topology.

One of the biggest, if not *the* biggest, open problem in algebraic topology is computing the homotopy groups of spheres. So why do people care?

Say we're building a space $Y$ by attaching a cell to $X$ via a map $f\colon S^{n-1} \to X$. This defines a pullback

$$\begin{array}{ccc} D^n & \dashrightarrow & Y \\ \uparrow & & \downarrow \\ S^{n-1} & \xrightarrow{f} & X, \end{array}$$

where the homotopy type of $Y$ depends only on the homotopy type of $f$; all ways of attaching $n$-cells to $X$ are determined by $\pi_{n-1}X$. If $X$ is itself built from attaching cells to smaller cells, we can ask about the basic building blocks $\pi_n S^m$.

**Definition 5.1.** For a space $X$, the $n$-th **stable homotopy group** is

$$\pi_n^s X = \operatorname*{colim}_i \pi_{n+i} \Sigma^i X.$$

It's not immediately apparent, but the stable homotopy groups are usually easier to think about. One reason why is that stable homotopy groups $\pi_*^s$ form a homology theory (the homology theory for the sphere spectrum).

Once we know the stable homotopy groups, the regular homotopy groups can be determined by a spectral sequence whose input is the stable homotopy groups, and whose output is the regular (unstable) homotopy groups. The general attitude of people who work with this spectral sequence is that it's not too hard to run.

To compute stable homotopy groups, there's a number theory trick floating around in the background. We use the following theorem.

**Theorem 5.2.** *If* $f_*\colon H_*(X;\mathbb{Z}/p) \to H_*(Y;\mathbb{Z}/p)$ *is an isomorphism for all* $p$*, then* $H_*(X) \to H_*(Y)$ *is an isomorphism as well.*

*If in addition the spaces are simply connected, then this gives by the Hurewicz theorem an isomorphism* $\pi_*(X) \to \pi_*(Y)$ *as well.*

So the idea is to work "mod $p$." What does this mean? On homotopy groups, this is tensoring with $\mathbb{Z}/p$, but this can actually be made sense of as an operation on spaces.

Unfortunately, this loses too much information. But if you think some about number theory, it turns out the right thing to do is to localize at $p$. And if we're localizing at $p$ for all $p$, then we'd better look at the rational case as well.

The rational case is very well understood. Rationally, the homotopy groups of spheres are the same as the rational homotopy groups of an Eilenberg-MacLane space. In fact, the sphere spectrum is rationally the same as $H\mathbb{Q}$ (written $\mathbb{S} \otimes \mathbb{Q} \simeq_{\mathbb{Q}} H\mathbb{Q}$).

So what we want to study instead is the localization of the sphere spectrum $\mathbb{S}$ at a prime $p$, denoted $\mathbb{S}_{(p)}$. To be more general, we localize at any space $E$ by saying that $f\colon X \sim_E Y$ if and only if $[Y, E] \xrightarrow[\cong]{f^*} [X, E]$. The localization of $X$ at $E$ is written $L_E X$.

**Theorem 5.3.** $\mathbb{S}_{(p)} \simeq \operatorname{colim}\left(\cdots \to L_{E(2)}\mathbb{S} \to L_{E(1)}\mathbb{S} \to L_{E(0)}\mathbb{S}\right)$

Where $L_{E(i)}$ is the **Morava $E$-theory**. It depends on a prime $p$, which is suppressed from the notation. So to compute the homotopy groups of $\mathbb{S}_{(p)}$, we want instead to understand the homotopy groups of $L_{E(i)}$. This is closely related to formal group laws.

We know how to do the zeroth level: $L_{E(0)} = H\mathbb{Q}$. The interesting stuff starts at the first level: $L_{E(1)}$ is related to complex topological $K$-theory $KU$. We might say that $L_{E(1)}$ is the next best approximation to $\mathbb{S}_{(p)}$ after $L_{E(0)}$, so we study $L_{E(1)}$ instead. The homotopy groups of these are given by the image of the J-homomorphism.

This is the beginning of the field of **Chromatic Homotopy Theory**. The idea is that you take a ray of white light $\mathbb{S}_{(p)}$ and put it through a prism (the colimit) to study all of the colors $L_{E(i)}\mathbb{S}$ separately.

## 5.1 The J-homomorphism

**Definition 5.4.** For a space $X$, the $n$-th **stable homotopy group** is

$$\pi_n^s X = \operatorname*{colim}_i \pi_{n+i}\Sigma^i X.$$

**Definition 5.5.** The **stable homotopy groups of spheres** or the **$i$-th stable stem** is the $i$-th stable homotopy group of $S^0$. It is often written just $\pi_i^s$ rather than $\pi_i^s S^0$.

The J-homomorphism is built from a sequence of homomorphisms $\pi_i O(n) \to \pi_{i+n} S^n$ such that the following diagrams commute

$$\begin{array}{ccc} \pi_i O(n) & \xrightarrow{J_{i,n}} & \pi_{i+n} S^n \\ \downarrow & & \downarrow \Sigma \\ \pi_i O(n+1) & \xrightarrow{J_{i,n+1}} & \pi_{i+n+1} S^{n+1} \end{array}$$

We may then take the colimit along the vertical maps to get the J-homomorphism

$$J \colon \pi_i O \to \pi_i^s S^0 = \pi_i \mathbb{S}.$$

Recall that $\pi_i O$ is periodic, with

$$\pi_i O = \begin{cases} \mathbb{Z}/2 & i \equiv 0 \pmod 8, \\ \mathbb{Z}/2 & i \equiv 1 \pmod 8, \\ 0 & i \equiv 2 \pmod 8, \\ \mathbb{Z} & i \equiv 3 \pmod 8, \\ 0 & i \equiv 4 \pmod 8, \\ 0 & i \equiv 5 \pmod 8, \\ 0 & i \equiv 6 \pmod 8, \\ \mathbb{Z} & i \equiv 7 \pmod 8. \end{cases}$$

Notice that $O$ has two connected components, one of which is $SO$.

**Theorem 5.6.** *The image of $J|_{SO}$ is a direct summand of the stable homotopy group $\pi_n^s$*

Our goal is to compute bounds on the size of the image of J. We'll start with a lower bound.

**Definition 5.7.** The **join** of two spaces $X$ and $Y$, written $X * Y$, is the space

$$X \times Y \times I /_{\sim}$$

Where $(x, y_0, 0) \sim (x, y_1, 0)$ for all $y_0, y_1 \in Y$ and $(x_0, y, 1) \sim (x_1, y, 1)$ for all $x_0, x_1 \in X$.

**Example 5.8.** The join of two line segments is a tetrahedron. The join of a point $\{a\}$ and a space $X$ is the cone $CX$ on $X$.

$$\{a\} * X = CX$$

The join of $\{a, b\}$ and $X$ is the unreduced suspension $SX$ of $X$.

$$\{a, b\} * X = SX$$

The join of $n + 1$ copies of the zero sphere is $S^n$.

$$S^n \cong S^0 * S^0 * \cdots * S^0$$

The join of an $m$-sphere and an $n$-sphere is an $(m + n + 1)$-sphere.

$$S^n * S^m \cong S^{n+m+1}$$

For any spaces $X$ and $Y$, the map $X \times Y \times I \to S(X \times Y)$ factors through the join.

$$\begin{array}{ccc} X \times Y \times I & \longrightarrow & S(X \times Y) \\ & \searrow \quad \nearrow_{h_{X,Y}} & \\ & X * Y & \end{array}$$

We name the map $h_{X,Y} \colon X \times Y \to S(X \times Y)$ for future use.

**Definition 5.9.** Given a map $f \colon X \times Y \to Z$, the **Hopf construction of** $f$ is the map

$$X * Y \xrightarrow{h_{X,Y}} S(X \times Y) \xrightarrow{Sf} SZ.$$

Now given $\gamma \in SO(n)$, $\gamma$ acts on $\mathbb{R}^n$ and preserves norms. Hence, it induces $\widetilde{\gamma} \colon S^{n-1} \to S^{n-1}$. Any class $[f] \in \pi_i(SO(n))$ for some $i > 0$, is represented by $f \colon S^i \to SO(n)$. This gives (by uncurrying) a map

$$S^i \times S^{n-1} \to S^{n-1}$$

which by the Hopf construction becomes a map

$$\widetilde{f} \colon S^{n+i} \to S^n.$$

This represents a class in $\pi_{n+i}(S^n)$.

**Definition 5.10.** We define the **J-homomorphism** $J \colon \pi_i(SO(n)) \to \pi_{n+i}(S^n)$ by $J[f] := [\widetilde{f}]$, where $\widetilde{f}$ is as constructed above.

Moreover, it's not hard to check that the diagram below commutes.

$$\begin{array}{ccc} \pi_i O(n) & \xrightarrow{J} & \pi_{i+n} S^n \\ \downarrow & & \downarrow S \\ \pi_i O(n+1) & \xrightarrow{J} & \pi_{i+n+1} S^{n+1} \end{array}$$

Therefore, $J$ induces a map from homotopy groups of the infinite orthogonal group to the $i$-th stable stem $\pi_i O \to \pi_i^s$.

There is an inclusion $U(n) \subseteq O(2n)$. Therefore, we have a homomorphism

$$\pi_i U \to \pi_i O \xrightarrow{J} \pi_i^s$$

By composing with the **Adams $e$ homomorphism** $e\colon \pi_i^s \to \mathbb{Q}/\mathbb{Z}$, we have a homomorphism

$$\pi_i U \to \pi_i O \xrightarrow{J} \pi_i^s \xrightarrow{e} {}^{\mathbb{Q}}/_{\mathbb{Z}}.$$

Hence, if a generator of $\pi_i U$ is sent to $\frac{a}{b}$ (in reduced terms), the order of the image of $\pi_i U \to \pi_i^s$ is at least $b$.

**Remark 5.11.** In fact, these denominators $b$ turn out to be Bernoulli numbers.

Over the next few lectures, we will construct the $e$ invariant and exploit the following theorem to learn about stable homotopy groups.

**Theorem 5.12.** $\pi_i^s = \mathrm{im}(J) \oplus \ker(e)$

## 5.2 The Chern Character and $e$ invariant

**Proposition 5.13.** *The set* $\mathrm{Vect}^1(X)$ *of line bundles on* $X$ *is a group under tensor product. With this structure, the first Chern class* $c_1\colon \mathrm{Vect}^1(X) \to H^2(X)$ *is a homomorphism. This is an isomorphism if* $X$ *is a CW complex.*

**Construction 5.14.** The **chern character** is a ring homomorphism

$$\mathrm{ch}\colon K^0(X) \to H^*(X;\mathbb{Q})$$

In particular, this means that $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$. For line bundles $L$, the chern character is $\mathrm{ch}(L) = e^{c_1(L)}$.

Given a sum of line bundles $E = L_1 \oplus \cdots \oplus L_n$, the chern character is

$$\begin{aligned} \mathrm{ch}(L_1 \oplus \cdots \oplus L_n) &= \sum_{i=1}^{n} \mathrm{ch}(L_i) \\ &= \sum_{i=1}^{n} e^{c_1(L_i)} \\ &= n + \sum_{j=1}^{\infty} \frac{1}{j!}(c_1(L_1)^j + c_1(L_2)^j + \ldots + c_1(L_n)^j) \\ &= n + \sum_{j=1}^{\infty} \frac{1}{j!} S_j \left(\sigma_1(t_1, \ldots, t_n), \sigma_2(t_1, \ldots, t_n), \ldots, \sigma_j(t_1, \ldots, t_n)\right) \end{aligned}$$

where $t_i = c_1(L_i)$. But notice that by the Whitney sum formula,

$$\begin{aligned} c_j(L_1 \oplus \ldots \oplus L_n) &= \sigma_j(t_1, \ldots, t_n) \\ c(L_1 \oplus \ldots \oplus L_n) &= (1 + t_1)(1 + t_2) \cdots (1 + t_n) \\ &= n + \sum_{j=1}^{\infty} \frac{1}{j!} c_j(E) \end{aligned}$$

**Definition 5.15.** The **chern cheracter** of a bundle $E$ is a ring homomorphism $\mathrm{ch}\colon K^0(X) \to H^*(X; \mathbb{Q})$ defined by

$$\mathrm{ch}(E) := \dim(E) + \sum_{j=1}^{\infty} \frac{1}{j!} c_j(E)$$

The Chern character also descends to a ring homomorphism on from reduced K-theory to reduced cohomology,

$$\mathrm{ch}\colon \widetilde{K}^0(X) \to \widetilde{H}^*(X; \mathbb{Q}).$$

**Proposition 5.16.** *The Chern character* $\widetilde{K}^0(S^{2n}) \to \widetilde{H}^*(S^{2n}; \mathbb{Q})$ *is the inclusion of* $\mathbb{Z}$ *into* $\mathbb{Q}$.

*Proof sketch.* For $n = 0$, the calculation is easy. Now, for arbitrary $n$, we check that this diagram below commutes.

$$\begin{array}{ccc} \widetilde{K}^0(S^{2n}) & \xrightarrow{\cong} & \widetilde{K}^0(S^{2n+2}) \\ \big\downarrow \mathrm{ch} & & \big\downarrow \mathrm{ch} \\ \widetilde{H}^*(S^{2n}; \mathbb{Q}) & \xrightarrow{\cong} & \widetilde{H}^*(S^{2n+2}; \mathbb{Q}) \end{array}$$

□

**Remark 5.17.** A different proof of Bott periodicity gives a generator for $\widetilde{K}^0(S^{2n})$; namely, $[(H-1)^{\otimes n}]$, where $H$ is the tautological bundle on $S^2$. Our proof of Bott periodicity should also give us a way to determine this, but it's not immediately clear.

**Proposition 5.18.** *The map* $\widetilde{K}^0(X) \otimes \mathbb{Q} \to \widetilde{H}^*(X; \mathbb{Q})$ *is an isomorphism if* $X$ *is a finite CW complex.*

Let $m > n$. Suppose $f \colon S^{2m-1} \to S^{2n}$. This defines an element in $\pi_{2m-1}(S^{2n})$. The mapping cone $C_f$ of $f$ is the same as attaching a cell to $S^{2n}$ along $f$; $C_f = S^{2n} \cup_f D^{2m}$. There is a cofiber sequence

$$S^{2n} \hookrightarrow C_f \to S^{2m}.$$

This gives two short exact sequences in $K$-theory and cohomology, and there are maps between these given by the Chern character.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{K}^{(}(S^{2m}) & \longrightarrow & \widetilde{K}^0(C_f) & \longrightarrow & \widetilde{K}^0(S^{2n}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle ch} & & \downarrow{\scriptstyle ch} & & \downarrow{\scriptstyle ch} & & \\
0 & \longrightarrow & \widetilde{H}^*(S^{2m};\mathbb{Q}) & \longrightarrow & \widetilde{H}^*(C_f;\mathbb{Q}) & \longrightarrow & \widetilde{H}^*(S^{2n};\mathbb{Q}) & \longrightarrow & 0
\end{array}$$

The generator $\alpha' \in \widetilde{K}^0(S^{2m})$ is sent to the generator $a' \in \widetilde{H}^*(S^{2m};\mathbb{Q})$. The commutativity of the diagram means that the image $\alpha$ of $\alpha'$ in $\widetilde{K}^0(C_f)$ is sent to the image $a$ of $a'$ in $\widetilde{H}^*(C_f;\mathbb{Q})$.

Likewise, there is a generator $\beta' \in \widetilde{K}^0(S^{2n})$ that is sent to a generator $b' \in \widetilde{H}^*(S^{2n};\mathbb{Q})$. The preimage of $\beta'$ in $\widetilde{K}^0(C_f)$ is some $\beta \in \widetilde{K}^0(C_f)$, and the preimage of $b'$ is some $b \in \widetilde{H}^*(C_f;\mathbb{Q})$. The choice of $b$ is determined by the cell structure of $C_f := S^{2n} \cup_f D^{2m}$, and is not a choice on our part.

The only thing that we can conclude about the image of $\beta \in \widetilde{K}^0(C_f)$ under the Chern character is that $\mathrm{ch}(\beta) = b + ra$ for some $r \in \mathbb{Q}$.

$$\begin{array}{ccccc}
\widetilde{K}^0(S^{2m}) \ni \alpha' & \longmapsto & \alpha \in \widetilde{K}^0(C_f) \ni \beta & \longmapsto & \beta' \in \widetilde{K}^0(S^{2n}) \\
\downarrow & & \downarrow \qquad\qquad \downarrow & & \downarrow \\
\widetilde{H}^*(S^{2m};\mathbb{Q}) \ni a' & \longmapsto & a \in \widetilde{H}^*(C_f;\mathbb{Q}) \ni b + ra & \longmapsto & b' \in \widetilde{H}^*(S^{2n};\mathbb{Q})
\end{array}$$

**Definition 5.19.** The ***e*-invariant of** $f \colon S^{2m-1} \to S^{2n}$ is the image of $r$ in $\mathbb{Q}/\mathbb{Z}$, with $r$ as above.

**Proposition 5.20.** *The* $e$*-invariant is well-defined.*

*Proof.* Suppose $\widetilde{\beta} = \beta + c\alpha$ for some $c \in \mathbb{Z}$. Then

$$\mathrm{ch}(\widetilde{\beta}) = \mathrm{ch}(\beta + c\alpha) = \mathrm{ch}(\beta) + c\,\mathrm{ch}(\alpha) = b + ra + ca = b + (r+c)a$$

Hence, $r$ has changed by an integer, so it's image in $\mathbb{Q}/\mathbb{Z}$ is unchanged. □

**Proposition 5.21.** *$e$ is a homomorphism.*

**Remark 5.22.** We can use the $e$-invariant to show that the two definitions of the Hopf invariant from before are actually the same. Recall that the Hopf invariant was defined as the integer $h$ such that $b^2 = ha$, and we also had $\beta^2 = c\alpha$.

We want to show that $h = c$. It suffices to show that $\mathrm{ch}(\beta^2) = h\,\mathrm{ch}(\alpha)$. To that end, calculate

$$\mathrm{ch}(\beta^2) = \mathrm{ch}(\beta)^2 = (b + ra)^2 = b^2 = ha = h\,\mathrm{ch}(\alpha)$$

So $h = c$.

Recall that we have a homomorphism $J_{\mathbb{C}} \colon \pi_i U \to \pi_i O \xrightarrow{J} \pi_i^s$, where the first map is the inclusion $U(n) \subseteq O(2n)$.

**Theorem 5.23.** *If $f \colon S^{2k-1} \to U(n)$ is a generator of $\pi_{2k-1} U$ then*

$$e \circ J_{\mathbb{C}}([f]) = \pm \beta_k / k,$$

*where $k$ is the $k$-th Bernoulli number.*

This theorem in particular implies that the order of the group $\pi_i^s$ is at least the denominator of $\beta_k/k$.

To prove this theorem, we need a few lemmas.

**Lemma 5.24.** *There is a Thom isomorphism $\Phi$ for $K$-theory: for a bundle $E \to B$, with Thom class $c \in \widetilde{K}^0(\mathrm{Th}(E))$, we have*

$$\log(\Phi^{-1}(\mathrm{ch}(c))) = \sum_j \alpha_j \,\mathrm{ch}^j(E)$$

*where*

- $\log$ *means the power series for the natural log;*
- $\alpha_j \in \mathbb{Q}$ *is a rational number defined by*

$$\sum_j \frac{\alpha_j}{j!} y^j = \log \frac{e^j - 1}{y};$$

- $\mathrm{ch}^j(E)$ *is the part of* $\mathrm{ch}(E)$ *in degree* $2j$*.*

**Remark 5.25.** In fact, $\alpha_j = \beta_j / j$ by messing around with power series.

**Lemma 5.26.** *$C_{Jf}$ is the Thom space of the bundle $E_f \to S^{2k}$ determined by the clutching function $f$.*

*Proof of Theorem 5.23.* Observe that $f\colon S^{2m-1} \to U(n)$ is a generator for $\pi_{2m-1}U(n)$. Moreover, since $f$ is a clutching function, $[E_f]$ is a generator for $K^0(S^{2m})$. Therefore, $\mathrm{ch}([E_f]) = \mathrm{ch}^m(E)$ is a generator of $\widetilde{H}^{2m}(S^{2n})$. Then by Lemma 5.24,

$$\log \Phi^{-1}\,\mathrm{ch}([E_f]) = \alpha_m h,$$

where $h$ is the image of $[E_f]$ in reduced cohomology.

On the other hand, we know that the Thom class $c$ is equal to $\beta \in \widetilde{K}^0(C_{Jf})$ by Lemma 5.26. Therefore,

$$\log \Phi^{-1}\,\mathrm{ch}(c) = \log \Phi^{-1}\,\mathrm{ch}(\beta) = \log \Phi^{-1}(b + ra) = \log(\Phi^{-1}(b) + r\Phi^{-1}(a)).$$

This number $r$ is the $e$-invariant of $J_{\mathbb{C}}[f]$ by the discussion preceeding Definition 5.19. By degree considerations, $\Phi^{-1}(b)$ lands in degree zero, and $\Phi^{-1}(a)$ lands in degree $m$. Moreover, $\Phi^{-1}$ sends generators to generators, so

$$\log(\Phi^{-1}(b) + r\Phi^{-1}(a)) = \log(1 + rh) = rh,$$

the last line by the power series for log.

Therefore, $\alpha_m h = rh$. Hence, $\alpha_m = r = e(J_{\mathbb{C}}[f])$. Finally, by Remark 5.25, $\alpha_m = \beta_m/m$. □

To complete the proof of Theorem 5.23, we need to prove the lemmas.

*Proof of Lemma 5.24.* The key observation is that we may think of the Thom space of a bundle $E$ as $\mathrm{Th}(E) \cong P(E \oplus \varepsilon^1)/P(E)$. The intuition for this is to think of $P(E)$ as the sphere bundle on $E$, and the bundle $P(E \oplus \varepsilon^1)$ as filling in the sphere bundle on $E$.

By the Leray-Hirsch theorem, $K^*(P(E \oplus \varepsilon^1))$ is a free $K^*(B)$-module with basis $\varepsilon^1, L, \ldots, L^n$, where $L$ is the tautological line bundle over $P(E \oplus 1)$. Likewise, $K^*(P(E))$ is a free $K^*(B)$-module with basis $\varepsilon^1, L_0, \ldots, L_0^{n-1}$, where $L_0$ is the restriction of $L$ to $P(E)$. This gives a short exact sequence

$$0 \to \widetilde{K}^*(T(E)) \to K^*(P(E \oplus \varepsilon^1)) \xrightarrow{\rho} K^*(P(E)) \to 0.$$

What is the kernel of $\rho$? It is generated by some polynomial in $L$ of degree $n$. We may find this polynomial by writing a monic polynomial of degree $n$ – if there was another one, we could subtract the two and get a polynomial of lower degree, but no polynomials of lower degree in the kernel of $\rho$.

$E$ over $P(E)$ splits as $L_0 \oplus E'$ with $E'$ of rank $n-1$. $E$ over $P(E \oplus \varepsilon^1)$ splits as $L_0 \oplus E''$. We know that $\lambda^n(E') = 0$ because $E'$ has rank $n-1$. We can also write

$$\lambda^n(E) = \lambda^n(L \oplus E'')$$

By the identities for λ-operations,

$$0 = \lambda^n(E') = (-1)^n \sum_{i=0}^{n} (-1)^i \lambda^{n-i}(E) L_0^i.$$

This is an identity in $K^*(P(E))$, so the degree $n$ monic polynomial

$$(-1)^n \sum_{i=0}^{n} (-1)^i \lambda^{n-1}(E) L^i$$

generates the kernel of $\rho$.

Since all terms of the short exact sequence are $K^*(B)$-modules, this represents the Thom class $U \in K^*(\mathrm{Th}(E))$ in K-theory.

Notice that none of the above relies on the fact that we're working in K-theory, only that we have a cohomology theory that vanishes in even degrees. So we may write a similar polynomial in cohomology using Chern classes:

$$c_n(E') = \sum_{i=0}^{n} (-1)^i c_{n-i}(E) x^i.$$

Hence, the Thom class $u \in H^*(\mathrm{Th}(E))$ is represented by this polynomial.

Messing around with power series proves the lemma. (See Inna's Notes). □

# 6 Student Presentations

## 6.1 Yun Liu: Clifford Algebras

We work over $\mathbb{R}$.

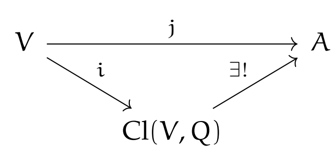
**Definition 6.1.** Given a real vector space $V$, and a quadratic form $Q$ on $V$, the **Clifford algebra** $\mathrm{Cl}(V, Q)$ is

$$\mathrm{Cl}(V, Q) = {}^{T(V)}/_{I_Q}$$

where $T(V)$ is the tensor algebra on $V$ and

$$I_Q = \langle v \otimes v - Q(v)1 \mid v \in V \rangle.$$

The Clifford algebra $\mathrm{Cl}(V, Q)$ satisfies the following universal property. It is the algebra such that for any real algebra $A$ and linear map $j \colon V \to A$ such that $j^2(v) = Q(v)1_A$, there is a unique $i \colon \mathrm{Cl}(V, Q) \to A$ such that the following commutes.



There is an involution $\alpha^*\colon \mathrm{Cl}(V,Q) \to \mathrm{Cl}(V,Q)$ on any Clifford algebra induced by $\alpha\colon v \mapsto -v$. The notation is $\alpha(x) = x^*$.

There is a $\mathbb{Z}/2$ grading on any Clifford algebra $\mathrm{Cl}(V,Q)$, with components

$$\mathrm{Cl}^i(V,Q) = \{x \in \mathrm{Cl}(V,Q) \mid \alpha(x) = (-1)^i x\}$$

for $i = 0, 1$.

**Definition 6.2.** We define several particularly important Clifford algebras.

- $\mathrm{Cl}(1) := \langle e \mid |e| = 1, e^2 = 1, e^* = -e\rangle$
- $\mathrm{Cl}(-1) := \langle f \mid |f| = 1, f^2 = 1, f^* = f\rangle$
- $\mathrm{Cl}(n) := \mathrm{Cl}(1)^{\otimes n}$
- $\mathrm{Cl}(-n) := \mathrm{Cl}(-1)^{\otimes n}$

**Remark 6.3.** Since we are working over $\mathbb{R}$, the quadratic form $Q$ induces a bilinear form on $V$, $\langle -,-\rangle\colon V \times V \to \mathbb{R}$. Then we can choose an orthogonal bass $e_1, \ldots, e_n$ for $V$ with respect to this bilinear form. Then just knowing $\dim(V) = n$ is enough to construct the Clifford algebra; we set $Q(e_i) = 1$. This corresponds to $\mathrm{Cl}(n)$. Likewise, if we take $Q(f_i) = -1$, then it corresponds to $\mathrm{Cl}(-n)$.

**Definition 6.4.** Two unital associative algebras $R$ and $S$ are called **Morita equivalent** if their categories of left-modules are equivalent: $R$-**Mod** $\simeq$ $S$-**Mod**. We write $R \simeq_M S$ when $R$ and $S$ are Morita equivalent.

**Theorem 6.5.** *$R$ and $S$ are Morita equivalent if there is an $(R,S)$-bimodule ${}_RM_S$ and an $(S,R)$-bimodule ${}_SN_r$ such that*

$$_RM_S \otimes_S {}_SN_R \cong {}_RR_R$$

$$_SN_R \otimes_R {}_RM_S \cong {}_SS_S$$

**Fact 6.6.** *Consider the two modules ${}_{\mathrm{End}(\mathbb{R}^n)}\mathbb{R}^n_{\mathbb{R}}$ and ${}_{\mathbb{R}}\mathbb{R}^n_{\mathrm{End}(\mathbb{R}^n)}$. There are equivalences*

$$_{\mathrm{End}(\mathbb{R}^n)}\mathbb{R}^n_{\mathbb{R}} \otimes_{\mathbb{R}} {}_{\mathbb{R}}\mathbb{R}^n_{\mathrm{End}(\mathbb{R}^n)} \cong \mathrm{End}(\mathbb{R}^n)$$

$$_{\mathbb{R}}\mathbb{R}^n_{\mathrm{End}(\mathbb{R}^n)} \otimes_{\mathrm{End}(\mathbb{R}^n)} {}_{\mathrm{End}(\mathbb{R}^n)}\mathbb{R}^n_{\mathbb{R}} \cong \mathbb{R}$$

**Lemma 6.7.** *$\mathrm{Cl}(1) \otimes \mathrm{Cl}(1)$ is Morita equivalent to $\mathbb{R}$.*

$$e \otimes 1 \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathrm{End}(\mathbb{R}^2)$$

$$1 \otimes f \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in \mathrm{End}(\mathbb{R}^2)$$

**Corollary 6.8.** $\mathrm{Cl}(m+n)$ *is Morita equivalent to* $\mathrm{Cl}(m) \otimes \mathrm{Cl}(n)$*, for all integers* $n$ *and* $m$*.*

**Example 6.9.** $\mathbb{C} \cong \mathrm{Cl}(-1)$

**Example 6.10.** $\mathbb{H} \cong \mathrm{Cl}(-2)$

**Example 6.11.** $\mathrm{Cl}(-3) \xrightarrow{\sim} \mathbb{H} \otimes \mathrm{Cl}(1)$ with $\mathbb{H}$ in degree 0 and $\mathrm{Cl}(1)$ in degree 1.

$$\begin{aligned} f_1 &\mapsto i \otimes e \\ f_2 &\mapsto j \otimes e \\ f_3 &\mapsto k \otimes e \end{aligned}$$

**Example 6.12.** $\mathrm{Cl}(3) \xrightarrow{\sim} \mathbb{H} \otimes \mathrm{Cl}(-1)$ with $\mathbb{H}$ in degree 0 and $\mathrm{Cl}(-1)$ in degree 1.

$$\begin{aligned} e_1 &\mapsto i \otimes f \\ e_2 &\mapsto j \otimes f \\ e_3 &\mapsto k \otimes f \end{aligned}$$

Now, combining the previous examples, we can see that

$$\mathrm{Cl}(-4) \simeq_M \mathrm{Cl}(-3) \otimes \mathrm{Cl}(-1) \simeq \mathrm{Cl}(1) \otimes \mathbb{H} \otimes \mathrm{Cl}(-1) \simeq \mathrm{Cl}(1) \otimes \mathrm{Cl}(3) \simeq \mathrm{Cl}(4)$$

So the $\mathrm{Cl}(n)$ construction is 8-periodic. Does this remind you of Bott periodicity for real vector bundles?

To make this precise, we need a few more definitions.

**Definition 6.13.** Given a topological space $X$, a **vector space object over** $X$ is a space $V$ with a map $V \to X$, together with three continuous maps

$$+\colon V \times_X V \to V \qquad 0\colon X \to V \qquad \times\colon R \times V \to V$$

such that each fiber of $V \to X$ is a vector space under these operations.

**Definition 6.14.** The **germ** of a vector bundle $E \to X$ over $x \in X$ is a pair $(U, V)$ where $U$ is a neighborhood of $X$ and $V$ is a vector bundle over $U$. If $U' \subseteq U$ is a smaller neighborhood, then we demand that $(U', V') \sim (U', V|_{U'})$.

**Definition 6.15.** A **quasi-bundle** $V \to E$ is a vector space object $V$ equipped with a vector bundle germ $V_x$ at each $x \in X$, and an inclusion $i\colon V_x \to V_{\langle x \rangle}$ where $V_{\langle x \rangle}$ is the germ of $V$ at $x$, satisfying some conditions roughly analogous to that of vector bundles.

Using these quasi-bundles, we can define K-theory. Then the periodicity of Clifford algebras gives a Bott periodicity theorem for this new quasi-bundle K-theory.

## 6.2 Sujit Rao: Elementary Bott Periodicity

This section outlines an elementary proof of Bott Periodicity.

**Theorem 6.16.** *If* $X$ *is compact, then the external product* $\mu\colon K^0(X)\otimes K^0(S^2)\to K^0(X\times S^2)$ *is an isomorphism.*

An outline of this proof is as follows.

(1) Classify vector bundles over $X\times S^2$ by clutching functions.

(2) Approximate clutching functions by a homotopy to Laurent polynomial clutching functions.

(3) "Linearize" a polynomial clutching function.

(4) Decompose a bundle on $X\times S^2$ using a linear clutching function.

(5) Define $\mu^{-1}$ using this decomposition.

**Definition 6.17** (Notation)**.** Let $X$ be compact, and view $S^2$ as $\mathbb{CP}^1$. Then we write the upper and lower hemispheres as

$$D_0 := \{z \mid |z|\le 1\} \qquad D_\infty := \{z \mid |z|\ge 1\}.$$

Their intersection $D_0\cap D_\infty$ is $S^1$, and we have projections

$$\pi_0\colon X\times D_0\to X$$
$$\pi_\infty\colon X\times D_\infty\to X$$
$$\pi\colon X\vee S^1\to X$$

and a map $S\colon X\to X\times S^2$ given by $x\mapsto (x,(1,0))$. Let $H$ be the tautological bundle over $\mathbb{CP}^1$. Let $\eta$ be the dual bundle of $H$.

Now we begin step 1.

**Definition 6.18.** Given a bundle $p\colon E\to X$, a **clutching function** is a bundle automorphism $f\colon E\times S^1\to E\times S^1$. Denote by $[E,f]$ the bundle $\pi_0^*(E)\cup_f \pi_\infty^*(E)$.

**Proposition 6.19.** *Every bundle* $p\colon E\to X\times S^2$ *is isomorphic to* $[s^*(E),f]$ *for some automorphism* $f\colon S^*(E)\times S^1\to S^*(E)\times S^1$.

*Proof sketch.* Since $\pi_0$ is a homotopy equivalence, then $E|_{X\times D_0}$ is isomorphic to a pullback of a bundle $E_0\to X$. Likewise for $\pi_\infty$, we get a bundle $E_\infty\to X$. Then let $h_\alpha\colon E|_{X\times D_\alpha}\to E_\alpha\times D_\alpha$, then $E\cong [S^*(E),h_0\circ h_\infty^{-1}]$ with appropriate restrictions. □

**Proposition 6.20.** *If* $f,g\in \mathrm{End}(E\times S^1)$ *are both clutching functions, and* $f\simeq g$ *via an always-invertible homotopy, then* $[E,f]\cong[E,g]$.

For step 2, we need the following lemma that we will not prove. It requires that $X$ is compact.

**Lemma 6.21.** *If* $f\colon X \times S^1 \to \mathbb{C}$ *is continuous, then there is a continuous* $a_n\colon X \times S^1 \to \mathbb{C}$ *such that* $(a_n)_{n\in\mathbb{N}}$ *converges to* $f$ *uniformly, and* $a_n(x,-)$ *is a Laurent polynomial.*

The proof of this lemma requires some analysis, so we will omit it for now.

**Definition 6.22.** A **Laurent Polynomial Clutching Function** (LPCF) is a clutching function of the form

$$(e,z) \mapsto \left(\sum_{k=-n}^{n} f_k(e)z^k, z\right)$$

for $f_k \in \mathrm{End}(E)$.

**Proposition 6.23.** *Every bundle* $p\colon E \to X \times S^2$ *is isomorphic to* $[S^*(E), f]$ *where* $f$ *is a Laurent polynomial clutching function.*

*Proof sketch.* It suffices to show that LPCFs are dens in $\mathrm{End}(S^*(E) \times S^1)$. For trivial bundles, use Lemma 6.21. In general, take a partition of unity then take convex combinations. ☐

**Proposition 6.24.**

*(a)* $[E, fz^n] \cong [E, f]\widehat{\otimes} H^n$

*(b)* $[E_1, f_1] \oplus [E_2, f_2] \cong [E_1 \oplus E_2, f_1 \oplus f_2]$

Combined with the fact that any bundle is isomorphic to one of the form $[S^*(E), f]$ for $f$ a LPCF, the proposition above lets us pull apart any bundle into a sum of tensor products of bundles, where each factor of the tensor product is either a bundle on $X$ or a bundle on $S^2$, which is just a power of $H$.

To proceed, we linearize polynomial clutching functions (step 3).

**Proposition 6.25.** *Let* $E$ *be a bundle over* $X$*, and* $f = f_0 + f_1 z + \ldots + f_n z^n$ *be a polynomial clutching function. Define a clutching function for* $E^{\oplus(n+1)}$ *by*

$$L^n(f) := \begin{bmatrix} f_0 & f_1 & \cdots & f_{n-1} & f_n \\ -z & 1 & \cdots & 0 & 0 \\ 0 & -z & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & -z & 1 \end{bmatrix}$$

*In particular,*

$$L^n(f)\big((e_1,\dots,e_{n+1}),z\big) = \bigg(f_0(e_1) - ze_2, f_1(e_1) + e_2 - ze_3, \dots, f_n(e_1) + e_n\bigg).$$

*Then* $[E^{\oplus(n+1)}, L^n(f)] \cong [E^{\oplus(n+1)}, [f] \oplus I_n]$*, where* $I_n$ *is the identity matrix.*

**Proposition 6.26.**

*(a)* $[E^{\oplus(n+2)}, L^{n+1}(f)] \cong [E^{\oplus(n+1)}, L^n(f)] \oplus [E, 1]$

*(b)* $[E^{\oplus(n+2)}, L^{n+1}(zf)] \cong [E^{\oplus(n+1)}, L^{n+1}(f)] \oplus [E, z]$.

Now for step 4: decomposing a bundle on $X \times S^2$.

**Proposition 6.27.** *If* $az + b$ *is a clutching function, then* $[E, az + b] \cong [E, z + c]$ *for some* $c$*.*

**Proposition 6.28.** *Given* $[E, f]$ *where* $f = z + b$*, then* $E = E_+ \oplus E_-$ *for some bundles* $E_+$ *and* $E_-$*, and* $[E, f] \cong [E_+, 1] \oplus [E_-, z]$*.*

*Proof Sketch.* Define

$$p_0 = \frac{1}{2\pi i}\int_{|z|=1} (z+b)^{-1}\, dz \in \mathrm{End}(E)$$

Then notice that

$$\frac{(z+b)^{-1}}{(w-z)} + \frac{(w+b)^{-1}}{z-w} = (w+b)^{-1}(z+b)^{-1} = (z+b)^{-1}(w+b)^{-1},$$

the last equality because the left side is symmetric in $z$ and $w$ (note that $b$ is an endomorphism of $E$, so they don't necessarily commute!).

Then $fp_0 = p_0 f$. To show that $p_0^2 = p_0$, note that $(z+b)$ is invertible for $1 - \varepsilon \leq |z| \leq 1 + \varepsilon$. Then

$$p_0^2 = \frac{1}{(2\pi i)^2}\int_{|z|=r_1}\int_{|w|=r_2}\left(\frac{(z+b)^{-1}}{(w-z)} + \frac{(w+b)^{-1}}{z-w}\right) dw\, dz$$

where $1 - \varepsilon \leq r_2 < r_1 < 1 + \varepsilon$. Somehow one of the terms goes away and

$$p_0^2 = \frac{1}{(2\pi i)^2}\int_{|z|=r_1}\int_{|w|=r_2}\frac{(w+b)^{-1}}{z-w}\, dw\, dz = p_0.$$

This implies that $p_0$ has constant rank, so we may define $E_+ = \operatorname{im} p_0$ and $E_- = \ker p_0$.

Finally,

$$[E, z+b] \cong [E_+, (z+b)|_{E_+}] \oplus [E_-, (z+b)|_{E_-}] \cong [E_+, z] \oplus [E_-, 1].$$

□

The final step is to define the inverse to $\mu$ as follows.

**Definition 6.29.**

$$\nu_n([E, u]) = [E^{\oplus(2n+1)}, L^{2n}(f_n)]_+ \otimes (\eta^{n-1} - \eta_n) \oplus E \otimes \eta^n \in K^0(X) \otimes K^0(S^2)$$

**Proposition 6.30.** $\nu_n = \nu_{n+1}$ *where both are defined, and gives a* $\nu$ *which is equal to* $\mu^{-1}$.

### 6.3 Oliver Wang: Even periodic theories

**Definition 6.31.** A generalized cohomology theory $E^*$ is an **even periodic ring theory** if

(a) $E^*(X)$ is a graded commutative ring, and the induced morphisms are morphisms of graded rings;

(b) $E^m(\mathrm{pt}) = 0$ when $m$ is odd;

(c) there is $u \in E^2(\mathrm{pt})$ and $u^{-1} \in E^{-2}(\mathrm{pt})$ such that $uu^{-1} = 1$.

The third condition says that we have a degree 2 unit in the cohomology theory, and therefore $E^m(\mathrm{pt}) \cong E^{m+2}(\mathrm{pt})$ as an abelian group. This isomorphism is given by multiplication by $u$. In fact, if $X$ is any space, then $X \to \mathrm{pt}$ gives $E^*(\mathrm{pt}) \to E^*(X)$ sending $u$ to a degree 2 unit. Therefore, $E^m(X) \cong E^{m+2}(X)$.

**Example 6.32.** Unreduced $K$-theory of complex vector bundles is an even periodic ring theory. $K^m(\mathrm{pt}) = \widetilde{K}^m(S^0) = 0$ when $m$ is odd. The element $H - 1 \in \widetilde{K}^0(S^2) = \widetilde{K}^2(S^0) = K^2(\mathrm{pt})$ is the element $u$ in degree 2.

**Example 6.33.** Another example of an even periodic ring theory is called **ordinary periodic cohomology**. Let $A$ be a commutative ring, and $X$ a finite CW-complex. Define

$$HP^*(X; A) := H^*(X; A) \otimes_A A[u, u^{-1}],$$

as the tensor product of graded rings, where $\deg(u) = 2$ and $\deg(u^{-1}) = -2$.

$$HP^n(X; A) = \bigoplus_{p+2q=n} H^p(X; A)u^q.$$

Even periodic theories behave well when evaluated on CW-complexes with even dimensional cells. We'll be interested in $\mathbb{CP}^n$ and $\mathbb{CP}^\infty$ in particular. For notation, set $E^0 := E^0(\mathrm{pt})$.

**Proposition 6.34.** *Let* $E^*$ *be an even periodic ring theory.*

*(a) If* $m$ *is odd,* $E^m(\mathbb{CP}^n) = 0$ *and* $E^0(\mathbb{CP}^n) \cong E^0[x]/\langle x^{n+1}\rangle$.

*(b) These isomorphisms can be taken such that the inclusion* $\mathbb{CP}^n \hookrightarrow \mathbb{CP}^{n+1}$ *induce a ring homomorphism*

$$E^0[x]\big/\langle x^{n+2}\rangle \to E^0[x]\big/\langle x^{n+1}\rangle$$

*given by* $x \mapsto x$.

*(c)* $E^0(\mathbb{CP}^\infty) \cong E^0[[x]]$ *and* $E^m(\mathbb{CP}^\infty) = 0$ *for* $m$ *odd.*

*Proof.* We will prove (b) $\implies$ (c).

$$\begin{aligned} E^0(\mathbb{CP}^\infty) &= [\mathbb{CP}^\infty, E_0] \\ &= [\operatorname*{colim}_n \mathbb{CP}^n, E_0] \\ &= \lim[\mathbb{CP}^n, E_0] \\ &= \lim E^0[x]\big/\langle x^{n+1}\rangle \\ &= E^0[[x]], \end{aligned}$$

where $E_0$ is the zeroth space of the spectrum $E$ representing the even periodic ring theory $E^*$. The last step is where we use part (b). □

**Proposition 6.35.** *Let* $X = \mathbb{CP}^\infty \times \ldots \times \mathbb{CP}^\infty$ *be the* $n$*-fold product of* $\mathbb{CP}^\infty$*, and* $p_i \colon X \to \mathbb{CP}^\infty$ *be the projection onto the* $i$*-th term. Then*

$$E^0(X) = E^0[[x_1, \ldots, x_n]]$$

*where* $x_i = p_i^*(x)$.

Now we move on to formal group laws.

**Definition 6.36.** Let A be a commutative ring. A **(1-dimensional, commutative) formal group law** is a power series $F \in A[[x, y]]$ such that

(a) $F(x, 0) = F(0, x) = x$,

(b) $F(x, y) = F(y, x)$,

(c) $F(F(x, y), z) = F(x, F(y, z))$.

This is kind of a weird definition, but formal group laws show up in various places. There is a formal group law associated to any elliptic curve. It also shows up in the context.

Let $p_1, p_2 \colon \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ be the two projections. Let $\mu \colon \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ be the classifying map of the bundle $p_1^*\gamma \otimes p_2^*\gamma$ where $\gamma \to \mathbb{CP}^\infty$ is the universal bundle.

**Proposition 6.37.** $\mu^*(x) = F(x_1, x_2) \in E^0[[x_1, x_2]]$ *is a formal group law.*

*Proof.* The composite

$$\mathbb{CP}^\infty \times \{pt\} \hookrightarrow \mathbb{CP}^\infty \times \mathbb{CP}^\infty \xrightarrow{\mu} \mathbb{CP}^\infty$$

is homotopic to the identity; the first map sends the bundle $\gamma \to \mathbb{CP}^\infty \times \{pt\}$ to $p_1^*(\gamma) \otimes p_2^*(\gamma)$ and the second sends this to $\gamma$.

This composite corresponds to the map on cohomology

$$\begin{array}{ccccc} E^0[[x]] & \longleftarrow & E[[x_1, x_2]] & \longleftarrow & E^0[[x]] \\ F(x,0) & \longleftarrow\!\shortmid & F(x_1, x_2) & \longleftarrow\!\shortmid & x \end{array}$$

but it is homotopic to the identity, so $F(x, 0) = x$. This demonstrates the identity condition on formal group laws.

Let $\tau\colon \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty \times \mathbb{CP}^\infty$ be the map that swaps the two coordinates. To check commutativity, consider the composite

$$\mathbb{CP}^\infty \times \mathbb{CP}^\infty \xrightarrow{\tau} \mathbb{CP}^\infty \times \mathbb{CP}^\infty \xrightarrow{\mu} \mathbb{CP}^\infty$$

which induces on cohomology

$$\begin{array}{ccccc} E^0[[x_1, x_2]] & \longleftarrow & E^0[[x_1, x_2]] & \longleftarrow & E^0[[x]] \\ & & & & \\ F(x_2, x_1) & \longleftarrow\!\shortmid & F(x_1, x_2) & \longleftarrow\!\shortmid & x \end{array}$$

The map $\mu \circ \tau$ is homotopic to just $\mu$, so $F(x_1, x_2) = F(x_2, x_1)$.

To check associativity, note that $\mu(\mathrm{id} \times \mu)$ and $\mu(\mu \times \mathrm{id})\colon \mathbb{CP}^\infty \times \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ have the same pullback bundle. □

**Definition 6.38.** Let $L \to X$ be a complex line bundle, and let $f\colon X \to \mathbb{CP}^\infty$ be the classifying map. Then the **Chern class of** $L$ is $c_1^E(L) := f^*(x) \in E^0(X)$.

If $L_1$, $L_2$ are line bundles with classifying maps $f_1, f_2$, then

$$c_1^E(L_1 \otimes L_2) = F(c_e^E(L_1), c_1^E(L_2))$$

because the classifying map of $L_1 \otimes L_2$ is

$$X \xrightarrow{f_1 \times f_2} \mathbb{CP}^\infty \times \mathbb{CP}^\infty \xrightarrow{\mu} \mathbb{CP}^\infty.$$

**Example 6.39.** What is the formal group law that comes from K-theory?

Let $t = H - 1$. Then $K^0(\mathbb{CP}^n) \cong \mathbb{Z}[t]/\langle t^{n-1} \rangle$, and $K^0(\mathbb{CP}^\infty) \cong \mathbb{Z}[[t]]$. Then we have

$$K^0(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong \mathbb{Z}[[t_1, t_2]],$$

where $t_1 = p_1^*(t) = c_1^K(p_1^*\gamma)$ and $t_2 = p_2^*(t) = c_1^K(p_1^*\gamma)$.
Then

$$\mu^*(t+1) = \mu^*(H) = [p_1^*(\gamma) \otimes p_2^*(\gamma)] = (1+t_1)(1+t_2) = 1 + t_1 + t_2 + t_1 t_2.$$

Therefore, $\mu^*(t) = t_1 + t_2 + t_1 t_2$.
This is called the **multiplicative formal group law.**

**Example 6.40.** For periodic cohomology, $HP^0(pt) \cong A$. Therefore,

$$HP^0(\mathbb{CP}^\infty; A) \cong A[[x]].$$

Recall that the Chern class $c_1\colon \operatorname{Vect}^1_{\mathbb{C}}(X) \to H^2(X;\mathbb{Z})$ sends $L_1 \otimes L_2$ to $c_1(L_1) + c_1(L_2)$. Then

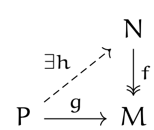
$$c_1^{HP}(L_1 \otimes L_2) = c_1^{HP}(L_1) + c_1^{HP}(L_2)$$

This means that $F(x_1, x_2) = x_1 + x_2$. This is called the **additive formal group law.**

**Remark 6.41.** We may define something called the **height** of a formal group laws, which is a nonnegative integer associated to a formal group law. The two formal group laws of the lowest heights are the multiplicative and additive formal group laws. In the filtration for chromatic homotopy theory, we may study the $n$-th filtered part by studying formal group laws of height $n$. So this is why formal group laws are interesting.

## 6.4 Shruthi Sridhar: Serre–Swan

**Definition 6.42.** An $R$-module $P$ is projective if for every surjection $f\colon N \twoheadrightarrow M$ and $g\colon P \to M$, there is a (not necessarily unique) $h\colon P \to N$ such that $fh = g$.




This definition isn't the most useful; a more useful condition is the following.

**Proposition 6.43.** *An $R$-module $P$ is free if and only if there is some $P'$ such that $P \oplus P'$ is a free module.*

**Example 6.44.** Any free module is projective. If $R$ and $S$ are rings, then $R \times 0$ and $0 \times S$ are projective modules over $R \times S$, yet neither is free.

**Question 6.45.** Can we find projective $R$-modules $P$ such that $P \oplus R^\ell \cong R^k$?

**Theorem 6.46** (Swan)**.** *Let $X$ be a compact Hausdorff space. The category of $\mathbb{R}$-vector bundles over $X$ is equivalent to the category of finitely generated projective modules over $C(X)$.*

This is often called the Serre-Swan theorem because Serre proved the corresponding algebraic fact.

Notice that if $p\colon E \to X$ is a vector bundle, then the space $\Gamma(E)$ of sections of $E$ is a module over $C(X)$; given $\alpha \in C(X)$ and $s \in \Gamma(E)$, then $(\alpha s)(x) = \alpha(x)s(x)$.

**Example 6.47.** If $E$ is a trivial bundle of rank $n$, then $\Gamma(E) = C(X)^n$.

Given $p_1\colon E_1 \to X$ and $p_2\colon E_2 \to X$, we will show that

$$\mathrm{Hom}(E_1, E_2) \cong \mathrm{Hom}_{C(X)}(\Gamma(E_1), \Gamma(E_2)).$$

To prove this, we need a few lemmas.

The following lemma doesn't require that $X$ is compact Hausdorff, only that it is normal.

**Lemma 6.48.** *Let $X$ be a normal topological space. Given any section $s$ of $E$ on $U \ni x$, there is $s' \in \Gamma(E)$ such that $s$ and $s'$ agree on some neighborhood of $x$.*

*Proof.* Use the normalcy assumption to get a smooth bump function around $x$, and multiply this by $s$. □

**Corollary 6.49.** *For all $x \in X$, and any bundle $E \to B$ of rank $n$, there are $s_1, \dots, s_n \in \Gamma(E)$ spanning $\Gamma(E; U)$ for some neighborhood $U$ of $y$.*

**Lemma 6.50.** *Given bundle maps $f, g\colon E_1 \to E_2$, if $\Gamma(f) = \Gamma(g)$, then $f = g$.*

*Proof.* Given $e \in E_1$ with $p_1(e) = x$, there is some section $s$ over $U \ni x$ such that $s(x) = e$. Then by Lemma 6.48, there is $s' \in \Gamma(E)$ such that $s'(x) = e$.

$$f(e) = f(s'(x)) = (\Gamma(f)(s))(x) = (\Gamma(g)(s'))(x) = g(e).$$

□

**Lemma 6.51.** *If $F\colon \Gamma(E_1) \to \Gamma(E_2)$ then there is a unique $f\colon E_1 \to E_2$ such that $\Gamma(f) = F$.*

*Proof sketch.* Let $I_x$ be the ideal of $C(X)$ of those functions that vanish at $x$. Then $\Gamma(E)/I_x\Gamma(E) \cong p^{-1}(x)$ via the map $s \mapsto s(x)$.

Then the map $F$ induces a map on quotients

$$F\colon {}^{\Gamma(E_1)}\!/\!_{I_x\Gamma(E_1)} \to {}^{\Gamma(E_2)}\!/\!_{I_x\Gamma(E_2)}.$$

Hence, this gives a map $p_1^{-1}(x) \to p_2^{-1}(x)$, from which we define $f\colon E_1 \to E_2$. Then we must check that this $f$ is continuous, and $F = \Gamma(f)$. □

**Corollary 6.52.** $E_1 \cong E_2$ *if and only if* $\Gamma(E_1) \cong \Gamma(E_2)$.

*Proof.* Injectivity is Lemma 6.48, and surjectivity is Lemma 6.51. ☐

To prove Theorem 6.46, it remains to show that the modules $\Gamma(E)$ are projective. To do so, we will show the following.

**Lemma 6.53.** *If* $X$ *is compact Hausdorff, then for any rank* $k$ *bundle* $E \to X$, *there is another bundle* $E' \to X$ *such that* $E \oplus E' = \varepsilon^n$. *Then*

$$\Gamma(E) \oplus \Gamma(E') \cong C(x)^n$$

*Proof.* For all $x \in X$, choose a local basis $s_{x,1}, \dots, s_{x,k}$ of $E|_{U_x}$ for a neighborhood $U_x$ of $x$. By compactness, we may find finitely many $s_1, \dots, s_n$ spanning the fiber of $E$ over $x$, for all $x$. Note that this is not a basis, because $n > k$ and and the rank of $E$ is $k$. Then define

$$\Gamma(\varepsilon^n) = C(X)^n \to \Gamma(E)$$

by $e_i \mapsto s_i$. This induces $X \times \mathbb{R}^n \to E$, and therefore $E \oplus E' = \varepsilon^n$. ☐

**Theorem 6.54.** *For all finitely generated projective modules over* $C(X)$, *there is a bundle* $p\colon E \to X$ *such that* $P \cong \Gamma(E)$.

*Proof.* $P$ is a finitely generated projective module, there is a finitely generated free module $F$ and a projection $g\colon F \to F$ such that $g^2 = g$ and $P = \operatorname{im}(g)$.

In our case, $F = C(X)^n \cong \operatorname{im}(g) \oplus \ker(g)$ gives $g\colon \Gamma(\varepsilon^n) \to \Gamma(\varepsilon^n)$. This induces $f\colon \varepsilon^n \to \varepsilon^n$ with $\operatorname{im}(f) = E$. Then $\operatorname{im}(f)$ is a subbundle if and only if the dimension of the fiber of $\operatorname{im}(f)$ over any point $x$ is locally constant. ☐

This lemma concludes the proof of Swan's theorem.

**Example 6.55** (Non-example)**.** An example when the image of $f\colon \varepsilon^n \to \varepsilon^n$ is not a subbundle. If $X = [0,1]$, and $E = X \times \mathbb{R}$, then $f\colon E \to E$ given by $(x, y) \mapsto (x, xy)$ is not locally constant, and hence not a bundle.

An application of Swan's theorem is the following: we can find stably free projective $C(X)$-modules $P$ such that $P \oplus C(X)^\ell \cong C(X)^k$. Let $\tau^n$ be the tangent bundle of $S^n$, with $\tau^n \oplus \gamma^1 = \varepsilon^{n+1}$. Then $\Gamma(\tau^n) \oplus C(X) \cong C(X)^{n+1}$ when $n \neq 0, 1, 3, 7$. But $\Gamma(\tau^n)$ is not free itself, so this answers question 6.45.

## 6.5 Elise McMahon: Equivariant K-theory I

Let $G$ be a compact topological group throughout today.

**Definition 6.56.** A $G$-**space** $X$ is a topological space with a group action, i.e. $G \times X \xrightarrow{\mu} X$ such that $g \cdot (g' \cdot x) = (gg') \cdot x$ and $e \cdot x = x$.

**Definition 6.57.** A G-**bundle** is a map of G-spaces $p: E \to X$, where $E$ and $X$ are G-spaces and $p$ is a G-map: $p(g \cdot x) = g \cdot p(x)$, such that

- $p$ is a complex vector bundle on $X$
- for all $g \in G$ and all $x \in X$, the group action $g: E_x \to E_{gx}$ is a homomorphism of vector spaces.

**Example 6.58.** Let $M$ be a G-module. Then $M \times X \to X$ is a G-bundle. This is an example of a trivial bundle.

**Example 6.59.** Let $E \to X$ be any vector bundle. Then $E \otimes E \otimes \cdots \otimes E$ becomes an G-bundle over $X$ for the symmetric group $S_k$, where $X$ has trivial action of the symmetric group.

**Definition 6.60.** $K_G(X)$ is the associated abelian group to the semi-group of G-bundles on $X$.

**Remark 6.61.**

1. Elements of $K_G(X)$ are formal differences of G-bundles $E_0 - E_1$, modulo the equivalence relation $E_0 - E_1 \sim E_0' - E_1'$ if there is a G-bundle $F$ such that $E_0 \oplus E_1 \oplus F \cong E_0' \oplus E_0' \oplus F$.
2. $K_G(X)$ forms a commutative ring under tensor product of bundles.
3. $K_G(-)$ is a contravariant functor from compact G-spaces to commutative rings.

**Example 6.62.** If G is trivial, then a G-bundle is an ordinary vector bundle, and $K_G(X) = K(X)$.

**Lemma 6.63.** *If $G$ acts on $X$ freely, then the projection $\mathrm{pr}: X \to X /\!/ G$ induces an isomorphism $K_0(X /\!/ G) \to K_G(X)$.*

*Proof Sketch.* If G acts on X freely, and $E \to X$ is a G-bundle, then $E /\!/ G \to X /\!/ G$ is a vector bundle. □

**Definition 6.64.** $R(G)$ is the free abelian group generated by isomorphism classes of representations of G, modulo the relation $[W] + [V] \sim [W \oplus V]$.

**Fact 6.65.** $K_G(\mathrm{pt}) = R(G)$*; any* $G$*-bundle over a point is a* $G$*-module.*

The theorem we aim to prove is the following.

**Theorem 6.66.** $K^0(BG)$ *is isomorphic to the representation ring of* $G$.

The idea of the proof is to use the following map. For any space $Y$, let $Y_G = Y \times E_G /\!/ G$. Then we have a map

$$K_G(X) \xrightarrow{\alpha} K(X_G)$$

$$[F] \mapsto [F_G].$$

To prove the theorem, we will prove something more general. For this, we need to introduce pro-objects.

**Definition 6.67.** Let **C** be a category and let $S$ be a directed set. Then Pro(**C**) is the category whose objects are inverse systems $\{A_\alpha\}_{\alpha\in S}$ of objects of **C**.

A morphism of Pro(**C**) $\{A_\alpha\}_{\alpha\in S} \to \{B_\beta\}_{\beta\in T}$ is $(\theta, f_\beta)$ where $\theta\colon T \to S$ and $f_\beta\colon A_{\theta_\beta} \to B_\beta$ is a morphism of **C**, such that

- if $\beta \subseteq \beta'$ in $T$, then for some $\alpha \in S$ with $\alpha \geq \theta\beta, \theta\beta'$, then the following commutes

$$\begin{array}{ccccc} & & A_{\theta\beta} & \xrightarrow{f_\beta} & B_\beta \\ & \nearrow & & & \uparrow \\ A_\alpha & & & & \\ & \searrow & & & \\ & & A_{\theta\beta'} & \xrightarrow{f'_\beta} & B_{\beta'} \end{array}$$

- $(\theta, f_\beta) \sim (\theta', f_{\beta'})$ if for all $\beta$, there is some $\alpha \in S$ such that $\alpha \geq \theta\beta, \theta\beta'$ and the following commutes.

$$\begin{array}{ccc} A_\alpha & \longrightarrow & A_{\theta\beta} \\ \downarrow & & \downarrow f_\beta \\ A_{\theta'\beta'} & \xrightarrow{f_{\beta'}} & B_\beta \end{array}$$

The motivation is that topological groups correspond to pro-groups under the map $A \mapsto A/I_\alpha$, where $\{I_\alpha\}$ is the family of open subgroups of $A$. The inverse functor is $\{A_\alpha\} \mapsto \lim_\alpha A_\alpha$.

**Definition 6.68.** $E_G = \lim_n E_G^n$, where $E_G^n = G * \cdots * G$ is the topological join of $n$ copies of $G$.

**Definition 6.69.** $B_G^n = E_G^n /\!/ G$ is the union of $n$ contractible open subsets.

Now, $\alpha_n\colon E_G^n \to \mathrm{pt}$ induces a map

$$K_G^*(\mathrm{pt}) \xrightarrow{\alpha_n} K_G^*(E_G^n) \xrightarrow{\xi} \mathbb{Z}.$$

Recall that $K_G^*(pt) = R(G)$ and $K_G^*(E_G^n) = K^*(B_G^n)$, because G acts on $E_G^n$ freely.

This $\alpha_n$ factorizes as follows.

$$\begin{array}{ccc} R(G) & \xrightarrow{\alpha_n} & K_G^*(E_G^n) \\ & \searrow \quad \nearrow & \\ & R(G)/\!/I_G^n & \end{array}$$

where $I_G$ is the kernel of the composite of the previous two maps.

It is not hard to show that $\alpha_n$ is natural, so we have more generally, that $X \times E_G^n \xrightarrow{pr} X$ induces $\alpha_n$, where

$$\begin{array}{ccc} K_G^*(X) & \xrightarrow{\alpha_n} & K_G^*(X \times E_G^n) \\ & \searrow \quad \nearrow_{\alpha_n} & \\ & K_G^*(X)/\!/I_G^n \cdot K_G^*(X) & \end{array}$$

**Definition 6.70.** If R is a commutative ring and I an ideal of R, and M is an R-module, then M can be given the **I-adic topology** defined by taking the basis of a neighborhood of zero to be submodules of $I^n \cdot M$.

**Definition 6.71.** The **Hausdorff completion** of M with respect to the I-adic topology is $\widehat{M} := \lim_n (M/I^n M)$. If the name of the module is too long to cover with a hat, we write $M^\wedge := \widehat{M}$.

In particular, $K_G^*(X)$ is an $R(G) = K_G^*(pt)$-module. So we have the following theorem.

**Theorem 6.72.** *If $K_G^*(X)$ is finitely generated as an $R(G)$-algebra, then $\alpha_n \colon K_G^*(X)/I_G^n \cdot K_G^*(X) \to K_G^*(X \times E_G^n)$ induces an isomorphism of pro-rings.*

This means that for all $n$, there is some $k$ and $\beta \colon K_G^*(X \times E_G^{n+k}) \to K_G^*(X)/I_G^n K_G^*(X)$ and the following diagram commutes.

$$\begin{array}{ccc} K_G^*(X)/I_G^{n+k} \cdot K_G^*(X) & \xrightarrow{\alpha_{n+k}} & K_G^*(X \times E_G^{n+k}) \\ \downarrow & \swarrow_{\beta_n} & \downarrow \\ K_G^*(X)/I_G^n \cdot K_G^*(X) & \xrightarrow{\alpha_n} & K_G^*(X \times E_G^n) \end{array}$$

**Corollary 6.73.** $K_G^*(X)^\wedge \xrightarrow{\cong} \lim_n K_G^*(X \times E_G^n)$ *as rings.*

So in particular, if X is a point, then $K_G^*(X)^\wedge = R(G)^\wedge$ and

$$\lim_n K_G^*(E_G^n) = K_G^*(E_G) = K^*(BG).$$

**Corollary 6.74.** *If* $G$ *is finite,* $R(G) = R(G)^{\wedge}$ *and* $R(G) \cong K^*(BG)$.

We will outline a proof of Theorem 6.72 in the case when $G = \mathbb{T}$ is the circle group.

**Lemma 6.75.** *Let* $\mathbb{T}$ *denote the circle group, and let* $G$ *be a compact Lie group, so* $K_G^*(X)$ *is a finitely generated* $R(G)$*-algebra. Let* $\theta\colon G \to T$ *be a homomorphism such that* $\alpha$ *acts on* $E_T$*. Then the homomorphism*

$$\alpha_n\colon K_G^*(X)/I_{\mathbb{T}}^n \cdot K_G^*(X) \to K_G^*(X \times E_G^n)$$

*induced by the projection* $X \times E_{\mathbb{T}}^n \to X$ *is an isomorphism of pro-rings.*

*Proof sketch.* Identify $E_{\mathbb{T}}^n = T * \cdots * T$ with $S^{2n-1}$ inside $\mathbb{C}^n$ on which $\mathbb{T}$ acts as a subgroup of the multiplicative group.

There is a short exact sequence

$$0 \to K_G^*(X \times D^{2n}, X \times S^{2n-1}) \xrightarrow{\psi} K_G(X \times D^{2n}) \to K_G^*(X \times S^{2n-1}) \to 0$$

where $\psi$ is multiplication by

$$\lambda_{-1}[\mathbb{C}^n] = \sum_{i=1}^{n} (-1)^i \lambda^i [\mathbb{C}^n] = (1-\rho)^n$$

where $1-\rho$ is the Thom class and $\rho$ is the standard 1-dimensional representation of $\mathbb{T}$.

Letting $\zeta = 1-\rho$, $K = K_G^*(X)$ and ${}_{\zeta^n}K = \{x \in K \mid \zeta^n x = 0\}$, we have another exact sequence

$$0 \to K/\zeta^n \cdot K \xrightarrow{\alpha_n} K_G^*(X \times S^{2n-1}) \to {}_{\zeta^n}K \to 0$$

$\zeta$ generates the augmentation ideal $I_{\mathbb{T}}$, so $K_G^*(X)$ is a finitely generated module over $R(G)$, and $R(G)$ is a Noetherian ring, so there is some $k$ such that $\zeta^k = \zeta^{k+1} = \zeta^{k+2} = \cdots$. So in the following diagram, we can see that the last vertical map is zero.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K/\zeta^{n+k} \cdot K & \xrightarrow{\alpha_{n+k}} & K_G^*(X \times S^{2n-1}) & \longrightarrow & {}_{\zeta^{n+k}}K & \longrightarrow & 0 \\
 & & \downarrow & \overset{\beta_n}{\swarrow} & \downarrow & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & K/\zeta^n \cdot K & \xrightarrow{\alpha_n} & K_G^*(X \times S^{2n-1}) & \longrightarrow & {}_{\zeta^n}K & \longrightarrow & 0
\end{array}$$

Hence, the composite $gf$ is zero, and so we can define $\beta_n$ as in the diagram. □

## 6.6 Brandon Shapiro: Equivariant K-theory II

This section will discuss operations arising from equivariant K-theory. The references here are Segal's *Equivaraint K-theory* and Atiyah's *Power Operations in K-theory*.

Let G be a compact topological group.

**Definition 6.76.** A G-vector bundle on a G-space $X$ is a vector bundle $p\colon E \to X$ such that $E$ is a G-space, $p$ is a G-map, and G acts on $E$ by maps restricting to linear maps on each fiber.

**Example 6.77.** For any G-module $M$, there is a G-bundle $\underline{M} := X \times M$. This is called a **trivial** $G$**-bundle**.

**Example 6.78.** Let $S_k$ be the symmetric group on $k$ letters. Assume that $X$ has trivial $S_k$-action. For any vector bundle $E \to X$, $E^{\otimes k}$ is an $S_k$-vector bundle.

**Definition 6.79.** A **morphism of** $G$**-vector bundles** $f\colon E \to E'$ is a G-map that restricts to G-linear maps of fibers $f_x\colon E_x \to E'_x$.

**Lemma 6.80.**

*(a) The image of a morphism of* $G$*-vector bundles is a sub-bundle.*

*(b) A morphism of* $G$*-vector bundles is an isomorphism if it is an isomorphism on each fiber.*

**Example 6.81.** Given any two G-vector bundles $E$ and $F$, there is a G-vector bundle $\mathrm{Hom}(E, F)$ with fibers

$$\mathrm{Hom}(E, F)_x = \mathrm{Hom}(E_x, F_x).$$

The G-action on this bundle is $(g \cdot \phi)(h) = (g \cdot \phi)(g^{-1}h)$.

If G is finite, then

$$\frac{1}{|G|}\sum_{g\in G} g\colon \mathrm{Hom}(E, F) \to \mathrm{Hom}(E, F)$$

defines a morphism of G-vector bundles. The image is $\mathrm{Hom}_G(E, F)$, the subbundle whose fibers are G-invariant maps $E_x \to F_x$.

In fact, since the maps in $\mathrm{Hom}_G(E, F)$ are G-equivariant, the action of G is trivial on this bundle. Hence, $\mathrm{Hom}_G(E, F) \to X$ is an ordinary vector bundle.

**Definition 6.82.** The isomorphism classes of G-vector bundles form a semigroup under direct sum; $K_G(X)$ is the group completion of this semigroup.

**Example 6.83.** $K_G(pt) = R(G)$. The pullback along $X \to pt$ gives a natural map $R(G) \to K_G(X)$ via $[M] \mapsto [\underline{M}]$.

Any vector bundle on $X$ can be given the trivial $G$-action, so this defines a homomorphism $K(X) \to K_G(X)$.

**Proposition 6.84.** *If $G$ acts on $X$ trivially, then $\mu\colon R(G) \otimes K(X) \to K_G(X)$ given by*

$$[M] \otimes [E] \mapsto [\underline{M} \otimes E]$$

*is a ring isomorphism.*

*Proof sketch.* Let $\{M_i\}_{i \in I}$ be the simple $G$-modules. Define $\nu\colon K_G(X) \to R(G) \otimes K(X)$ by

$$[E] \mapsto \sum_{i \in I} [M_i] \otimes [\mathrm{Hom}_G(\underline{M}_i, E)] .$$

This is an inverse to $\mu$. □

**Lemma 6.85.** *$E \mapsto E^{\otimes k}$ induces a natural function $K(X) \to K_{S_k}(X)$.*

**Remark 6.86.** This is not trivial! It is hard to see that this is well-defined. Moreover, this is *not* additive, although it is multiplicative.

Given any $\alpha \in R_k' := \mathrm{Hom}(R(S_k), \mathbb{Z})$, we may define $\widehat{\alpha}\colon K(X) \to K(X)$ as the composite

$$\widehat{\alpha}\colon K(X) \xrightarrow{(-)^{\otimes k}} K_{S_k}(X) \xrightarrow{\nu} R(S_k) \otimes K(X) \xrightarrow{\alpha \otimes 1} \mathbb{Z} \otimes K(X) \cong K(X).$$

For any vector bundle $E \to X$, this is given by

$$[E] \mapsto \sum_{i \in I} \alpha([M_i]) \left[\mathrm{Hom}_{S_k}(\underline{M}_i, E^{\otimes k})\right].$$

**Example 6.87.** Let $M$ be the trivial representation of $S_k$, and define $\sigma^k \in \mathrm{Hom}(R(S_k), \mathbb{Z})$ by defining it on the basis of simple modules

$$\sigma^k(N) = \begin{cases} 1 & N \cong M \\ 0 & N \text{ is any other simple } S_k\text{-module.} \end{cases}$$

Then we have

$$\widehat{\sigma}^k([E]) = \left[\mathrm{Hom}_{S_k}(\underline{M}, E^{\otimes k})\right] = [\mathrm{Sym}^k(E)].$$

**Example 6.88.** Now let $M$ be the alternating representation of $S_k$. Then if $\lambda^k \in \mathrm{Hom}(R(S_k), \mathbb{Z})$ is defined on the basis of simple modules by

$$\lambda^k(N) = \begin{cases} 1 & N \cong M \\ 0 & N \text{ is any other simple } S_k\text{-module,} \end{cases}$$

we have

$$\widehat{\lambda}^k([E]) = \left[\textstyle\bigwedge^k(E)\right].$$

**Definition 6.89.** For any $S_k$-module $M$, define $\Pi_M(V) := \mathrm{Hom}_{S_k}(M, V^{\otimes k})$ for any vector space $V$. This defines a functor $\Pi_M(-)$ on vector spaces.

let $T_n$ be a diagonal matrix $T_n = \mathrm{diag}(t_1, \ldots, t_n)$, considered as a linear transformation $\mathbb{C}^n \to \mathbb{C}^n$.

**Proposition 6.90.** $\mathrm{tr}(\Pi_M(T_n))$ *is a symmetric function in* $t_1, \ldots, t_m$.

This gives a map $\Delta_{n,k} \colon \mathrm{Hom}(R(S_k), \mathbb{Z}) \to \mathrm{Sym}[t_1, \ldots, t_n]$ by

$$\alpha \mapsto \sum_{i \in I} \alpha([M_i]) \,\mathrm{tr}(\Pi_{M_i}(T_n)).$$

We may extend this to a function

$$\Delta \colon \sum_{k=1}^{\infty} \mathrm{Hom}(R(S_k), \mathbb{Z}) \to \mathrm{Sym}$$

by $\Delta(\lambda^k) = e^k$.

**Theorem 6.91.** $\Delta$ *is a ring isomorphism.*

We may use this to define the Adams operations by evaluating the Newton polynomials on $\lambda^1, \ldots, \lambda^k$.

$$\psi^k := Q_k(\lambda^1, \ldots, \lambda^k)$$

By definition, $\Delta(\psi^k)$ is the $k$-th power sum.

## 6.7 David Mehrle: KR-theory

Any real number is a fixed point of complex conjugation on $\mathbb{C}$. This seemingly innocuous statement has many interesting generalizations to vector bundles and K-theory.

Let $X$ be a topological space. Let $E \to X$ be a $\mathbb{C}$-vector bundle on $X$. Since each fiber of $E$ is a $\mathbb{C}$-vector space, we may define a conjugation on $E$ fiberwise; the fixed points of this conjugation define a new $\mathbb{R}$-vector bundle $E_{\mathbb{R}} \to X$.

Conversely, given any $\mathbb{R}$-vector bundle $F \to X$, we may define a $\mathbb{C}$-vector bundle $F \otimes_{\mathbb{R}} \underline{\mathbb{C}} \to X$, where $\underline{\mathbb{C}}$ represents the trivial $\mathbb{R}$-vector bundle $\mathbb{C} \times X$ of rank 2.

This suggests that we should study not only vector bundles, but vector bundles with involution over $\mathbb{Z}/2$-spaces. Such an object would generalize both real vector bundles and complex vector bundles.

Notice that any $\mathbb{Z}/2$-space $X$ is just a topological space $X$ with an action of $\mathbb{Z}/2$ given by sending the generator to some homeomorphism $\tau \colon X \to X$ such that $\tau^2 = \mathrm{id}_X$. In the following, $\tau$ will always denote the action of the generator of $\mathbb{Z}/2$ on a $\mathbb{Z}/2$-space; if there are multiple $\mathbb{Z}/2$-spaces in question, all actions will be written as a homeomorphism $\tau$ unless it is unclear from context.

**Example 6.92.** An important example of $\mathbb{Z}/2$-spaces are spheres. There are many different actions of $\mathbb{Z}/2$ on these spaces. Let $\mathbb{R}^{p,q}$ be $\mathbb{R}^{p+q}$ with $\mathbb{Z}/2$-action

$$(x_1, \dots, x_p, y_1, \dots, y_q) \mapsto (x_1, \dots, x_p, -y_1, \dots, -y_q).$$

Let $S^{p,q}$ be the quotient of the unit disc in this space by the unit sphere in this space:

$$S^{p,q} := D(\mathbb{R}^{p,q})/S(\mathbb{R}^{p,q}),$$

with inherited $\mathbb{Z}/2$-action. Note that $S^{p,q}$ is topologically the $(p+q)$-sphere with a specified action of $\mathbb{Z}/2$.

In particular, $S^{1,0}$ is the circle with trivial $\mathbb{Z}/2$-action, and $S^{0,1}$ is the circle with $\mathbb{Z}/2$-action by reflection.

**Definition 6.93.** A **vector bundle with involution** over $(X, \tau)$ is a $\mathbb{Z}/2$-space $E$ and a map $p\colon E \to X$ such that

(a) $p\colon E \to X$ is a complex vector bundle;

(b) the projection $p\colon E \to X$ commutes with the $\mathbb{Z}/2$-action:

$$\begin{array}{ccc} E & \xrightarrow{\tau} & E \\ \downarrow p & & \downarrow p \\ X & \xrightarrow{\tau} & X; \end{array}$$

(c) the map $E_x \to E_{\tau(x)}$ is anti-linear: for any $\vec{v} \in E_x$ and $z \in \mathbb{C}$, we have $\tau(z\vec{v}) = \bar{z}\tau(\vec{v})$.

$$\begin{array}{ccc} \mathbb{C} \times E_x & \longrightarrow & E_x \\ \downarrow \tau & & \downarrow \tau \\ \mathbb{C} \times E_{\tau(x)} & \longrightarrow & E_{\tau(x)} \end{array}$$

**Definition 6.94.** A **morphism of vector bundles with involution** $\phi\colon E \to F$ is a morphism $\phi$ of complex vector bundles that commutes with the involutions: $\phi(\tau(\vec{v})) = \tau(\phi(\vec{v}))$ for any $\vec{v} \in E$.

**Remark 6.95.**

(a) Although this looks almost like it, this is *not* a $\mathbb{Z}/2$-vector bundle. For a $\mathbb{Z}/2$-vector bundle, the map $E_x \to E_{\tau(x)}$ is assumed to be $\mathbb{C}$-linear, not antilinear.

(b) This is not standard terminology. Atiyah calls these "real spaces" and "real vector bundles" but this is a confusing term and we will avoid it. His terminology is created by analogy with algebraic geometry; if X is the set of complex points of a real algebraic variety, it has an involution given by complex conjugation whose fixed points are the real points of the variety $X$.

**Example 6.96.** Complex projective space $\mathbb{CP}^n$ has an action of $\mathbb{Z}/2$ given by conjugation

$$[z_0, z_1, \ldots, z_n] \mapsto [\bar{z}_0, \bar{z}_1, \ldots, \bar{z}_n],$$

which is well-defined because conjugation is anti-linear. The tautological line bundle $H$

$$H := \left\{ (\vec{v}, \ell) \;\middle|\; \ell \in \mathbb{CP}^n, \vec{v} \in \ell \right\}$$

is a bundle with involution, with involution given by conjugating both the vector and the line.

In fact, the universal bundle $\gamma_n \to \mathrm{Gr}_n(\mathbb{C}^\infty)$ is again a bundle with involution, with the same conjugation action.

Just as we had $KO$-theory for $\mathbb{R}$-vector bundles and $KU$-theory for $\mathbb{C}$-vector bundles, there is a $K$-theory for vector bundles with involution over $\mathbb{Z}/2$-spaces. Recall that for a compact connected space $X$, $KU^0(X)$ is defined as $[X_+, \mathbb{Z} \times BU]$. We want to define something similar for vector bundles with involution.

From the previous example, $\mathrm{Gr}_n(\mathbb{C}^\infty)$ has an action of $\mathbb{C}^\infty$ by conjugation.

**Definition 6.97.** If $X$ and $Y$ are $\mathbb{Z}/2$-spaces, and $f, g \colon X \to Y$ are $\mathbb{Z}/2$-equivariant, a $\mathbb{Z}/2$**-homotopy** between $f$ and $g$ is a $\mathbb{Z}/2$-equivariant map $H \colon X \times I \to Y$ (where $I$ has trivial $\mathbb{Z}/2$-action) such that $H|_{X \times \{0\}} = f$ and $H|_{X \times \{1\}} = g$.

The set of $\mathbb{Z}/2$-homotopy classes of $\mathbb{Z}/2$-maps is written $[X, Y]_{\mathbb{Z}/2}$.

**Proposition 6.98** (Classification of Bundles with Involution)**.** *Let $X$ be a $\mathbb{Z}/2$-space. There is a bijection between rank $n$ vector bundles with involution on $X$ and $[X, \mathrm{Gr}_n(\mathbb{C}^\infty)]_{\mathbb{Z}/2}$.*

Recall that $BU(n) \simeq \mathrm{Gr}_n(\mathbb{C}^\infty)$. The $\mathbb{Z}/2$-action on $\mathrm{Gr}_n(\mathbb{C}^\infty)$ gives an action on $BU(n)$ by conjugation such that the inclusion $BU(n) \hookrightarrow BU(n+1)$ is $\mathbb{Z}/2$-equivariant. This in turn gives a $\mathbb{Z}/2$ action on the colimit

$$BU = \operatorname*{colim}_n BU(n).$$

We use this action to define the $K$-theory of bundles with involution, called $KR$-theory.

**Definition 6.99.** For a compact, connected $\mathbb{Z}/2$-space $X$,

$$KR^{0,0}(X) := [X_+, \mathbb{Z} \times BU]_{\mathbb{Z}/2}.$$

(The indices will be explained momentarily).

If you prefer, there is also a more concrete definition of $\mathrm{KR}^{0,0}(X)$.

**Proposition 6.100.** $\mathrm{KR}^{0,0}(X) = [X_+, \mathbb{Z} \times \mathrm{BU}]_{\mathbb{Z}/2}$ *is isomorphic to the free abelian group generated by isomorphism classes of vector bundles with involution over* $X$*, subject to the relation* $[E] + [F] = [E \oplus F]$.

KR-theory interpolates between the K-theory of $\mathbb{R}$-vector bundles $\mathrm{KO}(X)$ and the K-theory of $\mathbb{C}$-vector bundles $\mathrm{KU}(X)$.

Notice that any vector bundle with involution is already a complex vector bundle, so by forgetting the involution we obtain a homomorphism

$$\mathrm{KR}^{0,0}(X) \to \mathrm{KU}^0(X)$$

sending the class of the bundle with involution $E$ to the class of the underlying $\mathbb{C}$-vector bundle $E$. Another description of this is via classifying maps: a homotopy class $[f] \in [X_+, \mathbb{Z} \times \mathrm{BU}]_{\mathbb{Z}/2}$ defines a class $[f] \in [X_+, \mathbb{Z} \times \mathrm{BU}]$ simply by forgetting the $\mathbb{Z}/2$-equivariance.

There's another way to relate KR and KU too: if $X$ is any space, let $E \to X \times \{\pm 1\}$ be a vector bundle with involution over $X \times \{\pm 1\} \cong X \sqcup X$ with the swap action. Notice that in this scenario, $E$ is uniquely determined by its restriction to $X \times \{+1\}$. Therefore, we have the isomorphism:

**Proposition 6.101.** $\mathrm{KR}^{0,0}(X \times \{\pm 1\}) \cong \mathrm{KU}^0(X)$.

On the other hand, given a bundle with involution $E \to X$, we may take the $\mathbb{Z}/2$-fixed points $X^{\mathbb{Z}/2}$ of $X$ and then restrict the bundle to these. This gives a homomorphism

$$\mathrm{KR}^{0,0}(X) \to \mathrm{KR}^{0,0}(X^{\mathbb{Z}/2}).$$

This next lemma shows that $\mathrm{KR}^{0,0}(X^{\mathbb{Z}/2}) \cong \mathrm{KO}^0(X^{\mathbb{Z}/2})$, so restriction to the fixed points defines a homomorphism $\mathrm{KR}^{0,0}(X) \to \mathrm{KO}^0(X^{\mathbb{Z}/2})$.

**Proposition 6.102.** *If* $X$ *is has trivial* $\mathbb{Z}/2$*-action, then* $\mathrm{KR}(X) \cong \mathrm{KO}(X)$.

*Proof sketch 1.* We prove a stronger result: there is an equivalence of categories between the category of $\mathbb{R}$-vector bundles on $X$ and vector bundles with involution over $X$. The pseudo-inverse functors in this equivalence are defined on objects as follows.

For an $\mathbb{R}$-vector bundle $E \to X$,

$$E \mapsto E \otimes_{\mathbb{R}} \underline{\mathbb{C}},$$

where $\underline{\mathbb{C}}$ is the trivial 2-dimensional $\mathbb{R}$-vector bundle $\mathbb{C} \times X$.

For a vector bundle with involution $F \to X$,

$$F \mapsto F^{\mathbb{Z}/2}$$

where $F^{\mathbb{Z}/2}$ is the set of $\mathbb{Z}/2$-fixed points of $F$. □

*Proof sketch 2.* Any fixed point of $\mathrm{Gr}_n(\mathbb{C}^\infty)$ under conjugation is an element of $\mathrm{Gr}_n(\mathbb{R}^\infty)$. Therefore, restricting $f\colon X_+ \to \mathrm{Gr}_n(\mathbb{C}^\infty)$ to the fixed points of $X$ gives a new map $f\colon X_+^{\mathbb{Z}/2} \to \mathrm{Gr}_n(\mathbb{R}^\infty)$, which defines an element of $KO^0(X) := [X_+, \mathbb{Z} \times BO]$. □

**Example 6.103.** If $X = \mathrm{pt}$ is just a point with trivial $\mathbb{Z}/2$-action, then $KR^{0,0}(\mathrm{pt}) = KO^0(\mathrm{pt}) = \mathbb{Z}$ by the previous proposition.

So why is this called $KR^{0,0}(X)$? What is $KR^{p,q}(X)$? Recall that K-theory of $\mathbb{C}$-vector bundles and $\mathbb{R}$-vector bundles are defined from the spectra

$$KU = \mathbb{Z} \times BU, U, \mathbb{Z} \times BU, U, \ldots$$
$$KO = \mathbb{Z} \times BO, {}^{U}/_{O}, {}^{Sp}/_{U}, Sp, \mathbb{Z} \times B\,Sp, {}^{U}/_{Sp}, {}^{O}/_{U}, O, \mathbb{Z} \times BO, \ldots$$

as the associated cohomology theories:

$$KU^n(X) := [X_+, KU_n],$$
$$KO^n(X) := [X_+, KO_n].$$

This begs the question: does $KR^{0,0}$ come from a spectrum as well?

Unlike $KU$ or $KO$, $KR$ is a $\mathbb{Z}/2$-equivariant spectrum, which is a different type of object entirely. Recall that a spectrum is a sequence of spaces $E_0, E_1, E_2, \ldots$ together with maps $S^1 \wedge E_n \to E_{n+1}$. To define a $\mathbb{Z}/2$-spectrum, however, note that there are two actions of $\mathbb{Z}/2$ on the 1-sphere, which we denoted by $S^{1,0}$ and $S^{0,1}$. We must consider suspensions with respect to both.

**Definition 6.104.** A **$\mathbb{Z}/2$-spectrum** $E$ is a collection of $\mathbb{Z}/2$-spaces $E_{p,q}$ for all $p, q \in \mathbb{N}$, together with $\mathbb{Z}/2$-equivariant maps

$$S^{1,0} \wedge E_{p,q} \to E_{p+1,q}$$
$$S^{0,1} \wedge E_{p,q} \to E_{q,p+1}$$

**Remark 6.105.** Smashing with $S^{p,q}$ has a right adjoint $\Omega^{p,q}$ defined by $\Omega^{p,q}(X) = \mathrm{Map}_*(S^{p,q}, X)$. This is analogous to the loop-space functor.

**Definition 6.106.** The $\mathbb{Z}/2$-spectrum $KR$ is the spectrum with spaces

$$KR_{p,q} = \begin{cases} U & p+q \text{ odd} \\ \mathbb{Z} \times BU & p+q \text{ even,} \end{cases}$$

each equipped with the $\mathbb{Z}/2$-action given by conjugation. The maps

$$S^{1,0} \wedge KR_{p,q} \to KR_{p+1,q}$$
$$S^{0,1} \wedge KR_{p,q} \to KR_{q,p+1}$$

are given alternatively by the Bott map and the identity.

With these spectra, we may similarly compare $KR$, $KU$, and $KO$. For any $\mathbb{Z}/2$-spectrum $E$, there are two ways to obtain an ordinary spectrum: we may either forget all of the nontrivial $\mathbb{Z}/2$-actions, or take (homotopy) fixed points.

For the first, there is a forgetful functor $U$ from $\mathbb{Z}/2$-spectra to ordinary spectra defined on objects as follows. Given a $\mathbb{Z}/2$-spectrum $E$, the spectrum $U(E)$ has spaces

$$U(E) = E_{0,0}, E_{1,0}, E_{2,0}, E_{3,0}, \ldots$$

The image of $KR$ under this functor is $KU$.

The other way to produce an ordinary spectrum out of a $\mathbb{Z}/2$-spectrum is to take (homotopy) fixed points. There is a way to make the following statement rigorous, but this is all we'll say for now.

**Theorem 6.107.** $KO$ *is the fixed point spectrum of* $KR$*.*

**Definition 6.108.** For a compact connected $\mathbb{Z}/2$-space $X$, $KR^{p,q}(X) := [X, KR_{p,q}]_{\mathbb{Z}/2}$.

Much as with ordinary K-theory, we have

$$KR^{p,q}(X) = [X_+, KR_{p,q}]_{\mathbb{Z}/2} = [X_+, \Omega^{p,q} KR_{0,0}]_{\mathbb{Z}/2} = [S^{p,q} \wedge X_+, KR_{0,0}] = KR^{0,0}(S^{p,q} \wedge X_+).$$

From this definition of the $\mathbb{Z}/2$-spectrum $KR$, it's not too hard to read off the periodicity theorem.

**Theorem 6.109** (Atiyah)**.** $KR^{p,q}(X) \cong KR^{p+1,q+1}(X)$

**Remark 6.110** (References)**.** For a good introduction to KR-theory, see Atiyah *On K-theory and Reality*. A reference for the classification of bundles with involution and their relation to Grassmannians is Edelson *Real Vector Bundles and Spaces with Free Involutions*. For equivariant spectra, see Schwede *Lectures on Equivariant Stable Homotopy Theory*.