Math 6670: Algebraic Geometry

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1 Introduction

> Algebra is the offer made by the devil to the mathematician. The devil says: 'I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvelous machine.'
>
> – Sir Michael Atiyah

Algebraic geometry is *not* just commutative algebra in disguise. We might make a spectrum of topics from topology to noncommutative algebra, with fields falling in between as follows.

Topology	Differential Geometry	Complex Geometry	Algebraic Geometry	Commutative Algebra	Noncommutative Algebra

Algebraic geometry and commutative algebra allow us to deal with singular objects, whereas differential and complex geometry deal only with smooth things. An example of something non-smooth in algebraic geometry is solutions to the equations $xy = 0$ or $y^2 = x^3$, both with singularities at the origin.

Our main reference will be Ravi Vakil's *The Rising Sea*, although we won't follow it linearly. We'll work with the 19th century version of algebraic varieties in complex affine and projective space and then explain why we want to go beyond these.

Administrative

- There is a course webpage here here.
- There will be homework if you need or want a grade. Posted online.

2 Varieties and their Dimension Theory

2.1 Algebraic subsets of $\mathbb{C}^n$ and the Nullstellensatz

Consider the following two sets and maps between them:

$$\left\{\text{subsets of } \mathbb{C}^n\right\} \underset{V}{\overset{I}{\rightleftarrows}} \left\{\text{ideals in } \mathbb{C}[x_1, \dots, x_n]\right\}.$$

where

$$X \mapsto I(X) := \left\{ p \in \mathbb{C}[x_1, \ldots, x_n] \,\middle|\, p(\vec{v}) = 0 \forall\, \vec{v} \in X \right\}$$
$$J \mapsto V(J) := \left\{ \vec{v} \in \mathbb{C}^n \,\middle|\, \forall p \in J, p(\vec{v}) = 0 \right\}$$

We have the following containments:

$$V(I(X)) \supseteq X \qquad I(V(J)) \geq J,$$

but these are *not* necessarily equal.

Definition 2.1. If $V(I(X)) = X$, then X is an **algebraic subset** of $\mathbb{C}^n$.

Example 2.2 (Non-example)**.** $\mathbb{Z} \subseteq \mathbb{C}^1$. This cannot be an algebraic subset, because any polynomial which vanishes on all of $\mathbb{Z}$ is necessarily zero.

But we can do this in complex geometry, because there is a holomorphic function which vanishes exactly on the integers, namely $\sin(\pi x)$.

Example 2.3 (Another non-example)**.** $\mathbb{R} \subseteq \mathbb{C}^1$. This cannot be an algebraic subset of $\mathbb{C}^1$, but it is also not an example from complex geometry; any analytic function that vanishes on $\mathbb{R}$ vanishes on $\mathbb{C}$.

Definition 2.4. An ideal $I \leq A$ is **radical** if $p^n \in I \implies p \in I$.

Theorem 2.5 (Nullstellensatz)**.** $I(V(J)) = I$ *if and only if* I *is a radical ideal.*

More specifically, $I(V(J)) = \sqrt{I} := \left\{ p \in \mathbb{C}[x_1, \ldots, x_n] \,\middle|\, \exists\, n \in \mathbb{N}, p^n \in J \right\}$.

Example 2.6 (Non-example)**.** Note that this theorem only works because we have an algebraically closed field $\mathbb{C}$. If we take the ideal $\langle X^2 + 1 \rangle \leq \mathbb{R}[X]$, there are no points in $\mathbb{R}$ where this polynomial vanishes, so $I(V(I)) = \mathbb{R}[X]$, yet $\langle X^2 + 1 \rangle$ is a radical ideal.

2.2 Operations on Ideals

Fact 2.7. *Let* Γ *be a set of ideals in* $\mathbb{C}[x_1, \ldots, x_n]$. *We have that*

$$V\left(\bigcap_{J \in \Gamma} J \right) \supseteq \bigcup_{J \in \Gamma} V(J)$$

But again, this is not always an equality.

Example 2.8.

$$V\left(\bigcap_{n \in \mathbb{Z}} \langle x - n \rangle \right) = V(0) = \mathbb{C} \supsetneq \bigcup_{n \in \mathbb{Z}} V(\langle x - n \rangle) = \bigcup_{n \in \mathbb{Z}} \{n\} = \mathbb{Z}$$

This example shows that the correspondence between subsets of $\mathbb{C}^n$ and ideals of $\mathbb{C}[x_1, \ldots, x_n]$ is not a lattice equality, at least not if we take union to be the lattice join on subsets of $\mathbb{C}^n$.

Fact 2.9. *Let* Γ *be finite set of ideals in* $\mathbb{C}[x_1, \ldots, x_n]$. *Then*

$$V\left(\sum_{J \in \Gamma} J\right) = \bigcap_{J \in \Gamma} V(J).$$

Theorem 2.10. $V(I \cap J) = V(I) \cup V(J)$

Proof. We already know that $V(I) \cup V(J) \subseteq V(I \cap J)$ by Fact 2.7.

Let's first show that $V(I) \cup V(J) \supseteq V(IJ)$. Take $\vec{z} \notin V(I) \cup V(J)$. We want to show that $z \notin V(IJ)$. If $\vec{z} \notin V(I) \cup V(J)$, then there are $f \in I$, $g \in J$ such that $f(\vec{z}) \neq 0$, $g(\vec{z}) \neq 0$. Hence, $fg(\vec{z}) \neq 0$ as well. Hence, $\vec{z} \notin V(IJ)$. (Here, we're sneakily using the fact that $\mathbb{C}$ doesn't have zerodivisors.)

Now we know that $V(IJ) \supseteq V(I \cap J)$, so we have $V(I \cap J) \subseteq V(I) \cup V(J)$. $\square$

Remark 2.11. Since the collection of algebraic sets is closed under finite union, arbitrary intersection, and $V(0) = \mathbb{C}^n$, and $V(1) = \emptyset$, the algebraic sets form the closed sets in a topology.

Hence, for X to be an algebraic subset of $\mathbb{C}^n$, we only need that X is in the image of $V(-)$.

Definition 2.12. This topology is called the **Zariski topology**.

Example 2.13. In $\mathbb{C}$, the Zariski-closed sets are the finite sets, and all nonempty open sets are dense.

Example 2.14. Inside $V(\langle xy \rangle) \subseteq \mathbb{C}^2$, the open set $\{y \neq 0\}$ is not dense.

Definition 2.15. Let $I, J \lhd A$ be ideals. The **colon ideal** $(I \colon J)$ is

$$(I \colon J) := \left\{ a \in A \;\middle|\; aJ \leq I \right\}.$$

These are some kind of division of ideals, as the following example shows.

Example 2.16.

$$\begin{aligned} \langle xy \rangle \colon \langle x \rangle &= \langle y \rangle \\ \langle xy \rangle \colon \langle y \rangle &= \langle x \rangle \\ \langle x^2 \rangle \colon \langle x \rangle &= \langle x \rangle \\ \langle x \rangle \colon \langle x \rangle &= \langle 1 \rangle = A \end{aligned}$$

Definition 2.17. Let $I \lhd A$ be an ideal, and let $x \in A$. The **saturation of** I **with respect to** x is the ideal

$$(I \colon \langle x^\infty \rangle) := \left\{ a \in A \;\middle|\; \exists n, ax^n \in I \right\}.$$

Theorem 2.18. *If* I *is a radical ideal, then* $I\colon J = I(V(I) \setminus V(J))$.

Proof. Let $f \in I(V(I) \setminus V(J))$, which means that f vanishes on $V(I) \setminus V(J)$. Equivalently, $fg = 0$ on $V(I)$ for all $g \in J$. Now, since I is a radical ideal, this is equivalent to $fg \in I$. By definition, $f \in I\colon J$. □

The following corollary is by definition, since the Zariski closure of X is $V(I(X))$.

Corollary 2.19. $V(I\colon J)$ *is the Zariski closure of* $V(I) \setminus V(J)$.

Example 2.20. Consider the ideal $I = \langle x, z\rangle \cap \langle y, z - x^2\rangle$. This describes the union of a parabola in the xz-plane and a line. When we chop out the plane that contains the parabola, we are left with just the line. In algebra, this is expressed as follows:

$$I\colon \langle y\rangle = \langle x, z\rangle.$$

Fact 2.21 (Commutative Algebra Fact)**.** *Let* $I \leq \mathbb{C}[x_1, \ldots, x_n]$. *Then* I *is radical if and only if* I *is the intersection of the collection* Γ *of prime ideals containing* I.

Moreover, if $Q \in \Gamma$*, then* $Q = I\colon \left(\bigcap_{P \in \Gamma \setminus \{Q\}} P\right)$.

On the geometry side, we have $V(I) = \bigcup_{P \in \Gamma} V(P)$.

Let $I \leq \mathbb{C}[x_1, \ldots, x_n]$ be an ideal. Then, using the Nullstellensatz, we can think of the vanishing set of functions in I as follows:

$$V(I) = \bigcup_{\substack{M \geq I \\ M \text{ maximal}}} V(M)$$

Fact 2.22. *The following are equivalent.*

(a) M *is maximal;*

(b) $M = \langle \{x_i - \lambda_i\}_{i=1}^n \rangle$ *for some* $\vec{\lambda} \in \mathbb{C}^n$*;*

(c) $\mathbb{C}[x_1, \ldots, x_n]/M \cong \mathbb{C}$*;*

(d) $\mathbb{C}[x_1, \ldots, x_n]/M$ *is a field.*

Definition 2.23. $\mathbb{C}$-**Alg** is the category of commutative unital rings R with a homomorphism $\mathbb{C} \to R$ sending 1 to 1.

Definition 2.24. The $\mathbb{C}$**-points** of a $\mathbb{C}$-algebra R is

$$\mathrm{Hom}_{\mathbb{C}\textbf{-Alg}}(R, \mathbb{C}).$$

Taking the $\mathbb{C}$-points is a contravariant representable functor $\mathbb{C}$-**Alg** $\to$ **Sets**.

Definition 2.25. A functor $T\colon \mathbf{C} \to \mathbf{D}$ is called **faithful** if it is injective on homsets; that is, for all $X, Y \in \mathrm{Ob}(\mathbf{C})$,

$$\mathrm{Hom}_{\mathbf{C}}(X, Y) \to \mathrm{Hom}_{\mathbf{D}}(T(X), T(Y))$$

is injective. T is called **full** if this map is surjective.

Remark 2.26. The $\mathbb{C}$-points functor $\mathrm{Hom}_{\mathbb{C}\text{-}\mathbf{Alg}}(-, \mathbb{C})$ is not full. We could change the codomain to be the category **Top** of topological spaces, but even then it isn't full. Part of the point of schemes is to find the correct target for this functor.

2.3 Subvarieties of Projective Space

Definition 2.27. **Complex projective space** is, as a set,

$$\mathbb{CP}^n := (\mathbb{C}^{n+1} \setminus \{\vec{0}\})/\mathbb{C}^\times$$

where $\mathbb{C}^\times$ acts on $\mathbb{C}^{n+1}$ by scaling.

A point in $\mathbb{CP}^n$ is written as an equivalence class

$$[z_0, \dots, z_n] = [\lambda z_0, \dots, \lambda z_n]$$

for any $\lambda \in \mathbb{C}$.

Remark 2.28. We may decompose projective space as

$$\begin{aligned}\mathbb{CP}^n &= \left\{[1, z_1, \dots, z_n]\right\} \sqcup \left\{[0, 1, z_2, \dots, z_n]\right\} \sqcup \dots \sqcup \left\{[0, \dots, 0, 1]\right\} \\ &= \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \dots \sqcup \mathbb{C}^0\end{aligned}$$

This of course shows that $\mathbb{CP}^n = \mathbb{C} \sqcup \mathbb{CP}^{n-1}$.

This is often clunky. Another useful decomposition is as an open cover by sets $U_i = \left\{[z_0, \dots, z_n] \in \mathbb{CP}^n \;\middle|\; z_i \neq 0\right\}$

$$\mathbb{CP}^n = \coprod_{i=0}^{n} U_i$$

Example 2.29. $\mathbb{CP}^1 = S^2$ is a sphere, which consists of two copies U_0, U_1 of the complex plane $\mathbb{C} \cong D^2 = S^2 \setminus \{\mathrm{pt}\}$ glued together along a copy of the punctured complex plane $\mathbb{C}^\times = D^2 \setminus \{\mathrm{pt}\}$.

Remark 2.30. Why is $\mathbb{CP}^n$ compact with respect to the usual topology and the Zariski topology?

For the usual topology, consider the diagram

$$\begin{tikzcd}
 & & \mathbb{C}^{n+1} \setminus \{\vec{0}\} \arrow[dr, two heads, "\phi"] & \\
 & & & \mathbb{CP}^n \\
\left\{\vec{v} \mid |\vec{v}|^2 = 1\right\} \arrow[r, equal] & S^{2n+1} \arrow[uur, hook] \arrow[urr, two heads, "\psi"] & &
\end{tikzcd}$$

Here, ϕ is the quotient map by $\mathbb{C}^\times$ and ψ is the quotient map by $\{e^{i\theta} \mid \theta \in \mathbb{R}\}$. As the a quotient of a compact space, namely S^{2n+1}, $\mathbb{CP}^n$ is compact.

Since the Zariski topology is a coarsening of the usual topology, $\mathbb{CP}^n$ is also compact in the Zariski topology.

Definition 2.31. An ideal $I \leq \mathbb{C}[z_0, \ldots, z_n]$ is **homogeneous** if it is generated by homogeneous polynomials.

Definition 2.32. Let $X \subseteq \mathbb{CP}^n$. The **affine cone over** X is the union of $\{\vec{0}\}$ and the preimage of X in $\mathbb{C}^{n+1} \setminus \{\vec{0}\}$. We denote this by $\widehat{X}$.

$I(\widehat{X})$ is automatically invariant under the action of $\mathbb{C}^\times$ on $\mathbb{C}^{n+1} \setminus \{\vec{0}\}$, which means it must be homogeneous.

Definition 2.33. The **irrelevant ideal** of $\mathbb{C}[z_0, \ldots, z_n]$ is $\langle z_0, \ldots, z_n \rangle$.

Remark 2.34. This ideal is called the irrelevant ideal because it corresponds to the point $\vec{0} \in \mathbb{C}^{n+1}$, which is irrelevant once we pass to projective space and chop off $\{\vec{0}\}$. In $\mathbb{CP}^n$, it corresponds to the empty subset.

There is a correspondence in projective space

Definition 2.35. The **projectivization** of a subset X of $\mathbb{C}^{n+1}$ is the subset $\mathbb{P}X$ of $\mathbb{CP}^n$ given by

$$\mathbb{P}X := {}^{X \setminus \{\vec{0}\}}\big/_{\mathbb{C}^\times}.$$

Theorem 2.36 (Projective Nullstellensatz)**.** *If* J *is a homogeneous ideal of* $\mathbb{C}[z_1, \ldots, z_n]$ *and* $J \neq \langle 1 \rangle$*, then*

$$I\left(\widehat{\mathbb{P}V(J)}\right) = \sqrt{J}.$$

Given an inhomogeneous ideal $J \leq \mathbb{C}[z_1, \ldots, z_n]$, we can create a subset of $\mathbb{CP}^n$ related to $V(J)$.

$$\begin{tikzcd}
V(I) \arrow[d, hook] & \subseteq & \mathbb{C}^n \arrow[d, hook] & (z_1, \ldots, z_n) \arrow[d, maps to] \\
\overline{V(I)} & \subseteq & \mathbb{CP}^n & [1, z_1, \ldots, z_n]
\end{tikzcd}$$

the inclusion $\mathbb{C}^n \hookrightarrow \mathbb{CP}^n$ corresponds to $\mathbb{CP}^n = \mathbb{C}^n \sqcup \mathbb{CP}^{n-1}$.

Definition 2.37. Let $J \leq \mathbb{C}[z_1, \ldots, z_n]$ be an inhomogeneous ideal. The **homogenization of** $\sqrt{J}$ is the
$$I\left(\widehat{\overline{V(J)}}\right) \leq \mathbb{C}[z_0, \ldots, z_n].$$
Here, $\overline{V(J)}$ means the Zariski closure.

Example 2.38. Consider $I = \langle z_2 - z_1^2 \rangle$. This describes a parabola in $\mathbb{C}^2$, to which we must add a point in $\mathbb{CP}^1$ to get a Riemann sphere inside $\mathbb{CP}^2 = \mathbb{C}^2 \sqcup \mathbb{CP}^1$.
$$V(I) = \left\{(z_1, z_2) \mid z_2 = z_1^2\right\} \subseteq \mathbb{C}^2$$
In projective space, this corresponds to
$$X = \left\{[1, z_1, z_2] \mid z_2 = z_1^2\right\}$$
This is the same as
$$X = \left\{(z_0, z_1, z_2) \,\middle|\, \left(\tfrac{z_2}{z_0}\right) = \left(\tfrac{z_1}{z_0}\right)^2,\ z_0 \neq 0\right\}$$
The closure of this is
$$\left\{[z_0, z_1, z_2] \mid z_2 z_0 = z_1^2\right\}.$$
What did we add at ∞ (i.e. in the copy of $\mathbb{CP}^1$)? To answer this, we will intersect the Intersecting with the copy of $\mathbb{CP}^1$ corresponds to adding ideals and then taking the radical, and $\mathbb{CP}^1 = \left\{[z_0, z_1, z_2] \mid z_0 = 0\right\}$ corresponds to the ideal $\langle z_0 \rangle$.
$$\sqrt{\langle z_2 z_0 - z_1^2 \rangle + \langle z_0 \rangle} = \sqrt{\langle z_2 z_0 - z_1^2, z_0 \rangle} = \langle z_1, z_0 \rangle$$
This corresponds to the point $[0, 0, 1]$ in $\mathbb{CP}^2$, which is the one point we added.

Example 2.39. Consider $I = \langle z_1 z_2 - 1 \rangle$ corresponding to a hyperbola in $\mathbb{C}^2$. If we homogenize this, we get the ideal $\langle z_1 z_2 - z_0^2 \rangle$ corresponding to a Riemann sphere in $\mathbb{CP}^2$

The intersection with the copy of $\mathbb{CP}^1$ inside $\mathbb{CP}^2 = \mathbb{C}^2 \sqcup \mathbb{CP}^1$ corresponds to the ideal
$$\sqrt{\langle z_1 z_2 - z_0^2 \rangle + \langle z_0 \rangle} = \sqrt{\langle z_1 z_2 - z_0^2, z_0 \rangle} = \langle z_1, z_0 \rangle \cap \langle z_2, z_0 \rangle.$$
This means we added two points: $[0, 0, 1]$ and $[0, 1, 0]$.

Example 2.40. Let $I = \langle (z_1^2 + z_2^2) - r^2 + Az_1 + Bz_2 \rangle$. Then the homogenization is
$$J = \langle z_1^2 + z_2^2 - z_0^2 r^2 + Az_1 z_0 + Bz_2 z_0 \rangle$$
The added point at infinity is calculated by
$$\langle z_1^2 + z_2^2 - z_0^2 r^2 + Az_1 z_0 + Bz_2 z_0, z_0 \rangle = \langle z_1 + iz_2, z_0 \rangle \cap \langle \langle z_1 - iz_2, z_0 \rangle.$$

Then

$$\mathbb{P}V(J) = \{[0, 1, -i], [0, 1, +i]\}$$

These two points lie on all ellipses!

Exercise 2.41. If $J \leq \mathbb{C}[z_0, \ldots, z_n]$ is an inhomogeneous ideal, what is $\mathbb{P}V(J)$? Here $\mathbb{P}$ stands for removing $\vec{0}$ and projecting to $\mathbb{CP}^n$. What is the relation between the ideals J and $I\left(\widehat{\mathbb{P}V(J)}\right)$?

Remark 2.42. Let $J = \langle x_1^2 x_2 \rangle$. The vanishing set of J is the union of the x_1 and x_2 axes in $\mathbb{CP}^2$.

$\sqrt{J} = \langle x_1 x_2 \rangle = \langle x_1 \rangle \cap \langle x_2 \rangle$. The vanishing set of this is also the union of the axes x_1 and x_2, but they're not the same ideal!

We imagine that $\mathbb{P}V(J)$ is "fuzzier" than $\mathbb{P}V(\sqrt{J})$. If we look not at the vanishing set, but where $x_1^2 x_2$ is very small, then we learn either that x_1^2 or x_2 is very small. However, knowing that x_1^2 is small isn't as impressive as knowing that x_1 is small. Hence the fuzz.

2.4 Some Classic Morphisms

Let $R = \mathbb{C}[x_0, \ldots, x_n]$ and let $S = \mathbb{C}[y_0, \ldots, y_m]$, with ideals $I \leq R$, $J \leq S$. A homogeneous map $R/I \to S/J$ is the same as a map $R \to R/J$ with kernel zero.

Maps $g \colon S/J \to \mathbb{C}$ correspond to maximal ideals, which are points of $V(J)$. A map $f \colon R/I \to S/J$ therefore induces a map $V(J) \to V(I)$, since precomposing with f gives a map $g \circ f \colon R/I \to \mathbb{C}$.

Example 2.43. Consider $R = \mathbb{C}[x, y]$, $S = \mathbb{C}[t]$, $I = J = 0$. Then $V(I) = \mathbb{C}^2$ and $V(J) = \mathbb{C}$. The map

$$\begin{aligned} \mathbb{C}[x, y] &\longrightarrow \mathbb{C}[t] \\ x &\longmapsto t^2 \\ y &\longmapsto t^3 \end{aligned}$$

corresponds to the map on varieties

$$\begin{aligned} \mathbb{C}^2 &\longleftarrow \mathbb{C} \\ (t^2, t^3) &\longleftarrow\!\shortmid\; t \end{aligned}$$

with image $\{(x, y) \mid x^2 = y^3\}$. Note that the first map $\mathbb{C}[x, y] \to \mathbb{C}[t]$ factors through $\mathbb{C}[x, y]/\langle x^2 - y^3 \rangle$, and the ideal in the denominator describes the resulting curve.

Example 2.44. Consider the inclusion of the hyperbola into the plane and then projection onto the line.

$$\{(x,y) \mid xy = 1\} \hookrightarrow \mathbb{C}^2 \twoheadrightarrow \mathbb{C}$$

This is not a surjection onto $\mathbb{C}$, because it misses the origin, but it is an epimorphism because it has dense image.

These maps correspond to

$$\mathbb{C}[y] \hookrightarrow \mathbb{C}[x,y] \twoheadrightarrow \mathbb{C}[x,y]/\langle xy-1\rangle.$$

Remark 2.45. Why is the set of prime ideals called a spectrum? Let $T\colon \mathbb{C}^n \to \mathbb{C}^n$ be a linear transformations with minimal polynomial p. Then consider the map $\mathbb{C}[x] \to \mathrm{End}(\mathbb{C}^n)$, $x \mapsto T$. Although $\mathrm{End}(\mathbb{C}^n)$ is not quite commutative, it kind of looks like $\mathbb{C}[x]$.

Then

$$\operatorname{Specm}\left(\mathbb{C}[x]/\langle p\rangle\right) = V(\langle p\rangle) = \text{eigenvalues of } T,$$

this is the spectrum of T.

What about the projective version of this? Given $V(I) \to V(J)$, we want a map $\mathbb{P}V(I) \to \mathbb{P}V(J)$ that realizes $\mathbb{P}$ as a functor. But there may be $\vec{v} \in V(I)$ such that $\vec{v} \mapsto 0$! These are called basepoints of the rational map $\mathbb{P}V(I) \to \mathbb{P}V(J)$, and turn out not to be so much of a problem.

Definition 2.46. The **Segre embedding** is the map $\mathbb{CP}^{n-1} \times \mathbb{CP}^{m-1} \to \mathbb{CP}^{nm-1}$ from

$$\begin{aligned} \mathbb{C}^n \times \mathbb{C}^m &\longrightarrow \mathbb{C}(nm) \\ \begin{pmatrix} \text{column} & & \text{row} \\ \text{vector} & , & \text{vector} \end{pmatrix} &\longmapsto \text{product} \end{aligned}$$

This corresponds to the map

$$\begin{aligned} \mathbb{C}\left[\{z_{ij} \mid i,j = 1,\dots,n\}\right] &\longrightarrow \mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n] \\ z_{ij} &\longmapsto x_i y_j \end{aligned}$$

This is only graded if $\deg(x_i) = \deg(y_i) = 1$ and $\deg(z_{ij}) = 2$.

What is the image of this map? In terms of matrices, the product of a row and column vector is a rank 1 matrix, so the image of the Segre embedding is

$$\mathbb{P}\left(\text{rank 1 matrices}\right) = \mathbb{P}V(\langle \text{all } 2\times 2 \text{ minors}\rangle).$$

Definition 2.47. The **Veronese embedding** $\mathbb{CP}^{n-1} \to \mathbb{CP}^{n^k-1}$ comes from the map

$$\begin{aligned} \mathbb{C}^n &\longrightarrow (\mathbb{C}^n)^{\otimes k} \\ \vec{v} &\longmapsto \vec{v} \otimes \cdots \otimes \vec{v} \end{aligned}$$

This corresponds to the map

$$\begin{aligned} \mathbb{C}[\nu_1, \ldots, \nu_n] &\longleftarrow \mathbb{C}\left[\{z_{i_1 \ldots i_k}\}\right] \\ \prod_{j=1}^{k} \nu_{i_j} &\longleftarrow\!\shortmid\; z_{i_1 \ldots i_k} \end{aligned}$$

2.5 Hilbert Functions

For this section, let $R = \mathbb{C}[z_0, \ldots, z_n]$.

Definition 2.48. Let S be a graded ring. Its **Hilbert function** is $h_S \colon \mathbb{N} \to \mathbb{N}$

$$h_I(d) := \dim (S)_{\deg = d}\,.$$

Example 2.49. Let $R = \mathbb{C}[z_0, \ldots, z_n]$. Then $h_R(d)$ is the number of monomials in $z_0, \ldots, z_n$ of degree d.

$$h_R(d) = \binom{n+d}{d}$$

More generally, we can define a Hilbert function h_M when M is a finitely generated ($\mathbb{Z}$-)graded R-module.

Definition 2.50. Let M be a graded R-module and let $j \in \mathbb{Z}$. The j-shifted R-module M is $M[j]$ with d-th graded piece $(M[j])_d := M_{d-j}$.

WARNING: If $M = R$, then this is *not* a polynomial algebra.

Example 2.51. Consider R as an R-module and let $j \in \mathbb{Z}$.

$$h_{R[j]}(d) = h_R(d-j) = \binom{n+d-j}{n}.$$

Theorem 2.52 (Hilbert Syzygy Theorem)**.** *Let* $R = \mathbb{C}[z_1, \ldots, z_n]$*. If* M *is a finitely generated, graded* R*-module then there is a finitely generated, graded resolution of length* $\leq n+1$

$$0 \to F_{n+1} \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0 \tag{2.1}$$

where each F_i *is a free, finitely generated, graded* R*-module.*

Remark 2.53. Each F_i being finitely generated doesn't mean that it's R^n for some n; the generators may be shifted in degree.

Corollary 2.54. *Let* $R = \mathbb{C}[z_0, \ldots, z_n]$*. If* M *is a graded, finitely generated* R*-module, then* h_M *is eventually polynomial.*

Proof sketch. We can compute the dimension of M as the alternating sum of dimensions of $F_0, \ldots, F_n$, using the exact sequence (2.1), and each F_i has a finite number of generators. Because this is a finite resolution, there is a maximum degree of a generator, and then h_M is polynomial after that point. $\square$

Example 2.55. Let $R = \mathbb{C}[z_0, \ldots, z_n]$. $h_R(d)$ is polynomial for $d \geq 0$, since

$$h_R(d) = \binom{n+d}{d} = \frac{(n+d)(n+d-1)\cdots(d+1)}{n!}$$

Definition 2.56. The **Hilbert Polynomial** HP_M of an R-module M is the polynomial coming from the Hilbert function after its input is sufficiently large.

Exercise 2.57. Let $f\colon \mathbb{Z} \to \mathbb{Z}$ be polynomial, for example $\binom{n}{2} := \frac{n(n-1)}{2}$. What is the Taylor formula

$$f(d) = \sum_{i=0}^{\deg(f)} c_i \binom{d+i}{i}?$$

How do we compute the c_i?

Definition 2.58. If $h_M(d) = C\binom{d+m}{m} + (\text{lower order terms})$ for $d \gg 0$, then m is called the **Hilbert dimension** $\mathrm{Hdim}(M)$ of M, and C is called the **degree**.

WARNING: the degree in this sense is *not* the degree of the Hilbert polynomial.

Example 2.59. Let $S = R/\langle p\rangle$, for p a homogeneous polynomial of degree C. Then we have a short exact sequence

$$0 \longrightarrow R[C] \xrightarrow{\cdot p} R \twoheadrightarrow {}^{R}\!/_{\langle p\rangle} \longrightarrow 0$$

where $R[C]$ denotes R shifted in degree by C. We have that

$$\begin{aligned} h_{R/\langle p\rangle} &= h_R(d) - h_{R[C]}(d) \\ &= \binom{d+n}{n} - \binom{d+n-C}{n} \\ &= C\binom{d+n-1}{n-1} + (\text{lower order terms}) \end{aligned}$$

The degree of the hypersurface $\mathbb{P}V(R/\langle p\rangle)$ is C, which is the degree of this Hilbert polynomial.

Example 2.60. Let $R = \mathbb{C}[x_0, \dots, x_n]$. Then

$$h_R(d) = \binom{d+n}{n}$$

and so $\mathrm{Hdim}(R) = n$.

Example 2.61. Let $J = \langle x_1^2 x_2 \rangle$. Let's compute h_S for $S = \mathbb{C}[x_0, \dots, x_n]/J$. In S, there are

- $(d+1)$ monomials of degree d of the form $x_0^{d-k} x_1^k$
- d monomials of degree d of the form $x_0^{d-k} x_2^k$ for $k > 0$
- $(d-1)$ monomials of degree d of the form $x_0^{d-k-1} x_1 x_2^k$ for $k > 0$

for $d \geq 2$. For $d = 0$, there is one monomial, namely 1. For $d = 1$, there are two: x_1 and x_2.

Hence, the Hilbert function of S is

$$h_S = \begin{cases} 1 & d = 0 \\ 2 & d = 1 \\ 3d & d \geq 2. \end{cases}$$

This is eventually polynomial, $h_S(d) = 3d$ for $d \geq 2$. Hence,

$$\mathrm{Hdim}(S) = 1$$

and the degree of S is 3.

Theorem 2.62. $\mathrm{Hdim}(R/J)$ *depends only on* $V(J)$.

Proof. It is equivalent to show that $\mathrm{Hdim}(R/J) = \mathrm{Hdim}(R/\sqrt{J})$. Let $S = R/J$.

If $J = \sqrt{J}$, then we're done. Otherwise, there is some $r \notin J$ with $r^k \in J$. So r is nilpotent in R/J. The lowest homogeneous component of r is also nilpotent because, writing $r = t +$ (higher degree terms), then

$$r^k = t^k + \text{(higher degree terms)} \in J.$$

So we may assume that r is homogeneous.

Let m be the least integer such that $r^m \in J$. Let $s = r^{m-1}$. We know $s \notin J$, yet $s^2 \in J$. Let d be the degree of s in S.

Now consider the short exact sequence

$$0 \to \langle s \rangle \to S \to {}^{S}\!/_{\langle s \rangle} \to 0.$$

This demonstrates $h_S = h_{\langle s \rangle} + h_{S/\langle s \rangle}$.

We have another exact sequence

$$0 \to \operatorname{ann}_S(s) \to S \xrightarrow{\cdot s} S[d] \to {}^{S}\!/_{\langle s \rangle}[d] \to 0.$$

We must include degree shiftings (recall that $S[d]$ is S shifted in degree, not a polynomial ring!) to make multiplication by s a graded map. This gives us the equation

$$h_{\operatorname{ann}_S(s)}(t) - h_S(t) + h_S(t-d) - h_{S/\langle s \rangle}(t-d).$$

We have that $\langle s \rangle \leq \operatorname{ann}_S(s)$ since $s^2 = 0$. Therefore,

$$\begin{aligned} h_S(t) &= h_{S/\langle s \rangle}(t) + h_{\langle s \rangle}(t) \\ &\leq h_{S/\langle s \rangle}(t) + h_{\operatorname{ann}_S(s)}(t) \\ &= h_{S/\langle s \rangle}(t) + h_S(t) - h_S(t-d) + h_{S/\langle s \rangle}(t-d) \end{aligned}$$

Rearranging terms, we get

$$h_S(t-d) \leq h_{S/\langle s \rangle}(t) + h_{S/\langle s \rangle}(t-d)$$

On the right hand side, because the polynomials are $\mathbb{N}$-valued, there cannot be any cancellation of degrees. Moreover, the two summands on the right-hand-side are the same degree, so

$$\operatorname{Hdim}(S) \leq \operatorname{Hdim}(S/\langle s \rangle).$$

But on the other hand, $S/\langle s \rangle$ is a quotient of S, so

$$\operatorname{Hdim}(S/\langle s \rangle) \leq \operatorname{Hdim}(S).$$

Therefore, $\operatorname{Hdim}(S) = \operatorname{Hdim}(S/\langle s \rangle)$.

If the new ideal $J' = J + \langle s \rangle$ is not yet radical, then repeat, giving an ascending chain of ideals contained in $\sqrt{J}$. This must terminate since R is Noetherian, and it terminates at $\sqrt{J}$.

Therefore, $\operatorname{Hdim}(R/J) = \operatorname{Hdim}(R/\sqrt{J})$. □

Example 2.63. Consider $J = \langle x_1^3 x_2 \rangle$, and let $r = x_1 x_2$. Then the greatest power of r not in J is $s = x_1^2 x_2^2$. We have

$$h_S(t-4) \leq h_{S/\langle s \rangle}(t) + h_{S/\langle s \rangle}(t-4).$$

Considering leading terms, this looks like

$$4t + \ldots \leq (3t + \ldots) + (3t + \ldots),$$

but the degrees of these polynomials (and therefore the Hilbert dimensions) are the same.

Recall that an ideal J is radical if and only if it is the intersection of the minimal primes over it. The corresponding geometric fact is that

$$V(J) = \bigcup_{\substack{P \geq J \\ P \text{ minimal prime}}} V(P).$$

Theorem 2.64. *If J is a radical ideal, then the Hilbert dimension of R/J is the maximum of the Hilbert dimensions of R/P, for P a minimal prime over J.*

$$\operatorname{Hdim}(R/J) = \max_{\substack{P \geq J \\ P \text{ minimal prime}}} \operatorname{Hdim}(R/P).$$

Proof. Let $S = R/I$.

If I is not prime, then there is some product $ab \in I$ such that $a, b \notin I$. The same is true for their lowest degree terms in the grading, so we may assume that they are homogeneous.

Now consider

$$Sa \cap Sb \to {}^{R}\!/_{I} \to {}^{R}\!/_{I + \langle a \rangle} \oplus {}^{R}\!/_{I + \langle b \rangle}.$$

We should check that $Sa \cap Sb$ is indeed the kernel of the second map. If $s = ma = mb \in S_a \cap Sb$, then $s^2 = manb = mnab = 0$ in R/I. Hence, $s = 0$ since I is a radical ideal.

Then

$$h_{R/I} \leq h_{R/_{I+\langle a \rangle}} + h_{R/_{I+\langle b \rangle}} \leq h_{R/I} + h_{R/I}$$

Therefore, because these polynomials are $\mathbb{N}$-valued,

$$\operatorname{Hdim}(R/I) = \max\left\{ \operatorname{Hdim}\left({}^{R}\!/_{I + \langle a \rangle}\right), \operatorname{Hdim}\left({}^{R}\!/_{I + \langle b \rangle}\right) \right\}$$

Now we may replace $I + \langle a \rangle$ and $I + \langle b \rangle$ by their radicals.

We may repeat this process, giving ascending chains

$$\begin{tikzcd}
 & & I \arrow[dl] \arrow[dr] & & \\
 & \sqrt{I + \langle a \rangle} \arrow[dl] \arrow[d] & & \sqrt{I + \langle b \rangle} \arrow[d] \arrow[dr] & \\
\iddots & \vdots & & \vdots & \ddots
\end{tikzcd}$$

These chains must terminate since R is Noetherian, and they terminate at prime ideals. $\square$

Example 2.65. Consider the union of a plane and a line corresponding to

$$\langle x_1 \rangle \cap \langle x_2 x_3 \rangle = \langle x_1 x_2, x_1 x_3 \rangle.$$

The monomials not in this ideal yet in one of the two primes are x_1^k, $x_2^a x_3^b$. The Hilbert function is

$$h_S(t) = \binom{t+2}{2} + \binom{t+1}{1} - 1$$

Theorem 2.66. *Let* I *be a homogeneous and prime ideal. If* $V(J) \subsetneq V(I)$*, then*

$$\mathrm{Hdim}(R/J) < \mathrm{Hdim}(R/I).$$

Proof. By the Nullstellensatz, $J \gneq I$. Let $j \in J \setminus I$ be homogeneous. Then j is not a zerodivisor in R/I, since R/I is a domain because I is prime. Let d be the degree of j in R/I.

Now $J \geq I + \langle j \rangle > I$. Hence,

$$h_{R/J} \leq h_{R/_{I+\langle j \rangle}}. \tag{2.2}$$

Let $S = R/I$. We have a short exact sequence

$$0 \to S \xrightarrow{\cdot j} S \to {}^{S}/_{\langle j \rangle} \to 0$$

yielding

$$h_S - h_{S[-d]} = h_{S/_{\langle j \rangle}}.$$

Combining this with (2.2), we see that

$$\mathrm{Hdim}(R/J) \leq \mathrm{Hdim}(R/I). \qquad \square$$

2.6 Bézout's Theorem

Let k be a field.

Theorem 2.67 (Bézout)**.** *Let* $p, q \in k[x, y, z]$ *be homogeneous, coprime polynomials. Let* $S = k[x, y, z]/\langle p, q \rangle$*. Then* $\mathrm{Hdim}(S) = 0$ *and* $\deg(S) = \deg(p)\deg(q)$*.*

Remark 2.68. Note that under the assumptions of this theorem, $V(\langle p, q \rangle) = V(p) \cap V(q)$.

If we omit the assumption that p, q are coprime, then say $p = ra, q = rb$. Then $V(p) = V(r) \cup V(a)$, and $V(q) = V(r) \cup V(b)$, and

$$V(\langle p, q \rangle) = V(r) \cup (V(a) \cap V(b)).$$

This is supposed to be a (generalization of) statement about the intersection of plane curves, so we really want $V(\langle p, q \rangle) = V(p) \cap V(q)$.

Proof of Theorem 2.67. Let $R = k[x, y, z]$. The Hilbert polynomial of R is

$$h_R(d) = \binom{d+2}{2}$$

for $d \geq 0$. p is not a zerodivisor, so using the exact sequence

$$0 \to R[-\deg p] \xrightarrow{\cdot p} R \to {}^{R}/_{\langle p \rangle} \to 0$$

we can compute

$$h_{R/\langle p \rangle} = (\deg(p))d + C,$$

for some constant C. Since p and q are coprime, then q is not a zerodivisor in $R/\langle p \rangle$. Therefore, we may use a similar short exact sequence to conclude

$$h_{R/\langle p,q \rangle} = (\deg p)(\deg q). \qquad \square$$

Example 2.69. Note that we don't require k to be algebraically closed.

Consider $p = y - x^2$, and $q = y + 1$. We can homogenize these to get $yz - x^2$ and $y + z$. Then

$$\mathbb{R}[x, y, z]/_{\langle yz - x^2, y + z \rangle} \cong \mathbb{R}[x, y]/_{\langle x^2 + y^2 \rangle}.$$

The Hilbert polynomial of this ring is constant after degree 2; the monomials of degree $d \geq 2$ are the classes of x^d and $x^{d-1}y$. So the theorem holds, even though $\mathbb{R}$ isn't algebraically closed.

Example 2.70. Consider $p = y$ and $q = yz - x^2$. These are the projectivizations of $y = 0$ and $y = x^2$, respectively. The only point of intersection here is at $(0, 0)$, and this also holds projectively, where the only point of intersection is $[0, 0, 1]$.

Let's see what Bezout's theorem says. We have

$$\langle p, q \rangle = \langle y, yz - x^2 \rangle = \langle y, x^2 \rangle.$$

The degree of the quotient ring is here

Exercise 2.71. Let $R = k[x_0, \ldots, x_n]$. Let $p_1, \ldots, p_d \in R$ be homogeneous. Let $I = \langle p_1, \ldots, p_d \rangle$.

(a) Show that $\mathrm{Hdim}(R/I) \geq n - d$.

(b) If $\mathrm{Hdim}(R/I) = n - d$, show that $\deg(R/I) = \prod_{i=1}^{d} \deg(p_i)$.

Definition 2.72. If $\deg(R/I) = \prod_{i=1}^{d} \deg(p_i)$, then we say that $V(R/I)$ is a **complete intersection.**

Example 2.73. Let $X \subseteq M_{2\times 3}(\mathbb{C}) \cong \mathbb{C}^6$ be the closure of the set of matrices of rank 1. This is determined by the equations that all 2×2 minors vanish.

$$X = \left\{ \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \;\middle|\; ae - bd = bf - ce = af - cd = 0 \right\}$$

Hence, X is the vanishing set of the ideal $I = \langle ae - bd, bf - ce, af - cd \rangle$.

Claim that $\langle ae - bd, bf - ce \rangle$ is not a prime ideal. The element

$$(af - cd)b = a(bf - ce) + c(ae - bd)$$

is in this ideal, yet neither of the factors are. Likewise, $(af - cd)e$ is in this ideal, yet neither of its factors are. Hence, this ideal is not prime.

But we can rewrite this ideal as the intersection of its minimal primes:

$$\langle ae - bd, bf - ce \rangle = \langle ae - bd, bf - ce, af - cd \rangle \cap \langle b, e \rangle.$$

This intersection can be interpreted as follows: The set of matrices

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

such that $ae - bd = bf - ce = 0$ is the union of the set X of all matrices of rank 1 with the set of matrices of the form

$$\begin{bmatrix} a & 0 & c \\ d & 0 & f \end{bmatrix}.$$

The degree of $\langle ae - bd, bf - ce \rangle$ is $2^2 = 4$, since it is defined by two quadratic polynomials. The degree of $\langle b, e \rangle$ is $1^2 = 1$, since it is generated by two linear polynomials. Therefore, the degree of X must be three.

Yet there are three defining equations for the ideal $I = \langle ae - bd, bf - ce, af - cd \rangle$, and each is degree two. In this case, Exercise 2.71 says that the degree of X is $2^3 = 8$.

Of course, the issue here is that $\mathrm{Hdim}(X) > 5 - 3 = 2$. Actually, the Hilbert dimension of X is 3.

Theorem 2.74 (Bertini)**.** *Let* $R = k[x_0, \ldots, x_n]$. *Let* $J \leq R$ *be homogeneous, and let* k *be an infinite field. Then there exist linear polynomials* $f = \sum_{i=0}^{n} k_i x_i \in R$ *such that* $f + J$ *is not a zerodivisor in* R/J.

Corollary 2.75. *If in addition* J *is either prime with* $\dim(J) > 1$ *or radical, then* $J + \langle f \rangle$ *is again prime or radical.*

Remark 2.76. The geometric interpretation of Theorem 2.74 is that the intersection of $V(J)$ with a random hyperplane drops dimension by 1, and preserves degree. Therefore, $\deg(R/J)$ is the number of ways to intersect $V(J)$ with a random complimentary plane.

Question 2.77 (Open since 1974)**.** Let (M, N) be a pair of $n \times n$ complex matrices, considered as an element in $\mathbb{C}^{2n^2}$. Let I be the ideal generated by the entries of $MN - NM$; this is an ideal generated by n^2 equations. Then $V(I)$ is the set of pairs of commuting matrices; it has dimension $n^2 + n$. Is $\sqrt{I} = I$?

This question is asking whether or not there are secret equations that hold for commuting matrices, but can't be determined by the fact just that they commute?

Theorem 2.78 (Knutson)**.** *Let J be the ideal generated by off-diagonal entries of $MN - NM$. Then*

(a) $J = \sqrt{J}$*;*

(b) $J = I \cap Q$*, where* I *is the ideal generated by all entries of* $MN - NM$ *and* Q *is another ideal;*

(c) $V(R/I)$ *is a complete intersection.*

2.7 Krull Dimension

Let k be a field and $R = k[x_1, \ldots, x_n]$. Let I be a homogeneous ideal, and write $S = R/I$.

Definition 2.79. The **Krull dimension** $\dim S$ of S is the maximum length of a chain

$$S \twoheadrightarrow D_k \xrightarrow{\neq} D_{k-1} \xrightarrow{\neq} \cdots \xrightarrow{\neq} D_0 \xrightarrow{\neq} 0$$

such that each D_i is a domain. The first map might be an equality, but thereafter they are not.

Theorem 2.80. $\dim R/J = \dim R/\sqrt{J} = \max\limits_{P \geq J \text{ prime}} \dim R/P.$

Proof. Nilpotents in R/J come from elements of $\sqrt{J}$. Therefore, $\ker({}^R/_J \twoheadrightarrow D_k) \geq \sqrt{J}$. So sequences for R/J correspond to sequences for $R/\sqrt{J}$.

$$\begin{tikzcd} {}^R/_J \arrow[rr, two heads] \arrow[dr] & & D_k \arrow[r, two heads] & \cdots \\ & {}^R/_{\sqrt{J}} \arrow[ur, two heads] & & \end{tikzcd}$$

The kernel of $R \to R/J \to D_k$ is prime, and contains J, and therefore, contains one of the minimal primes $P \geq J$. The chain

$${}^R/_J \twoheadrightarrow {}^R/_P \twoheadrightarrow D_k \xrightarrow{\neq} \cdots$$

is longer, unless $R/P \twoheadrightarrow D_k$ is an isomorphism. In this case,

$$\dim {}^{R}/_{J} = \dim(D_k) = \dim {}^{R}/_{P} \qquad \square$$

Here's another equivalent definition of Krull dimension. We make two conventions: one for graded and one for ungraded rings.

Definition 2.81. The **Krull dimension** $\dim(R/I)$ of R/I is the maximum length d of a strictly ascending chain of prime ideals containing I,

$$I \subseteq P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_d \subseteq R$$

For an ungraded ring R, we allow the P_i to be arbitrary, and for a graded ring R, we require that they are homogeneous.

Tautologically, we have that the ungraded Krull dimension is at least the same as the graded Krull dimension.

Theorem 2.82. *Let R be a graded ring with homogeneous ideal I. The Hilbert dimension of R/I is one less than the graded Krull dimension of R/I.*

Proof. We will show that each is greater than or equal to the other.

To see that $\mathrm{Hdim}(R/I) \geq \dim(R/I) - 1$, recall that we proved that for a chain

$$I \subseteq P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_k \subseteq R,$$

we have $\mathrm{Hdim}(R/P_i) < \mathrm{Hdim}(R/P_{i+1})$. Then use induction.

Conversely, to see that $\mathrm{Hdim}(R/I) \leq \dim(R/I) - 1$, write $\sqrt{I} = \bigcap P_i$ as the intersection of its minimal prime ideals. Note that $R/\sqrt{I}$ has the same Hilbert dimension and Krull dimension as R/I.

Claim that each P_i is automatically homogeneous. Indeed given $xy \in I$ with $x, y \notin I$, write $I = (I + \langle x \rangle) \cap (I + \langle y \rangle)$ and repeat on the factors. This terminates because R is Noetherian, and, taking radicals, gives a decomposition of $\sqrt{I}$ as the intersection of its minimal primes.

We already showed that there exists some P with the same Krull dimension as $\sqrt{I}$; by the above, this means that they must have the same Hilbert dimension as well.

Take $P_d = P$, and let $a \notin P_d$ be homogeneous of positive degree (unless $P_d = \langle x_0, \ldots, x_n \rangle$).

Let P_{d-1} be a maximal dimension prime component of $P_d + \langle a \rangle$. We proved earlier that

$$\mathrm{Hdim}(R/P_{d-1}) = \mathrm{Hdim}(R/P_d) - 1.$$

Now continue by induction. This shows that $\mathrm{Hdim}(R/I) \leq \dim(R/I) - 1$. $\square$

Definition 2.83. Let $I \leq k[x_1, \ldots, x_n]$ be an inhomogeneous prime ideal, and define its **homogenization** $\widetilde{I} \leq k[x_0, \ldots, x_n]$ by the following. Consider the pullback of $k[x_0, \ldots, x_n]$-modules

$$\begin{tikzcd}
b^{-1}(I) \arrow[r] \arrow[d] & I \arrow[d, hook] \\
k[x_0, \ldots, x_n] \arrow[r, "b"] & k[x_1, \ldots, x_n] \\
x_0 \arrow[r, mapsto] & 1
\end{tikzcd}$$

Then

$$\widetilde{I} := \bigoplus_i \left(b^{-1}(I) \cap k[x_0, \ldots, x_n]_{\deg = i} \right)$$

The geometric interpretation of this is that in projective space $\mathbb{P}^n$, we have $\mathbb{P}V(\widetilde{I}) = \overline{V(I)}$.

Lemma 2.84. *Let* $I \leq k[x_1, \ldots, x_n]$ *be a (not necessarily homogeneous) prime ideal. Then*

(a) $I = b(\widetilde{I})$.

(b) $\widetilde{I}$ *is also prime.*

Proof.

(a) We have that $b(\widetilde{I}) \leq I$ because $b(\widetilde{I}) \subseteq b(b^{-1}(I)) \leq I$. Conversely, $I \leq b(\widetilde{I})$ using homogenization of polynomials.

(b) Assume $xy \in \widetilde{I}$. We may assume as before that x and y are homogeneous. Then $b(x)b(y) = b(xy) \in I$. Therefore, either $b(x)$ or $b(y) \in I$ since I is prime. Without loss, assume $b(x) \in I$ But x is the homogenization of $b(x)$ multiplied by some power of x_0

$$x = \big(\text{homogenization of } b(x)\big) \cdot x_0^i$$

We also know that, $\widetilde{I} \colon \langle x_0 \rangle = \widetilde{I}$, and that the homogenization of $b(x)$ is in $\widetilde{I}$, so therefore $x \in \widetilde{I}$. □

Theorem 2.85. *For a homogeneous ideal* I*, the ungraded Krull dimension of* R/I *is equal to the graded Krull dimension of* R/I.

3 Geometric Operations on Varieties

3.1 Blowing up

Definition 3.1. Define the **tautological bundle** over $\mathbb{CP}^n$ as

$$\widetilde{\mathbb{C}}^n := \left\{ (\vec{v}, \ell) \in \mathbb{C}^n \times \mathbb{CP}^n \;\middle|\; \vec{v} \in \ell \right\}.$$

This comes with a projection $\widetilde{\mathbb{C}}^n \to \mathbb{CP}^{n-1}$ given by $(\vec{v}, \ell) \mapsto \ell$.

There is another map $\widetilde{\mathbb{C}}^n \to \mathbb{C}^n$ given by $(\vec{v}, \ell) \mapsto \vec{v}$. Generically, this map is injective when $\vec{v} \neq \vec{0}$, but when $\vec{v} = \vec{0}$, there is a whole $\mathbb{CP}^{n-1}$ worth of lines in the fiber above $\vec{0}$.

We have an inclusion

$$\widetilde{\mathbb{C}}^n \hookrightarrow \mathbb{C}^n \times \mathbb{CP}^{n-1}$$

and, if $\mathbb{C}^n$ has coordinates $v_1, \ldots, v_n$ and $\mathbb{CP}^{n-1}$ has homogeneous coordinates $x_1, \ldots, x_n$, then the equations that demand $\vec{v} \in \ell$ are

$$v_i x_j - v_j x_i = 0 \;\text{ for all } i, j = 1, \ldots, n.$$

So the ideal that defines $\widetilde{\mathbb{C}}^n$ inside $\mathbb{C}[v_1, \ldots, v_n, x_1, \ldots, x_n]$ is

$$\langle v_i x_j - v_j x_i \mid i, j = 1, \ldots, n \rangle.$$

Under the analytic topology (not the Zariski topology), we may consider $\widetilde{\mathbb{C}}^n$ as a quotient of $\mathbb{C}^n \setminus B(\vec{0}, 1)$ by the action of the unit circle (multiplication by $e^{i\theta}$) on the boundary. Here, $B(\vec{0}, 1)$ is the ball of radius 1 centered at the origin.

The inclusion of $\mathbb{CP}^{n-1}$ into $\widetilde{\mathbb{C}}^n$ may be seen as follows. $\mathbb{CP}^{n-1}$ is diffeomorphic to the unit sphere modulo this action of the unit circle.

Definition 3.2. Given an algebraic subset X of $\mathbb{C}^n$, the **proper/strict transform** or the **blowup** of X is

$$\widetilde{X} := \overline{\pi^{-1}(X \setminus \{0\})}.$$

This fits into a diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d] & \widetilde{\mathbb{C}}^n \arrow[d, "\pi"] \\ X \arrow[r, hook] & \mathbb{C}^n \end{tikzcd}$$

but this diagram is not a pullback. The **total transform** is the pullback of X along π.

Example 3.3. Let $X = V(\langle ab \rangle) \subseteq \mathbb{C}^2 = \operatorname{Specm} \mathbb{C}[a, b]$. This corresponds to the axes in $\mathbb{C}^2$. The blowup $\widetilde{X}$ is then two lines that don't meet, but pass over each other.

The corresponding ring for $\widetilde{\mathbb{C}}^n$ is

$$\mathbb{C}[a, b, p, q] \big/ \langle aq - pb \rangle$$

with a, b in degree zero, and p, q are in degree one, where the ideal is generated by all 2×2 minors of $\left[\begin{smallmatrix} a & b \\ p & q \end{smallmatrix}\right]$.

To get the ring corresponding to $\widetilde{X}$, we must quotient by a few more relations. In particular, if we just took the pushout,

$$\begin{tikzcd}
\mathbb{C}[a,b]/\langle a, b \rangle & \mathbb{C}[a,b] \arrow[l] \arrow[d] \\
P \arrow[from=u] & \mathbb{C}[a,b,p,q]/\langle aq - bp \rangle \arrow[l]
\end{tikzcd}$$

then we would get

$$\mathbb{C}[x, y, p, q] \big/ \langle aq - bp, ab \rangle.$$

But this ideal is not prime, and we can't just take a pushout because $\widetilde{X}$ is not a pullback. Moreover, the ideal $I = \langle aq - bp, ab \rangle$ is not prime, since it contains neither a nor b. But,

$$\begin{aligned}
\langle aq - bp, ab \rangle &= \langle aq - bp, a \rangle \cap \langle aq - bp, b \rangle \\
&= \langle a, p \rangle \cap \langle a, b \rangle \cap \langle q, b \rangle
\end{aligned}$$

We don't want $\langle a, b \rangle$ at all; this isn't part of the blowup because it's singular at the origin, and the blowup has no singularities.

Therefore, the ring corresponding to $\widetilde{X}$ is

$$\mathbb{C}[a, b, p, q] \big/ \langle a, p \rangle \cap \langle q, b \rangle$$

Construction 3.4. An algorithm to determine the ideal corresponding to the blowup $\widetilde{X}$ is as follows. Recall the saturation ideal from Definition 2.17. The blowup is

$$\overline{\pi^{-1}(X \setminus \{0\})} = \overline{\pi^{-1}(X) \setminus \pi^{-1}(0)} = \mathbb{P}V\left(I \colon \langle x_1, \ldots, x_n \rangle^\infty\right),$$

where I is the ideal of $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ defining the blowup, generated by the 2×2 minors of

$$\begin{bmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{bmatrix}.$$

Example 3.5. In the previous example (Example 3.3), we can compute

$$\langle a, p\rangle \cap \langle q, b\rangle = \langle aq - bp, ab\rangle \colon \langle a, b\rangle^{\infty}$$

Definition 3.6. The **blowup** of $\mathbb{C}^n$ along $\mathbb{C}^k$ is the product of $\mathbb{C}^k$ and the blowup of $\mathbb{C}^{n-k}$ at $\vec{0}$.

Give $\mathbb{C}^n$ coordinates $x_1, \ldots, x_n$ and say that $\mathbb{C}^k$ is the first k-coordinates of $\mathbb{C}^n$. Then let I be the ideal generated by all 2×2 minors of the matrix

$$\begin{bmatrix} x_{k+1}, \ldots, x_n \\ y_{k+1}, \ldots, y_n \end{bmatrix}$$

inside $\mathbb{C}[x_1, \ldots, x_k, x_{k+1}, \ldots, x_n, y_{k+1}, \ldots, y_n]$ with x_i in degree zero and y_i in degree 1. We may think of this ring as representing $\mathbb{C}^k \times \mathbb{C}^{n-k} \times \mathbb{CP}^{n-k-1}$. The blowup corresponds to the quotient of this ring by I.

3.2 Specm **and** Projm

Let $S = \mathbb{C}[x_1, \ldots, x_n]/I$, where I is an inhomogeneous ideal. We think of each maximal ideal as corresponding to a point in $\mathbb{C}^m$ by the Nullstellensatz.

Definition 3.7. Let $\mathrm{Specm}(S)$ be the collection of maximal ideals of S.

Now let S be an $\mathbb{N}$-graded ring.

Definition 3.8. The **irrelevant ideal** of S is

$$S_{>0} := \bigoplus_{i>0} S_i.$$

Definition 3.9. Let $\mathrm{Projm}(S)$ be the collection of ideals which are maximal among homogeneous ideals not containing the irrelevant ideal.

If we did this construction on $S = \mathbb{C}[x_0, \ldots, x_n]$, each such ideal of $\mathrm{Projm}(S)$ corresponds to a point in projective space $\mathbb{CP}^n$.

Remark 3.10. Another description of $\mathrm{Projm}(S)$ is as follows. By analogy to the construction of projective n-space from affine $n+1$ space, we rip out the origin and quotient by the action of $\mathbb{C}^{\times}$.

Given an $\mathbb{N}$-graded ring, let $\mathbb{C}^{\times}$ act on $r \in S$ of degree d by $z \cdot r = z^d r$, and extend this by linearity.

The ring $S_0 \cong S/S_{>0}$ corresponds to the origin in $\mathrm{Specm}(S)$, so we remove that piece from $\mathrm{Specm}(S)$ and then quotient by the action of $\mathbb{C}^{\times}$.

$$\mathrm{Projm}(S) \cong \left(\mathrm{Specm}(S) \setminus \mathrm{Specm}({}^{S}/_{S_{>0}})\right)/\mathbb{C}^{\times} \tag{3.1}$$

Proposition 3.11. *If S is ungraded, then let $S_0 = S$ to give it an $\mathbb{N}$-grading. If we adjoin a new variable ℓ of degree one to get $S[\ell]$, then*

$$\operatorname{Projm}(S[\ell]) \cong \operatorname{Specm}(S)$$

Note that if we just take $\operatorname{Projm}(S)$ with everything in degree zero, then $\operatorname{Projm}(S) = \emptyset$.

Proof. Use the description (3.1).

$$\operatorname{Projm}(S[\ell]) = (\operatorname{Specm}(S[\ell]) \setminus \operatorname{Specm}(S))/\mathbb{C}^\times.$$

Now, each maximal ideal of $S[\ell]$ corresponds to the kernel of a surjective homomorphism from $S[\ell] \to \mathbb{C}$. To choose such a homomorphism, we may choose a homomorphism from $S \to \mathbb{C}$ and a destination for ℓ. Hence, as sets,

$$\operatorname{Specm}(S[\ell]) \cong \operatorname{Specm}(S) \times \mathbb{C}$$

We may likewise consider $\operatorname{Specm}(S)$ as those homomorphisms $S[\ell] \to \mathbb{C}$ sending $\ell \mapsto 0$. So we have

$$\begin{aligned}
\operatorname{Projm}(S[\ell]) &= (\operatorname{Specm}(S[\ell]) \setminus \operatorname{Specm}(S))/\mathbb{C}^\times \\
&= (\operatorname{Specm}(S) \times \mathbb{C}) \setminus (\operatorname{Specm}(S) \times \{0\})/\mathbb{C}^\times \\
&= (\operatorname{Specm}(S) \times \mathbb{C}^\times)/\mathbb{C}^\times \\
&= \operatorname{Specm}(S)
\end{aligned}$$

□

Example 3.12. This is an example of a space that is described as $\operatorname{Projm}(S)$, but is not itself a projective space or affine space.

Consider the blowup $\widetilde{\mathbb{C}}^n$.

$$\widetilde{\mathbb{C}}^n = \left\{ (\vec{v}, \ell) \in \mathbb{C}^n \times \mathbb{CP}^{n-1} \;\middle|\; \vec{v} \in \ell \right\}$$

We may also write this as the variety corresponding the ideal I generated by all 2×2 minors of $\left[\begin{smallmatrix} x_1, \ldots, x_n \\ y_1, \ldots, y_n \end{smallmatrix}\right]$. In the new language, using Projm, we have

$$\widetilde{\mathbb{C}}^n = \operatorname{Projm} \mathbb{C}\left[x_1^{(0)}, \ldots, x_n^{(0)}, y_1^{(1)}, \ldots, y_n^{(1)}\right]/I$$

Here, the superscript $x_i^{(j)}$ means that the generator x_i has degree j.

Example 3.13. There are also spaces that are not $\operatorname{Specm}(S)$ or $\operatorname{Projm}(S)$ of anything, but are still perfectly reasonable. For example, $\mathbb{C}^2 \setminus \{0\}$. This is not $\operatorname{Projm}(S)$ for any S.

3.3 Blowups, continued

Let $S = \mathbb{C}[x_1, \ldots, x_k]/I$ and let $J \leq S$ be an ideal generated by $g_1, \ldots, g_{n-k}$. **The blow up of** $\mathrm{Specm}(S)$ **along** $\mathrm{Specm}(S/J)$ is defined as follows.

First, re-embed $\mathrm{Specm}(S) \subseteq \mathbb{C}^k$ into $\mathbb{C}^n$ as the graph of

$$g = (g_1, \ldots, g_{n-k})\colon \mathrm{Specm}(S) \to \mathbb{C}^{n-k}.$$

This lives inside $S \times \mathbb{C}^{n-k} \subseteq \mathbb{C}^k \times \mathbb{C}^{n-k}$, and moreover the intersection of the graph of g with $\mathbb{C}^k \times 0 \subseteq \mathbb{C}^k \times \mathbb{C}^{n-k}$ is exactly $\mathrm{Specm}(S/J)$.

Now we blow up $\mathbb{C}^n$ along $\mathbb{C}^k$ and take the proper transform of $\mathrm{Specm}(S)$.

$$\begin{tikzcd} \mathbb{C}[x_1, \ldots, x_n] \arrow[r, two heads] \arrow[d, hook] & S \arrow[d] \\ \mathbb{C}[x_1, \ldots, x_n, y_{k+1}, \ldots, y_n]/K \arrow[r, "\phi"] & S[t] \end{tikzcd}$$

where K is the ideal generated by all 2×2 minors of the matrix $\left[\begin{smallmatrix} x_{k+1} \cdots x_n \\ y_{k+1} \cdots y_n \end{smallmatrix}\right]$, and the map ϕ is defined by

$$\begin{aligned} \phi(x_i) &= x_i & i &= 1, \ldots, k \\ \phi(x_{k+i}) &= g_i & i &= 1, \ldots, (n-k) \\ \phi(y_{k+i}) &= g_i t & i &= 1, \ldots, (n-k) \end{aligned}$$

Definition 3.14. The image of ϕ is called the **blowup algebra**

$$B(S, J) = S \oplus tJ \oplus t^2 J^2 \oplus \ldots = \bigoplus_d t^d J^d \leq S[t]$$

with t in degree 1 and S in degree zero.

Definition 3.15. The **blowup** of $\mathrm{Specm}(S)$ along $\mathrm{Specm}(S/J)$ is $\mathrm{Projm}(B(S, J))$.

Theorem 3.16. *If I is prime, then the blowup algebra $\subseteq S[t]$ is a domain, so it gives the proper transform.*

Example 3.17. Let $S = \mathbb{C}[x, y]$ and let $J = \langle x^2, y \rangle$. The blowup algebra of this is the image of

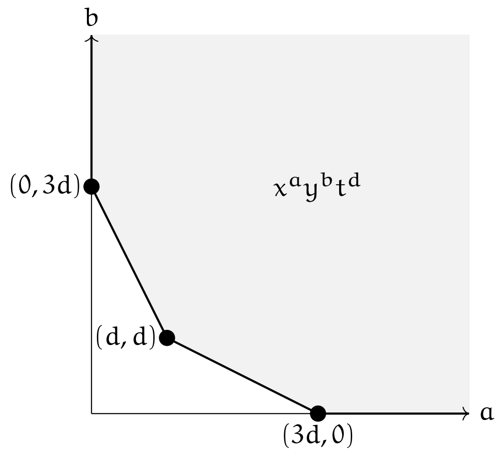
$$\mathbb{C}[x, y, a, b]/\langle x^2 b - ya \rangle$$

inside $\mathbb{C}[x, y, t]$ under the map

$$\begin{aligned} x &\mapsto x, \\ y &\mapsto y, \\ a &\mapsto x^2 t, \\ b &\mapsto yt. \end{aligned}$$

Example 3.18. Let $S = \mathbb{C}[x, y]$ and let $J = \langle x^3, xy, y^3 \rangle$. The blowup algebra in this case generated by all $x^a y^b t^d$ with (a, b) above the line segments connecting $(0, 3d)$, (d, d) and $(3d, 0)$.

The monomials in the blowup algebra with t-degree d correspond to all lattice points in the shaded region below.



Example 3.19. Consider $S = \mathbb{C}[x, y]/\langle xy \rangle$ and $J = \langle x, y \rangle$. The only monomials in S are x and y. The blowup algebra is isomorphic to its preimage in $\mathbb{C}[x, y, a, b]$:

$$B(J, S) \cong \mathbb{C}[x, y, a, b]/\langle xy, bx, ya, ab \rangle,$$

and in t-degree d has monomials $x^i t^d$ and $y^j t^d$ for $i, j > 0$.

The ideal generating this blowup decomposes as

$$\langle xy, bx, ya, ab \rangle = \langle x, a \rangle \cap \langle y, b \rangle.$$

Geometrically, this is blowing up the union of the coordinate axes at the origin, to get two lines that don't intersect yet project onto the coordinate axes.

3.4 Associated Graded Rings

Definition 3.20. Let J be an ideal in a ring S. Then the **Rees algebra** of S is the $\mathbb{Z}$-graded S-algebra defined by

$$\operatorname{Rees}(S, J) := \bigoplus_{n \in \mathbb{Z}} J^n t^n \subseteq S^{[}t, t^{-1}],$$

where J^n is understood to mean S if $n \leq 0$.

$$\operatorname{Rees}(S, J) = \ldots \oplus St^{-1} \oplus St^{-1} \oplus S \oplus Jt \oplus J^2t^2 \oplus \ldots$$

If S contains a field, say $\mathbb{C} \hookrightarrow S$, then there is a graded map

$$\mathbb{C}[t^{-1}] \to \mathrm{Rees}(S, J).$$

This makes $\mathrm{Rees}(S, J)$ into a torsion-free $\mathbb{C}[t^{-1}]$-module.

What does this mean geometrically? It gives a map to affine 1-space over $\mathbb{C}$:

$$\pi\colon \operatorname{Specm}(\mathrm{Rees}(S, J)) \to \operatorname{Specm}(\mathbb{C}[t^{-}1]).$$

Moreover, there is an action of $\mathbb{C}^\times$ on both $\operatorname{Specm}(\mathrm{Rees}(S, J))$ and $\operatorname{Specm}(\mathbb{C}[t^{-1}])$ by $z \cdot f = z^d f$ for f homogeneous of degree d, and π is $\mathbb{C}^\times$ equivariant. Hence, all fibers of π are isomorphic, except possibly for the zero fiber.

Example 3.21. Let $S = \mathbb{C}[x, y]$ and $J = \langle xy \rangle$. Then $\operatorname{Specm}(S/J)$ represents the union of the coordinate axes in $\mathbb{C}^2$. The Rees algebra is

$$\mathrm{Rees}(S, J) = \mathbb{C}[x, y, t^{-1}, txy] \subseteq \mathbb{C}[x, y, t^{\pm 1}].$$

It's tough to describe the geometry of a subalgebra, but it's easy to describe the geometry of a quotient. So let's rewrite

$$\mathrm{Rees}(S, J) \cong \mathbb{C}[x, y, t^{-1}, v] / \langle t^{-1}v - xy \rangle$$

This receives a map from $\mathbb{C}[t^{-1}]$, corresponding to the projection

$$\pi\colon \operatorname{Specm}(\mathrm{Rees}(S, J)) \to \operatorname{Specm}(\mathbb{C}[t^{-1}]).$$

A fiber above any $\lambda \in \mathbb{C}$ corresponds to setting $t^{-1} = \lambda$ in $\mathbb{C}[t^{-1}]$ and then taking the pushout

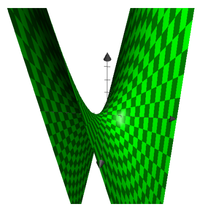
$$\begin{tikzcd}
 & \mathbb{C}[t^{-1}] \arrow[r, two heads] \arrow[d] & \mathbb{C}[t^{-1}]/\langle t^{-1} - \lambda \rangle \arrow[d] & \\
\mathrm{Rees}(S, J) = & \mathbb{C}[x, y, t^{-1}, v]/\langle t^{-1}v - xy \rangle \arrow[r] & \mathbb{C}[x, y, t^{-1}, v]/\langle t^{-1} - \lambda, t^{-1}v - xy \rangle & \cong \quad \mathbb{C}[x, y]
\end{tikzcd}$$

What if instead we computed the fiber over zero? In this case, $\lambda = 0$ and we have the quotient

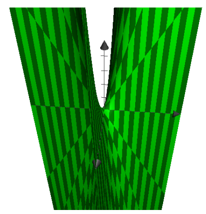
$$\mathbb{C}[x, y, t^{-1}, v] / \langle t^{-1}, t^{-1}v - xy \rangle \cong \mathbb{C}[x, y] / \langle xy \rangle .$$

So above any generic (read: nonzero) point, $\operatorname{Specm}(\mathrm{Rees}(S, J))$ looks like $\mathbb{C}^2$. But above zero, we get the union of coordinate axes, which is S/J again.

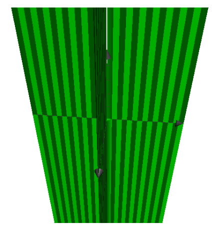
We may think of this as a family of hyperboloids $v = txy$ in three-space, parameterized by t. As $t \to \infty$, (and therefore $t^{-1} \to 0$), this becomes the union of the xv- and yv-planes.



$t = 1$



$t = 5$



$t = 1000$

From this example, we can learn about quotients of the Rees algebra.

Fact 3.22.

(a) For any nonzero $\lambda \in \mathbb{C}$, ${}^{\mathrm{Rees}(S,J)}/_{\langle t^{-1}-\lambda\rangle} \cong S$

(b) ${}^{\mathrm{Rees}(S,J)}/_{\langle t\rangle} \cong {}^{S}/_{J} \oplus {}^{J}/_{J^2} \oplus {}^{J^2}/_{J^3} \oplus \dots$

We have a name for $\mathrm{Rees}(S,J)/\langle t\rangle$.

Definition 3.23. For any ideal J, the **associated graded ring to the J-adic filtration of S** is

$$\mathrm{gr}_J(S) := {}^{S}/_{J} \oplus {}^{J}/_{J^2} \oplus {}^{J^2}/_{J^3} \oplus \dots$$

with S/J in degree zero, J/J^2 in degree 1, etc.

Definition 3.24. $\mathrm{Specm}(\mathrm{gr}_J(S))$ is called **the normal cone to** $V(J) \subseteq \mathrm{Specm}(S)$.

Remark 3.25. Usually, we have a map $S \to S/J$. For $\mathrm{gr}_J(S)$, we have a map $\mathrm{gr}_J(S) \to S/J$ given by taking the quotient module by J. There is also a map $S/J \to \mathrm{gr}_J(S)$ that puts S/J in degree zero. So there are maps both ways between the normal cone and $V(J)$. These work much like a section/retraction pair, and so the normal cone plays the role of tubular neighborhoods in differential topology.

Why do we study $\mathrm{gr}_J(S)$?

Example 3.26. If V is a finite dimensional vector space, and

$$V = V_0 \geq V_1 \geq \dots \geq V_k,$$

then we write

$$\mathrm{gr}\, V := {}^{V_0}/_{V_1} \oplus {}^{V_1}/_{V_2} \oplus \dots \oplus {}^{V_k}/_{0}.$$

Notice that $\dim(V) = \dim(\mathrm{gr}\, V)$. This is kind of silly, until we work with associated graded rings for algebras.

Now let J be an ideal in a graded ring S such that $J \leq S_{>0}$. Assume moreover $\dim(S_d)$ finite for each d, so there is a Hilbert function h_S. Then $\mathrm{gr}_J(S)$ has two gradings: one from the grading on S, and one from the usual one on $\mathrm{gr}_J(S)$. Moreover,

$$h_{\mathrm{gr}_J(S)} = h_S$$

with respect to the grading coming form that on S; this comes from the previous example with

$$S_d \geq S_d \cap J \geq S_d \cap J^2 \geq \ldots \geq S_d \cap J^{d+1}.$$

Example 3.27. Let $S = \mathbb{C}[x, y, z]/\langle xz - y^2 \rangle$. (This is known as the "second Veronese of $\mathbb{P}^1$.") Let $J = \langle y \rangle$. We have

$$\mathrm{Rees}(S, J) \cong {}^{\mathbb{C}[x, y, z, t^{-1}, j]}\big/_{\langle xz - y^2, t^{-1}j - y\rangle} \cong {}^{\mathbb{C}[x, z, t^{-1}, j]}\big/_{\langle xz - (t^{-1}j)^2\rangle}$$

Therefore,

$$\mathrm{gr}_J(S) \cong {}^{\mathrm{Rees}(S, J)}\big/_{\langle t^{-1}\rangle} \cong {}^{\mathbb{C}[x, z, j]}\big/_{\langle xz\rangle}.$$

This is homogeneous in j.

Exercise 3.28. Consider $\mathbb{C}[x_1, \ldots, x_n]$ and any polynomial $p(x_1, \ldots, x_n)$. Let $S = \mathbb{C}[x_1, \ldots, x_n]/\langle p \rangle$. Let f be the sum of the terms of p which have the lowest x_i-degree. Then

$$\mathrm{gr}_{\langle x_i \rangle} S \cong {}^{\mathbb{C}[x_1, \ldots, x_n]}\big/_{\langle f \rangle}.$$

Moreover,

$$\mathrm{gr}_{\langle x_1 \rangle} \mathrm{gr}_{\langle x_2 \rangle} \ldots \mathrm{gr}_{\langle x_1 \rangle} S \cong {}^{\mathbb{C}[x_1, \ldots, x_n]}\big/_{\langle m \rangle}$$

where $\langle m \rangle$ is a principal ideal generated by a single monomial m.

If I is any ideal with $I^\infty = 0$, and $\{i_1, i_2 \ldots, i_n\} = \{1, 2, \ldots, n\}$, then

$$\mathrm{gr}_{\langle x_{i_1} \rangle} \mathrm{gr}_{\langle x_{i_2} \rangle} \cdots \mathrm{gr}_{\langle x_{i_n} \rangle} \left({}^{\mathbb{C}[x_1, \ldots, x_n]}\big/_{I} \right)$$

is the quotient of $\mathbb{C}[x_1, \ldots, x_n]$ by a monomial ideal.

3.5 Singular Loci

We'll begin this section with some motivation from differential topology. Let $f\colon \mathbb{R}^n \to \mathbb{R}^k$ be a smooth map.

Definition 3.29. A **regular point** of f is $x \in \mathbb{R}^n$ such that $Df|_x\colon T_x\mathbb{R}^n \to T_{f(x)}\mathbb{R}^k$ is surjective.

Definition 3.30. A **regular value** of f is $y \in \mathbb{R}^k$ such that all $x \in f^{-1}(y)$ are regular points.

Theorem 3.31 (Sard)**.** *Most values in the image of* f *are regular values.*

This theorem in particular says that space filling curves don't happen in differential topology.

Theorem 3.32. $f^{-1}(y)$ *is smooth at all regular points* $x \in f^{-1}(y)$*. In particular, if* y *is a regular value of* f*, then* $f^{-1}(y)$ *is smooth.*

Example 3.33. Consider $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = xy$. Then

$$Df(x, y) = (y, x)$$

This is surjective as a map $\mathbb{R}^2 \to \mathbb{R}^2$, unless $x = y = 0$.

Example 3.34. Consider $\det\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

The derivative here is

$$D\det(a, b, c, d) = (d, -c, -b, a),$$

which is surjective unless $a = b = c = d = 0$.

We have an exact sequence

$$T_x f^{-1}(f(x)) \to T_x\mathbb{R}^n \to T_{f(x)}\mathbb{R}^k \to 0$$

The tangent space to the fiber $f^{-1}(f(x))$ is dimension $n - k$. In general, the preimage of a regular value is a complete intersection.

Example 3.35 (Non-example)**.** Consider the Klein bottle as a 2-manifold inside $\mathbb{R}^4$. There is no map $\mathbb{R}^4 \to \mathbb{R}^2$ that would have the Klein bottle as the zero set of some equations, because $\mathbb{R}^4$ and $\mathbb{R}^2$ are orientable but the Klein bottle is not.

Definition 3.36 (Nonstandard!)**.** A **semi-regular point** of f is a point x for which the rank of $Df|_x$ is maximized over all $x \in f^{-1}(f(x))$.

Theorem 3.37 (Improvement on Theorem 3.32)**.** $f^{-1}(y)$ *is smooth at all semi-regular points* $x \in f^{-1}(y)$

Now let's look at the algebraic geometry version of this.

Definition 3.38. Let $f\colon \mathbb{C}^n \to \mathbb{C}^k$. We say that f is **algebraic** if each f_i is polynomial.

Note that any arbitrary algebraic set in $\mathbb{C}^n$ is the preimage of a point under an algebraic function $f\colon \mathbb{C}^n \to \mathbb{C}^k$.

Definition 3.39. The **Jacobian** of an algebraic $f\colon \mathbb{C}^n \to \mathbb{C}^k$ is

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_1} & \frac{\partial f_k}{\partial x_2} & \cdots & \frac{\partial f_k}{\partial x_n} \end{bmatrix}$$

Definition 3.40. The **singular locus** of $f^{-1}(y)$ is defined as follows. Let M be the maximum possible rank of the Jacobian of f over all points in $f^{-1}(y)$. Then the singular locus of $f^{-1}(y)$ is

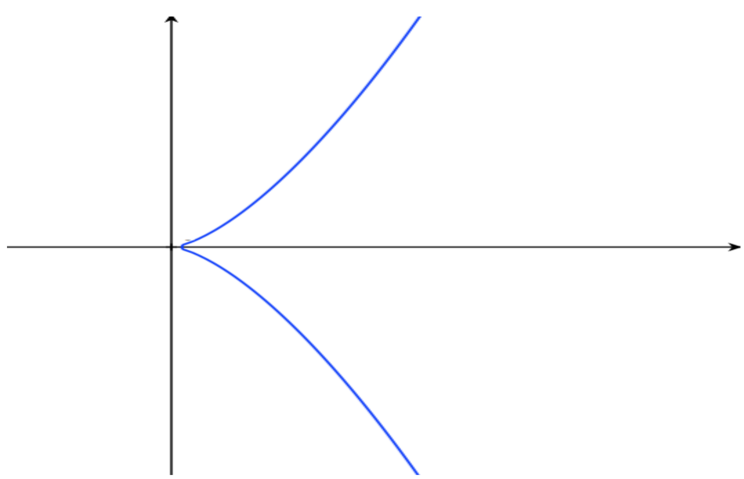
$$\{x \in f^{-1}(y) \mid \operatorname{rank} Df|_x < M\}$$

Remark 3.41. Actually, the singular locus is an algebraic set itself! The condition that the rank of $Df|_x$ is less than the maximum is equivalent to the condition that all $M \times M$ minors of $Df|_x$ are zero.

Example 3.42. Consider $f\colon \mathbb{C}^2 \to \mathbb{C}$ given by $f(x,y) = y^2 - x^3$. The derivative of f is

$$Df(x,y) = (-3x^2, 2y).$$

The maximum possible rank is 1, and it's only less than 1 where $(x,y) = (0,0)$. Hence, most fibers of f are smooth, except $f^{-1}(0)$ is singular at $(0,0)$.



Example 3.43. Consider the map $f\colon \mathbb{C}^2 \times \mathbb{C}^3 \to \mathbb{C}^2$ given by

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \longmapsto \begin{bmatrix} ae - bd \\ bf - ce \end{bmatrix}$$

The derivative of this map is

$$Df = \begin{bmatrix} e & -d & 0 & -b & a & 0 \\ 0 & f & -e & 0 & -c & b \end{bmatrix}$$

And this sometimes has rank 2 on $f^{-1}(0,0)$:

$$Df|_{\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

In fact, 2 is the maximum possible rank. The singular locus is where the rank is less than 2. Let's look at the ideal generated by all possible 2×2 minors of the determinant:

$$J = \langle ef, e^2, ec, eb, de, bf, dc - af, db, be, ae, bc, b^2, ab \rangle$$

The radical of J is

$$\sqrt{J} = \langle e, b, dc - af \rangle.$$

Hence, the singularities of f are where the two components $ae - bd$ and $bf - ce$ intersect.

3.6 (co)Tangent Spaces and Singularities

Recall that for any $p \in V(I) \subseteq \mathbb{C}^n$, there is a maximal ideal $M_p \geq I$ corresponding to p, with $V(M) = \{p\}$.

Definition 3.44. The **tangent cone** to $\text{Specm}(R)$ is $\text{Specm}\left(\text{gr}_M \left({}^{R}/_{I}\right)\right)$.

$$\text{gr}_M \left({}^{R}/_{I}\right) = {}^{R}/_{M} \oplus {}^{M}/_{M^2+I} \oplus {}^{M^2}/_{M^3+I} \oplus \dots$$

Definition 3.45. The **Zariski cotangent space** to $p \in V(I)$ is ${}^{M}/_{M^2+I}$, where M is the maximal ideal corresponding to p.

Remark 3.46. To understand why this is called the Zariski cotangent space, consider the case that $I = 0$ and $M = \langle x_1, \dots, x_n \rangle$. Then the Zariski cotangent space at the origin is the $\mathbb{C}$-algebra generated by $x_1, \dots, x_n$ with relations $x_i^2 = 0$. Hence, the Zariski cotangent space is spanned by the variables x_i, and we think of these x_i as functions on $\mathbb{C}^n$.

Definition 3.47. The **multiplicity of** $V(I)$ **at** x is the degree of $\text{gr}_M \left({}^{R}/_{I}\right)$. (Recall that the degree is the leading coefficient of the Hilbert function.)

Example 3.48. Consider $\mathbb{C}[x,y]/I$ where $I = \langle xy \rangle$. What is the multiplicity of a point on the x-axis? As long as it's not the origin, the multiplicity is 1. At the origin, the multiplicity is 2.

Example 3.49. Consider $\mathbb{C}[x,y]/\langle y^2 - x^3\rangle$. The multiplicity of any point not on the cusp is one, but what about at the cusp $(x = y = 0)$?

The Rees algebra of this is

$$\mathbb{C}[x,y,t^{-1},u,v]\Big/\langle y^2 - x^3, t^{-1}u - x, t^{-1}v - y, v^2 - t^{-1}u^3\rangle$$

At $t^{-1} = 0$, we have

$$\mathbb{C}[x,y,u,v]\Big/\langle x,y,v^2\rangle \cong \mathbb{C}[u,v]\Big/\langle v^2\rangle.$$

Hence, the multiplicity of this ring here is 2. The tangent cone looks like a double line along $y = 0$.

Definition 3.50. $V(I)$ is **regular** at a point p if $T_pV(I)$ is a vector space. Otherwise, $V(I)$ is **singular** at p.

Earlier we defined singularity as the points where the Jacobian has less than full rank. To be consistent, we should prove that this agrees with the new notion of singularity.

Lemma 3.51. *Let* $p \in V(I) \subseteq \mathbb{C}^n$, $\dim T_pV(I) = k$. *Then there is* $Y \supseteq V(I)$ *defined by* $\langle j_1, \ldots, j_{n-k}\rangle$ *with the same tangent space at* p.

Proof. Let M be the maximal ideal corresponding to I. Want to produce $j_1, \ldots, j_{n-k} \in I$. Consider the kernel of the map between cotangent spaces

$$K = \ker\left(M/_{M^2} \twoheadrightarrow M/_{M^2+I}\right).$$

This map of cotangent spaces is dual to the inclusion $T_pV(I) \hookrightarrow T_p\mathbb{C}^n$.

Pick $j_1, \ldots, j_{n-k} \in I$ to give a basis of K. Let $Y = V(\langle j_1, \ldots, j_{n-k}\rangle)$. By the choice of $j_1, \ldots, j_{n-k}$,

$$M/_{M^2+\langle j_1,\ldots,j_{n-k}\rangle} \cong M/_{M^2+I},$$

so Y and $V(I)$ have the same tangent space at p. □

Example 3.52. Let $I = \langle x\rangle \cap \langle x-1, y\rangle = \langle x(x-1), xy\rangle$. This is the variety $\{(x,y) \mid x = 0\} \cup \{(x,y)\}$. At some point p on the line $x = 0$, the tangent space has dimension 1.

Lemma 3.53. *Let* $Y = V(\langle j_1, \ldots, j_{n-k}\rangle)$ *as in the previous lemma. Let* $\vec{j} = (j_1, \ldots, j_{n-k})\colon \mathbb{C}^n \to \mathbb{C}^{n-k}$. *Then* $T_pY = \ker D\vec{j}$.

Proof sketch. Want to show that

$$0 \to T_pY \to T_p\mathbb{C}^n \xrightarrow{D\vec{j}} T_{\vec{j}(p)}\mathbb{C}^{n-k} \to 0$$

is exact. Since everything in sight is a finite-dimensional vector space, then the dual sequence is also exact:

$$0 \leftarrow {}^{M}/_{M^2+J} \leftarrow {}^{M}/_{M^2} \xleftarrow{(D\vec{j})^T} {}^{M'}/_{M'^2} \leftarrow 0$$

We can check that this latter sequence is exact. □

Example 3.54. What's the singular locus of $\widetilde{\mathbb{C}}^2$?

$$\widetilde{\mathbb{C}}^2 = \text{Projm}\left({}^{\mathbb{C}[x,y,u,v]}/_{\langle xv - yu\rangle}\right)$$

The Jacobian of $f(x,y,u,v) = xv - yu$ is

$$Df = (v, -u, -y, x).$$

This has rank less than the maximum when $u = v = x = y = 0$, but this never happens since we're working in projective space. Hence, there are no singular points.

Example 3.55. What about $\mathbb{C}^2$ blown up at $\langle x, y^2\rangle$? The blowup algebra is isomorphic to

$${}^{\mathbb{C}[x,y,u,v]}/_{\langle vx - uy^2\rangle}.$$

The Jacobian of $f(x,y,u,v) = vx - uy^2$ is

$$Df = (v, -2uy, -y^2, x)$$

which has lower than the maximum rank when $x = y = 0 = v$. Hence, there are singularities when $x = y = v = 0$.

3.7 Toric Varieties

Definition 3.56. A **rational polyhedral cone** C in $\mathbb{R}^n$ is one defined by finitely many $\vec{w}_1, \ldots, \vec{w}_n \in \mathbb{Q}^n$

$$C = \left\{\vec{v} \mid \vec{v} \cdot \vec{w}_i \geq 0\right\}$$

Fact 3.57. $(C \cap \mathbb{Z}^n, +)$ *is a finitely generated abelian group.*

Definition 3.58. An **affine Toric variety** is Specm $\mathbb{C}[(C \cap \mathbb{Z}^n, +)]$.

Remark 3.59. These are called Toric varieties because they have an action of the n-torus $(\mathbb{C}^\times)^n$.

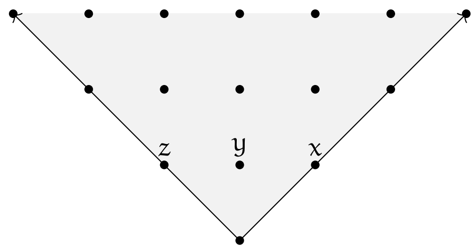
Example 3.60. Consider $C = \{x \geq 0 \mid x \in \mathbb{R}\}$. Then $C \cap \mathbb{Z} = \mathbb{N}$, and we have $\mathbb{C}[\mathbb{N}] \cong \mathbb{C}[x]$. The corresponding toric variety is line.

Example 3.61. Let C be the first quadrant. Then $\mathbb{C} \cap \mathbb{Z}^2 = \mathbb{N}^2$. The corresponding monoid algebra is

$$\mathbb{C}[\mathbb{N}^2] = \mathbb{C}[x, y]$$

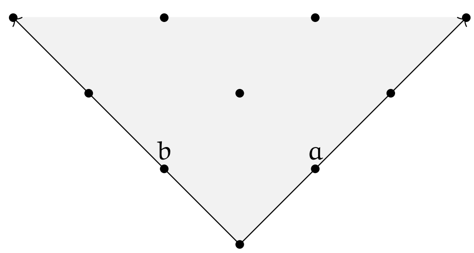
and the toric variety is the plane.

Example 3.62. Consider the cone C with $C \cap \mathbb{Z}^2$ as follows:



The corresponding monoid algebra is $\mathbb{C}[x, y, z]/\langle xz - y^2 \rangle$.

Example 3.63. Consider the cone C with $C \cap \mathbb{Z}^2$ as follows:



The corresponding monoid algebra is $\mathbb{C}[C \cap \mathbb{Z}] = \mathbb{C}[a^2, ab, b^2] \cong \mathbb{C}[a, b]^{\mathbb{Z}/2}$. Then $\operatorname{Specm} \mathbb{C}[C \cap \mathbb{Z}] \cong \mathbb{C}/(\mathbb{Z}/2)$.

Definition 3.64. Let $P \subseteq \mathbb{R}^n$ be defined by finitely many affine-linear inequalities $\{\vec{v} \mid \vec{v} \cdot \vec{w} \geq c_i\}$ for $w_i \in \mathbb{Z}^n$ and $c_i \in \mathbb{Z}$.

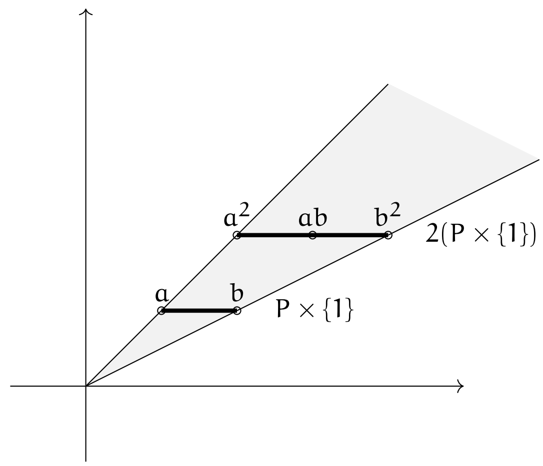
The **Toric Variety** associated to P is

$$\mathrm{TV}_P := \operatorname{Projm} \mathbb{C}\left[\overline{\mathbb{R}_+ (P \times \{1\})} \cap \mathbb{Z}^{n+1}\right].$$

The grading comes from

Example 3.65. Take $P = \{1\} \subseteq \mathbb{R}$. Then TV_P is a point.

Example 3.66. If $P = [1,2] \subseteq \mathbb{R}$, then $\overline{\mathbb{R}_+(P \times \{1\})}$ is the shaded region below.



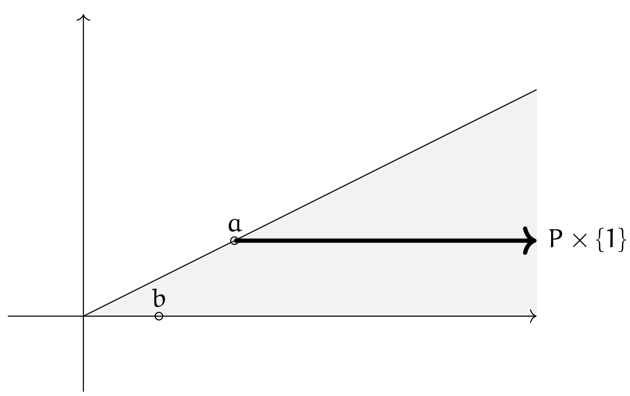
$$\mathrm{TV}_P = \operatorname{Projm} \mathbb{C}[a^{(1)}, b^{(1)}] = \mathbb{CP}^1$$

Example 3.67. If P is the standard n-simplex

$$\left\{ (r_1, \dots, r_n) \mid r_i \geq 0, \sum_i r_i \leq 1 \right\},$$

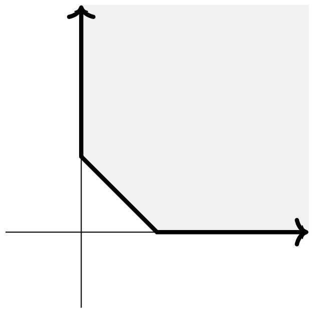
then $\mathrm{TV}_P \cong \mathbb{CP}^n$.

Example 3.68. If $P = [2, \infty)$, then $\overline{\mathbb{R}_+(P \times \{1\})}$ is the shaded region below.

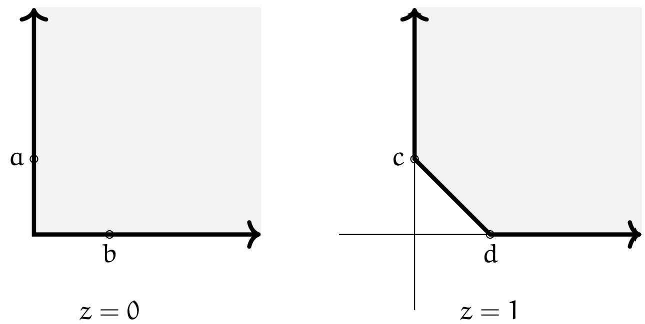


$$\mathrm{TV}_P = \operatorname{Projm} \mathbb{C}[a^{(1)}, b^{(0)}] \cong \operatorname{Specm} \mathbb{C}[b^{(0)}] = \mathbb{C}.$$

Example 3.69. Consider the polytope $P \subseteq \mathbb{R}^2$ below.



Slices of the cone on P in $\mathbb{R}^3$ look like



Notice that, as vectors, $a + d = b + c$. This gives the relations $ad - bc$ in the monoid algebra on this cone.

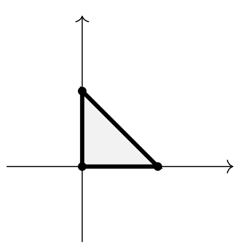
$$\mathrm{TV}_P = \mathrm{Projm}\left(\mathbb{C}[a^{(0)}, b^{(0)}, c^{(1)}, d^{(1)}] \big/ \langle ad - bc \rangle\right) = \widetilde{\mathbb{C}}^2.$$

Example 3.70. If P is a square in $\mathbb{R}^2$, then

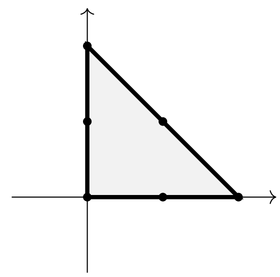
$$\mathrm{TV}_P = \mathrm{Projm}\left(\mathbb{C}[a^{(1)}, b^{(1)}, c^{(1)}, d^{(1)}] \big/ \langle ad - bc \rangle\right) \cong \mathbb{CP}^1 \times \mathbb{CP}^1.$$

This is consistent with Example 3.66, which says that if P is an interval, $\mathrm{TV}_P = \mathbb{CP}^1$. Here, P is the product of an interval with an interval, so $\mathrm{TV}_P = \mathbb{CP}^1 \times \mathbb{CP}^1$. This is more generally true.

Example 3.71. We know that this polytope gives $\mathbb{CP}^2$.



But this polytope also give $\mathbb{CP}^2$.



Theorem 3.72 (Ehrhart 1960's). *Let P be the convex hull of finitely many points in $\mathbb{Z}^n$. Then $h_{TV_P}(d)$ is polynomial for $d \geq 0$, and*

$$h_{TV_P}(d) = \#\{\text{lattice points in } d \cdot P\}$$
$$h_{TV_P}(-d) = (-1)^{\dim P}\#\{\text{lattice points in interior of } d \cdot P\}$$

Remark 3.73. The degree of $h_{TV_P}(d)$ is the volume of P in simplex units: how many unit 2-simplicies does it take to fill P?

Definition 3.74. Call P **smooth** if TV_P is regular.

Theorem 3.75. *TV_P is regular iff each corner is a cone isomorphic to $\mathbb{N}^{\dim P}$ as monoids, or via $GL_{\dim P}(\mathbb{Z})$ transformations.*

Definition 3.76. If $S = \bigoplus_{i \in \mathbb{N}} S_i$ is a graded ring, the n**-th Veronese** is

$$\mathrm{Ver}_n(S) = \bigoplus_{i \in \mathbb{N}} S_{1+in}.$$

Remark 3.77. $\mathrm{Projm}(S) \cong \mathrm{Projm}(\mathrm{Ver}_n(S))$.

Theorem 3.78. *Let R be a polynomial ring and I a graded ideal of R. Let $S = R/I$. Then $\mathrm{Ver}_n(S)$ is generated in degree 1 with relations in degree 2.*

Conjecture 3.79. *If P is smooth, then*

$$\mathbb{C}\left[\overline{\mathbb{R}_+(P \times \{1\})} \cap \mathbb{Z}^{n+1}\right]$$

is generated in degree 1 with relations in degree 2.

Theorem 3.80. *Let C be a rational polynomial cone, let $D \subseteq C$ be nonempty such that $C + D \subseteq D$. Let*

$$R = \mathbb{C}\left[\overline{\mathbb{R}_+(C \times \{1\})} \cap \mathbb{Z}^{n+1}\right],$$
$$I = \mathbb{C}\left[\overline{\mathbb{R}_+(D \times \{1\})} \cap \mathbb{Z}^{n+1}\right]$$

Then

(a) I *is an ideal of* R*, and*

(b) TV_D *is the blowup of* TV_C *along this ideal.*

Proof sketch. Compute the blowup algebra. □

Example 3.81. Take C to be the $z = 0$ slice of Example 3.69, and let D be the $z = 1$ slice. TV_D is the blowup of $TV_C = \mathbb{C}^2$ at a point.

4 Schemes

4.1 Non-closed points

Recall that $\mathrm{Specm}(S)$ is the set of maximal ideals in S, and an ideal is maximal if and only if the quotient by it is a field.

When S is finitely generated over $\mathbb{C}$ (that is, $\S \cong \mathbb{C}[x_1, \dots, x_n]/I$), the Nullstellensatz implies that all of these quotient fields are $\mathbb{C}$.

When we have a map $S \to T$ of such algebra, there is a corresponding map $\mathrm{Specm}(T) \to \mathrm{Specm}(S)$.

What if S, T do not contain $\mathbb{C}$?

Example 4.1. Consider the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$. $\mathrm{Specm}(\mathbb{Z})$ consists of all ideals $\langle p \rangle$ where p is prime. $\mathrm{Specm}(\mathbb{Q}) = \{\langle 0 \rangle\}$, and there are way too many choices for the map $\mathrm{Specm}(\mathbb{Q}) \to \mathrm{Specm}(\mathbb{Z})$. Which one should we take? It's ambiguous.

Definition 4.2. The **spectrum** of a commutative ring S is the set of its prime ideals, denoted $\mathrm{Spec}(S)$.

Given $\phi \colon S \to T$, there is a map $\phi^* \colon \mathrm{Spec}(T) \to \mathrm{Spec}(S)$ given by $\phi^*(I) = \{a \in S \mid \phi(a) \in I\}$.

Example 4.3. Consider the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$. Now,

$$\mathrm{Spec}(\mathbb{Z}) = \big\{ \langle p \rangle \bigm| p \text{ prime} \big\} \cup \{\langle 0 \rangle\}$$

and $\mathrm{Spec}(\mathbb{Q}) = \{\langle 0 \rangle\}$. The map $\mathrm{Spec}(\mathbb{Q}) \to \mathrm{Spec}(\mathbb{Z})$ is now clear.

Definition 4.4. The **Zariski Topology** on $\mathrm{Spec}(A)$ for a commutative ring A has one closed set for each ideal I, consisting of all prime ideals containing I.

$$\big\{P \geq I \bigm| P \text{ prime ideal}\big\}$$

Remark 4.5. Recall that we defined a Zariski-closed set in $\mathrm{Specm}(S)$ is the set of all maximal ideals containing I, for some ideal I. In $\mathrm{Specm}(S)$, each maximal ideal M gave a closed set with one element, namely $\{M\}$. Hence, we say that "points are closed," and M is a **closed point**.

In Spec(S), each prime P gives a closed set $\{Q \geq P \mid Q \text{ prime}\}$. This contains P, but may contain other ideals as well. Hence, the closure of the point $P \in \operatorname{Spec}(S)$ may be larger than just P itself. So we say that $P \in \operatorname{Spec}(S)$ is a **non-closed point**.

Example 4.6.

$$\operatorname{Spec} \mathbb{C}[x] = \{\langle x - \lambda \rangle \mid \lambda \in \mathbb{C}\} \cup \{\langle 0 \rangle\}.$$

The point 0 is not closed, and we picture it as suffusing the whole space. In fact, $\langle 0 \rangle \subseteq \langle x - \lambda \rangle$ for all $\lambda \in \mathbb{C}$, so the closure of $\langle 0 \rangle$ is all of $\operatorname{Spec} \mathbb{C}[x]$. Hence, we say that $\langle 0 \rangle$ is a **generic point** of this topological space.

Definition 4.7. A **generic point** P of Spec(S) is a minimal prime P of S.

Example 4.8.

$$\operatorname{Spec}\left(\mathbb{C}[x,y] / \langle xy \rangle \right) = \{\langle x - \lambda, y \rangle, \langle y - \lambda, x \rangle, \langle x \rangle, \langle y \rangle\}_{\lambda \in \mathbb{C} \setminus \{0\}}$$

The generic points of this space are $\langle x \rangle$ and $\langle y \rangle$.

Example 4.9. Consider the ring $A = \mathbb{C} \oplus \mathbb{C}[x]$. This corresponds to the disjoint union of a point and a line.

Then

$$\operatorname{Spec}(A) = \{\langle 0, 1 \rangle, \langle (1,0), (0, x - \lambda) \rangle, \langle (1,0) \rangle\}_{\lambda \in \mathbb{C}}$$

The ideal $\langle (0,1) \rangle$ is both a minimal prime and a maximal ideal. Hence, it is both a closed point and a generic point.

Example 4.10. Consider $\mathbb{C}[x] \to \mathbb{C}[y]$ given by $x \mapsto y^2$. As a map on the spectra, this corresponds to the map $\mathbb{C} \to \mathbb{C}$ given by $z \mapsto z^2$. This is one-to-one over zero, and two-to-one elsewhere.

4.2 Localization at a Point

Let A be a commutative ring and P a prime ideal inside A. Consider the composite

$$A \to A/P \to \operatorname{Frac}\left(A/P\right)$$

where Frac(D) is the fraction field of a domain D. This gives a map backwards on spectra

$$\operatorname{Spec}\left(\operatorname{Frac} A/P\right) \to \operatorname{Spec}\left(A/P\right) \to \operatorname{Spec}(A).$$

If k is a field, Spec(k) has just one element as a set, namely 0. So we think of Spec(k) geometrically as just a single point, and the map $\operatorname{Spec}(k) \to \operatorname{Spec}(A)$ geometrically is a point in the space Spec(A).

Definition 4.11. Given a prime ideal P, the **localization of** S **at** P is the set $S_P = \{\frac{s}{t} \mid t \notin P\}$.

What do the prime ideals of S_P look like? If $\langle r \rangle \leq S_P$ is not the whole ring, then $r \in P$. Hence, $\langle r \rangle \subseteq P$, so the only maximal ideal of S_P is the image of P under $S \to S_P$. So $\operatorname{Spec}(S_P)$ has only one closed point,

For any ring S, we have a map $S \to S_P$. This gives a map on spectra $\operatorname{Spec}(S_P) \to \operatorname{Spec}(S)$ as usual. In this case, the unique closed point of $\operatorname{Spec}(S_P)$ maps to $P \in \operatorname{Spec}(S)$.

Definition 4.12. A **local ring** is a ring with a unique maximal ideal.

Example 4.13. Let's localize $\mathbb{C}[x]$ at 0. This localization is

$$A = \left\{ \frac{p(x)}{q(x)} \,\middle|\, x \nmid q \right\}.$$

The spectrum $\operatorname{Spec}(A)$ of this localization consists of a generic point and a closed point; we have lost all of the others that existed in $\operatorname{Spec}\mathbb{C}[x]$.

Example 4.14. Consider the inclusion $\mathbb{R}[x] \to \mathbb{C}[x]$. The prime ideals of $\mathbb{C}[x]$ are 0 and $\langle x - \lambda \rangle$ for $\lambda \in \mathbb{C}$. The prime ideals of $\mathbb{R}[x]$ contain 0 and $\langle x - r \rangle$ for $r \in \mathbb{R}$, but also irreducible quadratics $\langle (x - z)(x - \bar{z}) \rangle$ for $z \in \{a + bi \in \mathbb{C} \mid b > 0\}$.

Each point of $\operatorname{Spec}\mathbb{R}[x]$ of the form $\langle (x - z)(x - \bar{z}) \rangle$ splits into two ideals $\langle x - z \rangle$ and $\langle x - \bar{z} \rangle$ in $\operatorname{Spec}\mathbb{C}[x]$. So the map $\operatorname{Spec}\mathbb{C}[x] \to \operatorname{Spec}\mathbb{R}[x]$ two-to-one almost everywhere.

This corresponds to the fact that $\mathbb{C}[x]$ is free of rank 2 over $\mathbb{R}[x]$, or rather $\dim_{\mathbb{R}(x)} \mathbb{C}(x) = 2$.

Remark 4.15. In classical algebraic geometry, polynomials $f \in \mathbb{C}[x_1, \ldots, x_n]$ are functions on $\mathbb{C}^n$. We want to have the same intuition for elements of S as functions on $\operatorname{Spec} S$.

Well, given $f \in S$, and $P \in \operatorname{Spec}(S)$, the value of this function f at the point P is the image of $f \in S$ under

$$S \to {}^{S}/_{P} \to \operatorname{Frac}\left({}^{S}/_{P}\right).$$

(They don't land in the same place!)

Example 4.16. Depending on whether or not we input the generic point into a function $f \in \mathbb{C}[x]$, we land in either $\mathbb{C}$ or $\mathbb{C}(x)$.

For a more abstract example, $45 \in \mathbb{Z}$ is a function on $\operatorname{Spec}(\mathbb{Z})$ that takes values in $\mathbb{Z}/\langle p \rangle$ for p a prime number, or in $\mathbb{Q}$ when we take the generic point of $\mathbb{Z}$.

Example 4.17. Consider $S = \mathbb{C}[x]/\langle x^2 \rangle$. This has a unique prime ideal, namely $\langle x \rangle$. We cannot distinguish the spectrum of S from the spectrum of a field – it only has one point. And evaluating $a + bx \in S$ as a function on $\langle x \rangle$, we get $a \in \mathbb{C}$. In particular, b might be any value, and so ring elements might not be distinguished by their values at points.

Definition 4.18. $\mathrm{Spec}(S)$ has a **distinguished open set** D_f for each $f \in S$, $f \neq 0$, given by

$$D_f := \{P \mid f \in P\} \subseteq \mathrm{Spec}(S).$$

Definition 4.19. The **Zariski topology** on $\mathrm{Spec}(S)$ is generated by the distinguished open sets.

4.3 Presheaves

Definition 4.20. Let X be a topological space with topology τ. We may consider τ as a category whose objects are open sets of X and morphisms are inclusions. Let $\mathbf{C}$ be a category. A **presheaf on** X **with values in C** is a contravariant functor

$$\mathcal{F}\colon \tau \to \mathbf{C}.$$

This means that for all open $U \subseteq X$, there is an object of $\mathbf{C}$,

$$\Gamma(U; \mathcal{F}) := \mathcal{F}(U)$$

called the **sections of** $\mathcal{F}$ **over** U, and for each inclusion $V \hookrightarrow U$, we have a **restriction map**

$$\mathcal{F}(U) \to \mathcal{F}(V)$$

which is the identity for $U = V$, and for each $W \hookrightarrow V \hookrightarrow U$, the following diagram of restrictions commutes:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[rr] \arrow[dr] & & \mathcal{F}(V) \arrow[dl] \\ & \mathcal{F}(W). & \end{tikzcd}$$

Example 4.21. The set of all smooth functions on R is a presheaf, whose value on an open set U is $C^\infty(U; \mathbb{R})$. The restriction maps are restriction of domains.

Example 4.22. The contravariant functor taking any nonempty open set $U \subseteq \mathbb{R}$ to set $\mathbb{R}$ is the presheaf of constant functions on $\mathbb{R}$. It takes $\emptyset$ to 0. We will later see that this is not a sheaf.

Locally constant functions on $\mathbb{R}$ fit together into a presheaf

$$U \mapsto \mathbb{R}^c$$

where c is the number of connected components of U. We will later see that this is a sheaf.

Example 4.23. Consider the sheaf $\mathcal{F}$ of $\mathbb{C}$-analytic functions on subsets of $\mathbb{CP}^1$. For $\mathbb{C} \subseteq \mathbb{CP}^1$, $\mathcal{F}(\mathbb{C})$ is an infinite-dimensional vector space. But by Liouville's theorem, $\mathcal{F}(\mathbb{CP}^1) \cong \mathbb{C}$.

So the functions on the whole space can't distinguish it from a point, but the functions on open sets can. This is one of the reason that presheaves are worth thinking about.

Example 4.24. Let $X = \mathrm{Spec}(S)$, and $U \subseteq X$ open. If U is a distinguished open set D_f, define a presheaf $\mathcal{F}$ by

$$U \mapsto S[f^{-1}].$$

Example 4.25. Rational functions on $\mathbb{C}$ with the Zariski topology. An open set U under this topology is a finite collection of points – the vanishing of a set of polynomials. In this case, the value of this presheaf on U is the rational functions whose poles lie in points of U.

Let U, V be open subsets of X. Let $\mathcal{F}$ be a presheaf on X. Consider the commuting diagram

$$\begin{tikzcd} & \mathcal{F}(U) \arrow[dr] & \\ \mathcal{F}(U \cup V) \arrow[ur] \arrow[dr] & & \mathcal{F}(U \cap V) \\ & \mathcal{F}(V) \arrow[ur] & \end{tikzcd}$$

From the diagram above, we get a map from $\mathcal{F}(U \cup V)$ to the fiber product of $\mathcal{F}(U)$ and $\mathcal{F}(V)$ over $\mathcal{F}(U \cap V)$:

$$\mathcal{F}(U \cup V) \to \mathcal{F}(U) \times_{\mathcal{F}(U \cap V)} \mathcal{F}(V). \tag{4.1}$$

Definition 4.26. The **sheaf axiom** states that (4.1) should always be an isomorphism. In other words, $\mathcal{F}$ preserves fibered products.

Remark 4.27. A better sheaf axiom is the same story, but with an arbitrary number of open sets instead of just two: for any open set $U \subseteq X$ and any open cover $\{U_i\}$ of U, the following diagram is an equalizer

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j).$$

Definition 4.28. A **sheaf** is a presheaf that satisfies the sheaf axiom.

Example 4.29. Let $\beta \colon E \to X$ be a complex vector bundle. Then we may define a presheaf $\mathcal{F}$ whose value on $U \subseteq X$ is the space of sections of E over U:

$$\mathcal{F}(U) := \{\sigma \colon U \to E|_U \mid \beta\sigma = \mathrm{id}_U\}.$$

This is why we call $\mathcal{F}(U)$ the sections over U. This is a sheaf.

If $E = X \times \mathbb{C}^n$, then $\mathcal{F}(U)$ is the set of continuous functions $U \to \mathbb{C}^n$.

Example 4.30. If $X = \mathbb{C}$, let

$$\mathcal{F}(U) = \begin{cases} \mathbb{C} & 3 \in U, \\ 0 & 3 \notin U. \end{cases}$$

This is an example of a **skyscraper sheaf.**

4.4 Operations on Sheaves

Definition 4.31. Let X be a topological space. Given two sheaves $\mathcal{F}_1, \mathcal{F}_2$, define

(a) **the direct sum** $(\mathcal{F}_1 \oplus \mathcal{F}_2)(U) = \mathcal{F}_1(U) \oplus \mathcal{F}_2(U)$,

(b) **the tensor product** $(\mathcal{F}_1 \otimes \mathcal{F}_2)(U) = \mathcal{F}_1(U) \otimes \mathcal{F}_2(U)$.

Remark 4.32. $\mathcal{F}_1 \oplus \mathcal{F}_2$ is a sheaf is both $\mathcal{F}_1$ and $\mathcal{F}_2$ are sheaves. However, the infinite direct sum is not a sheaf, although the infinite product is.

The tensor product of sheaves is a presheaf, but it is rarely a sheaf. We will need to take the sheafification in general to make it a sheaf.

Example 4.33. The **constant sheaf** $\mathcal{K}$ (sometimes $\underline{K}$) with values in K. For an open set U, $\mathcal{K}(U)$ is the set of locally constant maps $U \to K$.

If K is a group, $\mathcal{K}$ is a sheaf of groups, and if K is a ring, $\mathcal{K}$ is a sheaf of rings.

If K has a nondiscrete topology, then we may replace $\mathcal{K}(U)$ by the set of continuous maps $U \to K$. If K has a discrete topology, then continuous maps *are* locally constant maps.

Why is this a sheaf? Given an open cover $\{U_i\}_{i \in I}$ of U, and functions $f_i \colon U_i \to K$ that agree on intersections, is there $f \colon U \to K$ such that $f|_{U_i} = f_i$?

Yes, we may define $f(x)$ on any $x \in U$ by taking any $U_i \ni x$ and setting $f(x) = f_i(x)$. Then it's easy to check that f is continuous.

Definition 4.34. Given a map $p \colon Y \to X$ of spaces, we the **sheaf of sections** of p is

$$\mathcal{F}_Y(U) := \left\{ s \colon U \to Y \;\middle|\; ps = \mathrm{id}_U \right\}$$

Example 4.35. Consider the projection $p\colon X \times K \to X$ where X and K are topological spaces – K may be discrete (e.g. a ring, abelian group, etc.).

Let $\mathcal{F}$ be the sheaf of sections of p. Over any open set U, a section $s\colon U \to X \times K$ is determined by its projections. So $s = (\mathrm{id}_U, f)$ where $f\colon U \to K$ is an arbitrary continuous map. Hence, $\mathcal{F} = \mathcal{K}$, where $\mathcal{K}$ is the constant sheaf at K.

Remark 4.36. We can ask for the sheaf of sections in any category, not just topological spaces. For example, we could have manifolds, or schemes, or sets. We'll be interested in schemes.

Example 4.37. In algebraic geometry, consider the Zariski topology on $\mathrm{Spec}(R)$ for a ring R. There are open sets $D_f = \mathrm{Spec}(R_f) = \{P \in \mathrm{Spec}(R) \mid f \notin P\}$. As in Remark 4.15, we think of this as the points $x \in X$ where $f(x) \neq 0$.

There is a sheaf $\mathcal{O}_X$ such that $\mathcal{O}_X(D_f) = R_f$, and restriction maps $\mathcal{O}_X(D_f) \to \mathcal{O}_X(D_{fg})$ are localizations $R_f \to R_{fg}$.

4.5 Stalks and Sheafification

Definition 4.38. Given a presheaf $\mathcal{F}$, define the **stalk** of $\mathcal{F}$ at $x \in X$ by

$$\mathcal{F}_x := \operatorname*{colim}_{U \ni x} \mathcal{F}(U).$$

An element of the stalk is a section $s \in \mathcal{F}(U)$, where we identify $s \in \mathcal{F}(U)$ and $s' \in \mathcal{F}(V)$ if there is some $W \subseteq U \cap V$ such that $s|_W = s'|_V$

Lemma 4.39. *If $\mathcal{F}$ is a sheaf, then a section $s \in \mathcal{F}(U)$ is determined by its images in the stalks $\mathcal{F}_x$ for all $x \in U$. In other words, there is an injective map*

$$\mathcal{F}(U) \hookrightarrow \prod_{x \in U} \mathcal{F}_x.$$

Proof. Suppose given two sections $s, s' \in \mathcal{F}(U)$ such that $s_x = s'_x \in \mathcal{F}_x$ for each $x \in U$. Then $s|_{V_x} = s'|_{V_x}$ for each $V_x \subseteq U$ such that $V_x \ni x$.

Letting x range over all points $x \in U$, we get an open cover of U by such sets V_x. Then $s|_{V_x} = s'|_{V_x}$ for all x, so $s = s'$ by the sheaf axiom. □

Remark 4.40. This gives us a different way of thinking about sheaves. A sheaf $\mathcal{F}$ has sections over $\mathcal{F}(U)$ given by functions $U \ni x \mapsto s_x \in \mathcal{F}_x$ which locally come from $s \in \mathcal{F}(V)$ for some $V \subseteq U$.

In fact, we can give $\overline{\mathcal{F}} = \sqcup_{x \in X} \mathcal{F}_x$ a topology called the **espace étalé** so that $\mathcal{F}(U)$ is the sheaf of continuous sections of $\pi\colon \overline{\mathcal{F}} \to X$.

Definition 4.41. Given a presheaf $\mathcal{F}$, the **sheafification** $\mathcal{F}^a$ of $\mathcal{F}$ where $\mathcal{F}^a(U)$ is the subset of $\prod_{x \in U} \mathcal{F}_x$ consisting of $(s_x)_{x \in U}$ such that for all $x \in U$, there is a section $t \in \mathcal{F}(V)$ for some $V \subseteq U$, $V \ni x$ with $t_x = s_x$.

Example 4.42. Let K be a set (or abelian group, ring, etc.). Consider the presheaf $\mathcal{F}$ that is constantly K for any $U \subseteq X$, $\mathcal{F}(U) = K$. We may consider this as the set of constant maps $U \to K$.

The sheafification of $\mathcal{F}$ is $\mathcal{F}^a = \mathcal{K}$, where $\mathcal{K}$ is the locally constant sheaf on X from before. Notice that if U_1, U_2 are connected disjoint open subsets of X, $\mathcal{F}(U_1 \cup U_2) = K$, yet $\mathcal{F}^a(U_1 \cup U_2) = K \times K$.

Remark 4.43. Sheafication is a left-adjoint to the forgetful functor from sheaves to presheaves. So if $\mathcal{G}$ is a sheaf, maps $\mathcal{F} \to \mathcal{G}$ are the same as maps $\mathcal{F}^a \to \mathcal{G}$ (notice that maps of sheaves are the same as maps of presheaves). This gives a universal property for the sheafification.

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{G} \\ \mathcal{F}^a \arrow[ur, dashed, "\exists !"'] & \end{tikzcd}$$

If $\mathcal{F}$ is already a sheaf, then $\mathrm{id}\colon \mathcal{F} \to \mathcal{F}$ satisfies this universal property and so $\mathcal{F} = \mathcal{F}^a$.

4.6 Limits and Colimits of sheaves

Remark 4.44. The previous remark implies formally that the forgetful functor $U\colon \mathbf{Sh}(X) \to \mathbf{PSh}(X)$ preserves limits (products, equalizers, fiber products, kernels) because it is a right adjoint. So to form a limit of a collection of sheaves, it suffices to form a limit in the category of presheaves and the result will already be a sheaf.

On the other hand, to form a colimit (direct sums, cokernels, coequalizers, pushouts) in the category of sheaves, take the colimit in presheaves and then sheafify. As a left adjoint, the sheafification preserves colimits.

Example 4.45. Let $\{F_i\}_{i \in I}$ be a collection of sheaves of abelian groups. Consider the following object in the category of presheaves, where $U\colon \mathbf{Sh}(X) \to \mathbf{PSh}(X)$ is the forgetful functor:

$$\mathcal{F} = \bigoplus_{i \in I} U\mathcal{F}_i.$$

Maps $\mathcal{F} \to \mathcal{G}$ are the same as a collection of maps $\{\mathcal{F}_i \to \mathcal{G}\}_{i \in I}$. Then by the universal property, maps $\mathcal{F} \to \mathcal{G}$ are the same as maps $\mathcal{F}^a \to \mathcal{G}$, but sheafification preserves colimits. Hence, $\mathcal{F} = \mathcal{F}^a$.

$$\begin{tikzcd} \bigoplus_{i \in I} U\mathcal{F}_i \arrow[r] \arrow[d] & \mathcal{G} \\ (\bigoplus_{i \in I} \mathcal{F}_i)^a \arrow[ur, dashed] & \end{tikzcd}$$

Example 4.46. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of abelian groups. Then for any $\phi\colon \mathcal{F} \to \mathcal{G}$, the cokernel of ϕ in the category of presheaves is given by the presheaf

$$\operatorname{coker}^{\mathbf{PSh}(X)}(\phi)(U) = \mathcal{G}(U)/\operatorname{im}(\phi(U)).$$

The **cokernel** in the category of sheaves is then the sheafification of this.

Definition 4.47. Let $\phi\colon \mathcal{F} \to \mathcal{G}$ be a map of sheaves over X. We say that ϕ is **injective** or **surjective** if $\phi_x\colon \mathcal{F}_x \to \mathcal{G}_x$ is **injective** or **surjective** for all $x \in X$.

With this definition, injections are monomorphisms and surjections are epimorphisms.

Fact 4.48.

(a) If $\phi\colon \mathcal{F} \to \mathcal{G}$ is injective, then

$$\phi_*\colon \operatorname{Hom}_{\mathbf{Sh}(X)}(\mathcal{H}, \mathcal{F}) \to \operatorname{Hom}_{\mathbf{Sh}(X)}(\mathcal{H}, \mathcal{G})$$

is injective.

(b) If $\phi\colon \mathcal{F} \to \mathcal{G}$ is surjective, then

$$\phi^*\colon \operatorname{Hom}_{\mathbf{Sh}(X)}(\mathcal{G}, \mathcal{H}) \to \operatorname{Hom}_{\mathbf{Sh}(X)}(\mathcal{F}, \mathcal{H})$$

is injective.

Example 4.49. However, there are maps $\phi\colon \mathcal{F} \to \mathcal{G}$ that are surjective even when they are not surjective as maps of presheaves.

Consider $X = \mathbb{C} \setminus \{0\}$. Take a double cover of X as follows: let $Y \subseteq (\mathbb{C} \setminus \{0\}) \times \mathbb{C}$ be the algebraic set $Y = \{(f, g) \mid g^2 = f\}$. Any nonzero complex number *locally* has two square roots – positive or negative.

Yet, Y has no global sections because there is no global complex square root function. However, the map $Y \to X$ is surjective on sheaves of sections, but not surjective on global sections.

Example 4.50. Let $\mathcal{O}^{\times}_{\mathbb{C}\setminus\{0\}}$ be the sheaf of nonzero functions on $\mathbb{C} \setminus \{0\}$. There is a map

$$\mathcal{O}^{\times}_{\mathbb{C}\setminus\{0\}} \to \mathcal{O}^{\times}_{\mathbb{C}\setminus\{0\}}$$

that takes a function to its square. This is not surjective on global sections, yet surjective as a map of sheaves.

Definition 4.51. Let $\mathcal{R}$ be a sheaf of rings over X. A **sheaf of modules** $\mathcal{M}$ is a sheaf of abelian groups over X such that each $\mathcal{M}(U)$ is a module over $\mathcal{R}$ and moreover the following diagrams all commute.

$$\begin{tikzcd}
\mathcal{R}(U) \times \mathcal{M}(U) \arrow[r] \arrow[d] & \mathcal{M}(U) \arrow[d] \\
\mathcal{R}(V) \times \mathcal{M}(V) \arrow[r] & \mathcal{M}(V)
\end{tikzcd}$$

Definition 4.52. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be sheaves of modules over $\mathcal{R}$. The **tensor product** of $\mathcal{M}_1$ and $\mathcal{M}_2$ over $\mathcal{R}$ is the sheafification of the tensor product of $\mathcal{M}_1$ and $\mathcal{M}_2$ in the category of presheaves:

$$\mathcal{M}_1 \otimes \mathcal{M}_2 := (\mathcal{M}_1 \otimes^{\mathbf{PSh}(X)} \mathcal{M}_2)^a,$$

$$(\mathcal{M}_1 \otimes^{\mathbf{PSh}(X)} \mathcal{M}_2)(U) = \mathcal{M}_1(U) \otimes_{\mathcal{R}(U)} \mathcal{M}_2(U).$$

Example 4.53. Consider $\mathbb{CP}^1 = (\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^\times$ as the space of lines in $\mathbb{C}^2$, where $\mathbb{C}^\times$ acts by scaling. An open subset U of $\mathbb{CP}^1$ corresponds to $\widetilde{U} \subseteq \mathbb{C}^2 \setminus \{0\}$. For any $v \in \mathbb{C}^2 \setminus \{0\}$, there is a subset $\mathbb{C}v = \{\lambda v \mid \lambda \in \mathbb{C}\} \subseteq \mathbb{CP}^1$.

Define a sheaf $\mathcal{O}_{\mathbb{P}^1}(k)$ on $\mathbb{CP}^1$. that assigns to an open set U the set of polynomial functions $f\colon \widetilde{U} \to \mathbb{C}$ such that $f(tv) = t^k f(v)$.

$$\Gamma(U; \mathcal{O}_{\mathbb{P}^1}(k)) = \left\{ f\colon \widetilde{U} \to \mathbb{C} \;\middle|\; f(tv) = t^k f(v) \right\}$$

For example, sections of $\mathcal{O}_{\mathbb{P}^1}(-1)$ correspond to an assignment

$$\mathbb{P}^1 \ni x \mapsto v_x \in \mathbb{C}^2 \setminus \{0\}$$

such that v_x generates the line corresponding to x.

Observe that the global sections $\Gamma(\mathbb{CP}^1; \mathcal{O}_{\mathbb{P}^1}(k))$ is a finite-dimensional vector space of degree k homogeneous polynomials on $\mathbb{C}^2$. In particular, there are no nonzero polynomials of negative degree, so when $k < 0$, $\Gamma(\mathbb{CP}^1; \mathcal{O}_{\mathbb{P}^1}(k)) = 0$.

Then we have

$$\mathcal{O}_{\mathbb{P}^1}(n) \otimes \mathcal{O}_{\mathbb{P}^1}(m) = \mathcal{O}_{\mathbb{P}^1}(m+n).$$

4.7 Gluing

Definition 4.54. Given a (sub)basis B for the topology τ on X, we say that a contravariant functor $\mathcal{F}\colon B^{op} \to \mathbf{C}$ is a B**-presheaf** on X with values in $\mathbf{C}$.

A B**-sheaf** on X additionally has the property that for any $U \in B$, and any open cover $\{U_i\}_{i\in I}$ with $U_i \in B$, the following diagram is an equalizer.

$$\mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{\substack{i,j \\ U_i \cap U_j = \bigcup_k V_{i,j,k}}} \mathcal{F}(V_{i,j,k})$$

where $\{V_{i,j,k}\}$ is an open cover for $U_i \cap U_j$ for all k, with $V_{i,j,k} \in B$.

Proposition 4.55. *Any B-sheaf on X extends uniquely to a sheaf on X, and similarly for maps. In other words, there is an equivalence of categories B-$\mathbf{Sh}(X) \cong \mathbf{Sh}(X)$.*

We will later use this proposition to define Schemes.

Proof idea. For any $U \subseteq X$ open, define
$$\mathcal{F}(U) = \lim_{\substack{V \subseteq U \\ V \in B}} \mathcal{F}(V).$$
□

Corollary 4.56 (Gluing). *Given an open cover $\{U_i\}_{i \in I}$ of X and a collection of sheaves $\mathcal{F}_i$ over U_i and isomorphisms $\phi_{i,j} \colon \mathcal{F}_i|_{U_i \cap U_j} \xrightarrow{\cong} \mathcal{F}_j|_{U_i \cap U_j}$ satisfying **cocycle condition:***
$$\phi_{j,k} \circ \phi_{i,j} = \phi_{i,k}$$
on $U_i \cap U_j \cap U_j$, then there exists a unique sheaf $\mathcal{F}$ over X with isomorphisms $\psi_i \colon \mathcal{F}|_{U_i} \xrightarrow{\cong} \mathcal{F}_i$ that are compatible in the sense that the following diagram commutes.

$$\begin{tikzcd}
 & \mathcal{F}|_{U_i \cap U_j} \arrow[dl, "\psi_i"'] \arrow[dr, "\psi_j"] & \\
\mathcal{F}_i|_{U_j} \arrow[rr, "\phi_{i,j}"] & & \mathcal{F}_j|_{U_i}
\end{tikzcd}$$

4.8 Schemes

Definition 4.57. Given $f \colon X \to Y$ a map of topological spaces and a sheaf $\mathcal{F}$ on X, the **pushforward sheaf** $f_*(\mathcal{F})$ is the sheaf on Y with $f_*\mathcal{F}(U) := \mathcal{F}(f^{-1}(U))$.

Recall that a basis for the topology on $\mathrm{Spec}(R)$ is the set of $D_f = \mathrm{Spec}(R_f) = \{P \in \mathrm{Spec}(R) \mid f \notin P\}$.

Definition 4.58. We say that a topological space X is **quasi-compact** if every cover has a finite subcover.

Remark 4.59. In practice, **compact** refers to a space that is both Hausdorff and quasi-compact. The Zariski topology is very non-Hausdorff.

Lemma 4.60. *Let $X = \mathrm{Spec}(R)$. $X = \bigcup_i D_{f_i}$ if and only if $\{f_i\}$ generate the unit ideal. Hence, any cover of $\mathrm{Spec}(R)$ has a finite subcover and so $\mathrm{Spec}(R)$ is quasi-compact.*

Proof. $X = \bigcup_i D_{f_i}$ if and only if no prime ideal contains all f_i if and only if $\{f_i^{n_i}\}$ generate the unit ideal for some positive integers n_i if and only if $\{f_i\}$ generate the unit ideal. We may write
$$1 = \sum_i e_i f_i,$$
and the finite subcover of this open cover is the one given by D_{f_i} for those f_i appearing in the sum. □

Proposition 4.61. *An assignment* $\mathcal{O}_X(D_f) = R_f$ *is a* B*-sheaf for the basis* $B = \{D_f\}$ *of the topology on* X*.*

So given a ring R, we get a topological space $X = \mathrm{Spec}(R)$ together with a sheaf $\mathcal{O}_{\mathrm{Spec}(R)}$ defined on a basis of open sets by $\mathcal{O}_{\mathrm{Spec}(R)}(D_f) = R_f$. For an R-module M, we have a sheaf $\mathcal{F}_M$ of abelian groups. This is in fact a sheaf of modules over $\mathcal{O}_{\mathrm{Spec}(R)}$.

Definition 4.62. A **ringed space** $(X, \mathcal{O}_X)$ is a topological space X together with a sheaf $\mathcal{O}_X$ of (commutative) rings on X, called the **structure sheaf**.

Definition 4.63. An **isomorphism of ringed spaces** $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a homeomorphism of spaces $f\colon X \to Y$ such that $f^*\mathcal{O}_Y \cong \mathcal{O}_X$ as sheaves.

Example 4.64. Let $X = \{*\}$, and $\mathcal{O}_X = R$ any ring.

This example is bad, because earlier in this chapter we were talking about spaces where we wanted to associate fields to points, not just any ring. Hence, we refine this definition.

Definition 4.65. An **affine scheme** is a ringed space of the form $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ for a (commutative) ring R.

Definition 4.66. A **scheme** is a locally affine ringed space. That is, a ringed space $(X, \mathcal{O}_X)$ is a scheme if for each $x \in X$, there is a neighborhood $U \subseteq X$ containing x such that $(U, \mathcal{O}_X|_U) \cong (\mathrm{Spec}\, R, \mathcal{O}_{\mathrm{Spec}(R)})$ for some (commutative) ring R.

Proposition 4.67. *Given an affine scheme* $(X, \mathcal{O}_X)$*, let* $R = \mathcal{O}_X(X) = \Gamma(X; \mathcal{O}_X)$*. Then the adjective "affine" implies*

(a) for all principal open sets D_f*,* $D_f \cong \mathrm{Spec}\, R_f$ *as topological spaces, where* R_f *is the localization of* R *at* f*.*

(b) the stalk $\mathcal{O}_{X,x}$ *of* $\mathcal{O}_X$ *at* $x \in X$ *is a local ring with maximal ideal* M_x*.*

(c) The natural map $X \to \mathrm{Spec}(R)$ *given by* $x \mapsto \{r \in R \mid r_x \in O_{X,x}\}$ *is a homeomorphism, with* $r_x \in M_x \subseteq \mathcal{O}_{X,x}$*.*

Exercise 4.68. Let $X = \mathrm{Spec}(\mathbb{C}[x])$ and let $\mathcal{O}_X$ be the structure sheaf. The stalk of $\mathcal{O}_X$ at $\langle x \rangle \in X$ is the power series ring $\mathbb{C}[[x]]$.

Example 4.69 (Non-affine scheme)**.** Let R be the subring of $\mathbb{C}[z]$ given by those functions $f \in \mathbb{C}[z]$ such that $f(0) = f(1)$. Let $Z = \mathrm{Spec}(\mathbb{C}[z])$ and let $X = \mathrm{Spec}(R)$. Let $\phi\colon Z \to X$ be the map induced by the inclusion of the subring R into $\mathbb{C}[z]$.

We claim that $(X, \phi_*(\mathcal{O}_Z))$ is not an affine scheme. The stalk of $\mathcal{O}_X$ at the point $\langle x \rangle = \langle x - 1 \rangle$ is isomorphic to the ring

$$\big\{(f, g) \;\big|\; f, g \in \mathbb{C}[[z]],\ f(0) = g(0)\big\}.$$

This includes into the stalk of $\phi_*(\mathcal{O}_Z) = \{(f,g) \mid f,g \in \mathbb{C}[[z]]\}$, but this is not a local ring, contradicting Proposition 4.67(b). So this scheme is not affine.

Example 4.70 (Non-affine scheme)**.** Consider $Z = \mathbb{C}^2 \setminus 0$ and $X = \mathbb{C}^2$. There is an inclusion $Z \hookrightarrow X$ that is a morphism of ringed spaces, where $\mathcal{O}_Z$ and $\mathcal{O}_X$ are the sheaves of functions on Z and X, respectively. This is not an affine scheme: although it satisfies Proposition 4.67(b), it fails Proposition 4.67(c) because of **Hartogs's Theorem:**

$$\Gamma(Z; \mathcal{O}_X|_Z) = \Gamma(X; \mathcal{O}_X) = \mathbb{C}[x,y].$$

Example 4.71 (Non-affine scheme)**.** Write $\mathbb{A}^1_\mathbb{C}$ for the affine scheme $\operatorname{Spec}(\mathbb{C}[x])$ corresponding to the complex line $\mathbb{C}$, and $\mathbb{G}_m$ for the multiplicative group $\operatorname{Spec}(\mathbb{C}[z,z^{-1}])$ corresponding to $\mathbb{C}^\times$.

Then define $\mathbb{P}^1_\mathbb{C}$ to be the scheme obtained by the gluing $\mathbb{A}^1_\mathbb{C} \cup_{\mathbb{G}_m} \mathbb{A}^1_\mathbb{C}$ with transition function $z \mapsto z^{-1}$ on the overlap. The structure sheaf $\mathcal{O}_{\mathbb{P}^1}$ has sections

$$\Gamma(U; \mathcal{O}_{\mathbb{P}^1}) = \begin{cases} \Gamma(\mathbb{A}^1_\mathbb{C}; \mathcal{O}_{\mathbb{A}^1}) \text{ with coordinate } z & \infty \notin U, \\ \Gamma(\mathbb{A}^1_\mathbb{C}; \mathcal{O}_{\mathbb{A}^1}) \text{ with coordinate } z^{-1} & 0 \notin U, \\ \text{use sheaf axiom} & 0, \infty \in U. \end{cases}$$

This is not an affine scheme, because $\operatorname{Spec}(\mathbb{P}^1_\mathbb{C}, \mathcal{O}_{\mathbb{P}^1})$ is a point by Liouville's theorem.

Example 4.72. The **line with two origins** is the space X $\mathbb{A}^1_\mathbb{C}$ glued to $\mathbb{A}^1_\mathbb{C}$ along $\mathbb{G}_m$ with transition function $z \mapsto z$. There is one point repeated, namely the origin. There is a map $X \to \operatorname{Spec}(\Gamma(X; \mathcal{O}_X)) \cong \mathbb{A}^1$ that identifies the origin. In particular it is not a bijection, so X is not affine. In the analytic topology, it is not Hausdorff, while $\mathbb{A}^1_\mathbb{C}$ is.

Example 4.73. $\operatorname{Spec} \mathbb{C}[[z]] = \{\langle 0 \rangle, \langle z \rangle\}$ consists of just two points. $\operatorname{Spec} \mathbb{C}((z))$, meanwhile, has just one point $\langle 0 \rangle$. We may glue two of $\operatorname{Spec} \mathbb{C}[[z]]$ together along $\operatorname{Spec} \mathbb{C}((z))$ to get the smallest example of a non-affine scheme, occasionally called the **ravioli**.

$$\operatorname{Spec} \mathbb{C}[[z]] \cup_{\operatorname{Spec} \mathbb{C}((z))} \operatorname{Spec} \mathbb{C}[[z]]$$

This has three points: zero, and two copies of $\langle z \rangle$, one from each $\operatorname{Spec} \mathbb{C}[[z]]$.

4.9 Morphisms of Schemes

Definition 4.74. A **morphism of ringed spaces** $(f, f^\#) \colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair of a continuous map $f \colon X \to Y$ and a morphism of sheaves $f^\# \colon \mathcal{O}_Y \to f_*\mathcal{O}_X$.

Unpacking this definition, we must have that for all $U \subseteq Y$ open, there is a morphism $f^{\#}(U)\colon \Gamma(U; \mathcal{O}_Y) \to \Gamma(f^{-1}(U); \mathcal{O}_X)$, compatible with restriction. Note that $f^{\#}$ need not be related to f, except insofar as its codomain is $f_*\mathcal{O}_X$.

Example 4.75. For any map $R \to S$ of rings, there is a map of schemes $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$.

Example 4.76. Each $\Gamma(U; \mathcal{O}_X)$ is characteristic p(or, equivalently, accepts a map from $\mathbb{F}_p$). This happens if and only if there is a map $X \to \operatorname{Spec}(\mathbb{F}_p)$.

Define the **absolute Frobenius** $(X, \mathcal{O}_X) \to (X, \mathcal{O}_X)$ which is the identity map on points, and $\Gamma(U; \mathcal{O}_X) \to \Gamma(U; \mathcal{O}_X)$ is the map $x \mapsto x^p$. Since we're in characteristic p, this is a ring homomorphism.

We can describe this map of ringed spaces as the pair $(\mathrm{id}, x \mapsto x^p)$ – notice that we can't determine the map of sheaves just from the map of spaces.

Moreover, this may not be invertible, even though it is a bijection on spaces. Consider the absolute Frobenius map $\operatorname{Spec}(\mathbb{F}_p(x)) \to \operatorname{Spec}(\mathbb{F}_p(x))$ – here, the map of sheaves is not surjective.

Example 4.77. Let $\sigma\colon k \to k$ be a field endomorphism. This induces a map $(f, f^{\#})\colon \operatorname{Spec}(k) \to \operatorname{Spec}(k)$ such that $f^{\#} = \sigma$ and $f = \mathrm{id}$.

Example 4.78. Consider $X = \operatorname{Spec} \mathbb{C}((z))$ and $Y = \operatorname{Spec} \mathbb{C}[[z]]$. Identify points in these schemes by their residue fields. Notice that X has a unique point $\mathrm{pt}_{\mathbb{C}((z))}$, since it is a field, and Y has a two points: $\mathrm{pt}_{\mathbb{C}}$ and $\mathrm{pt}_{\mathbb{C}((z))}$.

Consider the map of spaces $f\colon X \to Y$ given by $\mathrm{pt}_{\mathbb{C}((z))} \mapsto \mathrm{pt}_{\mathbb{C}}$. Take the map of sheaves $f^{\#}\colon \mathcal{O}_Y \to \mathcal{O}_X$ that induces

$$\mathbb{C}[[z]] \cong \Gamma(Y; \mathcal{O}_Y) \to \Gamma(X; \mathcal{O}_X) \cong \mathbb{C}((z))$$

given by $z \mapsto z$. This morphism of ringed spaces $(f, f^{\#})$ doesn't come from any map $\mathbb{C}[[z]] \to \mathbb{C}((z))$. If it did, then we would have $f(\mathrm{pt}_{\mathbb{C}((z))}) = \mathrm{pt}_{\mathbb{C}((z))}$.

This last example is bad! We want all maps of affine schemes to be induced by ring maps. So we must fix our definition.

Definition 4.79. A **locally ringed space** is a ringed space whose stalks are all local rings.

Example 4.80. Any affine scheme $\operatorname{Spec}(R)$ is a locally ringed space, and indeed, any scheme is a locally ringed space because it is locally affine.

Definition 4.81. A **morphism of local rings** $\phi\colon (R, M_R) \to (S, M_S)$ is a ring homomorphism $\phi\colon R \to S$ such that $\phi(M_R) \subseteq M_S$.

Definition 4.82. A **morphism of locally ringed spaces** is a morphism of ringed spaces $(f, f^{\#})\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that the induced morphisms on stalks $f^{\#}_y\colon \mathcal{O}_{Y,y} \to (f_*\mathcal{O}_X)_x$ are morphisms of local rings.

Example 4.83. In Example 4.78, we con't have a morphism of local rings. To be local, we would need to have $\mathbb{C}[[z]] \ni z \mapsto 0 \in \mathbb{C}((z))$.

Definition 4.84. A **morphism of schemes** is a morphism of locally ringed spaces whose domain and codomain are both schemes.

Theorem 4.85. *If* R *and* S *are rings, then the functor* Spec *from rings to locally ringed spaces is fully faithful, that is,*

$$\operatorname{Spec}\colon \operatorname{Hom}(R,S) \to \operatorname{Map}(\operatorname{Spec}(S), \operatorname{Spec}(R))$$

is bijective.

Proof sketch. The inverse will be the global sections functor $(X, \mathcal{O}_X) \mapsto \Gamma(X; \mathcal{O}_X)$. Given $(f, f^\#)\colon \operatorname{Spec}(S) \to \operatorname{Spec}(R)$, we have a map

$$f^\#\colon \Gamma(\operatorname{Spec}(R), \mathcal{O}_{\operatorname{Spec}(R)}) \to \Gamma(\operatorname{Spec}(S), \mathcal{O}_{\operatorname{Spec}(S)}).$$

Since global sections of $\operatorname{Spec}(R)$ is isomorphic to R, and global sections of $\operatorname{Spec}(S)$ is isomorphic to S, we have a ring homomorphism $\phi\colon R \to S$.

To check that this is a bijection, we must check that $\operatorname{Spec}(\phi)(P) = f(P)$ for all $P \in \operatorname{Spec}(S)$. Consider the diagram

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & S \arrow[d] \\
R_{f(P)} \arrow[r] & S_P
\end{tikzcd}$$

where the bottom row is the map on stalks. Locality gives that $\phi^{-1}(P) = f(p)$. Hence, $\operatorname{Spec}(\phi)(P) = \phi^{-1}(P) = f(P)$. $\square$

Remark 4.86. This theorem says that we have an equivalence of categories between the opposite category of commutative rings and the category of affine schemes; once we restrict locally ringed spaces to the image of Spec.

We have a diagram of categories

$$\begin{tikzcd}
\text{Ringed Spaces} & \text{Locally Ringed Spaces} \arrow[l, "\text{fewer objects}", "\text{fewer homs}"'] & \text{Schemes} \arrow[l, "\text{fewer objects}", "\text{same homs}"'] & \text{Affine Schemes} \arrow[l, "\text{fewer objects}", "\text{same homs}"'] \\
 & & & \text{Rings}^{op} \arrow[ul, "\text{Spec}", "\text{fully faithful embedding}"'] \arrow[u, "\simeq"']
\end{tikzcd}$$

Example 4.87. The **affinization** of a scheme X is the morphism $X \to \operatorname{Spec}(\Gamma(X, \mathcal{O}_X))$.

If $X = \mathbb{CP}^1$, this is the map $\mathbb{CP}^1 \to \mathrm{pt} = \operatorname{Spec}(\mathbb{C})$. If $X = \widetilde{\mathbb{C}}^n$, this is the map $\widetilde{\mathbb{C}}^n \to \mathbb{C}^n$. If $X = \mathbb{C}^2 \setminus 0$, this is the map $\mathbb{C}^2 \setminus 0 \hookrightarrow \mathbb{C}^2$. If $X = \mathbb{C} \setminus 0$, this is the identity map because $\mathbb{C} \setminus \{0\}$ is described by the scheme $\operatorname{Spec}(\mathbb{C}[z, z^{-1}])$.

Example 4.88. Consider the ring homomorphism $R \to R/\sqrt{0}$. This gives a map of affine schemes $\operatorname{Spec}(R/\sqrt{0}) \to \operatorname{Spec}(R)$ that is bijective on points, but not an isomorphism on rings unless R is **reduced** (has no nilpotents, or equivalently $\sqrt{0} = 0$).

Definition 4.89. For any scheme X, we may define a new scheme X_{red} called the **reduction of** X with the same underlying topological space, but new structure sheaf $\mathcal{O}_X/\sqrt{0}$. A scheme X is **reduced** if $X = X_{\text{red}}$.

Definition 4.90. A morphism $X \to Y$ is an inclusion of a closed subscheme if $f^\#\colon \Gamma(U; \mathcal{O}_Y) \to \Gamma(f^{-1}(U); \mathcal{O}_X)$ is locally surjective.

4.10 Projective Schemes

Definition 4.91. If S is an $\mathbb{N}$-graded ring, then the ringed space $\operatorname{Proj}(S)$ is, as a set, those homogeneous prime ideals of S that don't contain the irrelevant ideal.

$$\operatorname{Proj}(S) := \left\{ P \leq S \;\middle|\; P \text{ homogeneous, prime, } P \not\supseteq S_+ \right\}$$

Closed subsets come from homogenous ideals I of S, and a subset $U \subseteq \operatorname{Proj}(S)$ is open if and only if U is the compliment of those prime ideals containing I.

We can define a presheaf on this space that sends the ideal $U = \{P \mid P \not\supseteq I\}$ to the degree zero component of S localized at the homogeneous elements of $S \setminus I$. We then define $\mathcal{O}_{\operatorname{Proj} S}$ as the sheafification of this presheaf.

Theorem 4.92. $\operatorname{Proj}(S)$ *is a scheme.*

Proof sketch. We must define an open cover of $\operatorname{Proj}(S)$ by affine schemes. For $g \in S_+$ homogeneous, let $U_g = \{P \mid P \not\ni g\}$.

First, claim that $\operatorname{Proj}(S) = \bigcup_{g \in S_+} U_g$. This is easy.

Second, claim that $(U_g, \mathcal{O}_{\operatorname{Proj}(S)}|_{U_g}) \cong \operatorname{Spec}((S_g)_{\deg=0})$. The map of spaces is given by

$$P \mapsto P_g \cap (S_g)_{\deg=0}.$$

We may then check that these are isomorphisms. □

Example 4.93. We may define the **Grassmannian** $\operatorname{Gr}(k, \mathbb{A}^n_{\mathbb{F}})$ of k-planes in affine n-space $\mathbb{A}^n_{\mathbb{F}} = \operatorname{Spec} \mathbb{F}[x_1, \ldots, x_n]$ for $\mathbb{F}$ a field. We may describe this as the quotient by $\mathrm{GL}(k)$ of $k \times n$ matrices over $\mathcal{F}$ having rank k. But we may describe a rank k matrix by the system of equations that amounts to the non-vanishing of the determinant of a $k \times k$ minor. For $\lambda \in \binom{[n]}{k}$, let A_λ be the minor of the $k \times n$ matrix A consisting of the columns in λ. Let

$$U_\lambda = \{A \mid \det(A_\lambda) \neq 0\} / \operatorname{GL}(k)$$

Note that $U_\lambda \cong \mathbb{A}^{k(n-k)}$, since an element A of U_λ may be written as a matrix where the columns in λ are the columns of the identity and the other columns are free. Hence,

$$\operatorname{Gr}(k, \mathbb{A}^n_{\mathbb{F}}) = \bigcup_{\lambda \in \binom{[n]}{k}} U_\lambda = \bigcup_{\lambda \in \binom{[n]}{k}} \mathbb{A}^n_{\mathbb{F}}$$

To check that this is a scheme, we need to check that the overlap maps

$$U_\lambda \leftarrow U_\lambda \cap U_\mu \to U_\mu$$

are algebraic.

In fact, $\operatorname{Gr}(k, \mathbb{A}^n)$ is $\operatorname{Proj}(-)$ of the ring of **Plücker coordinates**.

Example 4.94. The **Hilbert scheme** of $\mathbb{P}^n$ with Hilbert polynomial p is, as a set,

$$\operatorname{Hilb}_{\mathbb{P}^n}(p) := \{\text{closed subschemes of } \mathbb{P}^n \text{ with Hilbert polynomial } p\}.$$

It is a theorem due to Grothendieck that this can be made into (the closed points of) a scheme, and moreover this scheme is complete and separated (analogous to compact and Hausdorff). Mumford showed that these schemes are projective.

4.11 $\mathcal{O}_X$-modules

Definition 4.95. Let $\mathcal{R}$ be a sheaf of rings over X. A **sheaf of modules** $\mathcal{M}$ is a sheaf of abelian groups over X such that each $\mathcal{M}(U)$ is a module over $\mathcal{R}$ and moreover the following diagrams all commute.

$$\begin{tikzcd} \mathcal{R}(U) \times \mathcal{M}(U) \arrow[r] \arrow[d] & \mathcal{M}(U) \arrow[d] \\ \mathcal{R}(V) \times \mathcal{M}(V) \arrow[r] & \mathcal{M}(V) \end{tikzcd}$$

Example 4.96. If M is an R-module, then $\mathcal{F}_M$ is an $\mathcal{O}_{\operatorname{Spec}(R)}$-module, where $\mathcal{F}_M$ is the sheaf defined by $\mathcal{F}_M(D_f) = M_f$ for the principal open sets $D_f = \{P \in \operatorname{Spec}(R) \mid P \not\ni f\}$.

Example 4.97 (Scary (non)-example)**.** Let $R = \mathbb{C}[x]$ and define a sheaf $\mathcal{F}$ on $\operatorname{Spec}(R)$ by

$$\Gamma(U; \mathcal{F}) = \begin{cases} \Gamma(U; \mathcal{O}_{\operatorname{Spec}(R)}) & 0 \notin U \\ 0 & 0 \in U \end{cases}$$

Then $\mathcal{F} = \bigcap_{i=1}^{\infty} \mathcal{F}_{\langle x^i \rangle}$ is an intersection of the $\mathcal{O}_{\mathrm{Spec}(R)}$-modules $\mathcal{F}_{\langle x^i \rangle}$ corresponding to the ideals $\langle x^i \rangle$, considered as R-modules.

Definition 4.98. An $\mathcal{O}_X$-module $\mathcal{F}$ is **quasicoherent** if it is locally isomorphic to $\mathcal{F}_M$ for M a module over $\Gamma(U, \mathcal{O}_X)$.

$$\mathcal{F}|_U \cong \mathcal{F}_M$$

Definition 4.99. An $\mathcal{O}_X$-module $\mathcal{F}$ is **coherent** if it is locally isomorphic to $\mathcal{F}_M$ for M a module over $\Gamma(U, \mathcal{O}_X)$ and M is finitely generated as a $\Gamma(U, \mathcal{O}_X)$-module.

Example 4.100. Consider the map $\pi \colon \mathbb{A}^1_k \to \mathrm{pt}_k$. Then $\Gamma(\mathrm{pt}; \pi_* \mathcal{O}_{\mathbb{A}^1_k}) \cong k[x]$. This is quasicoherent on a point, but not coherent. The problem is that this map is not proper, which we'll encounter later.

4.12 Open and Closed Subschemes

Definition 4.101. If $Y \subseteq X$ is an open subset, then we may define $\mathcal{O}_Y := \mathcal{O}_X|_Y$. The resulting scheme $(Y, \mathcal{O}_Y)$ is an **open subscheme** of X.

Example 4.102. Let $X = \mathrm{Spec}(\mathbb{C}[x,y]/\langle x^2 \rangle)$ and let $Y = X_{\mathrm{red}} = \mathrm{Spec}(\mathbb{C}[x,y]/\langle x \rangle)$. In this case, Y is the whole space X, and therefore $\mathcal{O}_X|_Y = \mathcal{O}_X$, but $\mathcal{O}_{X_{\mathrm{red}}} \neq \mathcal{O}_X$. Hence, this is not an open subscheme.

Nevertheless, this is a closed subscheme, because $\mathcal{O}_{X_{\mathrm{red}}} = \mathcal{O}_X / \mathcal{F}_{\langle x \rangle}$, because

$$\mathbb{C}[x,y]/\langle x^2 \rangle \Big/ \langle x \rangle \cong \mathbb{C}[x,y] \Big/ \langle x \rangle.$$

Definition 4.103. A **closed subscheme** $X \hookrightarrow Y$ is defined by a quasicoherent sheaf $\mathcal{I}$ of ideals: $\mathcal{O}_X \cong \mathcal{O}_Y / \mathcal{I}$.

Remark 4.104 (Warning about subschemes.)**.** As schemes, $A \cap (B \cup C) \neq (A \cap B) \cup (A \cap C)$. The counterexample is three lines that meet at a point: $A = \mathrm{Spec}(\mathbb{C}[x,y]/\langle x \rangle)$, $B = \mathrm{Spec}(\mathbb{C}[x,y]/\langle x-y \rangle)$, and $C = \mathrm{Spec}(\mathbb{C}[x,y]/\langle y \rangle)$. So A corresponds to the ideal $\langle x \rangle$, B corresponds to the ideal $\langle x - y \rangle$, and C corresponds to the ideal $\langle y \rangle$.

Then $A \cap (B \cup C) = \langle x, y(x-y) \rangle = \langle x^2, y^2 \rangle$, while $(A \cap B) \cup (A \cap C) = \langle x, x-y \rangle \cap \langle x, y \rangle$, so these schemes are not the same.

4.13 Fibered Products

Definition 4.105. Let X, Y, S be schemes. Given $\psi \colon Y \to S$ and $\phi \colon X \to S$, the **fibered product** of X and Y over S is a scheme $X \times_S Y$ together with maps

$\pi_X \colon X \times_S Y \to X$ and $\pi_Y \colon X \times_S Y \to Y$ such that $\phi\pi_X = \psi\pi_Y$. Additionally, given any Z with maps $f \colon Z \to X$ and $g \colon Z \to Y$ such that $\phi f = \psi g$, there is a unique $h \colon Z \to X \times_S Y$ such that $\pi_X h = f$ and $\pi_Y h = g$.

$$\begin{tikzcd}
Z \arrow[drr, bend left, "f"] \arrow[dr, dashed, "h"] \arrow[ddr, bend right, "g"'] & & \\
& X \times_S Y \arrow[r] \arrow[d] & X \arrow[d, "\phi"] \\
& Y \arrow[r, "\psi"] & S
\end{tikzcd}$$

Recall that in the category of affine schemes, the fibered product of $\operatorname{Spec}(A)$ and $\operatorname{Spec}(B)$ over $\operatorname{Spec}(R)$ is $\operatorname{Spec}(A \otimes_R B)$.

Remark 4.106. We would like to have that $\mathbb{A}^1_k \times \mathbb{A}^1_k$ is $\mathbb{A}^2_k$, naively. But the only closed subsets in $\mathbb{A}^1_k$ are finite sets of points, so the closed subsets of $\mathbb{A}^1_k \times \mathbb{A}^1_k$ are finite unions of vertical and horizontal lines. Yet $\mathbb{A}^2_k$ has any plane curve as a closed subset, which is not a union vertical and horizontal lines.

Yet a closed point of $\mathbb{A}^1_{\mathbb{C}} \times \mathbb{A}^1_{\mathbb{C}}$ is a closed point of $\mathbb{A}^2_{\mathbb{C}}$.

Example 4.107. $\mathbb{A}^1_k \times_{\operatorname{Spec}(k)} \mathbb{A}^1_k = \operatorname{Spec}(k[x] \otimes_k k[y]) = \operatorname{Spec}(k[x,y]) \cong \mathbb{A}^2_k$.

Theorem 4.108. *Fibered products always exist in the category of schemes.*

Proof. First, notice that fibered products always exist for affine schemes.

Step (0): To show that the fibered product of affine schemes X and Y over an affine scheme S is also a fibered product in the category of schemes, recall that

$$\operatorname{Mor}_{\mathbf{Sch}}(A, \operatorname{Spec}(A)) \cong \operatorname{Hom}(A, \Gamma(Z, \mathcal{O}_Z))$$

for any scheme Z, not necessarily affine. Therefore, we now have the first step. Hence, the fibered product of affine schemes is also a fibered product of schemes.

Step (0.5): Next, claim that if $U \to S$ is an open embedding of affine schemes, then $(U \times_S Y) = (\psi^{-1}(U), \mathcal{O}_Y|_{\psi^{-1}(U)})$ is a fibered product of U and Y over S. Moreover, $U \times_S Y \to Y$ is an open embedding as well.

$$\begin{tikzcd}
U \times_S Y \arrow[r] \arrow[d, hook] & U \arrow[d, hook] \\
Y \arrow[r, "\psi"] & S
\end{tikzcd}$$

Now we use the fact that any scheme has an open cover by affines, and glue them together.

Step (1): If X and S are affine and Y is any scheme with an open embedding $Y \hookrightarrow Y' \to S$ with Y' affine. So we may take the fibered product of X and Y' over

S, and then the pullback along the open embedding $Y \hookrightarrow Y'$ to get a pullback of X and Y over S.

$$\begin{tikzcd}
W' \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & W \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d] \\
Y \arrow[r, hook] & Y' \arrow[r] & S
\end{tikzcd}$$

Then by general category theory nonsense, $W' = X \times_S Y$.

Step (2): If X and S are affine but Y is arbitrary, write $Y = \bigcup_{i \in I} Y_i$ as a union of open affines. Let $Y_{ij} = Y_i \cap Y_j$. Then $W_i = X \times_S Y_i$ exists for every i by the above, and so does $W_{ij} = X \times_S Y_{ij}$. Moreover, W_{ij} comes with canonical open embeddings $W_{ij} \hookrightarrow W_i$ and $W_{ij} \hookrightarrow W_j$. Define a scheme W by gluing the W_i along the W_{ij}'s.

Claim that W is a fibered product of X and Y over S. To show this, let Z be any scheme with maps $\alpha \colon Z \to X$ and $\beta \colon Z \to Y$ such that the diagram of solid arrows commutes.

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\gamma"] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "\beta"'] & & \\
& X \times_S Y \arrow[r] \arrow[d] & X \arrow[d, "\phi"] \\
& Y \arrow[r, "\psi"] & S
\end{tikzcd}$$

We want to find a map γ such that the diagram commutes. Let $Z_i = \beta^{-1}(Y_i)$ and $Z_{ij} = \beta^{-1}(Y_{ij})$. Then there is a unique map $Z_i \to W_i$ by the universal property of $W_i = X \times_S Y_i$. So there is a unique map $\gamma_i \colon Z_i \to W$. Similarly, we get a unique map $\gamma_{ij} \colon Z_{ij} \to W$. By the uniqueness, $\gamma_i|_{Z_{ij}} = \gamma_{ij} = \gamma_j|_{Z_{ij}}$. Once we check that the triple intersections also agree, we may glue the γ_i to get $\gamma \colon Z \to W$.

This shows that fibered products exist for X and S affine and Y arbitrary.

Step (3): Now assume that S is affine but X and Y are arbitrary. To construct the fibered product $X \times_S Y$, cover X with open affines X_j and repeat the argument in step (2) above with X and Y interchanged.

Step (4): Let X, Y, and S be any arbitrary schemes with maps $\phi \colon X \to S$ and $\psi \colon Y \to S$. Cover S by open affines S_m. Define $X_m = \phi^{-1}(S_m)$ and $Y_m = \psi^{-1}(S_m)$. Therefore, $X_m \times_{S_m} Y_m$ exists by step (3). Now if we have a diagram

$$\begin{tikzcd}
Z \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X_m \times_S Y \arrow[r] \arrow[d] & X_m \arrow[d] \\
& Y \arrow[r] & S
\end{tikzcd}$$

for any scheme Z, the image of $Z \to Y$ lands inside Y_m. Hence, $X_m \times_S Y$ is uniquely isomorphic to $X_m \times_{S_m} Y_m$. In particular, $X_m \times_S Y$ exists. Then apply the same argument as in step (2) to show that $X \times_S Y$ exists by gluing. $\square$

Now that we have constructed fibered products, here are some cool applications of them.

Definition 4.109. Let K be a field extension of k, and let $S' = \operatorname{Spec}(K)$, with $S = \operatorname{Spec}(k)$. Then $Y \times_S S'$ is the **base change** of Y from k to K.

Example 4.110. If $k = \mathbb{Q}$, $K = \mathbb{Q}(\sqrt{2})$, $Y = \operatorname{Spec}(\mathbb{Q}[x]/\langle x^2 - 2\rangle)$, then the base change of Y is $\operatorname{Spec}(\mathbb{Q}(\sqrt{2})[x]/\langle x^2 - 2\rangle)$. In particular, we have introduced solutions to this

Example 4.111. Let $f\colon X \to Y$ be a morphism of schemes, and let $y \in Y$ be any point. Let $k(y)$ be the residue field at y, and let $\operatorname{Spec}(k(y)) \to Y$ be the inclusion. Then the **fiber of** f **over** y is the fibered product $\operatorname{Spec}(k(y)) \times_Y X$.

To form the fibered product $X \times_S Y$, we need maps $X \to S$ and $Y \to S$. Although we need these maps, they don't appear in the notation, yet they are essential for the definition of the fibered product.

But in categories like manifolds or topological spaces, we define just a product of manifolds or spaces or whatnot without needing these maps. So how do we get an absolute version of the fibered product of schemes?

We want to be able to say $\mathbb{A}^2_k = \mathbb{A}^1_k \times \mathbb{A}^1_k$; as we have it now, $\mathbb{A}^2_k = \operatorname{Spec}(k[x] \otimes_k k[y]) = \mathbb{A}^1_k \times_{\operatorname{Spec}(k)} \mathbb{A}^1_k$.

We could define $X \times Y = X \times_{\operatorname{Spec}(\mathbb{Z})} Y$, but later we would find that

$$\dim(X \times_{\operatorname{Spec}(\mathbb{Z})} Y) = \dim(X) + \dim(Y) - \dim(\operatorname{Spec}(\mathbb{Z})) = \dim(X) + \dim(Y) - 1.$$

This is weird: the terminal object has dimension 1 instead of zero. Instead, we will force another object to be terminal.

Definition 4.112. Let S be a scheme. The **category of S-schemes $\mathbf{Sch}/S$** is the category whose objects are schemes X with a scheme morphism $X \to S$ and whose morphisms are morphisms of schemes commuting with the morphisms $X \to S$.

Definition 4.113. In the category of S-schemes, the **product of schemes** (over S) is $X \times_S Y$.

In this context, when we take $S = \operatorname{Spec}(\mathbb{C})$, then $\dim(X \times Y) = \dim(X) + \dim(Y)$. The new terminal object is $\operatorname{id}\colon S \to S$.

5 Other constructions

5.1 Functors of points

To any scheme X, we will associate a contravariant functor from schemes to sets. This will have the side effect of translating questions of existence in the category of schemes to questions of representability of a functor.

Example 5.1.

(a) The points of a topological space X are in bijection with $\mathrm{Mor}_{\mathbf{Top}}(\mathrm{pt}, X)$.

(b) The elements of a group G are in bijection with $\mathrm{Hom}_{\mathbf{Groups}}(\mathbb{Z}, G)$.

(c) The elements of a ring R are in bijection with $\mathrm{Hom}_{\mathbf{Rings}}(\mathbb{Z}[x], R)$.

(d) If **Hot** is the category of CW-complexes with homotopy classes of maps, then the initial object is still a point. But $\mathrm{Mor}_{\mathbf{Hot}}(\mathrm{pt}, X) = \pi_0(X)$, which is not homotopy equivalent to X when some of the components are not contractible. So we cannot recover X, even up to homotopy.

This last example is most like the category of schemes – there is no one scheme S from which we can recover any scheme from the functor represented by S. The solution is to use the Yoneda embedding to associate to X the representable functor $\mathrm{Mor}_{\mathbf{Sch}}(-, X)$. This embeds **Sch** inside $\mathbf{Fun}(\mathbf{Sch}^{\mathrm{op}}, \mathbf{Sets})$.

Definition 5.2. Let X be a scheme. The **functor of points** for X is the representable functor $h_X = \mathrm{Mor}_{\mathbf{Sch}}(-, X)$.

The function on objects $h\colon \mathbf{Sch} \to \mathbf{Fun}(\mathbf{Sch}^{\mathrm{op}}, \mathbf{Sets})$ is a functor: for any $\phi\colon X \to X'$, we have a natural transformation $\phi_*\colon h_X \to h_{X'}$ given by $g \mapsto \phi \circ g$.

Definition 5.3. The Y**-valued points of** X are the elements of $h_X(Y)$.

We still want to be able to recover the scheme X from the functor h_X, or this whole setup is useless. But the Yoneda Lemma lets us do exactly that.

Lemma 5.4 (Yoneda)**.** *If $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$ is a contravariant functor, then the natural transformations from F to $h_Y := \mathrm{Hom}_{\mathbf{C}}(-, Y)$ are in bijection with $F(X)$.*

If additionally $\mathrm{Hom}_{\mathbf{C}}(-, Y) \cong \mathrm{Hom}_{\mathbf{C}}(-, X)$ as functors, then $X \cong Y$ in $\mathbf{C}$. In other words, $h\colon X \mapsto \mathrm{Hom}_{\mathbf{C}}(-, X)$ is fully faithful.

Proof. Exercise. □

The whole point of this is that we can now do constructions in $\mathbf{Fun}(\mathbf{Sch}^{\mathrm{op}}, \mathbf{Sets})$. For any three such functors f, g, h, define a new functor $f \times_h g$ by

$$(f \times_h g)(Y) = f(Y) \times_{h(Y)} g(Y),$$

which exists because $f(Y)$, $g(Y)$ and $h(Y)$ are sets. On morphisms $\phi\colon Y \to X$, this functor is determined by the universal property of pullbacks.

$$(f \times_h g)(\phi)\colon (f \times_h g)(X) \to (f \times_h g)(Y).$$

Can we use this to define a fibered product of schemes?

Given $X \to S$ and $Y \to S$, and we form $h_X \times_{h_S} h_Y$, the question we want to answer is whether or not this functor is representable $h_X \times_{h_S} h_Y = h_Z$. Then by the Yoneda lemma, this scheme Z will be the fibered product of X and Y over S.

Example 5.5.

(a) Is the functor $\Gamma\colon X \to \Gamma(X; \mathcal{O}_X)$ representable? In the category **Sch**/ Spec($\mathbb{C}$), this is represented by $\mathrm{Mor}(X, \mathbb{A}^1_{\mathbb{C}})$. So Γ is representable. We think of a global section as a function on X, so this is analogous to the manifolds definition: functions on M are $\mathrm{Mor}(M, \mathbb{R})$.

(b) The functor Γ^* of invertible functions is represented by $\mathrm{Spec}(\mathbb{C}[x, x^{-1}])$.

In general, how do we tell if a functor $h\colon$ **Sch** $\to$ **Fun**(**Sch**$^{\mathrm{op}}$, **Sets**) is representable? The idea is that schemes are glued together from a cover of open affine schemes. We can repeat this inside the functor category: representable functors are glued together from a cover of open representable functors. Rephrased: locally representable functors are representable. What does this mean?

First, how does one glue functors? If $h = h_X$, then $h_X(Y) = \mathrm{Mor}(Y, X)$, and if $Y = \bigcup_{i \in I} Y_i$, then a morphism $f\colon X \to Y$ is determined by $f_i := f|_{Y_i}\colon Y_i \to X$. So $\mathrm{Mor}(Y, X)$ forms a sheaf on Y. Consequently, $\mathrm{Mor}_{\mathbf{Sch}}(-, X)$ form a sheaf on *any* scheme.

Definition 5.6. A functor $h\colon \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Sets}$ such that $h(Y)$ is a sheaf on Y for any Y is a **Zariski sheaf**.

The proposition below was proved in the paragraph above.

Proposition 5.7. *Representable functors are Zariski sheaves.*

The following is also not hard to check.

Proposition 5.8. *Given* $X \to S$ *and* $Y \to S$, $h_X \times_{h_S} h_Y$ *is a Zariski sheaf.*

Second, what does it mean for these to be open? If $U \hookrightarrow X$ is an open subscheme, we know that the fibered product $U \times XY$ always exists as the preimage of U under $Y \to X$. By the Yoneda lemma, we have a morphism $h_U \to h_X$. We want to call this an open map.

Definition 5.9. In general, we say $h' \to h$ expresses h' as an **open subfunctor** of h if for all representable functors h_Y and natural transformations $h_Y \to h$, $h_Y \times_h h'$ exists and is representable by $\widetilde{U}$, and $h_{\widetilde{U}} \to h_Y$ corresponds to a map where $\widetilde{U} \to Y$ is an open embedding.

Definition 5.10. We say that a collection of open subfunctors $h_i \to h$ covers h if for any representable functor h_X and natural transformations $h_X \to h$, there are $h_{U_i} \to h_X$ with U_i covering X and the following diagram commutes

$$\begin{tikzcd} h_{U_i} \arrow[r] \arrow[d] & h_X \arrow[d] \\ h_i \arrow[r] & h. \end{tikzcd}$$

Theorem 5.11. *Locally representable functors are representable. This means that if h has an open covering by representable Zariski sheaves, then h is representable.*

This gives us a new construction of the fibered product, as the scheme representing $h_X \times_{h_S} h_Y$.

5.2 Reduced Schemes

Definition 5.12. An affine scheme $\mathrm{Spec}(A)$ is **reduced** if A has no nonzero nilpotents.

A scheme X is **reduced** if it has an open cover by affine schemes, each of which is reduced.

Theorem 5.13. *Let X be an affine scheme over a ring R that is finitely generated over $\mathbb{Z}$. This comes with a map $X \to \mathrm{Spec}(\mathbb{Z})$. Let $X_{\mathbb{Q}}$ be the fiber over the generic point, and let X_p be the fiber the point $\mathrm{Spec}(\mathbb{Z}/\langle p \rangle)$ in $\mathrm{Spec}(\mathbb{Z})$.*

Then $X_{\mathbb{Q}}$ is reduced if and only if all but finitely many X_p are.

Example 5.14. Consider $\mathbb{C}[x] \hookrightarrow \mathbb{C}[x,y]/\langle x^2 y \rangle$. As a map on spaces, the codomain of this map looks like the coordinate axes, but the x-axis is a double point. We picture $\mathbb{C}[x]$ as a line. The associated map on spectra is the projection of the coordinate axes (with a double x-axis) onto the line.

The fibers over all points except $\langle x \rangle$ are non-reduced.

Example 5.15. Consider $\mathbb{Z} \to \mathbb{Z}[y]/\langle 2y^2 \rangle$. When we reduce mod p, there are two things that can happen. For $p = 2$, this becomes the inclusion $\mathbb{F}_2 \to \mathbb{F}_2[y]$. So the fiber over $\langle p \rangle$ is reduced. When $p > 2$, this is $\mathbb{F}_2 \to \mathbb{F}_2[y]/\langle 2y^2 \rangle$, in which case it is not reduced.

Proof of Theorem 5.13. It suffices to show that $R \otimes \mathbb{Q}$ has no nilpotents if and only if $R \otimes \mathcal{F}_p$ has none for all but finitely many p.

Suppose that we have $r \in R$ such that $r_{\mathbb{Q}} \in R \otimes \mathbb{Q}$ is nilpotent. In fact, we may assume $r_{\mathbb{Q}}^2 = 0$. Notice that $r_{\mathbb{Q}}^2 = (r^2)_{\mathbb{Q}} = 0$. Claim that this is equivalent to r^2 being a torsion element of R; indeed, if r^2 is torsion, then there is a natural number N such that $Nr^2 = 0$, so $(r^2)_{\mathbb{Q}} = r^2 \otimes 1 = Nr^2 \otimes \frac{1}{N}$. And conversely, if $(r^2)_{\mathbb{Q}} = 0$, then $r^2 \otimes 1 = 0$ in $R_{\mathbb{Q}}$. Hence, $Nr^2 \otimes \frac{1}{N} = 0$ in $\mathbb{R}_{\mathbb{Q}}$ for all N, but $\frac{1}{N} \neq 0$. So $Nr^2 = 0$ in R. Therefore, if $r_{\mathbb{Q}}^2 = 0$, then $r_{\mathbb{F}_p}^2 = 0$ for all p not dividing N.

Conversely, if $s_{\mathbb{F}_p} = 0$ for all but finitely many p, then we may multiply these finitely many primes together to get some N such that $Ns_{\mathbb{F}_p} = 0$ for all p. Therefore, $Ns \in \bigcap_{p \text{ prime}} pR$. Since R is finitely generated over $\mathbb{Z}$, $Ns = 0$. □

5.3 Frobenius Splittings

Notice that a ring R has no nonzero nilpotents, if and only if for all (and in particular, there exists) $n > 1$ such that $r^n = 0 \implies r = 0$ for all $r \in R$. We want to rewrite this second condition as $\ker(r \mapsto r^n) = 0$, but this doesn't quite work.

If n is a prime, then $R \geq \mathbb{F}_p$, and $r \mapsto r^n$ is additive. Hence, R being reduced corresponds in this case to $\ker(f\colon r \mapsto r^p) = 0$ if and only if there is a one-sided inverse to f.

Let's axiomatize this definition.

Definition 5.16. A function $\phi\colon R \to R$, $R \geq \mathbb{F}_p$ is a **(Frobenius) splitting** if

(a) $\phi(a+b) = \phi(a) + \phi(b)$

(b) $\phi(a^p b) = a\phi(b)$

(c) $\phi(1) = 1$.

Definition 5.17. If a field k has characteristic p, then we call k **perfect** if it has all p-th roots.

Example 5.18. Let $R = k[x_1, \ldots, x_n]$ for a perfect field k. For a monoial m, define

$$\phi(m) = \begin{cases} \sqrt[p]{m} & \text{if this exists} \\ 0 & \text{otherwise.} \end{cases}$$

This is called the **standard splitting of** R.

Easy theorem:

Theorem 5.19. *If there is a Frobenius splitting $\phi\colon R \to R$, then R is reduced.*

Proof. Assume first that there is some x such that $x^p = 0$. In this case, $\phi(x^p) = \phi(0) = 0$, but on the other hand, $\phi(x^p) = x\phi(1) = x$. Hence, $x = 0$.

If x is a nilpotent with $x^n = 0$, we will reduce to the case $x^p = 0$. If $n < p$, then certainly $x^p = 0$, and if $x^n = 0$ for $n > p$, then $(x^{n-1})^p = 0$. The argument above shows that $x^{n-1} = 0$ and $n - 1 > p$, then we may repeat the argument to show that $x^{n-1} = 0$, and in this case reduce to the case that $x^p = 0$, in which case $x = 0$. □

Definition 5.20. If $I \leq R$ is **(compatibly) split** if $\phi(I) \leq I$.

Proposition 5.21. *If $I \leq R$ is compatibly split, then ϕ descends to R/I.*

Proposition 5.22. *Let I and J be compatibly split and let K be an ideal. Then*

(a) $I = \sqrt{I}$*,*

(b) $I \cap J$ *is split,*

(c) $I + J$ *is split,*

(d) $I\colon K$ *is split,*

(e) prime components of I *are split.*

Example 5.23. The standard splitting in $k[x_1, \ldots, x_n]$

$$\phi(m) = \begin{cases} \sqrt[p]{m} & \text{if this exists} \\ 0 & \text{otherwise.} \end{cases}$$

splits the ideal $I = \langle \prod_i x_i \rangle$.

The previous proposition says that the prime components $\langle x_i \rangle$ are themselves split, and then that sum of any two of those are also split. Hence, any union of coordinate spaces is split.

Definition 5.24. If m is a monomial in $k[x_1, \ldots, x_n]$ for a perfect field k, define

$$\operatorname{tr}(m) = \begin{cases} \left(m \prod_{i=1}^n x_i\right)^{1/p} \Big/ \prod_{i=1}^n x_i & \text{if this } p\text{-th root exists,} \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\operatorname{tr}(1) = 0$, so this is not a splitting.

Theorem 5.25. *let $f \in k[x_1, \ldots, x_n]$, and let $\phi_f(g) = \operatorname{tr}(gf^{p-1})$. Then ϕ_f satisfies conditions (a) and (b) of the definition of a Frobenius splitting, and if $\phi_f(1) = 1$, then $\langle f \rangle$ is compatibly split.*

Example 5.26. Let $f = \prod_{i=1}^n x_i$. Then ϕ_f is the standard splitting.

Theorem 5.27 (Knutson,LMP)**.** *If $f \in \mathbb{Z}[x_1, \ldots, x_n]$ is of degree n with leading term $\prod_{i=1}^n x_i$ (under some term order). Then for all p, $\phi_f(1) = 1$ (ϕ_f is a splitting) and $\langle f \rangle \leq \mathcal{F}_p[x_1, \ldots, x_n]$ is compatible.*

Proof. Need to compute $\mathrm{tr}(f^{p-1})$. Note that f^{p-1} has leading term $\prod_{i=1}^n x_i^{p-1}$. When we compute $\mathrm{tr}(f^{p-1})$, it turns out to be

$$\sqrt[p]{\prod_i x_i^p} \Big/ \prod_i x_i = 1,$$

so $\phi_f(1) = \mathrm{tr}(f^{p-1}) = 1$. $\square$

Example 5.28. Some examples of f for which ϕ_f is a splitting as in the previous theorem.

Let f be the product of the $i \times i$ northwest determinants of the matrix

$$\begin{bmatrix} m_{11} & m_{12} & \cdots & 1 \\ m_{21} & & \iddots & \\ \vdots & \iddots & & \\ 1 & & & \end{bmatrix}$$

in $\binom{N}{2}$ variables. When $N = 3$, the matrix is

$$\begin{bmatrix} a & b & 1 \\ b & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and $f = a(bc - a)$.

5.4 Line bundles on projective schemes

Recall that if $X \hookrightarrow Y$ is the inclusion of a closed subscheme, then $\mathcal{O}_X = \mathcal{O}_Y / \mathcal{I}_Y$ where $\mathcal{I}_Y$ is a quasicoherent sheaf of ideals on Y.

Given a sheaf $\mathcal{F}$ on Y, it is possible (but unpleasant) to define a sheaf on X. For example, if X is a point in Y, then we might take $\mathcal{F}$ to be the stalk over that point, which is kind of gross as a limit. But if $\mathcal{F}$ is an $\mathcal{O}_Y$-module, then $\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{F}$ gives a nice sheaf on X.

How is this nice? If $\mathcal{F}$ is locally free of rank n, then so too is $\mathcal{O}_X \otimes \mathcal{O}_Y \mathcal{F}$.

Definition 5.29. Let $\mathcal{F}$ be an $\mathcal{O}_Y$-module. We say that $\mathcal{F}$ is **locally free of rank** n if it is isomorphic to a direct sum of copies of $\mathcal{O}_Y$ on open sets.

$$\mathcal{F}(U) \cong \bigoplus_{i=1}^n \mathcal{O}_Y$$

We also call this kind of sheaf an n**-dimensional vector bundle**.

For the remainder of this section, assume we are working over a field k.

Recall $\widetilde{\mathbb{C}}^n \to \mathbb{CP}^{n-1}$, where $\widetilde{\mathbb{C}}^n$ is the set of pairs $(\vec{v}, \ell)$ with $v \in \ell$; the map $\widetilde{\mathbb{C}}^n \to \mathbb{CP}^{n-1}$ forgets the line.

Given any vector space V, we may perform a similar construction to obtain a bundle $\tau\colon \widetilde{V} \to \mathbb{P}(V)$. Denote this line bundle on $\mathbb{P}(V)$ by $\mathcal{O}(-1)$.

Definition 5.30. $\mathcal{O}(-1)$ is the **tautological line bundle**. $\mathcal{O}(1)$ is by definition the dual of $\mathcal{O}(-1)$.

The sheaf of rings $\tau_*\mathcal{O}_{\widetilde{V}}$ on $\mathbb{P}(V)$ is moreover a sheaf of graded rings, and

$$\Gamma(\mathbb{P}(V); \tau_*\mathcal{O}_{\widetilde{V}}) = \mathrm{Sym}(V^*) = \bigoplus \Big\{ \text{degree } n \text{ polynomials} \Big\}.$$

Think about $\mathrm{Sym}(V^*)$ as global functions on V; it is a polynomial ring. This reflects the grading on $\tau_*\mathcal{O}_{\widetilde{V}}$.

Definition 5.31. $\mathcal{O}(k)$ is the degree k part of $\tau_*\mathcal{O}_{\widetilde{V}}$.

Fact 5.32. $\mathcal{O}(k) \cong \mathcal{O}(1)^{\otimes k}$.

Definition 5.33. $\mathcal{O}(-k) := O(-1)^{\otimes k}$ for $k > 0$.

Given any $X \hookrightarrow \mathbb{P}(V)$, we may pull back $\mathcal{O}(1)$ to get a line bundle/sheaf on X, usually also called $\mathcal{O}(1)$.

Remark 5.34. Sometimes, we call a sheaf **invertible** if it comes from a line bundle; this is because the sheaf associated to a line bundle has an inverse under tensor product.

For X nonempty, the map $\Gamma(\mathbb{P}(V); \mathcal{O}(1)) \cong V^* \to \Gamma(X; \mathcal{O}(1))$ induced by restriction is not zero. Indeed, pick $x \in X \hookrightarrow \mathbb{P}(V)$ and $f \in V^*$ such that $f|_x \neq 0$; the image of f in $\Gamma(X; \mathcal{O}(1))$ is nonzero. But this map may be neither surjective or injective.

If $X \hookrightarrow \mathbb{P}(V)$ factors through $\mathbb{P}(W)$ for a linear subspace W of V, then $V^* \to \Gamma(X; \mathcal{O}(1))$ factors through $V^*/W^\perp$, where $W^\perp = \{f \in V^* \mid f|_W = 0\}$.

In fact, there is a unique smallest W, called $\mathrm{Span}(X)$, such that

$$\ker(V^* \to \Gamma(X; \mathcal{O}(1))) = W^\perp.$$

Example 5.35 (Example where this is not surjective)**.** Consider the **rational normal curve** $X = \mathbb{P}^1 \to \mathbb{P}^n$ given by the n-th Veronese map

$$\mathrm{Ver}^n\colon [a, b] \mapsto [a^n, a^{n-1}b, a^{n-2}b^2, \ldots, b^n].$$

In this case, $\mathrm{Span}(X)$ is all of $\mathbb{P}^n$. Moreover,

$$(\mathrm{Ver}^n)^*(\mathcal{O}(1)) \cong \mathcal{O}(n).$$

Compose this with the quotient by a generic codimension 4 subspace to get a well-defined map $\gamma\colon \mathbb{P}^1 \to \mathbb{P}^3$. There may not actually be a map $\mathbb{P}^n \dashrightarrow \mathbb{P}^3$; it is only defined on the generic point and a large open subset of $\mathbb{P}^n$, but this large open subset contains $\mathbb{P}^1$. Hence, the composite $\gamma\colon \mathbb{P}^1 \to \mathbb{P}^3$ is well-defined degree n curve in $\mathbb{P}^3$.

On sheaves, this gives

$$\begin{array}{ccc} \mathbb{P}^1 & \xrightarrow{\gamma} & \mathbb{P}^3 \\ \mathcal{O}(n) & \longleftarrow & \mathcal{O}(1) \end{array}$$

Thinking about sections of these sheaves, $\Gamma(\mathbb{P}^3; \mathcal{O}(1))$ is 4-dimensional, yet $\Gamma(\mathbb{P}^1; \mathcal{O}(n))$ is $(n+1)$-dimensional. So this cannot be surjective.

So far, we have considered projective embeddings $X \hookrightarrow \mathbb{P}(V)$ and line bundles over X. The pullback map taking line bundles on $\mathbb{P}(V)$ to line bundles on X is far from injective, because there might be many ways to embed X in $\mathbb{P}(V)$. It is also far from surjective, because there are more bundles on X than those that come from pullbacks of bundles on $\mathbb{P}(V)$.

But given a line bundle $\mathcal{L}$ on X, let $W = \Gamma(X; \mathcal{L})$ (i.e. $W^* = V$ from before). When can we define $X \to \mathbb{P}(W^*)$?

Given $x \in X$, and if there is some $\vec{w}_x \in W$ such that $0 \neq \vec{w}_x|_x \in \mathcal{L}_x$, we may use it to define an element of W^*:

$$\vec{w} \mapsto \vec{w}|_x / \vec{w}_x|_x \in k.$$

The choice of $\vec{w}_x$ only changes this element up to scale, so no matter which $\vec{w}_x$ we choose, we get a well-defined element $\mathbb{P}(W^*)$. But we don't know that such a $\vec{w}_x \in W$ exists at all!

Definition 5.36. The **basepoints of** $\mathcal{L}$ are those $x \in X$ such that there is no $\vec{w}_x$ as above, or equivalently, for all $\vec{w} \in W$, $\vec{w}|_x = 0$.

So we only get a map from $X \setminus \{\text{basepoints}\}$ to $\mathbb{P}(W^*)$.

Example 5.37. Consider $\mathcal{L} = \mathcal{O}(-1)$ on $\mathbb{P}^1$. The only homogeneous polynomial of degree -1 is 0:

$$\Gamma(\mathbb{P}^1; \mathcal{O}(-1)) = 0.$$

Therefore, all of $\mathbb{P}^1$ is basepoints.

Of course, in this example $W^* = 0$, so $\mathbb{P}(W^*) = \emptyset$, so this is expected.

Example 5.38. Consider $\mathcal{O}_{\mathbb{P}^1}$. Global functions on $\mathbb{P}^1$ are constants by Liouville's theorem:

$$\Gamma(\mathbb{P}^1; \mathcal{O}_{\mathbb{P}^1}) = k.$$

There are no points in $\mathbb{P}^1$ where every constant function vanishes, so there are no basepoints. In this case, $W = k$ and $\mathbb{P}(W^*) = \mathbb{P}(k)$ is just a point, so $X \to P(W^*)$ is the unique such map.

Example 5.39. Consider the blowup of $\mathbb{P}^2$ at a point, $b\colon \widetilde{\mathbb{P}}^2 \to \mathbb{P}^2$. The line bundle $\mathcal{L} = b^*\mathcal{O}_{\mathbb{P}^2}(1)$ has no basepoints, and $W = k^3$. The map $\widetilde{\mathbb{P}}^2 \to \mathbb{P}^2$ is just b itself.

Definition 5.40. If $\mathcal{L}$ has no basepoints, and the induced map $X \to \mathbb{P}(\Gamma(X;\mathcal{L})^*)$ is an embedding, then $\mathcal{L}$ is called **very ample.**

For any very ample line bundle, we may reconstruct the embedding $X \hookrightarrow \mathbb{P}^n$. On the other hand, if $X \hookrightarrow \mathbb{P}^n$ is an embedding, we may pullback the tautological line bundle on $\mathbb{P}^n$ to construct a very ample bundle on X. In fact, starting with a very ample line bundle $\mathcal{L} \to X$ and constructing from it an embedding $X \hookrightarrow \mathbb{P}^n$, the pullback of the tautological line bundle along this embedding is isomorphic to $\mathcal{L}$.

How can we go from $\mathcal{L} \to X$ to a statement of the form "$X \setminus \{\text{basepoints}\} = \operatorname{Proj}(R)$"?

Definition 5.41. The **section ring/form ring** of a line bundle $\mathcal{L} \to X$ is

$$\bigoplus_{n=1}^{\infty} \Gamma(X;\mathcal{L}^{\otimes n}).$$

Definition 5.42. If $\mathcal{L} \to X$ is a line bundle, then we call $\mathcal{L}$ **ample** if $\mathcal{L}^{\otimes N}$ is very ample for some N.

Theorem 5.43 (Several theorems)**.**

(a) *If $\mathcal{L}$ is ample, then X is $\operatorname{Proj}(R)$ where R is the section ring.*

(b) *Otherwise, the section ring may be non-Noetherian and not finitely generated.*

(c) *If X is smooth and $\mathcal{L} = \bigwedge^{\text{top}} T^*X$, then the section ring is Noetherian.*

The last of these is a result from 2010.

5.5 Divisors

Definition 5.44. Let X be irreducible. A **geometric divisor** $D \subseteq X$ is a subscheme of pure codimension 1.

Remark 5.45. This is not a standard definition or terminology.

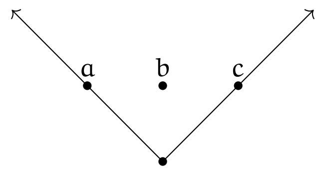
Example 5.46. Let $X = \mathbb{A}^2_k$. Then $D = V(x^2y)$ is a geometric divisor.

Example 5.47 (Non-examples)**.** Let $X = \mathbb{A}^2_k$. Then $D = V(x^2y, xy^2)$ and $D = V(x^2, xy)$ are not codimension 1, so they are *not* geometric divisors.

Example 5.48. If $\mathcal{I}$ is a locally principal, quasicoherent ideal sheaf, then the subscheme defined by $\mathcal{I}$ is a geometric divisor.

Definition 5.49. A **geometric Cartier divisor** is a geometric divisor defined by a locally principal, quasicoherent ideal sheaf $\mathcal{I}$.

Example 5.50 (Non-example)**.** Let X be the toric variety associated to the cone



In particular, $X = \operatorname{Spec} \mathbb{C}[a, b, c]/\langle ac - b^2 \rangle$. Notice that $D(b, c)$ is not Cartier, but $V(c)$ is.

Theorem 5.51. *Where* X *is smooth, a geometric divisor is a Cartier geometric divisor.*

Definition 5.52. If $D \subseteq X$ is a geometric divisor, then $\mathcal{F}_D$ is the sheafification of the presheaf with sections over U given by

$$\left\{ \frac{f}{\sigma} \;\middle|\; f \in \mathcal{O}_{U \cap X_{\text{reg}}},\ D|_{U \cap X_{\text{reg}}} = V(\sigma) \right\} \subseteq \mathcal{O}_{X \setminus D} = \left\{ \frac{f}{\sigma^n} \right\}$$

Example 5.53. Let $X = \mathbb{P}^n = \operatorname{Proj} \mathbb{C}[z_0, \ldots, z_n]$, and let $D = \mathbb{P}V(z_0) \cong \mathbb{P}^{n-1}$. Then $\Gamma(\mathbb{P}^n; \mathcal{F}_D)$ has basis $1, \frac{z_1}{z_0}, \frac{z_2}{z_0}, \ldots, \frac{z_n}{z_0}$. This is isomorphic to $\mathcal{O}(1)$.

Remark 5.54. $\Gamma(U; \mathcal{F}_D) \cong \mathcal{O}_{U \cap X_{\text{reg}}}$.

Lemma 5.55. *If* $\mathcal{D}$ *is a geometric Cartier divisor, this* $\mathcal{F}_D$ *is a line bundle.*

Since we have a notion of tensoring line bundles on X, there should be a corresponding notion on geometric Cartier divisors, which we call $+$. Similarly, there should be another operation corresponding to dualizing a line bundle, which we call $-$.

Definition 5.56. A **Weil divisor** is a formal $\mathbb{Z}$-linear combination of geometric divisors. A **Cartier divisor** is a formal $\mathbb{Z}$-linear combination of geometric Cartier divisors.

Example 5.57. Any rational function f on $\mathbb{CP}^1$ defines a Cartier divisor as the sum of the zeroes of f minus the poles of f. This is a **degree zero** Cartier divisor.

In general, the zeroes minus the poles of a general rational section of a line bundle defines a Cartier divisor.

Definition 5.58. The **Picard group** of a scheme X is the group of line bundles on X with operation $\otimes$, and inverses given by duals.

Definition 5.59. The **divisor class group** of a scheme X is the group of Cartier divisors on X modulo the degree zero Cartier divisors.

5.6 Ample Line Bundles

Given a curve γ embedding in a projective scheme X, and a very ample line bundle $\mathcal{L}$ on X, then we have a diagram

$$\begin{tikzcd} \mathcal{L}|_\gamma \arrow[r] \arrow[d] & \mathcal{L} \arrow[r] \arrow[d] & \mathcal{O}(1) \arrow[d] \\ \gamma \arrow[r, hook] & X \arrow[r, hook] & \mathbb{P}^N \end{tikzcd}$$

The Hilbert polynomial of γ has the form

$$\deg(\mathcal{L}|_\gamma)d^1 + g$$

where g is called the **arithmetic genus of** γ, and the degree of $\mathcal{L}$ on γ is a number greater than zero.

Theorem 5.60. *If $\mathcal{L}$ and $\mathcal{L}'$ are very ample on X, then*

$$\deg(\mathcal{L} \otimes \mathcal{L}'|_\gamma) = \deg(\mathcal{L}|_\gamma) + \deg(\mathcal{L}'|_\gamma).$$

Theorem 5.61. $\deg(\mathcal{L}|_\gamma)$ *extends to non-ample $\mathcal{L}$ as the number of zeros minus the number of poles of a rational section.*

Lemma 5.62. *If $\mathcal{L}$ is very ample, then* $\deg(\mathcal{L}|_\gamma) > 0$ *for all* $\gamma \hookrightarrow X$.

Theorem 5.63 (Kodaira). *If* $\deg(\mathcal{L}) > 0$ *for all* $\gamma \hookrightarrow X$*, then $\mathcal{L}$ is ample.*

For any scheme X, there is a group $A_1(X)$ generated by formal $\mathbb{Z}$-linear combinations of curves γ inside X modulo some equivalence relation. Then there is a homomorphism

$$\begin{aligned} A_1(X) \otimes \operatorname{Pic}(X) &\longrightarrow \mathbb{Z} \\ \gamma \otimes \mathcal{L} &\longmapsto \deg(\mathcal{L}|_\gamma). \end{aligned}$$