Math 7350: Differential Graded Algebras and Differential Graded Categories

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The latest version is online here.

Contents

Chapter 1

Introduction: "Handwaving"

1.1 References

All of these are linked to on the course webpage. We will frequently refer to the notes from two previous courses, Homological Algebra [HA1] and Homotopical Algebra [HA2]. For references on DG categories, see [Kel06, Kel93, Dri04, Toë11, Toë11, Tab05b, Tab05a, Kon98].

1.2 Some Definitions

Let A be a unital associative algebra over a field k. Let $\mathbf{A} = \mathbf{Mod}(A)$, the category of (left or right) modules over A.

Definition 1.2.1. Let $\mathbf{Com}(\mathbf{A})$ be the **category of complexes** in $\mathbf{A}$, that is, an object in$\mathbf{Com}(\mathbf{A})$ is a complex $C^\bullet$, that is, a diagram in $\mathbf{A}$

$$\cdots \longrightarrow C^n \xrightarrow{d^n} C^{n+1} \xrightarrow{d^{n+1}} C^{n+2} \xrightarrow{d^{n+2}} \cdots$$

with $d \in \mathrm{Mor}(\mathbf{A})$ and $d^{n+1} \circ d^n = 0$ for all n. A morphism $f^\bullet \colon C^\bullet \to D^\bullet$ of complexes is a collection of $f^n \in \mathrm{Mor}(\mathbf{A})$ such that the following commutes.

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C^n & \xrightarrow{d^n} & C^{n+1} & \xrightarrow{d^{n+1}} & C^{n+2} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f^{n+1}} & & \downarrow{\scriptstyle f^{n+2}} & & \\ \cdots & \longrightarrow & D^n & \xrightarrow{d^n} & D^{n+1} & \xrightarrow{d^{n+1}} & D^{n+2} & \longrightarrow & \cdots \end{array}$$

Definition 1.2.2. The **cohomology** of a complex $C^\bullet$ is the complex $H^\bullet(C)$, with $H^n(C) = \ker(d^n)/\operatorname{im}(d^{n-1})$. Cohomology defines a functor on complexes.

We want to study cohomology of complexes, rather than the complexes themselves. So we need to get rid of the "irrelevant information."

Definition 1.2.3. A morphism $f^\bullet \colon C^\bullet \to D^\bullet$ is called a **quasi-isomorphism** (**weak equivalence**) if $H(f^\bullet) \colon H^\bullet(C) \xrightarrow{\cong} H^\bullet(D)$.

Definition 1.2.4. The **derived category** $\mathcal{D}(\mathbf{Mod}(A))$ is the localization of $\mathbf{Com}(A)$ at the quasi-isomorphisms,

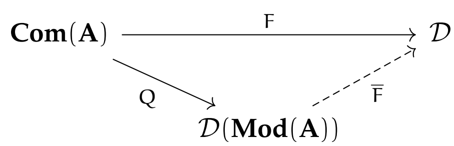
$$\mathcal{D}(\mathbf{Mod}(A)) := \mathbf{Com}(A)[\mathrm{Qis}^{-1}].$$

Here, Qis is the class of all quasi-isomorphisms.

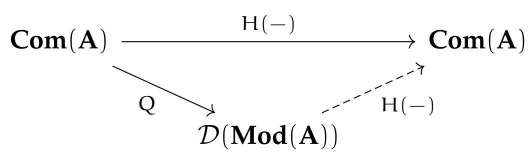
Remark 1.2.5 (Universal property of the derived category)**.** The pair

$$\Big(\mathcal{D}(\mathbf{Mod}(A)), Q\colon\ \mathbf{Com}(\mathbf{A}) \to \mathcal{D}(\mathbf{Mod}(A))\Big)$$

is universal among all pairs $(\mathbf{D}, F\colon\ \mathbf{Com}(\mathbf{A}) \to \mathbf{D})$ where $\mathbf{D}$ is an additive category and $F(f)$ is an isomorphism for all $f \in \mathbf{Com}(\mathbf{A})$.



Example 1.2.6.



1.3 The Role of Derived Categories

Derived categories have appeared in...

(1) Algebraic Geometry (due to Grothendieck, Verdier)

  (i) Grothendieck duality theory (rigid dualizing complexes, see HA I)

  (ii) "Tilting" theory. Beilinson's Derived Equivalence.

Let X be a projective algebraic variety over $\mathbb{C}$, for example $X = \mathbb{P}^n_{\mathbb{C}}$. Let $\mathbf{VB}(X)$ be the category of vector bundles over X, and let $\mathbf{Coh}(X)$ be the category of **coherent sheaves** on X.

Example 1.3.1. If X is affine, or maybe even $X = \mathbb{A}^n_{\mathbb{C}}$, and $A = \mathbb{C}[X]$, then $\mathbf{Coh}(X) = \mathbf{Mod}^{fg}(A)$ and $\mathbf{VB}(X) = \mathrm{Proj}^{fg}(A)$. Think of coherent sheaves as a generalization of vector bundles where we may allow singular points.

We might want to classify algebraic vector bundles over X; this was a classical question. To answer this question, we can instead classify algebraic coherent sheaves (see Example 1.3.1) over X. This is still a very difficult question, but we enlarge the category again and consider the **bounded derived category** $\mathcal{D}^b(\mathbf{Coh}(X))$. This last question we can answer, with Theorem 1.3.2.

$$\mathbf{VB}(X) \hookrightarrow \mathbf{Coh}(X) \hookrightarrow \mathcal{D}^b(\mathbf{Coh}(X))$$

Theorem 1.3.2 (Beilinson 1982)**.** *Let $X = \mathbb{P}^2_{\mathbb{C}}$ (or any n). There is a natural equivalence of triangulated categories*

$$\mathcal{D}^b\left(\mathbf{Coh}(\mathbb{P}^n_{\mathbb{C}})\right) \xrightarrow{\ \sim\ } \mathcal{D}^b\left(\mathbf{Rep}^{fd}(Q)\right),$$

where Q is the quiver

$$Q = \bullet \underset{x_n^{(0)}}{\overset{x_1^{(0)}}{\substack{\longrightarrow \\ \vdots \\ \longrightarrow}}} \bullet \underset{x_n^{(1)}}{\overset{x_1^{(1)}}{\substack{\longrightarrow \\ \vdots \\ \longrightarrow}}} \bullet \underset{x_n^{(2)}}{\overset{x_1^{(2)}}{\substack{\longrightarrow \\ \vdots \\ \longrightarrow}}} \bullet \cdots\cdots\cdots \bullet$$

So the derived category of coherent sheaves on projective spaces can be described by some complicated linear algebra.

Example 1.3.3. For example, if $n = 2$, then $\mathcal{D}^b\left(\mathbf{Coh}(\mathbb{P}^2_{\mathbb{C}})\right) \cong \mathcal{D}^b\left(\mathbf{Rep}^{fd}(Q)\right)$ where Q is the quiver

$$Q = \bullet \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \bullet \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \bullet$$

If $n = 1$, then Q is the **Kronecker Quiver**

$$Q = \bullet \rightrightarrows \bullet$$

Example 1.3.4 (More motivation)**.** For $n = 2$, $A = k[x, y]$, with k an algebraically closed field of characteristic zero. We want to classify ideals in A (as A-modules) in terms of linear algebra.

$$J \underset{\text{ideal}}{\subset} A \iff J \in \mathbf{Coh}(\mathbb{A}^2_k)$$

Then we can embed $\mathbb{A}^2_k$ inside $\mathbb{P}^2_k$, which contains a line ℓ_∞ at infinity.

$$\begin{array}{ccccc} \mathbb{A}^2_k & \hookrightarrow & \mathbb{P}^2_k & \hookleftarrow & \ell_\infty \\ J & \longmapsto & \widetilde{J} & & \widetilde{J}|_{\ell_\infty} = \mathcal{O}_{\mathbb{P}^1} \end{array}$$

We can embed coherent sheaves inside $\mathbb{P}^2$ as

$$\begin{aligned} \mathbf{Coh}(\mathbb{P}^2) &\hookrightarrow \mathcal{D}^b(\mathbf{Coh}(\mathbb{P}^2)) \\ \widetilde{J} &\longmapsto \left(\cdots \to 0 \to \widetilde{J} \to 0 \to \cdots\right) \end{aligned}$$

And then apply Theorem 1.3.2.

In general, if X is a smooth projective variety, we have the following theorem.

Theorem 1.3.5 (Bondal, Kapranov, Van der Bergh, et. al.)**.**

$$\mathcal{D}^b(\mathbf{Coh}(X)) \cong \mathcal{D}^b(\mathbf{DGMod}(A)),$$

where $\mathbf{DGMod}(A)$ *is dg-modules over the dg-algebra* A*.*

The problem with derived categories is that most invariants of X are *determined* by $\mathcal{D}^b(\mathbf{Coh}(X))$ but they cannot be computed directly from $\mathcal{D}^b(\mathbf{Coh}(X))$. To understand the derived category, we need to "represent" the derived category in the same way that differential forms "represent" de Rham cohomology. Therefore, we need to "enhance" $\mathcal{D}^b(\mathbf{Coh}(X))$ by replacing it by a **dg-category** $\mathbb{D}(\mathbf{Coh}(X))$ such that $H(\mathbb{D}(\mathbf{Coh}(X))) \cong \mathcal{D}^b(\mathbf{Coh}(X))$.

There are many different dg-models for $\mathcal{D}^b(\mathbf{Coh}(X))$. we need a way to get rid of the irrelevant information carried by $\mathbb{D}(\mathbf{Coh}(X))$. The best way is to put a **Quillen model structure** on the category **dgCat** of all (small) dg-categories, making it into a **model category.** This is referred to as the study of **noncommutative motives**.

Remark 1.3.6 (Goal)**.** Our goal is to understand this model structure on **dgCat**.

Part I

Quivers and Gabriel's Theorem

Chapter 2

Quivers

Definition 2.0.1. A **quiver** is a quadruple $Q = (Q_0, Q_1, s, t)$, where

- Q_0 is the set of **vertices**;
- Q_1 is the set of **arrows**;
- $s\colon Q_1 \to Q_0$ is the **source map**; and
- $t\colon Q_1 \to Q_0$ is the **target map**.

$$\overset{i}{\bullet} \xleftarrow[a]{} \overset{j}{\bullet}$$

$$s(a) = j, \;\; t(a) = i$$

Together, s, t are called the **incidence maps.**

Definition 2.0.2. A quiver is called **finite** if and only if $|Q_0| < \infty$ and $|Q_1| < \infty$.

Definition 2.0.3. A **path** in Q is a sequence $\vec{a} = (a_1, \ldots, a_m)$ such that $t(a_i) = s(a_{i-1})$ for all i.

$$\overset{1}{\bullet} \xleftarrow[a_1]{} \overset{2}{\bullet} \xleftarrow[a_2]{} \cdots \xleftarrow[a_m]{} \overset{m+1}{\bullet}$$

Definition 2.0.4. Write P_Q for the set of all paths in Q. Notice that s, t extend to maps $s, t\colon \mathcal{P}_Q \to Q_0$ by $s(\vec{a}) = s(a_m)$ and $t(\vec{a}) = t(a_1)$.

Definition 2.0.5. The **path algebra** kQ for a quiver Q over a field k is defined by

$$kQ = \mathrm{Span}_k(\mathcal{P}_Q),$$

with a product defined by concatenating paths:

$$\vec{a} \cdot \vec{b} = \begin{cases} \vec{a}\vec{b} & \text{if } t(\vec{b}) = s(\vec{a}) \\ 0 & \text{otherwise} \end{cases}$$

$$\bullet \longleftarrow \overset{\vec{a}}{\cdots} \;\relbar\!\relbar\; \overset{e}{\bullet} \longleftarrow \overset{\vec{b}}{\cdots} \;\relbar\!\relbar\; \bullet$$

Paths of length zero are by convention the vertices e_i for $i \in Q_0$. This product is associative, and satisfies relations (for example)

$$\vec{a}e = \vec{a}, e\vec{b} = \vec{b}, e\vec{a} = 0, \vec{b}e = 0, e^2 = e$$

What kind of algebras are the path algebras of quivers?

Example 2.0.6. If $|Q_0| = 1$ and $|Q_1| = r$, then the path algebra is the free on r variables, $kQ \cong k\langle x_1, x_2, \ldots, x_r\rangle$. If $r = 1$, then $kQ \cong k[x]$.

Example 2.0.7. If

$$Q = \left(\overset{1}{\bullet} \to \overset{2}{\bullet} \to \cdots \to \overset{n}{\bullet} \right),$$

then kQ is isomorphic to the algebra of lower triangular $n \times n$ matrices over k.

Exercise 2.0.8. If Q has at most one path between any two vertices, show that

$$kQ \cong \left\{ A \in \mathbb{M}_n(k) \mid A_{ij} = 0 \text{ if there is no path } j \to i \right\}$$

Example 2.0.9.

$$k\left(\overset{\infty}{\bullet} \xrightarrow{\ v\ } \overset{0}{\bullet} \circlearrowright x \right) \cong \begin{pmatrix} k[x] & k[x] \\ 0 & k \end{pmatrix} \subseteq \mathbb{M}_2(k[x])$$

The generators of kQ are e_∞, e_0, v, x, with relations

$$e_0 v = v e_\infty = v, \qquad e_0 x = x e_0 = x, \qquad e_\infty^2 = e_\infty,$$

$$v e_0 = e_\infty v = 0, \qquad e_\infty x = x e_\infty 1 = 0, \qquad e_0^2 = e_0.$$

The isomorphism is given on generators by

$$e_0 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad e_\infty \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$X \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \qquad v \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

2.1 Path Algebras

Let Q be a quiver and let $A = kQ$.

Proposition 2.1.1. $\{e_i\}_{i \in Q_0}$ *is a complete set of orthogonal idempotents:*

- $e_i^2 = e_i$ *(**idempotent**),*
- $e_i e_j = 0$ *for* $i \neq j$ *(**orthogonal**),*
- $\sum_{i \in Q_0} e_i = 1_A$ *(**complete**).*

Proposition 2.1.2. *For* $i, j \in Q_0$*, the space* Ae_i*,* e_jA *and* e_jAe_i *have the following bases:*

- Ae_i *= all paths starting at* i
- e_jA *= all paths ending at* j
- e_jAe_i *= all paths starting at* i *and ending at* j*.*

Proposition 2.1.3. *Decompositions of* A *into direct sums of projective ideals.*

(a) $A = \bigoplus_{i \in Q_0} Ae_i$ *as a left* A*-module* $\Longrightarrow$ Ae_i *is a projective left* A*-module.*

(b) $A = \bigoplus_{j \in Q_0} e_jA$ *as a right* A*-module*$\Longrightarrow$ e_jA *is a projective right* A*-module.*

Proposition 2.1.4. *For any left* A*-module* M *and right* A*-module* N*:*

(a) $\mathrm{Hom}_A(Ae_i, M) \cong e_iM$

(b) $\mathrm{Hom}_A(e_jA, N) \cong Ne_j$

Proof of (a). Any $f \in \mathrm{Hom}_A(Ae_i, M)$ is determined by $f(e_i) = x \in M$, by A-linearity. On the other hand, $e_i^2 = e_i$, so

$$e_i f(e_i) = f(e_i)^2 = f(e_i)$$

Hence, for any $x \in M$, if $f(e_i) = x$, then $e_i x = x$. The map is then

$$\begin{array}{rcl} \mathrm{Hom}_A(Ae_i, M) & \longrightarrow & e_iM \\ f & \longmapsto & f(e_i) \end{array}$$

□

Proposition 2.1.5. *If* $0 \neq a \in Ae_i$ *and* $0 \neq b \in e_iA$ *then* $ab \neq 0$ *in* A*.*

Proof. Write

$$a = cx + \dots$$

$$b = \tilde{c}y + \dots$$

where x is the longest path starting at i, and y is the longest path ending at i, with $C \neq 0 \neq \tilde{c}$.

$$a \cdot b = c\tilde{c}xy + \dots$$

This is nonzero because $c\tilde{c} \neq 0$. □

Proposition 2.1.6. *Each e_i is a **primitive** idempotent, meaning that each Ae_i is an indecomposable left A-module.*

Proof. If M is decomposable, then there is some submodule $N \subsetneq M$ such that $M \cong N \oplus K$. In this case, $\mathrm{End}_A(M)$ has at least one idempotent

$$e\colon \ M \xrightarrow{\mathrm{pr}} N \xhookrightarrow{i} M \ \in \mathrm{End}_A(M).$$

Thus, we need to check that $\mathrm{End}_A(Ae_i)$ has no nontrivial idempotents.

$$\mathrm{End}_A(Ae_i) = \mathrm{Hom}_A(Ae_i, Ae_i) \underset{(4)}{\cong} e_iAe_i$$

If $f\colon Ae_i \to Ae_i$ is idempotent in $\mathrm{End}_A(Ae_i) = e_iAe_i$, then $f^2 = f = fe_i$, so $f(f - e_i) = 0$. This implies by Proposition 2.1.1(5) that $f = 0$ or $f - e_i = 0$. Hence, $\mathrm{End}_A(Ae_i)$ has no nontrivial idempotents. □

Definition 2.1.7. Let

$$kQ_0 = \bigoplus_{i \in Q_0} ke_i \cong \underbrace{k \times \ldots \times k}_{|Q_0|}$$

$$kQ_1 = \bigoplus_{a \in Q_1} ka$$

Notice that kQ_1 is naturally an kQ_0-bimodule.

Definition 2.1.8. For any k-algebra S and any S-bimodule M, the tensor algebra T_SM is

$$T_SM = S \oplus M \oplus (M \otimes_S M) \oplus \ldots \oplus (M \otimes_S \cdots \otimes_S M) \oplus \ldots$$

is defined by the following universal property.

Given any k-algebra $f_0\colon S \to A$ and any S-bimodule map $f_1\colon M \to A$, there is a unique S-bimodule map $f\colon T_SM \to A$ such that $f|_S = f_0$ and $f|_M = f_1$.

Proposition 2.1.9. kQ *is naturally isomorphic to the tensor algebra* $kQ \cong T_S(V)$

Proof. Check the universal property. If $S = kQ_0$, $M = V = kQ_1$, then

$$f_0\colon kQ_0 \hookrightarrow kQ$$

$$f_1\colon kQ_1 \hookrightarrow kQ$$

$$f\colon T_S(V) \to kQ$$

f is surjective by definition of kQ, and f is injective by induction on the grading in $T_S(V)$. □

Corollary 2.1.10. kQ *is a graded* S*-algebra with grading determined by the length function on paths.*

Exercise 2.1.11.

(a) $\dim_k(kQ) < \infty$ if and only if Q has no (oriented) cycles.

(b) kQ is **prime** (i.e. $IJ \neq 0$ for any two 2-sided ideals $I, J \neq 0$) if and only if for all $i, j \in Q_0$, there is a path $i \to j$.

(c) kQ is left (resp. right) Noetherian $\iff$ if there is an oriented cycle at i, then at most one arrow starts (resp. ends) at i.

Example 2.1.12. Consider the quiver

$$Q = \left(\overset{\infty}{\bullet} \xrightarrow{\ \nu\ } \overset{0}{\bullet} \circlearrowright x \right)$$

The path algebra kQ is left Noetherian but *not* right Noetherian.

2.2 Representations of Quivers

Fix a field k, and let $Q = (Q_0, Q_1, s, t)$ be a quiver. Recall that $\mathcal{P}_Q$ is the set of all paths in Q, and s, t extend to maps $s, t \colon \mathcal{P}_Q \to Q_0$.

Definition 2.2.1. The **path category Q** is the category with objects Q_0 and

$$\mathrm{Hom}_{\mathbf{Q}}(i, j) = \{\vec{a} \in \mathcal{P}_Q : s(\vec{a}) = i,\ t(\vec{a}) = j\}.$$

Composition is given by concatenating paths.

Remark 2.2.2. We can modify this definition in two ways. First, we can make **Q** into a k**-category** (a category enriched in k-modules) $k\mathbf{Q}$ whose objects are Q_0 and

$$\mathrm{Hom}_{k\mathbf{Q}}(i, j) = k\big[\mathrm{Hom}_{\mathbf{Q}}(i, j)\big]$$

Second, we can also make **Q** a k**-linear category** (to be defined later).

Definition 2.2.3. A **representation** of Q is a functor $F \colon \mathbf{Q} \to \mathbf{Vect}_k$. The category of all such representations is a functor category, denoted

$$\mathbf{Rep}_k(Q) := \mathbf{Fun}(\mathbf{Q}, \mathbf{Vect}_k).$$

A representation $X \colon \mathbf{Q} \to \mathbf{Vect}_k$ is usually denoted as follows.

$$Q_0 \ni i \longmapsto X(i) = X_i$$

$$Q_1 \ni i \longmapsto X(a) = X_a$$

$$\left(\overset{i}{\bullet} \xrightarrow{\ a\ } \overset{j}{\bullet} \right) \longmapsto \left(X_i \xrightarrow{\ X_a\ } X_j \right)$$

Definition 2.2.4. If $Q = (Q_0, Q_1, s, t)$ is a quiver, define the **opposite quiver**

$$Q^{op} := (Q_0, Q_1, s^\circ = t, t^\circ = s).$$

Theorem 2.2.5. *There are natural equivalences of categories*

$$\mathbf{Rep}_k(Q) \simeq kQ\text{-}\mathbf{Mod} \quad \textit{(left } kQ\textit{-modules)}$$

$$\mathbf{Rep}_k(Q^{op}) \simeq \mathbf{Mod}\text{-}kQ \quad \textit{(right } kQ\textit{-modules)}$$

Proof. The functor $F\colon kQ\text{-}\mathbf{Mod} \to \mathbf{Rep}_k(Q)$ is given on objects by

$$M \mapsto X_M := (X_i, X_a)_{\substack{i \in Q_0 \\ a \in Q_1}}$$

where $X_i = e_i M$ and X_a is the morphism given by

$$\begin{array}{ccc} X_i & \xrightarrow{X_a} & X_j \\ \| & & \| \\ e_i M & \xrightarrow{a\cdot} & e_j M. \end{array}$$

(recall that $ae_i = a = e_j a$). The functor F is given on morphisms by

$$(f\colon M \to N) \longmapsto \left(f|_{e_i M}\colon e_i M \to e_i N\right)_{i \in Q_0}.$$

Conversely, $G\colon \mathbf{Rep}_k(Q) \to kQ\text{-}\mathbf{Mod}$ is given on objects by

$$X = (X_i, X_a)_{\substack{i \in Q_0 \\ a \in Q_1}} \longmapsto X = \bigoplus_{i \in Q_0} X_i.$$

Write $\varepsilon_i\colon X_i \hookrightarrow X$ and $\pi_i\colon X \twoheadrightarrow X_i$ for the canonical maps. Given a path $\vec{a} = (a_1, \ldots, a_n) \in \mathcal{P}_Q$, $x \in X$, define

$$\vec{a} \cdot x = \varepsilon_{t(a_1)} \circ X_{a_1} \circ X_{a_2} \circ \cdots \circ X_{a_n} \circ \pi_{s(a_n)} x$$

Now check that $F \circ G \simeq \mathrm{id}$, $G \circ F \simeq \mathrm{id}$. □

Example 2.2.6. Consider a quiver representation $S(i)$, where $i \in Q_0$. The representation $S(i)$ is defined by

$$S(i)_j = \begin{cases} k & (j = i) \\ 0 & (j \neq i) \end{cases}$$

and $S(i)_a = 0$ for all $a \in Q_1$. Every irreducible kQ-module looks like $S(i)$ for some $i \in Q_0$.

Example 2.2.7. The indecomposable projective kQ-modules are of the form Ae_i, which correspond to the quiver representation X with $X_j = e_j A e_i$ for $j \in Q_0$.

2.3 Homological Properties of Path Algebras

Recall Baer's definition of $\mathrm{Ext}^1_A(V, W)$.

Definition 2.3.1 (Baer)**.** Let A be a k-algebra and V, W two objects in $\mathbf{A} = \mathbf{Mod}(A)$.

$$\mathrm{Ext}^1_A(V, W) := \left\{ (\alpha, \beta) \in \mathrm{Mor}(\mathbf{A})^{\times 2} \;\middle|\; 0 \to W \xrightarrow{\alpha} X \xrightarrow{\beta} V \to 0 \right\} \Big/ \sim$$

where $(\alpha, \beta) \sim (\alpha', \beta')$ if and only if there is some $\phi \colon X \to X'$ such that $\alpha' = \phi \circ \alpha$ and $\beta' \circ \phi = \beta$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W & \xrightarrow{\alpha} & X & \xrightarrow{\beta} & V & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \phi} & & \| & & \\ 0 & \longrightarrow & W & \xrightarrow{\alpha'} & X' & \xrightarrow{\beta'} & V & \longrightarrow & 0 \end{array}$$

Remark 2.3.2. We can think of a quiver as a kind of finite non-commutative space. We can think of Example 2.0.9 as a kind of non-commutative "extension" of the affine line $\mathbb{A}^1_k = \mathrm{Spec}(k[x])$.

Recall that X is an affine variety over $\mathbb{C}$, and $A = \mathcal{O}(X)$ is a finitely generated commutative $\mathbb{C}$-algebra, and $X = \mathrm{Specm}(A)$; that is, the points of X correspond to irreducible representations of A, which have the form $A/\mathfrak{m}$, where $\mathfrak{m}$ is a maximal ideal of A.. Points of X are "homologically disjoint" in the sense that

$$\mathrm{Ext}^*_A\left({}^{A}/_{\mathfrak{m}_i}, {}^{A}/_{\mathfrak{m}_j} \right) = 0 \quad (i \neq j).$$

On the contrary, in the noncommutative case (for quivers), we will see that

$$\mathrm{Ext}^1_{kQ}(S(i), S(j)) \neq 0$$

if there is an arrow $i \to j$. Thus, the arrows play a role of "homological links" between the "points" in the quiver Q.

Theorem 2.3.3. *Let $A = kQ$. For any (left) A-module X, there is an exact sequence of A-modules:*

$$0 \to \bigoplus_{\rho \in Q_1} A e_{t(\rho)} \otimes_k e_{s(\rho)} X \xrightarrow{\;f\;} \bigoplus_{i \in Q_0} A e_i \otimes_k e_i X \xrightarrow{\;g\;} X \to 0 \tag{2.1}$$

where $g(a \otimes x) := ax$ *and* $f(b \otimes x) := ab \otimes x - a \otimes bx$.

Proof. First we show that g is surjective. This can be seen from the fact that any element of x can be written as

$$x = 1 \cdot x = \left(\sum_{i \in Q_0} e_i \right) x = \sum_{i \in Q_0} e_i x = g\left(\sum_{i \in Q_0} e_i \otimes e_i x \right).$$

The fact that $\text{im}(f) \subseteq \ker(g)$ is just a direct computation. Indeed,

$$g \circ f(a \otimes e_{t(\rho)} \otimes e_{s(\rho)} x) = g(a\rho \otimes x - a \otimes \rho x) = a\rho x - a\rho x = 0$$

To show that $\ker(g) \subseteq \text{im}(f)$, we first note that any $\xi \in \bigoplus_{i=1}^{n} Ae_i \otimes e_i X$ can be uniquely written as

$$\xi = \sum_{i=1}^{n} \sum_{\substack{\text{paths } \vec{a} \\ s(\vec{a})=i}} a \otimes x_a$$

where all but finitely many of the $x_{\vec{a}} \in e_{s(\vec{a})}X$ are zero. Let the **degree of** ξ be the length of the longest path $\vec{a}$ such that $x_{\vec{a}} \neq 0$. If $\vec{a}$ is a nontrivial path, we can factor it as $\vec{a} = a'\rho$, with $s(a') = t(\rho)$ and a' consisting of only a single edge. We then have that

$$a' \otimes x_{\vec{a}} = a' e_{s(a)} \otimes e_{s(a)} x = a' e_{t(\rho)} \otimes e_{s(a)}.$$

Then by definition,

$$f(a' \otimes x_{\vec{a}}) = a'\rho \otimes x_{\vec{a}} - a' \otimes \rho x_{\vec{a}} = \vec{a} \otimes x_{\vec{a}} - a' \otimes \rho x_{\vec{a}}$$

Now claim that for any ξ, the set $\xi + \text{im}(f)$ contains elements of degree zero. For if $\deg(\xi) = d$, then

$$\xi - f\left(\sum_{i=1}^{n} \sum_{\substack{s(a)=i \\ \ell(a)=d}} a' \otimes x_{\vec{a}} \right)$$

has degree strictly less than d. The claim then follows by induction on $\ell(a) = d$.

Now let $\xi \in \ker(g)$, and take an element $\xi' \in \xi + \text{im}(f)$ of degree zero. In other words,

$$\xi' = \sum_{i=1}^{n} e_i \otimes x_{e_i}.$$

If $g(\xi) = 0$, then because $g \circ f = 0$, we get

$$g(\xi) = g(\xi') = \sum_{i=1}^{n} e_i x_{e_i} \in \bigoplus_{i=1}^{n} e_i X.$$

This is zero if and only if each $e_i x_{e_i} = 0$. But $x e_i = 0$ implies that $\xi' = 0$, or that $\xi \in \text{im}(f)$. This demonstrates that $\ker(g) \subseteq \text{im}(f)$.

Finally, let's show that f is injective. Suppose $f(\xi) = 0$, yet $\xi \neq 0$. Then we can write

$$\xi = \sum_{\rho \in Q_1} \sum_{\substack{\text{paths } \vec{a} \\ s(\vec{a})=t(\vec{\rho})}} a \otimes x_{\rho,a} = b \otimes x_{\rho,b} + \dots,$$

where b is a path of maximal length. We then get

$$f(\xi) = \sum_{\rho,a} a\rho \otimes x_{\rho,a} - \sum_{\rho,a} a \otimes \rho x_{\rho,a} = b\rho \otimes x_{\rho,b} + \text{lower terms} = 0$$

Here the lower terms are of the form $c\rho \otimes x_{\rho,c}$, where c is a path shorter than b. Hence, nothing can cancel with the $b\rho \otimes x_{\rho,b}$ term, which contradicts our choice of b as the path of maximal length. □

Definition 2.3.4. The resolution (2.1) in Theorem 2.3.3 is called the **standard resolution** of A.

Remark 2.3.5.

(a) There is a compact way to express this resolution if we identify $kQ \cong T_S(V)$, where $S = \bigoplus_{i \in Q_0} ke_i$, $V = \bigoplus_{a \in Q_1} ka$ as S-bimodules.

The exact sequence (2.1) can be written for any tensor algebra T and any (left) T-module X.

$$0 \longrightarrow T \otimes_S V \otimes_S X \xrightarrow{f} T \otimes_S X \xrightarrow{g} X \longrightarrow 0 \tag{2.2}$$

Note that the standard resolution is *projective* because each $Ae_i \otimes_k e_j X$ is a direct summand of $A \otimes_k X$, which is a free A-module based on V.

(b) If $X = A$, then the standard resolution becomes an A-bimodule resolution.

Exercise 2.3.6. Check that the sequence (2.2) gives the standard resolution (2.1).

Definition 2.3.7. An algebra A is a (left or right) **hereditary algebra** if every submodule of a projective (left or right) A-module is projective.

Proposition 2.3.8. *For a k-algebra A, the following conditions are equivalent:*

(a) Every A-module X has projective dimension $\operatorname{pdim}_A(X) \leq 1$, that is, $\operatorname{Ext}^i_A(X, Y) = 0$ for all $i \geq 2$.

(b) A is a (left and right) hereditary algebra.

Proof.
Consider the exact sequence

$$0 \to X \to P \to {}^{P}/_{X} \to 0,$$

where X is an A-submodule of the projective A-module P. Apply the functor $\mathrm{Hom}_A(-, Y)$ to get the long exact sequence

$$\begin{array}{l} 0 \longrightarrow \mathrm{Hom}_A\left({}^{P}/_{X}, Y\right) \longrightarrow \mathrm{Hom}_A(P, Y) \longrightarrow \mathrm{Hom}_A(X, Y) \longrightarrow \\ \longrightarrow \mathrm{Ext}^1_A\left({}^{P}/_{X}, Y\right) \longrightarrow \mathrm{Ext}^1_A(P, Y) \longrightarrow \mathrm{Ext}^1_A(X, Y) \longrightarrow \\ \longrightarrow \mathrm{Ext}^2\left({}^{P}/_{X}, Y\right) \longrightarrow \mathrm{Ext}^2_A(P, Y) \longrightarrow \mathrm{Ext}^2_A(X, Y) \longrightarrow \cdots \end{array}$$

Since P is projective, we have that $\mathrm{Ext}^1(P, Y) = 0$, so this long exact sequence shows that

$$\mathrm{Ext}^1_A(X, Y) \cong \mathrm{Ext}^2_A\left({}^{P}/_{X}, Y\right).$$

If every A-module X has projective dimension at most one, then $\mathrm{Ext}^2_A\left({}^{P}/_{X}, Y\right) = 0$ and therefore $\mathrm{Ext}^1_A(X, Y) = 0$. Hence X is projective.

Conversely, if A is a hereditary algebra, then as X is a submodule of the projective module P, we have $\mathrm{Ext}^1(X, Y) = 0$. Hence, $\mathrm{Ext}^2_A\left({}^{P}/_{X}, Y\right) = 0$ for any A-module of the form P/X. But any A-module whatsoever is the quotient of a free module, and therefore of the form P/X. So any A-module has projective dimension at most 1.

□

Remark 2.3.9. Suppose that we want to apply $\mathrm{Hom}_A(-, Y)$ to the exact sequence

$$\xi\colon 0 \longrightarrow W \longrightarrow X \longrightarrow U \longrightarrow 0$$

to get a the long exact sequence. The connecting homomorphism in the long exact sequence above is defined as follows. Given $f \in \mathrm{Hom}_A(W, Y)$, let $\partial(f)$ be the class in $\mathrm{Ext}^1_A(U, Y)$ such that

$$\begin{array}{ccccccccc} \xi\colon 0 & \longrightarrow & W & \longrightarrow & X & \longrightarrow & U & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & \ulcorner & \downarrow & & \downarrow & & \\ f_*\xi\colon 0 & \longrightarrow & Y & \longrightarrow & W \oplus X & \longrightarrow & U & \longrightarrow & 0 \end{array}$$

where the square indicated is a pushout. Note that if $W = Y$, and $f = \mathrm{id}_W$, then $[f_*\xi] = [\xi] \in \mathrm{Ext}^1_A(U, Y)$.

Remark 2.3.10. Another way to say Proposition 2.3.8(a) is to say that the global dimension of kQ is at most 1, for any quiver Q.

Definition 2.3.11. If X is a finite-dimensional A-module, we define the **dimension vector** of X to be

$$\underline{\dim}_k(X) := (\dim_k X_1, \dots, \dim_k X_n) \in \mathbb{N}^n \subset \mathbb{Z}^n,$$

where $X_i = e_i X = \mathrm{Hom}_A(Ae_i, X)$.

Definition 2.3.12. For a finite quiver Q, with $|Q_0| = n$, the **Euler form** is a bilinear form $\langle -,- \rangle_Q \colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}^n$ given by

$$\langle \alpha, \beta \rangle_Q := \sum_{i \in Q_0} \alpha_i \beta_i - \sum_{a \in Q_1} \alpha_{t(a)} \beta_{s(a)}$$

Sometimes we also need a symmetric version of the Euler form, which is written $(-,-) \colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ and given by

$$(\alpha, \beta) := \langle \alpha, \beta \rangle_Q + \langle \beta, \alpha \rangle_Q.$$

Definition 2.3.13. The associated quadratic form $q \colon \mathbb{Z}^n \to \mathbb{Z}^n$ is called the **Tits form**.

$$q(\alpha) := \langle \alpha, \alpha \rangle_Q.$$

Lemma 2.3.14. *For any two finite-dimensional* A*-modules* $X, Y \in A\text{-}\mathbf{Mod}^{\mathrm{fg}}$*, we have*

$$\langle \underline{\dim} X, \underline{\dim} Y \rangle_Q = \dim_k \left(\mathrm{Hom}_A(X, Y) \right) - \dim_k \left(\mathrm{Ext}^1_A(X, Y) \right).$$

Proof. Apply the functor $\mathrm{Hom}_A(-, Y)$ to the standard resolution (2.1). Then we get a long exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(X, Y) \longrightarrow \bigoplus_{i \in Q_0} \mathrm{Hom}_A(Ae_i \otimes e_i X, Y) \longrightarrow \bigoplus_{a \in Q_1} \mathrm{Hom}_A(Ae_{t(a)} \otimes e_{s(a)} X, Y) \longrightarrow \mathrm{Ext}^1_A(X, Y) \longrightarrow 0$$

Recall that $\mathrm{Hom}_A(Ae_i, X) \cong e_i X$, for any A-module X, so applying this to the above sequence gives

$$\mathrm{Hom}_A(Ae_i \otimes e_i X, Y) \cong \mathrm{Hom}_K(e_i X, e_i Y) \cong (e_i Y) \otimes (e_i X)^* = Y_i \otimes X_i^*$$

This then implies that

$$\dim \left(\mathrm{Hom}_A(Ae_i \otimes e_i X, Y) \right) = \dim(Y_i) \dim(X_i) = (\underline{\dim}\, X)_i (\underline{\dim}\, Y)_i$$

Now apply this to the exact sequence, to see that

$$\dim \operatorname{Hom}_A(X,Y) - \sum_{i\in Q_0} (\underline{\dim}\, X)_i (\underline{\dim}\, Y)_i + \sum_{a\in Q_1} (\underline{\dim}\, X)_{s(a)} (\underline{\dim}\, Y)_{t(a)} - \dim_k \operatorname{Ext}^1_A(X,Y) = 0$$

Therefore, $\dim_A \operatorname{Hom}_A(X,Y) < \infty$ and $\dim_k \operatorname{Ext}^1_A(X,Y) < \infty$. Moving terms around gives us the desired conclusion.

$$\langle \underline{\dim} X, \underline{\dim} Y\rangle_Q = \dim_k \Big(\operatorname{Hom}_A(X,Y)\Big) - \dim_k \Big(\operatorname{Ext}^1_A(X,Y)\Big). \qquad \square$$

Corollary 2.3.15. *For any finite-dimensional* A*-module,*

$$\dim_k(\operatorname{End}_A(X)) = q(\underline{\dim}(X)) + \dim_k \operatorname{Ext}^1_A(X,X)$$

Remark 2.3.16. Since all of the higher Ext-groups vanish for $A = kQ$ by Proposition 2.3.8(a), the Euler form on dimension vectors is equal to the Euler characteristic.

$$\langle \underline{\dim}\, X, \underline{\dim}\, Y\rangle_Q = \chi_A(X,Y)$$

Definition 2.3.17 (Notation)**.** For any k-algebra A, define

$$\operatorname{Irr}(A) := \Big\{\text{isomorphism classes of irreducible } A\text{-modules}\Big\}$$
$$\operatorname{Ind}(A) := \Big\{\text{isomorphism classes of indecomposable projective } A\text{-modules}\Big\}$$

Lemma 2.3.18. *Let* A *be a* k*-algebra, and let* X, Y *be two simple* A*-modules. Then there is a nonsplit extension of* X *by* Y

$$0 \longrightarrow Y \xrightarrow{\alpha} Z \xrightarrow{\beta} X \longrightarrow 0 \tag{2.3}$$

if and only if $\operatorname{im}(\alpha)$ *is the only proper submodule of* Z*.*

Proof. This proof is basically just Schur's lemma.

First, assume that (2.3) splits, and $s\colon X \to Z$ is a section of β, that is, $\beta s = \mathrm{id}_X$. This means in particular that s is injective, so $X \cong \operatorname{im}(s) \subset Z$ is a proper submodule different from $\operatorname{im}(\alpha)$; $\beta(\operatorname{im}(\alpha)) = 0$ and $\beta(\operatorname{im}(s)) = X \neq 0$.

Conversely, assume that Z has a submodule $K \subset Z$ with $K \neq \operatorname{im}(\alpha)$. Then $\beta|_K\colon K \to X$ is an isomorphism, and $s = \beta|_K^{-1}$ is a splitting of (2.3). $\square$

Corollary 2.3.19. *Let* A *be a commutative* k*-algebra, and let* X, Y *be two nonisomorphic simple* A*-modules. Then* $\operatorname{Ext}^1_A(X,Y) = \operatorname{Ext}^1_A(Y,X) = 0$.

Proof. If $X \neq 0$ is simple, choose some nonzero $x \in X$ and define

$$\begin{aligned} \phi_x \colon A &\longrightarrow X \\ a &\longmapsto ax \end{aligned}$$

Since X is simple, ϕ_x is surjective and $\ker(\phi_x) = \mathfrak{m}_x$. For two simples X and Y, $X \cong Y \iff \mathfrak{m}_x = \mathfrak{m}_y$.

Now given $X \not\cong Y$, choose $a \in \mathfrak{m}_y \setminus \mathfrak{m}_x$ and assume that the sequence

$$0 \longrightarrow Y \xrightarrow{\alpha} Z \longrightarrow X \longrightarrow 0$$

splits. Define $\widehat{a} \colon Z \to Z$ by $z \mapsto az$. Then $\widehat{a}(\operatorname{im}\alpha) = 0$ and $\operatorname{im}(\widehat{\alpha})$ is a proper submodule not equal to $\operatorname{im}\alpha$, so it splits. □

Corollary 2.3.20. *If Q is a quiver and $i, j \in Q_0$, then there is a nonsplit extension of $S(i)$ by $S(j)$.*

$$0 \longrightarrow S(j) \longrightarrow X \longrightarrow S(i) \longrightarrow 0$$

if and only if there is some $a \colon i \to j$ in Q_1.

Exercise 2.3.21. Prove Corollary 2.3.20.

Theorem 2.3.22. *Assume Q has no oriented cycles (so that $\dim_k(A) < \infty$). Then*

(a) the assignments

$$\begin{aligned} \operatorname{Irr}(A) \xleftarrow[\sim]{\alpha} \; &Q_0 \xrightarrow[\sim]{\beta} \operatorname{Ind}(A) \\ [S(i)] \longleftarrow\!\shortmid \; &i \longmapsto [Ae_i] \end{aligned}$$

are bijections.

(b) For any $i, j \in Q_0$,

$$\operatorname{Ext}^1_A(S(i), S(j)) = \operatorname{Span}_k Q(i,j),$$

where $Q(i,j) = \{a \in Q_1 \mid s(a) = i, t(a) = j\}$.

Proof.

(a) The map α is injective. $\operatorname{Hom}_A(S(i), S(j)) = 0$ if $i \neq j$ or $S(i) \cong S(j)$ (that is, $i = j$).

If X is any simple A-module, and if $\dim_k X = 1$, then there is some $i \in Q_0$ such that $e_i X = X \implies X \cong S(i)$. If $\dim_k X > 1$, then it cannot be a simple module.

The map β is also injective: we will show $Ae_i \cong Ae_j \iff i = j$. Then if

$$f\colon Ae_i \to Ae_j$$

$$g\colon Ae_j \to Ae_i,$$

are inverse, we have $f \in \mathrm{Hom}(Ae_i, Ae_j) \cong e_iAe_j$ and $g \in \mathrm{Hom}(Ae_j, Ae_i) \cong e_jAe_i$. Then $fg = e_j \in e_jAe_iAe_j \subseteq Ae_iA$. But Ae_iA has a basis consisting of all paths that pass through e_i, so it must be that $j = i$. Surjectivity is an exercise.

(b) Apply Euler's formula to $S(i)$, $S(j)$. Then by definition, we have that

$$\langle \underline{\dim}\, S(i), \underline{\dim}\, S(j)\rangle = \delta_{ij} - \sum_{a\in Q_1} \delta_{i,s(a)}\delta_{j,t(a)}$$

On the other hand, by Euler's formula,

$$\langle \underline{\dim}\, S(i), \underline{\dim}\, S(j)\rangle = \dim_k \mathrm{Hom}_A(S(i), S(j)) - \dim_k \mathrm{Ext}^1_A(S(i), S(j))$$

Note that

$$\dim_k(\mathrm{Hom}_A(S(i), S(j))) = \delta_{ij},$$

so comparing terms with the other calculation, we see that

$$\dim \mathrm{Ext}_A(S(i), S(j)) = \sum_a \delta_{i,s(a)}\delta_{j,t(a)} = \#\Big\{\text{arrows } i \to j\Big\}$$

□

Remark 2.3.23.

(1) Theorem 2.3.22 shows that we can reconstruct Q from A-**Mod**, provided we know *a priori* that $A = kQ$ for some Q and $\dim_k A < \infty$.

(2) Theorem 2.3.22 fails if $\dim_k A = \infty$.

Chapter 3

Gabriel's Theorem

Throughout this section, assume that k is a field of characteristic zero.

Definition 3.0.1. A (unital associative) k-algebra A has **finite representation type** if A has only finitely many isomorphism classes of finite-dimensional indecomposable representations.

Example 3.0.2.

(a) $A = k$ has finite representation type; there is only one indecomposable representation, namely k. This is the path algebra of the quiver $Q = \bullet$ with one vertex.

(b) Consider $A = kQ$, where $Q = \overset{1}{\bullet} \longrightarrow \overset{2}{\bullet}$. Then

$$A \cong \begin{bmatrix} k & k \\ 0 & k \end{bmatrix},$$

and A has finite representation type. The indecomposable representations are

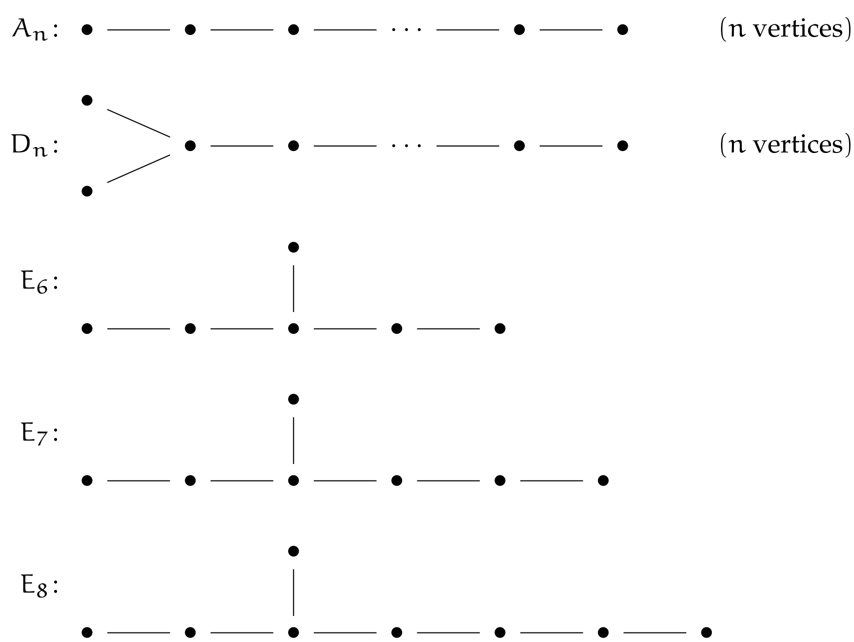
$$\overset{k}{\bullet} \longrightarrow \overset{0}{\bullet} = S(1) \cong \begin{pmatrix} k \\ 0 \end{pmatrix}$$

$$\overset{0}{\bullet} \longrightarrow \overset{k}{\bullet} = S(2) \cong \begin{pmatrix} 0 \\ k \end{pmatrix}$$

$$\overset{k}{\bullet} \xrightarrow{\text{id}} \overset{k}{\bullet} = I \cong \begin{pmatrix} k \\ k \end{pmatrix}$$

(c) Let $A = k[x] = kQ$, where Q is the quiver $Q = \bullet \circlearrowleft$. This doesn't have finite representation type; there are infinitely many irreducibles of the form $k[x]/\langle x - a \rangle$ with $a \in k$.

Theorem 3.0.3 (Gabriel 1972)**.** *Assume k is algebraically closed. Let Q be a finite quiver, possibly with loops and multiple edges. Let Γ_Q be the underlying graph of Q, that is, Q with orientation forgotten. Then $A = kQ$ has finite representation type if and only if Γ_Q is a Dynkin diagram of type ADE.*



Remark 3.0.4. The proof of Theorem 3.0.3 that we give is due to [JTR82]. The proof has three ingredients.

(1) Classical geometric representation theory (representation varieties)

(2) Noncommutative/homological algebra (Fitting Lemma, Ringel Lemma)

(3) Classification of graphs (due to Tits)

We will return to the proof of Gabriel's theorem in the next chapter after taking a look at these ingredients individually.

3.1 Representation Varieties

Example 3.1.1. Let A be a finitely generated associative k-algebra; fix $n \geq 1$. We want to classify all n-dimensional representations of A. Define (naively):

$$\begin{aligned}\operatorname{Rep}_k(A,n) &= \left\{\text{A-module structures on } k^n\right\} \\ &= \left\{\text{algebra homomorphisms } \rho\colon A \to \mathbb{M}_n(k)\right\}\end{aligned}$$

If $A = k\langle x_1,\dots,x_m\rangle/I$, for a 2-sided ideal $I \lhd k\langle x_1,\dots,x_n\rangle$, say with generators $I = \langle r_1,\dots,r_d\rangle$ with $r_i \in k\langle x_1,\dots,x_m\rangle$, then

$$\operatorname{Rep}_k(A,n) = \left\{ (X_1,\dots,X_m) \in \mathbb{M}_n(k)^m \cong \mathbb{A}^{n^2 m} \;\middle|\; \begin{array}{c} r_1(X_1,\dots,X_m) = 0 \\ \vdots \\ r_d(X_1,\dots,X_m) = 0 \end{array} \right\}$$

We need to define a *relative version* of $\operatorname{Rep}_n(A)$.

Let A be a finitely generated associative k-algebra, $S \subset A$ a finite-dimensional semisimple subalgebra. Fix a finite-dimensional S-module V ($\rho\colon S \to \operatorname{End}(V)$), and define

$$\begin{aligned}\operatorname{Rep}_S(A,V) &= \left\{\text{all A-module structures on V extending } \rho\right\} \\ &\cong \left\{ \text{k-algebra maps } f\colon A \to \operatorname{End}(V) \;\middle|\; \begin{array}{ccc} A & \xrightarrow{f} & \operatorname{End}(V) \\ \nwarrow & & \nearrow_{\rho} \\ & S & \end{array} \right\} \\ &\cong \left\{\text{S-algebra homomorphisms } \phi\colon A \to \operatorname{End}(V)\right\}\end{aligned}$$

Definition 3.1.2. Formally, we define $\operatorname{Rep}_S(A,V)$ by its **functor of points**

$$\begin{aligned}\operatorname{Rep}_S(A,V)\colon \mathbf{CommAlg}_k &\longrightarrow \mathbf{Sets} \\ B &\longmapsto \operatorname{Hom}_{\mathbf{Alg}_S}(A, \operatorname{End}_B(B \otimes V))\end{aligned}$$

Remark 3.1.3.

(1) The set $\operatorname{Hom}_{\mathbf{Alg}_S}(A, \operatorname{End}_B(B \otimes V))$ can be thought of geometrically as the set of families of representations parameterized by points of $\operatorname{Spec}(B)$.

(2) Note that $\operatorname{End}_B(B \otimes V)$ has a natural S-algebra structure

$$B \otimes -\colon \mathbf{Vect}_k \to B\text{-}\mathbf{Mod}$$

$$S \to \operatorname{End}(V) \to \operatorname{End}_B(B \otimes V)$$

(3) $$\begin{aligned}\mathrm{End}_B(B\otimes V) &\cong \mathrm{Hom}_B(B\otimes V, B\otimes V)\\ &\cong \mathrm{Hom}_k(V, B\otimes V)\\ &\cong B\otimes V\otimes V^*\\ &\cong B\otimes \mathrm{End}_k(V)\end{aligned}$$

(4) If $V = k^n$, $\mathrm{End}(V) = \mathbb{M}_n(k)$ and $B\otimes \mathrm{End}(V)\cong \mathbb{M}_n(B)$.

Proposition 3.1.4. $\mathrm{Rep}_S(A, V)$ *is a (co)representable functor, that is, there is a commutative* k*-algebra* $(S\setminus A)_V$ *such that*

$$\mathrm{Hom}_{\mathbf{CommAlg}_k}((S\setminus A)_V, B)\cong \mathrm{Hom}_{\mathbf{Alg}_S}(A, B\otimes_k \mathrm{End}(V)) \tag{3.1}$$

Remark 3.1.5. We should think of $(S\setminus A)_V$ as the coordinate ring $k[\mathrm{Rep}_S(A,V)]$ of the affine scheme $\mathrm{Rep}_S(A, V)$.

Corollary 3.1.6. $\mathrm{Rep}_S(A, V)$ *exists as an affine* k*-scheme* $\mathrm{Spec}(S\setminus A)_V$ *and we define the (relative) representation variety*

$$\mathrm{Rep}_S(A,V)_{\mathrm{red}} := \mathrm{Spec}\left({}^{(S\setminus A)_V}/_{\sqrt{0}}\right).$$

Proof of Proposition 3.1.4; [Ber74]. Consider (3.1). We need to show that the functor $-\otimes_k \mathrm{End}(V)\colon \mathbf{CommAlg}_S\to\mathbf{Alg}_S$ has a left adjoint $(S\setminus -)_V$. This will be a functor

$$(S\setminus -)_V\colon \mathbf{Alg}_S\longrightarrow \mathbf{CommAlg}_k\,.$$

Let's decompose $-\otimes\mathrm{End}(V)$ into a composition of functors.

$$\mathbf{CommAlg}_k \xhookrightarrow{\text{forgetful}} \mathbf{Alg}_k \xrightarrow{-\otimes_k \mathrm{End}(V)} \mathbf{Alg}_{\mathrm{End}(V)} \xrightarrow{\text{restrict}} \mathbf{Alg}_S$$

The forgetful functor $\mathbf{CommAlg}_k\to\mathbf{Alg}_k$ has a left adjoint, namely the abelianization $(-)_{\mathrm{ab}}$. The restriction functor also has a left adjoint given by a free product over S, denoted by $(-)*_S\mathrm{End}(V)$. Finally, $-\otimes\mathrm{End}(V)$ has a left adjoint, denoted $(-)^{\mathrm{End}(V)}$.

$$\mathbf{CommAlg}_k \underset{\text{forgetful}}{\overset{(-)_{\mathrm{ab}}}{\rightleftarrows}}_{\perp} \mathbf{Alg}_k \underset{-\otimes_k\mathrm{End}(V)}{\overset{(-)^{\mathrm{End}(V)}}{\rightleftarrows}}_{\perp} \mathbf{Alg}_{\mathrm{End}(V)} \underset{\text{restrict}}{\overset{(-)*_S\mathrm{End}(V)}{\rightleftarrows}}_{\perp} \mathbf{Alg}_S$$

□

What are these left-adjoint functors? The first is called **abelianization**, and given by

$$\begin{aligned}(-)_{\mathrm{ab}}\colon \mathbf{Alg}_k &\longrightarrow \mathbf{CommAlg}_k\\ A &\longmapsto A_{\mathrm{ab}} = {}^{A}/_{\langle[A,A]\rangle}.\end{aligned}$$

$$(-) *_S \operatorname{End}(V) \colon \mathbf{Alg}_S \longrightarrow \mathbf{Alg}_{\operatorname{End}(V)}$$

$$\left(S \xrightarrow{f} A\right) \longmapsto A *_S \operatorname{End}(V) = \operatorname{colim}\left(\begin{array}{ccc} & S & \xrightarrow{f} A \\ \rho \downarrow & & \\ \operatorname{End}(V) & & \end{array}\right)$$

$$(-)^{\operatorname{End}(V)} \colon \mathbf{Alg}_{\operatorname{End}(V)} \longrightarrow \mathbf{Alg}_k$$

$$\left(\operatorname{End}(V) \xrightarrow{f} A\right) \longmapsto A^{\operatorname{End}(V)} = \left\{ a \in A \;\middle|\; [a, f(x)] = 0 \text{ for all } x \in \operatorname{End}(V) \right\}$$

Lemma 3.1.7. *The functor* $- \otimes \operatorname{End}(V) \colon \mathbf{Alg}_{\operatorname{End}(V)} \to \mathbf{Alg}_k$ *is an equivalence of categories, with inverse*

$$\left(\operatorname{End}(V) \xrightarrow{f} A\right) \mapsto A^{\operatorname{End}(V)} = \left\{ a \in A \;\middle|\; [a, f(x)] = 0 \text{ for all } x \in \operatorname{End}(V) \right\}$$

Exercise 3.1.8. Prove Lemma 3.1.7.

Corollary 3.1.9. $(-)^{\operatorname{End} V}$ *is left adjoint to* $- \otimes \operatorname{End}(V)$

Hence,

$$(S \setminus A)_V = \left((A *_S \operatorname{End}(V))^{\operatorname{End}(V)}\right)_{ab}.$$

3.2 Algebraic Group Actions on a Variety

Let f be finitely generated k-algebra, and $S \subseteq A$ a finite-dimensional semisimple subalgebra. Let $(V, \rho_0 \colon S \to \operatorname{End}_k(V))$ be a finite-dimensional S-module.

We have defined an affine scheme $\operatorname{Rep}_S(A, V)$ parameterizing A-module structures on V extending the ρ_0. For a representation ϕ of A extending (V, ρ_0), denote the corresponding point in $\operatorname{Rep}_S(A, V)$ by x_ϕ.

Remark 3.2.1. $\operatorname{Rep}_S(A, V)$ can also be seen as the fiber of the restriction map $r \colon \operatorname{Rep}_k(A, V) \to \operatorname{Rep}_k(S, A)$ over the representation (V, ρ_0).

$$\begin{array}{ccc} \operatorname{Rep}_S(A, V) & \longrightarrow & \{*\} \\ \downarrow & & \downarrow (V, \rho_0) \\ \operatorname{Rep}_k(A, V) & \xrightarrow{r} & \operatorname{Rep}_k(S, A) \end{array}$$

Definition 3.2.2. Denote the group of S-module automorphisms of V by $GL_S(V) := \operatorname{Aut}_S(V)$.

We may define an action of $GL_S(V)$ on the scheme $\mathrm{Rep}_S(A, V)$ by

$$\begin{aligned} GL_S(V) \times \mathrm{Rep}_S(A, V) &\longrightarrow \mathrm{Rep}_{S(A,V)} \\ (g, \psi) &\longmapsto \big((g, \psi)\colon a \mapsto g\psi(a)g^{-1}\big) \end{aligned}$$

Note that for all $s \in S$, we have

$$(g, \psi)(s) = g\psi(s)g^{-1} = g\rho_0(s)g^{-1} = \rho_0(s)gg^{-1} = \rho_0(s) = \psi(s)$$

so this is well-defined.

Note that $k^\times \subset GL_S(V)$ acts trivially, which gives an induced action

$$G_S(V) = {}^{GL_S(V)}/_{k^\times}.$$

Proposition 3.2.3.

(a) *There is a one-to-one correspondence between the following:*

- *isomorphism classes of* A*-modules isomorphic to* V *as* S*-modules,*
- $GL_S(V)$*-orbits on* $\mathrm{Rep}_S(A, V)$*.*

(b) *For any* $x_\rho \in \mathrm{Rep}_S(A, V)$*:* $\mathrm{Stab}_{GL_S(V)}(x_\rho) = \mathrm{Aut}_A(V_\rho)$*.*

Exercise 3.2.4. Show that for any $x_\rho \in \mathrm{Rep}_S(A, V)$, the Zariski tangent space is isomorphic to the space of S-derivations of A taking values in $\mathrm{End}(V_\rho)$.

$$T_{x_\rho} \mathrm{Rep}_S(A, V) \cong \mathrm{Der}_S(A, \mathrm{End}(V_\rho))$$

Hint: Use adjunction.

$$T_{x_\rho} \mathbf{Rep}_S(A, V) := \mathrm{Der}_k((S \setminus A)_V, k_{\overline{\rho}}),$$

where $\overline{\rho}\colon (S \setminus A)_V \to k \iff \rho\colon A \to \mathrm{End}(V)$.

$$\mathrm{Der}_S(A, \mathrm{End}(V_\rho)) = \left\{ \Theta \in \mathrm{Hom}_k(A, \mathrm{End}_k(V)) \;\middle|\; \begin{array}{c} \Theta(a, b) = \rho(a)\Theta(b) + \Theta(a)\rho(b) \\ \Theta(s) = 0 \ \forall\, s \in S \end{array} \right\}$$

Example 3.2.5. Let $Q = (Q_0, Q_1, s, t)$ be a finite quiver, with $Q_0 = \{1, \ldots, n\}$. Let $A = kQ$, and let

$$S = \bigoplus_{i=1}^{n} ke_i \subseteq A.$$

Note that giving an S-module structure to a vector space V corresponds to a splitting $V = \bigoplus_{i=1}^n V_i$ with $V_i = e_i V$. Then up to isomorphism, S-modules are determined by dimension vectors $\alpha \in \mathbb{N}^n$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\alpha_i = \dim_k V_i$.

Then we write $\operatorname{Rep}_S(A, V) = \operatorname{Rep}(Q, \alpha) = \operatorname{Rep}(\alpha)$, the last if Q is understood. In particular,

$$\operatorname{Rep}_S(A, V) = \operatorname{Rep}(\alpha) = \prod_{a \in Q_1} \operatorname{Hom}_k(k^{\alpha_{s(a)}}, k^{\alpha_{t(a)}}) \cong \mathbb{A}_k^r$$

is an affine space, where $r = \sum_{a \in Q_1} \alpha_{s(a)} \alpha_{t(a)}$.

Remark 3.2.6. Ludwig Wittgenstein said there are no philosophical, only linguistic puzzles. While it would be nice to think something like the Poincaré conjecture is only a notational issue, that's insulting to a lot of geometers. But it's hard to prove that it *isn't* only an issue of notation – maybe we haven't found the right notation yet.

Definition 3.2.7 (Notation)**.** Let $x = (x_a)_{a \in Q_1} \in \operatorname{Rep}(\alpha)$ correspond to the representation $R(x)$ of Q. Write

$$\begin{aligned} \operatorname{GL}_S(V) &= \operatorname{GL}(\alpha) = \prod_{i=1}^{n} \operatorname{GL}_k(\alpha_i) \ni g = (g_1, \ldots, g_n) = (g_i)_{i \in Q_0} \\ G_S(V) &= \operatorname{GL}(\alpha)/k^\times \end{aligned}$$

The action is then given by

$$\begin{aligned} \operatorname{GL}(\alpha) \times \operatorname{Rep}(\alpha) &\longrightarrow \operatorname{Rep}(\alpha) \\ (g, x) &\longmapsto ((g \cdot x)_a)_{a \in Q_1} = \left(g_{t(a)} \cdot x_a \cdot g_{s(a)}^{-1} \right)_{a \in Q_1} \end{aligned}$$

$$\begin{array}{ccc} k^{\alpha_{s(a)}} & \xrightarrow{(g \cdot x)_a} & k^{\alpha_{t(a)}} \\ {\scriptstyle g_{s(a)}} \downarrow & & \uparrow {\scriptstyle g_{t(a)}^{-1}} \\ k^{\alpha_{s(a)}} & \xrightarrow{x_a} & k^{\alpha_{t(a)}} \end{array}$$

Proposition 3.2.3 says that there is a correspondence between isomorphism classes of representations X with $\underline{\dim}(X) = \alpha$ and $\operatorname{GL}(\alpha)$ orbits in $\operatorname{Rep}(\alpha)$.

$$X \longmapsto \mathcal{O}_X = \{x \in \operatorname{Rep}(\alpha) \mid R(x) \cong X\}$$

Remark 3.2.8. Our goal is to clarify the relation between algebraic (or homological) properties of representations and geometric properties of orbits.

3.3 (Practical) Algebraic Geometry

(1) $V \subseteq \mathbb{A}_k^r$ is closed if and only if there is $I \lhd k[x_1, \ldots, x_r]$ with $V = Z(I)$.

(2) $U \subseteq \mathbb{A}^r_k$ is **locally closed** if U is open in its closure $\overline{U}$. This holds if and only if U is the intersection of an open and closed subset in $\mathbb{A}^r_k$.

(3) A locally closed non-empty U is **irreducible** if any nonempty $V \subsetneq U$, which is open in U, is dense in U. $\emptyset \neq V \subseteq U$ open $\implies \overline{V} = U$.

(4) The **(Krull) dimension** of a locally closed $U \neq \emptyset$ is the supremum of lengths of all chains of irreducible closed subsets of U.

$$\dim U = \sup \left\{ n \in \mathbb{N} \,\middle|\, Z_0 \subset Z_1 \subset \ldots \subset Z_n \subset U;\ Z_i \text{ irreducible, closed} \right\}$$

Example 3.3.1. The dimension of $\mathbb{A}^r$ is r. Moreover, $\dim U = \dim \overline{U}$, and $\dim(U \cap V) = \max(\dim U, \dim V)$.

(5) A subset U is **constructible** if U is a finite union of locally closed subsets.

Example 3.3.2. The following set is constructible, but not locally closed.

$$(\mathbb{A}^2 \setminus \{x\text{-axis}\}) \cup \{(0,0)\} = \{x = yz \mid \text{ for some } z\}$$

Lemma 3.3.3 (Chevalley)**.** *If $\pi \colon X \to Y$ is a dominant morphism of irreducible varieties, then every irreducible component of the fiber $\pi^{-1}(y)$ has dimension*

$$\dim(\pi^{-1}(y)) \geq \dim(X) - \dim(Y).$$

Moreover, there is a nonempty open $U \subseteq Y$ such that for all $y \in U$,

$$\dim \pi^{-1}(y) = \dim(X) - \dim(Y)$$

The proof of this lemma is pure commutative algebra.

Theorem 3.3.4 (Chevalley)**.** *If $\pi \colon X \to Y$ is a morphism of varieties, then $\pi(X)$ is a constructible subset of Y. More generally, π maps constructible subsets to constructible subsets.*

Proof Sketch. Work by induction on the dimension of X. We may assume that X is irreducible. We may also assume that π is dominant, that is, $Y = \overline{\pi(X)}$. This then allows us to claim Y is irreducible. Then write using Lemma 3.3.3

$$\pi(X) = U \cup \pi(X \setminus \pi^{-1}(U));$$

$X \setminus \pi^{-1}(U)$ is constructible by induction because it has smaller dimension. Therefore, $\pi(X)$ is constructible. □

Corollary 3.3.5 (Chevalley)**.** *For any $\pi \colon X \to Y$, the local dimension $\dim_x \colon X \to \mathbb{Z}$, $x \mapsto \dim_x\left(\pi^{-1}(\pi(x))\right)$ is **upper-semicontinuous**, that is, for all $n \in \mathbb{Z}$, the set*

$$\{y \in X \mid \dim(y) \geq n\}$$

is closed.

Lemma 3.3.6 (Orbit Formula)**.** *If a connected algebraic group* G *acts on a variety* X*, then*

(a) all G*-orbits are irreducible and locally closed;*

(b) for each $x \in X$*,* $\operatorname{Stab}_G(x)$ *is a closed subgroup of* G*;*

(c) $\dim X = \dim G - \dim \operatorname{Stab}_G(x)$*;*

(d) $\overline{\mathcal{O}}_x \setminus \mathcal{O}_x$ *is a union of orbits of dimension strictly smaller that* $\dim \mathcal{O}_x$*.*

Proof. Fix $x \in X$, and consider

$$\begin{aligned} \mu\colon G &\longrightarrow X \\ g &\longmapsto g \cdot x. \end{aligned}$$

(a) Note that $\operatorname{im}(\mu) = \mathcal{O}_x$. Then by Chevalley's Theorem (Theorem 3.3.4) applied to μ, $\overline{\mathcal{O}}_x$ is irreducible and $\mathcal{O}_x$ is constructible. Then by Lemma 3.3.3, there is a nonempty open $U \subseteq \mathcal{O}_x$; this also shows U is open in $\overline{\mathcal{O}}_x$. Now consider

$$G \cdot U = \bigcup_{g \in G} g \cdot U,$$

where $g \cdot U = \{x \in X \mid x = gx \text{ for some } u \in U\}$. This set is G-stable, and $x \in GU \subseteq \mathcal{O}_x$. This implies that $GU = \mathcal{O}_x$.

Finally, gU is open in $\mathcal{O}_x$, so GU is open in $\mathcal{O}_x$, which implies GU is open in $\mathcal{O}_x$. But $GU = \mathcal{O}_x$, which implies that $\mathcal{O}_x$ is open in $\overline{\mathcal{O}}_x$ which by definition says that $\mathcal{O}_x$ is locally closed.

(b) $\operatorname{Stab}_G(x) = \mu^{-1}(x)$ is closed.

(c) By Lemma 3.3.3, the dimension of the fiber $\mu^{-1}(x)$ is $\dim G - \dim X$. But $\mu^{-1}(x)$ is exactly the stabilizer $\operatorname{Stab}_G(x)$. □

Remark 3.3.7. If G is a connected affine algebraic group, then G is an irreducible affine variety.

3.4 Back to quivers

Let $Q = (Q_0, Q_1, s, t)$ be a quiver. Let $A = kQ$ be its path algebra. Write $Q_0 = \{1, 2, \ldots, n\}$. Fix $\alpha \in \mathbb{N}^n$ with $\alpha = (\alpha_1, \ldots, \alpha_n)$. Recall

$$\operatorname{Rep}(\alpha) = \operatorname{Rep}(Q, \alpha) = \prod_{a \in Q_1} \operatorname{Hom}_k(k^{\alpha_{s(a)}}, k^{\alpha_{t(a)}}) \ni (x_a)_{a \in Q_1} = x.$$

$X = R(x)$ is the representation corresponding to $x \in \operatorname{Rep}(\alpha)$. We write $\mathcal{O}_X$ and $\mathcal{O}_x$ interchangeably. We have

$$\operatorname{GL}(\alpha) = \prod_{i=1}^{n} \operatorname{GL}_k(\alpha_i) \circlearrowright \operatorname{Rep}(\alpha)$$

There is a one-to-one correspondence between isomorphism classes of representations X of $\underline{\dim} X = \alpha$ and $\operatorname{GL}(\alpha)$-orbits in $\operatorname{Rep}(\alpha)$, given by

$$X \mapsto \mathcal{O}_X = \{x \in \operatorname{Rep}(\alpha) \mid R(x) \cong X\}.$$

Lemma 3.4.1 (Dimension Formula)**.** *For any representation X of Q with $\underline{\dim} X = \alpha$, we have both*

$$\dim \operatorname{Rep}(\alpha) - \dim \mathcal{O}_x = \dim_k \operatorname{End}_A(X) - q(\alpha) \tag{3.2}$$

$$\dim \operatorname{Rep}(\alpha) - \dim \mathcal{O}_x = \dim_k \operatorname{Ext}^1_A(X, X) \tag{3.3}$$

Proof. Recall from Lemma 2.3.14 that

$$\langle \underline{\dim} X, \underline{\dim} Y \rangle_Q = \dim_k \operatorname{Hom}_A(X, Y) - \dim_k \operatorname{Ext}^1_A(X, Y).$$

Then put $X = Y$ and we recover Eq. (3.3).

To prove Eq. (3.2), fix $x \in \operatorname{Rep}(\alpha)$ and consider $\mathcal{O}_X = \mathcal{O}_x$. Then, by the Orbit Formula (Lemma 3.3.6(c)),

$$\dim \mathcal{O}_X = \dim \operatorname{GL}(\alpha) - \dim \operatorname{Stab}_G(x) \tag{3.4}$$

Notice that

$$\operatorname{Stab}_G(x) = \operatorname{Aut}_A(X) \subseteq \operatorname{End}_A(X),$$

and moreover the inclusion $\operatorname{Aut}_A(X) \subseteq \operatorname{End}_A(X)$ is open as the inclusion of vector spaces. Therefore,

$$\dim \operatorname{Stab}_G(x) = \dim \operatorname{Aut}_A(X) = \dim \operatorname{End}_A(X) = \dim_k \operatorname{End}_A(X).$$

Moreover,

$$\operatorname{GL}(\alpha) = \prod_{i=1}^{n} \operatorname{GL}_k(\alpha_i) \subseteq \mathbb{A}_1^{\alpha_1^2} \times \cdots \times \mathbb{A}_k^{\alpha_n^2} = \mathbb{A}_k^s.$$

This is also an open inclusion, so

$$\dim \operatorname{GL}(\alpha) = \dim \mathbb{A}_k^s = \sum_{i=1}^{n} \alpha_i^2.$$

Substitute into (3.4) to see that

$$\dim \mathcal{O}_X = \dim \mathrm{GL}(\alpha) - \dim \mathrm{Stab}_G(x) = \sum_{i=1}^{n} \alpha_i^2 - \dim_k \mathrm{End}_A(X).$$

Finally, $\mathrm{Rep}(\alpha) = \mathbb{A}_k^r$, where $r = \sum_{a \in Q_1} \alpha_{s(a)}\alpha_{t(a)}$. Hence,

$$\dim \mathrm{Rep}(\alpha) = \sum_{a \in Q_1} \alpha_{s(a)}\alpha_{t(a)}$$

So put it all together now,

$$\begin{aligned}\dim \mathrm{Rep}(\alpha) - \dim \mathcal{O}_x &= \sum_{a \in Q_1} \alpha_{s(a)}\alpha_{t(a)} - \sum_{i=1}^{n} \alpha_i^2 + \dim_k \mathrm{End}_A(X) \\ &= \dim_k \mathrm{End}_A(X) - q(\alpha).\end{aligned}$$

□

Corollary 3.4.2. *If $\alpha \neq 0$, and $q(\alpha) \leq 0$, then* $\mathbf{Rep}(\alpha)$ *contains infinitely many orbits.*

Proof. Trivially, $\dim_k(\mathrm{End}_A(X)) > 0$, because $\mathrm{id}_X \in \mathrm{End}_A(X)$. If $q(\alpha) \geq 0$, then by the Dimension Formula (Lemma 3.4.1), $\dim \mathbf{Rep}(\alpha) > \dim \mathcal{O}_X = \dim \overline{\mathcal{O}}_X$. This implies that there are infinitely many orbits. □

Corollary 3.4.3. *$\mathcal{O}_X$ is open if and only if $\mathrm{Ext}_A^1(X, X) = 0$.*

Proof. By Lemma 3.4.1,

$$\begin{aligned}\mathrm{Ext}_A^1(X,X) = 0 &\iff \dim \mathcal{O}_X = \dim \mathrm{Rep}(\alpha) \\ &\iff \dim \overline{\mathcal{O}}_X = \dim \mathrm{Rep}(\alpha) \\ &\iff \overline{\mathcal{O}}_X = \mathrm{Rep}(\alpha).\end{aligned}$$

$\mathrm{Rep}(\alpha)$ is irreducible and a *proper* closed subset must have strictly smaller dimension.

($\Longleftarrow$). $\overline{\mathcal{O}}_X = \mathrm{Rep}(\alpha) \implies \mathrm{Ext}_A^1(X,X) = 0$

($\Longrightarrow$). $\mathrm{Ext}_A^1(X,X) = 0 \implies \overline{\mathcal{O}}_X = \mathrm{Rep}(\alpha) \implies \mathcal{O}_X$ open in $\mathrm{Rep}(\alpha)$, because $\mathcal{O}_X$ open in $\overline{\mathcal{O}}_X$. □

Corollary 3.4.4. *Up to isomorphism, there is at most one representation of* $\underline{\dim}(X) = \alpha$ *such that* $\mathrm{Ext}_A^1(X,X) = 0$.

Proof. Suppose that X, Y are two such representations, X and Y. By Corollary 3.4.3, $\mathcal{O}_X$ and $\mathcal{O}_Y$ are open. If $\mathcal{O}_X \cap \mathcal{O}_Y = \emptyset$, then $\mathcal{O}_X \subseteq \mathbf{Rep}(\alpha) \setminus \mathcal{O}_Y$. This implies that $\overline{\mathcal{O}}_X \subseteq \mathrm{Rep}(\alpha) \setminus \mathcal{O}_Y$, which contradicts irreducibility of $\mathcal{O}_X$.

Therefore, $\mathcal{O}_X \cap \mathcal{O}_Y \neq \emptyset$. This implies that $\mathcal{O}_X = \mathcal{O}_Y$, so $X \cong Y$. □

Definition 3.4.5. Consider representations X and Y. We say X **degenerates to** Y if $\mathcal{O}_Y \subseteq \overline{\mathcal{O}}_X$.

Lemma 3.4.6. *Assume that* $\xi\colon 0 \longrightarrow U \longrightarrow X \longrightarrow V \longrightarrow 0$ *is a nonsplit exact sequence. Then* $\mathcal{O}_{U\oplus V} \subseteq \overline{\mathcal{O}}_X \setminus \mathcal{O}_X$. *In particular,* X *degenerates to* $U \oplus V$.

Recall from Definition 2.3.1 that $[\xi] \in \operatorname{Ext}^1_A(V, U)$ is nonzero if and only if ξ is nonsplit.

Proof. First, we show that $\mathcal{O}_{U\oplus V} \subset \overline{\mathcal{O}}_X$. Let's identify U_i with a subspace X_i for each $i \in Q_0$. Via α, chose a basis in each U_i and extend it to a basis in X_i. Then $X \cong R(x)$, where $x = (x_a)_{a\in Q_1} \in \operatorname{Rep}(\alpha)$ such that

$$x_a = \begin{pmatrix} u_a & w_a \\ 0 & v_a \end{pmatrix}$$

where $U \cong R(u)$, $V \cong R(v)$, and $u = (u_a)_{a\in Q_1}$ and $v = (v_a)_{a\in Q_1}$.

Let's take $\lambda \in k^\times$ and define

$$g_\lambda = ((g_\lambda)_1, \ldots, (g_\lambda)_n) \in GL(\alpha) = \prod_{i=1}^{n} GL(\alpha_i)$$

so that for each $i = 1, 2, \ldots, n$,

$$(g_\lambda)_i = \begin{pmatrix} \lambda \mathrm{id}_{U_i} & 0 \\ 0 & \mathrm{id}_{V_i} \end{pmatrix}$$

Then

$$(g_\lambda \cdot x)_a = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_a & w_a \\ 0 & v_a \end{pmatrix} \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} u_a & \lambda w_a \\ 0 & v_a \end{pmatrix} \xrightarrow[\lambda\to 0]{} \begin{pmatrix} u_a & 0 \\ 0 & v_a \end{pmatrix}$$

Hence, if $x' = (x'_a)_{a\in Q_1}$ with

$$x'_a = \begin{pmatrix} u_a & 0 \\ 0 & v_a \end{pmatrix},$$

then $x' \in \overline{\mathcal{O}}_X$. But $\mathbb{R}(x) \cong U \oplus V$, so we conclude that $\mathcal{O}_{U\oplus V} \subseteq \overline{\mathcal{O}}_X$.

It remains to show that $\mathcal{O}_{U\oplus V} \cap \mathcal{O}_X \neq 0$. To show this, it suffices to show that $X \not\cong U \oplus V$ (this is not immediately obvious! see Remark 3.4.7 below). We will use the fact that U, V are finite-dimensional modules.

Consider $0 \to U \to X \to V \to 0$ and dualize it with U, that is, apply $\operatorname{Hom}_A(-, U)$. We get a long exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(V, U) \longrightarrow \operatorname{Hom}_A(X, Y) \longrightarrow \operatorname{Hom}_A(U, U) \xrightarrow{\partial} \operatorname{Ext}^1_A(V, U) \longrightarrow \cdots$$

By assumption, ξ is a nonsplit short exact sequence, so $[\xi] \neq 0$ in $\mathrm{Ext}^1_A(V, U)$. This tells us that $\partial([\xi]) \neq 0$ in $\mathrm{Ext}^1_A(V, U)$ as well; see Remark 2.3.9. In particular, $\operatorname{im} \partial \neq 0$.

So we may truncate the long exact sequence by replacing $\mathrm{Ext}^1_A(V, U)$ by $\operatorname{im} \partial \neq 0$, to get an exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(V, U) \longrightarrow \mathrm{Hom}_A(X, U) \longrightarrow \mathrm{Hom}_A(U, U) \xrightarrow{\partial} \underset{\substack{\not\shortparallel \\ 0}}{\operatorname{im} \partial} \longrightarrow 0$$

Therefore,

$$\dim_k \mathrm{Hom}_A(V, U) - \dim_k \mathrm{Hom}_A(X, U) + \dim_k \mathrm{Hom}_A(U, U) = \dim_k \operatorname{im}(\partial) > 0$$

Shuffling around some terms,

$$\begin{aligned} \dim_k \mathrm{Hom}_A(V, U) + \dim_k \mathrm{Hom}_A(U, U) &> \dim_k \mathrm{Hom}_A(X, U) \\ \implies \qquad \dim_k \mathrm{Hom}_A(U \oplus V, U) &> \dim_k \mathrm{Hom}_A(X, U) \end{aligned}$$

In particular, this shows that $X \not\cong U \oplus V$. □

Remark 3.4.7. Note that a short exact sequence $0 \to U \xrightarrow{f} X \xrightarrow{g} V \to 0$ being nonsplit does *not* automatically imply that $X \not\cong U \oplus V$. It just means that via the maps f and g, $X \not\cong \operatorname{im} f \oplus \operatorname{coim} g$. In fact, there exist nonsplit exact sequences of the form

$$0 \longrightarrow U \xrightarrow{\ f\ } U \oplus V \xrightarrow{\ g\ } V \longrightarrow 0$$

where f and g are not the canonical maps.

Exercise 3.4.8. Give an example of such a nonsplit exact sequence when U and V are representations of an algebra A over a field k.

Joke 3.4.9 (see Remark 3.4.7)**.** There are two people on a hot air balloon and they get blown off course. They see someone on the ground below them, so they go down to ask this person where they are. The person on the ground says "Why, you're on the balloon, of course!" Then one of the people on the balloon turns to the other and says "He must be a mathematician, because his answer is correct but absolutely useless."

Corollary 3.4.10 (Corollary to Lemma 3.4.6)**.** *If $\mathcal{O}_X$ is a maximal orbit (that is, has a maximal dimension among all orbits) and $X \cong U \oplus V$, then $\mathrm{Ext}^1_A(V, U) = 0$.*

Proof. Assume the contrary. If $\mathrm{Ext}^1_A(V, U) \neq 0$, then there is a nonsplit short exact sequence

$$\xi \colon 0 \longrightarrow U \longrightarrow E \longrightarrow V \longrightarrow 0$$

By Lemma 3.4.6, $\mathcal{O}_X = \mathcal{O}_{U \oplus V} \subseteq \overline{\mathcal{O}}_E \setminus \mathcal{O}_E$. This implies that $\dim \mathcal{O}_X < \dim \mathcal{O}_E$, which contradicts the maximality of $\dim \mathcal{O}_X$. □

Corollary 3.4.11 (Corollary to Lemma 3.4.6)**.** *$\mathcal{O}_X$ is closed if and only if X is semisimple.*

Recall that X is semisimple if for each submodule $U \subseteq X$, there is a submodule $V \subseteq X$ such that $U \oplus V = X$.

Proof. Assume first that $\mathcal{O}_X$ is closed. Assume that X is not semisimple. Then there is $U \subseteq X$ such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & X & \longrightarrow & V & \longrightarrow & 0 \\ & & & & & & \| & & \\ & & & & & & {}^{X}/_{U} & & \end{array}$$

is a nonsplit short exact sequence. By Lemma 3.4.6, $\mathcal{O}_{U\oplus V} \subseteq \overline{\mathcal{O}}_X \setminus \mathcal{O}_X = \emptyset$ because $\mathcal{O}_X$ is closed. This is a contradiction.

Conversely, see [Art69]. □

3.5 Modules over Hereditary Algebras

Definition 3.5.1. A k-algebra A is **hereditary** (or **Mod**(A) is hereditary) if one of the following equivalent conditions holds:

(a) Every submodule of a projective A-module is projective.

(b) The global dimension of A is at most 1; $\mathrm{gldim}(A) \leq 1$.

(c) For all $X, Y \in$ **Mod**(A), then $\mathrm{Ext}^i_A(X, Y) = 0$ for all $i \geq 1$.

(d) Every $X \in$ **Mod**(A) has a projective resolution of length at most 1.

Example 3.5.2.

(a) $A = k\langle x_1, \ldots, x_n\rangle$ free algebras of finite rank

(b) $A = kQ$ for Q any finite quiver

(c) $A = kG$ for G a finitely generated **virtually free** group. This means that G contains a subgroup H that is free, and the index of H in G is finite.

(d) When A is a commutative algebra, A is hereditary if A is a Dedekind domain.

(e) $A = \mathcal{O}(X)$ for X a smooth affine curve.

(f) $A_1(k) = {}^{k\langle x,y\rangle}/_{(xy-yx=1)}$ for k a field of characteristic zero.

Algebras in Examples (a)-(e) are **formally smooth**, that is, smooth in the category $\mathbf{Alg}_k$, while example (f) is not formally smooth.

Definition 3.5.3. Let A be an arbitrary associative algebra. A (left) module X over A has **finite length** if

$$\ell(X) = \sup\left\{ r \in \mathbb{N} \cup \{\infty\} \;\middle|\; X = X_0 \supsetneq X_1 \supsetneq \ldots \supsetneq X_r = 0 \right\} < \infty.$$

Theorem 3.5.4 (Krull-Schmidt). *Any* A*-module* X *of finite length can be written as a direct sum of indecomposable modules*

$$X = \bigoplus_{\alpha} X_{\alpha}^{m_{\alpha}}$$

with $m_\alpha > 0$ *for all* α*. The* ***isotypes*** $\{X_\alpha\}$ *and* ***multiplicities*** m_α *are determined (uniquely) by* X*.*

The proof is an easy induction on the length of X.

Lemma 3.5.5 (Fitting Lemma, first form). *Let* X *be an* A*-module of length* $n < \infty$*. Let* $f \in \mathrm{End}_A(X)$*. Then* $X = \mathrm{im}(f^n) \oplus \ker(f^n)$*.*

Proof. If a module has finite length, then it is Noetherian and Artinian. Therefore, the descending chain of submodules

$$X \supseteq \mathrm{im}(f) \supseteq \mathrm{im}(f^2) \supseteq \ldots \supseteq \mathrm{im}(f^{2n}) \supseteq \ldots$$

must stabilize, and likewise the ascending chain

$$\ker(f) \subseteq \ker(f^2) \subseteq \ldots \subseteq \ker(f^{2n}) \subseteq \ldots$$

must stabilize at some $N \leq n$. We may without loss take $N = n$. Then explicitly,

$$\mathrm{im}(f^n) = \mathrm{im}(f^{n+1}) = \ldots = \mathrm{im}(f^{2n}) = \ldots$$

$$\ker(f^n) = \ker(f^{n+1}) = \ldots = \mathrm{im}(f^{n+1}) = \ldots$$

Now let $x \in X$. Then consider $f^n(x) \in \mathrm{im}(f^n)$. There is some $y \in X$ such that $f^n(x) = f^{2n}(y)$. Hence, $f^n(x - f^n(y)) = 0$, which in turn implies that $x - f^n(y) \in \ker(f^n)$. Therefore, $x \in \mathrm{im}(f^n) + \ker(f^n)$.

It remains to show that $\mathrm{im}(f^n) \cap \ker(f^n) = 0$. Let $x \in \mathrm{im}(f^n) \cap \ker(f^n)$. Write $x = f^n(y)$ for some y, so $f^n(x) = f^{2n}(y) = 0$ since $x \in \ker(f^n)$. This shows that $y \in \ker(f^{2n}) = \ker(f^n)$. Hence, $x = f^n(y) = 0$. So the sum is direct. □

Corollary 3.5.6. *If* X *is a finite-dimensional indecomposable* A*-module over a* k*-algebra* A*, then any* $f \in \mathrm{End}_A(X)$ *is either invertible or nilpotent.*

Proof. Finite dimension implies finite length, so the previous lemma applies and we see that $X = \mathrm{im}(f^n) \oplus \ker(f^n)$. But X is indecomposable, so either $\mathrm{im}(f^n) = 0$ or $\ker(f^n) = 0$. If $\mathrm{im}(f^n) = 0$, then $f^n = 0$ so f is nilpotent. If $\ker(f^n) = 0$, then $\ker(f) = 0$ and f is injective. Since f is injective and X indecomposable, then f is invertible. □

Corollary 3.5.7 (Fitting Lemma, second form)**.** *Let* A *be a* k*-algebra over an algebraically closed field* k*. Then for any indecomposable* X *of finite* k*-dimension,*

$$\mathrm{End}_A(X) = k \cdot \mathrm{id}_X + \mathrm{Rad}(\mathrm{End}_A(X)),$$

where $\mathrm{Rad}(R)$ *is the largest nilpotent 2-sided ideal of* R*.*

Proof. Take $f \in \mathrm{End}_A(X)$. Since k is algebraically closed, f has an eigenvalue $\lambda \in k$. $f - \lambda \cdot \mathrm{id}_X$ is not invertible, since $\det(f - \lambda \cdot \mathrm{id}_X) = 0$. Therefore, by the previous corollary, $f - \lambda\, \mathrm{id}_X = \theta$ is nilpotent, so any $f \in \mathrm{End}_A(X)$ can be written as $f = \lambda \cdot \mathrm{id}_X + \theta$, with θ nilpotent. □

Example 3.5.8. Consider a representation of the quiver $Q = \bullet \circlearrowleft$ with dimension vector $\alpha = n$. In this case, we have a vector space V with an endomorphism $f \in \mathrm{End}_k(V)$ and the previous corollary is Jordan Normal Form of f. This representation is indecomposable when f is a Jordan block.

Exercise 3.5.9. A ring A is called **local** if for all $a, b \in A$ such that $a + b$ is a unit, then a is a unit or b is a unit.

(a) Assume A is commutative. Then A is local if and only if it has a unique maximal ideal.

(b) An A-module X is indecomposable of finite length if and only if $\mathrm{End}_A(X)$ is local.

Definition 3.5.10. An A-module X is called a **brick** if $\mathrm{End}_A(X) = k$.

Note that a brick is necessarily indecomposable (sometimes called **stable indecomposable**).

Lemma 3.5.11 (Happel-Ringel)**.** *Let* X *and* Y *be two indecomposable finite-dimensional modules over a hereditary algebra* A*. Assume that* $\mathrm{Ext}^1_A(Y, X) = 0$*. Then any nonzero* $\theta \in \mathrm{Hom}_A(X, Y)$ *is either monic or epic.*

Proof. Let's split $\theta\colon X \to Y$ into two exact sequences.

$$\xi\colon\ 0 \longrightarrow \mathrm{im}(\theta) \xrightarrow{\ \delta\ } Y \longrightarrow \mathrm{coker}(\theta) \longrightarrow 0$$

$$\eta\colon\ 0 \longrightarrow \ker(\theta) \longrightarrow X \xrightarrow{\ \theta\ } \mathrm{im}(\theta) \longrightarrow 0$$

Apply $\mathrm{Hom}_A(\mathrm{coker}(\theta), -)$ to the sequence η to get a long exact sequence, which contains the snippet

$$\cdots \longrightarrow \mathrm{Ext}^1_A(\mathrm{coker}\,\theta, \ker\theta) \longrightarrow \mathrm{Ext}^1_A(\mathrm{coker}\,\theta, X) \xrightarrow{\theta_*} \mathrm{Ext}^1_A(\mathrm{coker}\,\theta, \mathrm{im}\,\theta) \longrightarrow 0$$

The last term is zero because A is hereditary ($\mathrm{Ext}^i_A(X, Y) = 0$ for all $i \geq 2$).

We learn from this long exact sequence that θ_* is surjective, so $[\xi] = \theta_*[\zeta]$ for some $\zeta \in \mathrm{Ext}^1_A(\mathrm{coker}\,\theta, X)$.

$$\begin{array}{cccccccccc}
\zeta & & 0 & \longrightarrow & X & \xrightarrow{\alpha} & Z & \longrightarrow & \mathrm{coker}(\theta) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \theta_*} & & & & \downarrow{\scriptstyle \theta} & (\star) & \downarrow{\scriptstyle \gamma} & & \| & & \\
\xi & & 0 & \longrightarrow & \mathrm{im}(\theta) & \xrightarrow[\delta]{} & Y & \longrightarrow & \mathrm{coker}(\theta) & \longrightarrow & 0
\end{array}$$

The square $(\star)$ is Cocartesian (a pushout), so $\gamma\alpha = \delta\theta$ in $\mathrm{Hom}_A(X, Y)$. This is equivalent to sequence below being exact.

$$0 \longrightarrow X \xrightarrow{\binom{\alpha}{\theta}} Z \oplus \mathrm{im}(\theta) \xrightarrow{(\gamma, -\delta)} Y \longrightarrow 0 \tag{3.5}$$

Since $\mathrm{Ext}^1_A(Y, X) = 0$, (3.5) splits. Hence X and Y are direct summands of $Z \oplus \mathrm{im}(\theta)$. So by the Krull-Schmidt Theorem (Theorem 3.5.4), either X or Y (being indecomposable) must be a direct summand of $\mathrm{im}(\theta)$.

If θ is not monic, then $\ker\theta \neq 0$. Therefore, $\dim_k(\mathrm{im}(\theta)) < \dim_k(X)$ by exactness of η. Hence, X cannot be a direct summand of $\mathrm{im}\,\theta$.

If θ is not epic, then $\mathrm{coker}(\theta) \neq 0$. Therefore, $\dim_k(\mathrm{im}(\theta)) < \dim_k(Y)$ by exactness of ξ, so Y cannot be a direct summand of $\mathrm{im}\,\theta$.

If both cases hold, then we have a contradiction. Hence, θ must be either monic or epic (or both). □

Corollary 3.5.12. *Assume that k is algebraically closed, and let A be a hereditary k-algebra. If X is an indecomposable finite-dimensional A-module with $\mathrm{Ext}^1_A(X, X) = 0$, then X is a brick.*

Proof. Put $Y = X$ in Lemma 3.5.11. Then every $\theta \in \mathrm{End}_A(X)$ is either monic or epic; if it's one, then it must be both. So θ is an isomorphism. Then apply Schur's Lemma (Corollary 3.5.7) to conclude that $\theta = \lambda \cdot \mathrm{id}_X$ for some $\lambda \in k$. Hence, $\mathrm{End}_A(X) \cong k \cdot \mathrm{id}_X$, so X is a brick. □

Lemma 3.5.13 (Ringel)**.** *Let A be a hereditary algebra. Let X be a finite indecomposable A-module which is not a brick. Then X contains*

(a) a proper submodule U such that $\mathrm{End}_A(U) = k$ and $\mathrm{Ext}^1_A(U, U) \neq 0$; and,

(b) *a proper quotient* X/Y *such that* $\mathrm{End}_A(X/Y) = k$ *and* $\mathrm{Ext}^1_A(X/Y, X/Y) \neq 0$.

Notice that U *and* X/Y *are bricks.*

Proof. We prove only (a). The proof of (b) is similar.

It suffices to show that there is $U \subsetneq X$ indecomposable with $\mathrm{Ext}^1_A(U, U)$ nonzero. (Indeed, if U is a brick, we are done; otherwise we repeat the process to get $X \supsetneq U \supsetneq U_1 \supsetneq \ldots \supsetneq U_m$, which terminates because $\dim_k(X) < \infty$.)

Pick a nonzero endomorphism $\theta \in \mathrm{End}_A(X)$ such that $I := \mathrm{im}(\theta)$ has minimal dimension among all endomorphisms. Claim that θ is nilpotent, and $\theta^2 = 0$. Indeed, $\mathrm{im}(\theta^2) \subseteq \mathrm{im}(\theta)$. By minimality of $\dim(\mathrm{im}(\theta))$, $\mathrm{im}(\theta^2) = \mathrm{im}(\theta)$. Now consider the composite

$$I \hookrightarrow X \xrightarrow{\theta} I.$$

The condition $\mathrm{im}(\theta^2) = \mathrm{im}(\theta)$ says that this is an isomorphism. Hence, I is a direct summand of X, but this is a contradiction because X is indecomposable.

Now $\theta^2 = 0 \implies I = \mathrm{im}(\theta) \subseteq \ker(\theta)$. Decompose $\ker(\theta)$ into indecomposables by the Krull-Schmidt Theorem (Theorem 3.5.4):

$$\ker(\theta) = \bigoplus_{i=1}^{N} K_i.$$

with each K_i indecomposable. Let $\pi_i \colon \ker(\theta) \twoheadrightarrow K_i$ denote the canonical projections. Since $I \neq 0$, there is some $j \in \{1, \ldots, N\}$ such that

$$I \hookrightarrow \ker(\theta) \twoheadrightarrow K_j$$

is nonzero. Claim that $U = K_j$ is the required module.

First, K_j is proper: if $X = K_j$, then $X = K_j \subseteq \ker(\theta) \subseteq X \implies \ker\theta = X$. But $\theta \neq 0$, so this cannot be. Second, K_j is indecomposable by construction.

It remains to show that $\mathrm{Ext}^1_A(K_j, K_j) \neq 0$. We do this in three steps:

(1) Notice $\alpha \colon I \hookrightarrow \ker(\theta) \overset{\pi_j}{\twoheadrightarrow} K_j$ is injective. Indeed, consider the composite

$$X \xrightarrow{\theta} I \overset{\alpha}{\hookrightarrow} K_j \hookrightarrow X.$$

Notice that $\mathrm{im}(\alpha \circ \theta) = \alpha(\mathrm{im}(\theta))$. If α is not injective, then $\mathrm{im}(\alpha \circ \theta)$ has strictly smaller dimension than $\mathrm{im}(\theta) = I$, which contradicts minimality of $\dim_k(I)$.

(2) Consider the short exact sequence

$$0 \longrightarrow I \overset{\alpha}{\hookrightarrow} K_j \longrightarrow K_j/I \longrightarrow 0$$

and dualize it by K_j. The long exact sequence contains the following snippet:

$$\cdots \longrightarrow \mathrm{Ext}^1_A({}^{K_j}/_{I}, K_j) \longrightarrow \mathrm{Ext}^1_A(K_j, K_j) \xrightarrow{f} \mathrm{Ext}^1_A(I, K_j) \longrightarrow 0.$$

Since A is hereditary, f is onto, so it suffices to show that $\mathrm{Ext}^1_A(I, K_j) \neq 0$.

(3) Consider the pushout

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(\theta) & \hookrightarrow & X & \xrightarrow{\theta} & I & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_j} & & \downarrow{\scriptstyle g} & & \| & & \\ 0 & \longrightarrow & K_j & \xrightarrow{h} & Y & \longrightarrow & I & \longrightarrow & 0 \end{array}$$

If $\mathrm{Ext}^1_A(I, K_j) = 0$, then the bottom sequence splits; let $r\colon Y \to K_j$ be a retraction $r \circ h = \mathrm{id}_{K_j}$. Then $g \circ r\colon X \to K_j$ implies that K_j is a direct summand of X. This contradicts the indecomposability of X.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(\theta) & \hookrightarrow & X & \xrightarrow{\theta} & I & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_j} & \swarrow & \downarrow{\scriptstyle g} & & \| & & \\ 0 & \longrightarrow & K_j & \underset{r}{\overset{h}{\rightleftarrows}} & Y & \longrightarrow & I & \longrightarrow & 0 \end{array}$$

□

3.6 Classification of graphs

We classify graphs into three types: (simply laced) Dynkin diagrams, Euclidean (or extended Dynkin) diagrams, or "wild" graphs. In the first two cases, we associate to a graph a **root system**.

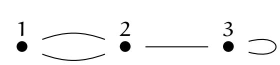
Definition 3.6.1. Let Γ be a finite graph (i.e. a quiver with the orientation forgotten). We permit Γ to have loops or multiple edges. Assume once and for all that the vertices of Γ are

$$V(\Gamma) = \{1, 2, \ldots, n\}.$$

The graph Γ is determined by

$$n_{ij} = n_{ji} := \#\{\text{edges between } i \text{ and } j\}.$$

Example 3.6.2.



$$\begin{aligned} n_{11} = n_{22} &= 0 \\ n_{33} &= 1 \\ n_{12} = n_{21} &= 2 \\ n_{23} = n_{32} &= 1 \\ n_{13} = n_{31} &= 0 \end{aligned}$$

Definition 3.6.3. Define for a finite graph Γ the map $q_\Gamma \colon \mathbb{Z}^n \to \mathbb{Z}$ by

$$q_\Gamma(\alpha) := \sum_{i=1}^{n} \alpha_i^2 - \sum_{i \leq j} n_{ij} \alpha_i \alpha_j.$$

Similarly, define a map $(-,-)_\Gamma \colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ by

$$(\varepsilon_i, \varepsilon_j)_\Gamma := \begin{cases} -n_{ij} & \text{if } (i \neq j) \\ 2 - 2n_{ii} & \text{if } (i = j), \end{cases}$$

where ε_i denotes the standard basis vector of $\mathbb{Z}^n$: zeroes except for 1 in the i-th position.

Remark 3.6.4. If Γ has no loops (that is, $n_{ii} = 0$ for all i), then

$$C_\Gamma = \|(\varepsilon_i, \varepsilon_j)_\Gamma\|$$

is called the (generalized) **Cartan matrix** of the graph Γ.

Example 3.6.5. If $\Gamma = \bullet \text{------} \bullet \text{------} \bullet \text{------} \bullet$, then

$$C_\Gamma = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

Remark 3.6.6. Giving Γ is equivalent to giving q_Γ, which is equivalent to giving $(-,-)_\Gamma$. Indeed, $q_\Gamma(\alpha) = \frac{1}{2}(\alpha, \alpha)_\Gamma$ and

$$(\alpha, \beta)_\Gamma = q_\Gamma(\alpha + \beta) - q_\Gamma(\alpha) - q_\Gamma(\beta).$$

Lemma 3.6.7. *If $\Gamma = \Gamma_Q$ is the graph underlying a quiver Q, then*

$$\begin{aligned} q_\Gamma(\alpha) &= \langle \alpha, \alpha \rangle_Q \\ (\alpha, \beta)_\Gamma &= \langle \alpha, \beta \rangle_Q \end{aligned}$$

where $\langle -,- \rangle_Q$ is the Euler form of Q.

Remark 3.6.8. Note that q_Γ and $(-,-)_\Gamma$ are independent of orientations, but $\langle -,- \rangle_Q$ depends on orientation of the quiver.

Definition 3.6.9.

(a) q_Γ is called **positive definite** if $q(\alpha) > 0$ for all $0 \neq \alpha \in \mathbb{Z}^n$.

(b) q_Γ is called **positive semidefinite** if $q(\alpha) \geq 0$ for all $\alpha \in \mathbb{Z}^n$.

(c) $\mathrm{Rad}(q_\Gamma) := \{\beta \in \mathbb{Z}^n \mid (\beta, -) = 0\}$ is the **radical** of q_Γ.

(d) Define a partial order on $\mathbb{Z}^n$ by $\alpha \geq \beta \iff \alpha - \beta \in \mathbb{N}^n$

(e) Call $\alpha \in \mathbb{Z}^n$ **sincere** if $\alpha_i \neq 0$ for all $i = 1, \dots, n$.

Lemma 3.6.10 (Key Lemma)**.** *Let* Γ *be a connected graph. Assume that there is* $\beta \in \mathrm{Rad}(q_\Gamma)$ *such that* $\beta \neq 0$ *and* $\beta \geq 0$*. Then*

(a) β *is sincere,*

(b) q_Γ *is positive semidefinite, and*

(c) for $\alpha \in \mathbb{Z}^n$*, we have* $q_\Gamma(\alpha) = 0 \iff \alpha \in \mathbb{Q}\beta \iff \alpha \in \mathrm{Rad}(q_\Gamma)$.

Proof.

(a) Note that $\beta \in \mathrm{Rad}(q_\Gamma) \iff (\beta, \varepsilon_i)_\Gamma = 0$ for all $i = 1, \dots, n$, which is equivalent to

$$(\beta, \varepsilon_i)_\Gamma = \sum_{j=1}^{n} (\varepsilon_i, \varepsilon_j)_\Gamma \beta_j = (2 - 2n_{ii})\beta_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} n_{ij}\beta_j = 0$$

If β is not sincere, then there is some i such that $\beta_i = 0$. Choose this i in the equation above. Then because all terms $n_{ij}\beta_j$ are positive,

$$\sum_{\substack{j=1 \\ j\neq i}}^{n} n_{ij}\beta_j = 0 \implies n_{ij}\beta_j = 0$$

for all $j \neq i$.

Hence, $\beta_j = 0$ whenever $n_{ij} \neq 0$. But Γ is connected, so there is $j = i_1 \neq i_1$ such that $n_{ii_1} \neq 0$. This implies that $\beta_{i_1} = 0$. Then take i_1 instead of i. Connectedness again gives $i_2 \neq i_1, i$ such that $\beta_{i_2} = 0$, and so on.

So by connectedness of Γ, $\beta = 0$. This contradicts the assumption that $\beta \neq 0$. Hence, β must be sincere.

(b) For any $\alpha \in \mathbb{Z}^n$, for any $\beta \in \mathrm{Rad}(q_\Gamma)$, we can rewrite $q_\Gamma(\alpha)$ as follows

$$q_\Gamma(\alpha) = \sum_{1\leq i<j\leq n} n_{ij} \frac{\beta_i\beta_j}{2} \left(\frac{\alpha_i}{\beta_i} - \frac{\alpha_j}{\beta_j} \right)^2 \tag{3.6}$$

which is well-defined from part (a): $\beta_i \neq 0$ for all i. It follows immediately that q_Γ is positive semidefinite.

(c) If $q_\Gamma(\alpha) = 0$, then (3.6) shows that

$$\frac{\alpha_i}{\beta_i} = \frac{\alpha_j}{\beta_j}$$

whenever $n_{ij} \neq 0$. Hence,

$$\alpha_i = \frac{\beta_i}{\beta_j}\alpha_j.$$

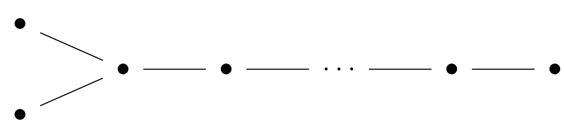
By connectedness of Γ, we again argue that $\alpha \in \mathbb{Q}\beta$. If $\alpha \in \mathbb{Q}\beta$, then $\alpha \in \mathrm{Rad}(q_\Gamma)$. Thus, $q_\Gamma(\alpha) = 0 \implies \alpha \in \mathbb{Q}\beta \implies \alpha \in \mathrm{Rad}(q_\Gamma)$.

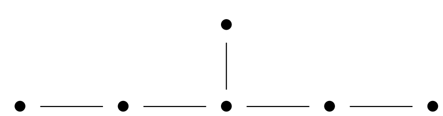
Conversely, $\alpha \in \mathrm{Rad}(q_\Gamma) \implies q_\Gamma(\alpha) = 0$ because $q_\Gamma(\alpha) = \frac{1}{2}(\alpha, \alpha)_\Gamma$. □

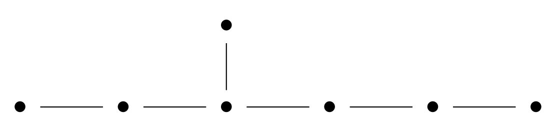
Exercise 3.6.11. Verify (3.6).

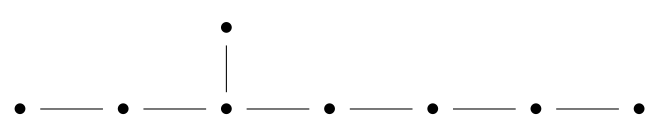
Definition 3.6.12. A graph Γ is called a (simply laced) **Dynkin** diagram if Γ is of the form A_n, D_n, E_6, E_7 or E_8.

A_n: (n vertices)

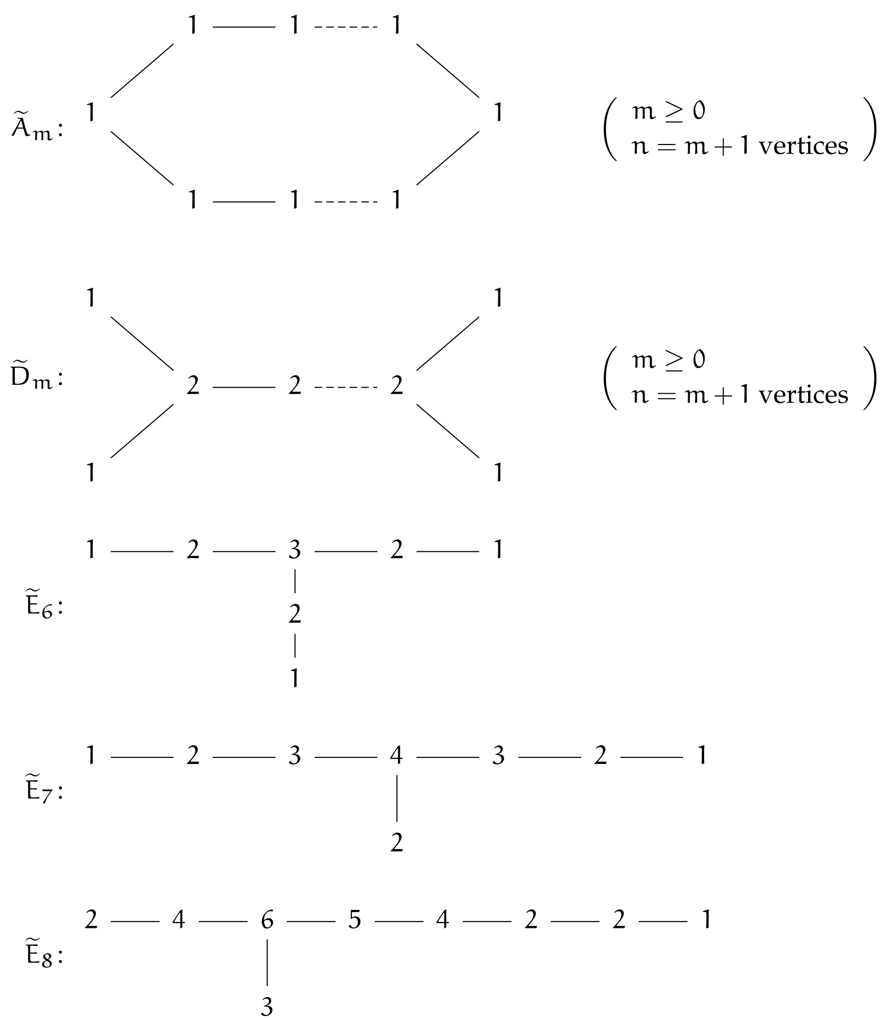
D_n:  (n vertices)

E_6: 

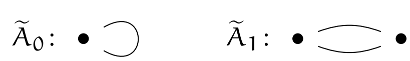
E_7: 

E_8: 

Definition 3.6.13. A graph Γ is called **Euclidean** or **extended Dynkin** if it has one of the following forms: (the numbers will be explained in Theorem 3.6.15)



Example 3.6.14.



Theorem 3.6.15 (Classification of graphs)**.** *Let* Γ *be a connected graph. Then*

(a) Γ *is Dynkin if* q_Γ *is positive definite*

(b) Γ *is Euclidean if* q_Γ *is positive semidefinite and* $\mathrm{Rad}(q_\Gamma) = \mathbb{Z} \cdot \delta$*, where* $\delta = (\delta_1, \ldots, \delta_n) \in \mathbb{Z}^n$ *is the vector with components indicated on the graph in Definition 3.6.13. (Note:* δ *is sincere and* $\delta \geq 0$ *in all cases.)*

(c) *If Γ is neither Dynkin nor Euclidean, then there exists some $\alpha \geq 0$ with $q_\Gamma(\alpha) < 0$ and $(\alpha, \varepsilon_i)_\Gamma \leq 0$ for all $i = 1, \ldots, n$.*

Proof sketch.

(b) By case-by-case inspection, one checks that $\delta \in \mathrm{Rad}(q_\Gamma)$. For example, if Γ has no loops or multiple edges (i.e. $\Gamma \neq \widetilde{A}_0$), then

$$2\delta_i = \sum_j \delta_j,$$

where the sum runs over all vertices j neighboring i. Indeed, by definition $\delta \in \mathrm{Rad}(q_\Gamma) \iff (\delta, \varepsilon_i)_\Gamma = 0$, if and only if

$$\sum_j \delta_j(\varepsilon_j, \varepsilon_i) = 0 \iff 2\delta_i - \sum_{j \neq i} n_{ij}\delta_j = 0.$$

By Lemma 3.6.10, for $\beta = \delta$, q_Γ is positive semidefinite and $\alpha \in \mathrm{Rad}(q_\Gamma) \iff \alpha \in \mathbb{Q}\delta$. But δ always has $\delta_i = 1$ for some i, so

$$\alpha \in \mathbb{Q}\delta \cap \mathbb{Z}^n \implies \alpha_j = p\delta_j \text{ for all } j, \text{ for some } p \in \mathbb{Q}$$

But we have that $\alpha_i = p \cdot 1$, so $p = \alpha_i \in \mathbb{Z}$. Therefore, $\mathrm{Rad}(q_\Gamma) = \mathbb{Z}\delta$.

(a) We may embed a Dynkin diagram Γ inside the corresponding Euclidean diagram $\widetilde{\Gamma}$ by adding an extra vertex. Check that $q_\Gamma(\alpha) = q_{\widetilde{\Gamma}}(\alpha) > 0$; $q_{\widetilde{\Gamma}}(\alpha)$ is positive on all nonzero non-sincere vectors.

(c) If Γ is not Dynkin or Euclidean, then it always contains a subgraph Γ' which is Euclidean. If $V(\Gamma) = V(\Gamma')$, then take $\alpha = \delta$, and if $V(\Gamma') \neq V(\Gamma)$, then take $i \in \Gamma \setminus \Gamma'$ and $\alpha = 2\delta' + \varepsilon_i$. □

Definition 3.6.16. If Γ is Euclidean, a vertex $i \in \Gamma$ is **extending** if $\delta_i = 1$.

Remark 3.6.17.

(a) Every Euclidean Γ has an extending vertex.

(b) By deleting an extending vertex, we get the corresponding Dynkin diagram. For example, $\widetilde{A}_{n+1} \rightsquigarrow A_n$.

3.7 Root Systems

Suppose Γ is Dynkin or Euclidean: that is, q_Γ is positive semidefinite.

Definition 3.7.1. $\alpha \in \mathbb{Z}^n$ is called a **root** of Γ if $\alpha \neq 0$ and $q_\Gamma(\alpha) \leq 1$. If $q_\Gamma(\alpha) = 1$, then α is called a **real root**. Otherwise $q_\Gamma(\alpha) = 0$ and α is called an **imaginary root**.

Definition 3.7.2. The set of all roots Δ_Γ is called the **root system** of Γ,

$$\Delta_\Gamma := \{\alpha \in \mathbb{Z}^n \mid \alpha \neq 0, q_\Gamma(\alpha) \leq 1\}.$$

Further define the sets of **real** and **imaginary roots**,

$$\Delta_\Gamma^{\text{re}} = \{\alpha \in \Delta_\Gamma \mid q_\Gamma(\alpha) = 1\}$$

$$\Delta_\Gamma^{\text{im}} = \{\alpha \in \Delta_\Gamma \mid q_\Gamma(\alpha) = 0\}$$

and the sets of **positive** and **negative** roots.

$$\Delta_\Gamma^+ := \{\alpha \in \Delta_\Gamma \mid \alpha \geq 0\}$$

$$\Delta_\Gamma^- := \{\alpha \in \Delta_\Gamma \mid \alpha \leq 0\}$$

Remark 3.7.3. One can define root systems for arbitrary graphs: if Γ has no loops, see [Kac94]. In general, for any Γ, see [Kač80].

Proposition 3.7.4 (Some properties of roots)**.**

(R1) *Every* ε_k, $k = 1, \dots, n$, *is a root of* Γ.

(R2) $\alpha \in \Delta_\Gamma \implies -\alpha \in \Delta_\Gamma$ *and* $\alpha + \beta \in \Delta_\Gamma$ *for all* $\beta \in \mathrm{Rad}(q_\Gamma)$.

(R3) $\Delta_\Gamma^{im} = \begin{cases} \emptyset & \Gamma \text{ is Dynkin,} \\ r \cdot \delta, r \in \mathbb{Z} \setminus \{0\} & \Gamma \text{ is Euclidean.} \end{cases}$

(R4) $\Delta_\Gamma = \Delta_\Gamma^+ \sqcup \Delta_\Gamma^-$

(R5) *If* Γ *is Euclidean, then* ${}^{(\Delta \cup \{0\})}/_{\mathbb{Z}\cdot\delta}$ *is finite.*

(R6) *If* Γ *is Dynkin, then* Δ_Γ *is finite.*

Proof.

(R1) $(\varepsilon_k)_i = \delta_{ik}$

$$q_\Gamma(\varepsilon_k) = \sum_{i=1}^{n} \delta_{ik}^2 - \sum_{i \leq 1} n_{ij} \delta_{ik} \delta_{jk} = 1 - (\text{something } \geq 0) \leq 1.$$

(R2) $$\begin{aligned} q_\Gamma(\alpha \pm \beta) &= q(\alpha) + q(\beta) \pm (\beta, \alpha_\Gamma) && \text{for all } \alpha, \beta \in \mathbb{Z}^n, \\ &= q(\alpha) \leq 1 && \text{if } \beta \in \mathrm{Rad}(q_\Gamma), \alpha \in \Delta_\Gamma \end{aligned}$$

(R3) By Lemma 3.6.10(c), for $\beta = \delta$.

(R4) Recall that $\beta \geq 0 \iff \beta \in \mathbb{N}^n$. For all $\beta \in \mathbb{Z}^n$, write

$$\operatorname{supp}(\beta) = \{i \mid 1 \leq i \leq n, \beta_i \neq 0\} \subseteq \{1, \dots, n\}.$$

For example, β is sincere if and only if $\operatorname{supp}(\beta) = \{1, 2, \dots, n\}$.

Observe that for nonzero α, we may write $\alpha = \alpha^+ - \alpha^-$, where $\alpha^\pm \geq 0$, $\alpha^\pm \neq 0$, and $\operatorname{supp}(\alpha^+) \cap \operatorname{supp}(\alpha^-) = \emptyset$. For example, $(1, -2, 3) = (1, 0, 3) - (0, 2, 0)$.

$$(\alpha^+, \alpha^-) = \sum_{i,j} \alpha_i^+ \alpha_j^- (\varepsilon_i, \varepsilon_j)_\Gamma = \sum_i \underbrace{\alpha_i^+ \alpha_i^-}_{=0} (2 - 2n_i i) - \sum_{i \neq j} \alpha_i^+ \alpha_j^- n_{ij} \leq 0$$

If $\alpha \in \Delta_\Gamma$, then

$$1 \geq q_\Gamma(\alpha) = q(\alpha^+ - \alpha^-) = q(\alpha^+) + q(\alpha^-) - (\alpha^+, \alpha^-) \geq q(\alpha^+) + q(\alpha^-)$$

Then $q(\alpha^+) \geq 0$, $q(\alpha^-) \geq 0$. Since $q_\Gamma(\beta) \geq 0$ for all $\beta \in \Gamma$, then either $q(\alpha^+) = 0$ or $q(\alpha^-) = 0$. Then either α^+ or α^- is an imaginary root. By (R3), either α^+ or α^- is sincere, WLOG say α^+ is sincere. Then $\operatorname{supp}(\alpha^+)$ is maximal, so $\operatorname{supp}(\alpha^-) = \emptyset$, so $\alpha^- = 0$. Therefore, $\alpha = \alpha^+$.

(R5) Let $i \in \Gamma$ be an extending vertex of Γ so that $\delta_i = 1$. Then for all $\beta \in \Delta_\Gamma \cup \{0\}$, by (R2), we have

$$\beta - \beta_i \cdot \delta \in \{\alpha \in \Delta \cup \{0\} \mid \alpha_i = 0\},$$

because $\delta \in \operatorname{Rad}(q_\Gamma)$. Call this set S_i. The above shows that for any $\beta \in \Delta_\Gamma \cup \{0\}$, $\beta \equiv s \pmod{\mathbb{Z} \cdot \delta}$ for some $s \in S_i$. So it suffices therefore to show that S_i is a finite set.

For $\delta \in \operatorname{Rad}(q_\Gamma)$, we have $\delta \pm \alpha \in \Delta_\Gamma$ for $\alpha \in \Delta_\Gamma$.

If $\alpha \in S_i$ (so $\alpha_i = 0$), then $(\delta \pm \alpha)_i = \delta_i = 1$, so $\delta \pm \alpha \in \Delta_\Gamma^+$, so

$$\delta \pm \alpha \geq 0 \implies -\delta \leq \alpha \leq \delta.$$

Hence,

$$S_i = \{\alpha \in \Delta \cup \{0\} \mid \alpha_i = 0\} \subseteq \{\alpha \in \mathbb{Z}^n \mid -\delta \leq \alpha \leq \delta\}.$$

This last set is clearly finite.

(R6) Extend Γ to $\widetilde{\Gamma}$ by adding an extending vertex i. Then

$$\Delta_\Gamma \subseteq \{\alpha \in \Delta_{\widetilde{\Gamma}} \mid \alpha_i = 0\},$$

which is finite by (R5). □

3.8 Proof of Gabriel's Theorem

We will divide this proof into two parts: finite representation type and infinite representation type.

Theorem 3.8.1. *Let* Q *be a quiver with* Γ_Q *Dynkin. Then the assignment*

$$\underline{\dim}\colon X \longmapsto \underline{\dim}(X)$$

induces a bijection between isomorphism classes of indecomposable finite dimensional representations of Q *and positive roots in* Δ_{Γ_Q}.

$$\left\{\begin{array}{c}\text{isomorphism classes of}\\ \text{indecomposable finite-}\\ \text{dimensional representations of } Q\end{array}\right\} \xrightarrow{\sim} \Delta^+_{\Gamma_Q}$$

We divide the proof into four parts.

Lemma 3.8.2 (Step 1)**.** *If* X *is an indecomposable representation of* Q*, then* X *is a brick, i.e.* $\mathrm{End}_Q(X) = k$*.*

Proof. Assume X is *not* a brick. By Ringel's Lemma (Lemma 3.5.13), there is some proper $U \subsetneq X$ such that $\mathrm{End}_k(U) = k$ and $\mathrm{Ext}^1_Q(U,U) \neq 0$.

Then, since Γ_Q is Dynkin,

$$\begin{aligned} 1 \le q_\Gamma(\underline{\dim}\, U) &= \dim_k\left(\mathrm{End}_Q(U)\right) - \dim_k\left(\mathrm{Ext}^1(U,U)\right) \\ &= 1 - \dim_k\left(\mathrm{Ext}^1_Q(U,U)\right). \end{aligned}$$

In particular, we have that $\dim\left(\mathrm{Ext}^1_Q(U,U)\right) \le 0$, so $\mathrm{Ext}^1_Q(U,U) = 0$. This contradicts Ringel's Lemma (Lemma 3.5.13). □

Lemma 3.8.3 (Step 2)**.** *If* X *is indecomposable, then it has no self-extensions and* $\underline{\dim}(X) \in \Delta^+_\Gamma$.

Proof. If X is indecomposable, then X is a brick, so

$$\begin{aligned} 0 < q_\Gamma(\underline{\dim}(X)) &= \dim\left(\mathrm{End}_Q(X)\right) - \dim\left(\mathrm{Ext}^1_Q(X,X)\right) \\ &= 1 - \dim_k\left(\mathrm{Ext}^1_Q(X,X)\right) \end{aligned}$$

In particular, this means that $\mathrm{Ext}^1_Q(X,X) = 0$. Therefore,

$$q_\Gamma(\underline{\dim}(X)) = \dim_k\left(\mathrm{End}_Q(X)\right) = 1.$$

So by Definition 3.7.2, $\underline{\dim}(X) \in \Delta^+_\Gamma$. □

Lemma 3.8.4 (Step 3)**.** *If X, X' are indecomposables such that $\underline{\dim} X = \underline{\dim} X'$, then $X \cong X'$.*

Proof. Since $\operatorname{Ext}^1_Q(X,X) = \operatorname{Ext}^1_Q(X',X') = 0$, then $\mathcal{O}_X$ and $\mathcal{O}_{X'}$ are both *open* orbits in $\operatorname{Rep}(\alpha)$, where $\alpha = \underline{\dim} X = \underline{\dim} X'$. If $\mathcal{O}_X \neq \mathcal{O}_{X'}$, then $\mathcal{O}_X \cap \mathcal{O}_{X'} \implies \mathcal{O}_{X'} \subseteq \mathbf{Rep}(\alpha) \setminus \mathcal{O}_X$. Therefore, $\dim \mathcal{O}_{X'} < \dim(\operatorname{Rep}(\alpha))$, which is a contradiction because $\mathcal{O}_X$ is open. □

Lemma 3.8.5 (Step 4)**.** *If $\alpha \in \Delta^+_{\Gamma_Q}$, then there is indecomposable X with $\underline{\dim} X = \alpha$.*

Proof. Fix $\alpha \in \Delta^+_{\Gamma_Q}$ and consider $\operatorname{Rep}(\alpha)$. Let X be such that $\mathcal{O}_X$ is an orbit in $\operatorname{Rep}(\alpha)$ of maximal dimension. Then claim that X is the required representation.

Assume that X is not indecomposable; this is the only thing we might need to check. We may write $X = U \oplus V$. Then by Lemma 3.4.6,

$$\operatorname{Ext}^1_Q(U,V) = \operatorname{Ext}^1_Q(V,U) = 0.$$

We have $\alpha = \underline{\dim}(X) = \underline{\dim}(U) + \underline{\dim}(V)$. So count dimensions in a clever way:

$$\begin{aligned}
1 &= q_\Gamma(\alpha) \\
&= q_\Gamma(\underline{\dim} U + \underline{\dim} V) \\
&= q_\Gamma(\underline{\dim} U) + q_\Gamma(\underline{\dim} V) + (\underline{\dim} U, \underline{\dim} V)_\Gamma \\
&= q_\Gamma(\underline{\dim} U) + q_\Gamma(\underline{\dim} V) + \langle \underline{\dim} U, \underline{\dim} V\rangle_\Gamma + \langle \underline{\dim} U, \underline{\dim} V\rangle_\Gamma \\
&= q_\Gamma(\underline{\dim} U) + q_\Gamma(\underline{\dim} V) + \dim_k \operatorname{Hom}(U,V) - \dim_k \operatorname{Ext}^1_Q(U,V) \\
&\qquad + \dim_k \operatorname{Hom}(V,U) - \dim_k \operatorname{Ext}^1_Q(V,U) \\
&= q_\Gamma(\underline{\dim} U) + q_\Gamma(\underline{\dim} V) + \dim_k \operatorname{Hom}(U,V) + \dim_k \operatorname{Hom}(V,U).
\end{aligned}$$

But we have that $q_\Gamma(\underline{\dim} U) \geq 1$ and $q_\Gamma(\underline{\dim} V) \geq 1$, because Γ is Dynkin. So the equality we have shows that $1 \geq 2$, which is absurd. Hence, X must be indecomposable. □

Proof of Theorem 3.8.1. By Lemma 3.8.3 and Lemma 3.8.4, $\underline{\dim}\colon X \mapsto \underline{\dim}(X)$ is well-defined and injective. Lemma 3.8.5 shows that this map is surjective as well. □

Theorem 3.8.6 (Gabriel 1972)**.** *Let Q be any quiver with Γ_Q connected. Then Q has finitely many isomorphism classes of indecomposables if and only if Γ_Q is Dynkin.*

Proof. ($\Longleftarrow$). If Γ_Q is Dynkin, then by Theorem 3.8.1, we have

$$\#\{\text{isomorphism classes of indecomposables}\} = |\Delta^+_{\Gamma_Q}| < |\Delta_{\Gamma_Q}|.$$

By property (R6) of Proposition 3.7.4, this is finite.

($\Longrightarrow$). Assume that Q has finitely many indecomposable representations up to isomorphism. By Krull-Schmidt (Theorem 3.5.4), there are finitely many isomorphism classes of finite-dimensional representations of a given dimension vector α. Therefore, $\operatorname{Rep}(\alpha)$ has only finitely many orbits.

Recall if $\alpha \geq 0$ such that $q_\Gamma(\alpha) \leq 0$, then $\operatorname{Rep}(\alpha)$ must have infinitely many orbits. Indeed,

$$\dim \operatorname{Rep}(\alpha) - \dim \mathcal{O}_X = \dim \operatorname{End}_Q(X) - q_\alpha > 0.$$

Therefore, $\dim \mathcal{O}_X < \dim \operatorname{Rep}(\alpha)$ implies that there are infinitely many orbits. Hence, we must have that $q_\Gamma(\alpha) > 0$, so by Theorem 3.6.15, Γ_Q is Dynkin. $\square$

Example 3.8.7. Consider the quiver $Q = \overset{1}{\bullet} \xrightarrow{a} \overset{2}{\bullet}$. This is a Dynkin quiver of type A_2. Then

$$q_{\Gamma_Q} \colon \mathbb{Z}^2 \to \mathbb{Z}$$

is given by

$$q_{\Gamma_Q} = \alpha_1^2 + \alpha_2^2 - \alpha_1\alpha_2 = \frac{1}{2}\alpha_1^2 + \frac{1}{2}\alpha_2^2 + \frac{1}{2}(\alpha_1 - \alpha_2)^2 > 0$$

$$\begin{aligned}
\Delta_Q &= \{\alpha \in \mathbb{Z}^2 \colon q_{\Gamma_Q}(\alpha \leq 1), \alpha \neq 0\}. \\
\Delta_Q^{\mathrm{im}} &= \{\alpha \in \mathbb{Z}^2 \mid q_{\Gamma_Q}(\alpha) = 0\} = \emptyset. \\
\Delta_Q^{\mathrm{re}} &= \{\alpha \in \mathbb{Z}^2 \mid q_{\Gamma_Q}(\alpha) = 1\} = \{\pm(1,0), \pm(0,1), \pm(1,1)\}. \\
\Delta_Q^{+} &= \{(1,0), (0,1), (1,1)\}
\end{aligned}$$

There are three indecomposable modules:

$$U = \left(\overset{k}{\bullet} \xrightarrow{\;0\;} \overset{0}{\bullet} \right) \qquad V = \left(\overset{0}{\bullet} \xrightarrow{\;0\;} \overset{k}{\bullet} \right) \qquad W = \left(\overset{k}{\bullet} \xrightarrow{\;1\;} \overset{k}{\bullet} \right)$$

We have that $\underline{\dim}\, U = (1,0)$, $\underline{\dim}\, V = (0,1)$ and $\underline{\dim}\, W = (1,1)$. Any finite dimensional representation

$$X = \left(X_1 \xrightarrow{X_a} X_2 \right)$$

can be written as a direct sum of U, V and W.

Chapter 4

Generalizations of Gabriel's Theorem

There are a few things that we should remark.

(1) Tame representation type

(2) Kac's Theorem

(3) Ringel-Hall Algebras

(4) Quantum Groups

4.1 Tame Representation Type

What can we say about representations of kQ when Γ_Q is Euclidean?

Definition 4.1.1. Q is of **finite type** if kQ has finitely many indecomposables (up to isomorphism).

Corollary 4.1.2 (Corollary to Theorem 3.8.6)**.** Q *is of finite type if and only if* Γ_Q *is a union of Dynkin diagrams of type* A, D, E.

Definition 4.1.3. Q is of **tame (affine) type** if the isomorphism classes of indecomposable kQ-modules can be split into discrete families each of which depends on a continuous parameter.

Example 4.1.4. If Q is of type $\widetilde{A}_0$, that is, $Q = \bullet \circlearrowleft$. Representations of Q are pairs (V, f) with $f \in \mathrm{End}_k(V)$. If k is an algebraically closed field of characteristic zero, then we put f into Jordan Normal Form

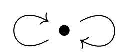
$$f = \begin{pmatrix} J_{n_1}(\lambda_1) & & & 0 \\ & J_{n_2}(\lambda_2) & & \\ & & \ddots & \\ 0 & & & J_{n_m}(\lambda_m) \end{pmatrix}$$

where each $J_n(\lambda)$ is a Jordan block

$$J_n(\lambda) = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix}.$$

Thus, indecomposable representations of kQ are in bijection with Jordan Blocks. So each representation corresponding to $J_n(\lambda)$ depends on a discrete parameter n and a continuous parameter λ.

Definition 4.1.5. If Q is neither finite type nor tame type, then Q is **wild**.

Example 4.1.6. The quiver $Q =$  is wild.

Theorem 4.1.7 (Generalized Gabriel)**.** *If* Q *is quiver with a connected graph* Γ_Q*, then* Q *is of tame type if and only if* Γ_Q *is Euclidean.*

There are two ways to prove this theorem:

(1) Auslander-Reiten sequences and Nakayama functors, or

(2) through proving Kac's Theorem and the (deformed) preprojective algebras of quivers.

4.2 Kac's Theorem

Given any finite quiver Q and $\alpha \in \mathbb{N}^n$, is there an indecomposable representation X of kQ of $\underline{\dim}\, X = \alpha$? If yes, how many such representations?

Kac gave a general answer to this question by relating this to Kac-Moody Lie Algebras. We need to first define a root system for any quiver.

Definition 4.2.1. Let Q be an arbitrary quiver. For each loop-free vertex $i \in Q_0$ ($n_{ii} = 0$), define the **reflection operator** $s_i \colon \mathbb{Z}^n \to \mathbb{Z}^n$ by

$$\alpha \mapsto s_i(\alpha) = \alpha - (\alpha, \varepsilon_i)\varepsilon_i.$$

The **Weyl group** of Q is the subgroup

$$W_Q = \langle s_i \mid n_{ii} = 0 \rangle \subseteq \mathrm{Aut}(\mathbb{Z}^n) = \mathrm{GL}_n(\mathbb{Z})$$

The **fundamental region of** W is

$$F = \left\{ \alpha \in \mathbb{Z}^n \;\middle|\; \alpha \neq 0, \mathrm{supp}(\alpha) \text{ connected}, (\alpha, \varepsilon_i) \leq 0 \text{ for all } i = 1, \dots, n \right\}$$

Definition 4.2.2. The **real roots** of Q are the W-orbits of the basis vectors ε_i, where i is loop free.

The **imaginary roots** of Q are the W-orbits of $\pm\alpha$ for $\alpha \in F$.

Some properties of general root systems.

Proposition 4.2.3.

(R1) $\alpha \in \Delta_Q \implies -\alpha \in \Delta_Q$

(R2) $\Delta_Q = \Delta_Q^+ \sqcup \Delta_Q^-$, *where* $\Delta_Q^+ = \{\alpha \geq 0\}$ *and* $\Delta_Q^- = \{\alpha \leq 0\}$.

(R3) $q(\alpha)$ *is W-invariant. If* α *is real, then* $q(\alpha) = q(\varepsilon_i) = 1$, *and if* α *is imaginary, then* $q(\alpha) \leq 0$.

Exercise 4.2.4.

(a) Prove the previous proposition.

(b) If Q is Dynkin or Euclidean, then Definition 4.2.1 agrees with Definition 3.7.2.

Theorem 4.2.5 (Kac)**.** *Let* Q *be any quiver with* $\alpha \in \mathbb{N}^n$. *Then*

(a) If there is an indecomposable representation X *of* Q *with* $\underline{\dim}\, X = \alpha$, *then* $\alpha \in \Delta_Q^+$.

(b) If $\alpha \in \Delta_Q^{re,+}$ *then there is a unique indecomposable* X *(up to isomorphism) of* $\underline{\dim}\, X = \alpha$.

(c) If $\alpha \in \Delta_Q^{im,+}$ *then there are infinitely many isomorphim classes of indecomposables with dimension vector* α.

4.3 Hall Algebra of a Quiver

Let Q be a quiver with no oriented cycles (so that $\dim_k(kQ) < \infty$). Fix the vertex set $Q_0 = \{1, \dots, n\}$.

Definition 4.3.1. We can associate to it a **Kac-Moody Lie algebra** $\mathfrak{g}_Q$ as follows: $\mathfrak{g}_Q$ is generated by

$$e_1, \ldots, e_n, \qquad f_1, \ldots, f_n, \qquad h_1, \ldots, h_n$$

subject to the **Serre relations:**

$$\begin{aligned} [h_i, h_j] &= 0 \\ [e_i, f_j] &= \delta_{ij} h_j \\ [h_i, e_j] &= c_{ij} e_j \\ [h_i, f_j] &= -c_{ij} f_j \\ \mathrm{ad}(e_i)^{1-c_{ij}}(e_j) &= 0 \qquad (i \neq j) \\ \mathrm{ad}(f_i)^{1-c_{ij}}(f_j) &= 0 \qquad (i \neq j) \end{aligned}$$

where $C = (c_{ij})_{i,j \in Q_0}$ is the associated **generalized Cartan matrix** with entries $c_{ij} = (\varepsilon_i, \varepsilon_j)_Q$.

Example 4.3.2. For finite type A_n, $\mathfrak{g} = \mathfrak{sl}_{n+1}(k)$.

Proposition 4.3.3. *The Kac-Moody algebra $\mathfrak{g}_Q$ decomposes as*

$$\mathfrak{g}_Q = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-,$$

where $\mathfrak{n}^+ = \mathrm{Span}_k\{e_i\}_{i \in Q_0}$, $\mathfrak{n}^- = \mathrm{Span}_k\{f_i\}_{i \in Q_0}$, *and* $\mathfrak{h} = \mathrm{Span}_k\{h_i\}_{i \in Q_0}$.

Definition 4.3.4. $\mathfrak{n}^\pm$ are called the **positive/negative parts** of $\mathfrak{g}$, and $\mathfrak{h}$ is the **Cartan subalgebra.**

We have moreover that the **universal enveloping algebra** of $\mathfrak{g}_Q$ decomposes as $U(\mathfrak{g}_Q) \cong U(\mathfrak{n}^+) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}^-)$.

Theorem 4.3.5 (Poincaré-Birkhoff-Witt)**.**

$$U(\mathfrak{n}^+) = {}^{k\langle e_1, \ldots, e_n \rangle}\big/_{\langle\langle \mathrm{ad}(e_i)^{1-c_{ij}} e_j = 0 \mid i \neq j \rangle\rangle}$$

Now fix $k = \mathbb{F}_q$ a finite field with $|k| = q$. Consider $\mathbf{Mod}(kQ)$ the category of finite-dimensional kQ-modules. Note that every object $V \in \mathbf{Mod}(kQ)$ is *finite* as a set. We define an associative algebra that encodes the homological structure of $\mathbf{Mod}(kQ)$.

Fix a commutative integral domain A that contains $\mathbb{Z}$ and an element ν such that $\nu^2 = q$.

Example 4.3.6. $A = \mathbb{Q}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$

Definition 4.3.7. Let $\mathcal{P}$ be the set of isomorphism classes of all kQ-modules in $\mathbf{Mod}(kQ)$ and define the **Ringel-Hall algebra** $\mathcal{H}_{A,\nu}(kQ)$ by

$$\mathcal{H}_{A,\nu}(kQ) = \bigoplus_{[V]\in\mathcal{P}} A[V],$$

where $A[V]$ is the free A-module based on V. The multiplication in $\mathcal{H}_{A,\nu}(kQ)$ is given by

$$[U]\cdot[V] := \nu^{\langle \underline{\dim} U, \underline{\dim} V\rangle_Q} \sum_{[W]\in\mathcal{P}} c^W_{U,V}(k)[W]$$

and the **structure constants** $c^W_{U,V}(k)$ are

$$c^W_{U,V}(k) := \#\left\{W^1 \subseteq W \;\; kQ\text{-submodules} \;\middle|\; {}^{W}\!/_{W^1} \cong U \text{ and } W^1 \cong V\right\}$$

Remark 4.3.8. The product can be written compactly as

$$[U]\cdot[V] = \sum_{W\in\mathcal{P}} \frac{|\operatorname{Ext}^1_Q(U,V)_W|}{|\operatorname{Hom}_Q(U,V)|},$$

where $\operatorname{Ext}^1_Q(U,V)_W \subset \operatorname{Ext}^1_Q(U,V)$ is the subset of all extensions of U by V with middle term isomorphic to W.

Proposition 4.3.9. *$\mathcal{H}_{A,\nu}(kQ)$ is an associative $\mathbb{N}^{|Q_0|}$-graded A-algebra with the identity element being $[0]$, the trivial representation.*

Exercise 4.3.10. Compute $\mathcal{H}_{A,\nu}(kQ)$ for the Kronecker quiver $Q = (\bullet \longrightarrow \bullet)$.

Remark 4.3.11. The grading is given by the dimension vector

$$\mathcal{H}_{A,\nu}(kQ) = \bigoplus_{\alpha\in\mathbb{N}^n} \mathcal{H}^{\alpha}_{A,\nu}(kQ)$$

where $\mathcal{H}^{\alpha}_{A,\nu}(kQ)$ is spanned over A by the classes $[V]$ with $\underline{\dim}\, V = \alpha$.

Lemma 4.3.12. *Assume that Q is Dynkin (connected with finite representation type). Let U, V, W be finite-dimensional kQ-modules. Then there is a polynomial $f^W_{U,V}(t) \in \mathbb{Z}[t]$ such that $f^W_{U,V}(q) = c^W_{U,V}(\mathbb{F}_q)$ for all q.*

Definition 4.3.13. Using the lemma, we define the **specialized Hall Algebra** $\mathcal{H}_{\mathbb{C}}(Q)$ of Q as the $\mathbb{C}$-algebra with basis as a $\mathbb{C}$-vector space given by the set of isomorphism classes of finite-dimensional $\mathbb{C}Q$-modules and multiplication

$$[U]\cdot[V] = \sum_{[W]\in\mathcal{P}} f^W_{U,V}(1)\cdot[W].$$

Theorem 4.3.14 (Ringel 1990)**.** *Let Q be Dynkin. Then $\mathcal{H}_{\mathbb{C}}(Q) \cong U(\mathfrak{n}^+_Q)$.*

Exercise 4.3.15 (Continued from Exercise 4.3.10)**.** Prove this theorem for the Kronecker quiver $Q = (\bullet \longrightarrow \bullet)$.

What happens to $\mathcal{H}$ for arbitrary non-Dynkin quivers and $t \neq 1$?

Definition 4.3.16. Assume now that Q only has no oriented cycles; Q is not necessarily Dynkin. Consider for $k = \mathbb{F}_q$ the A-subalgebra $\mathcal{C}_{A,\nu}(kQ) \subseteq \mathcal{H}_{A,\nu}(kQ)$ generated by the classes of **simple** kQ-modules $\{S_i(k)\}_{i \in Q_0}$. This is the **composition algebra** of Q.

Remark 4.3.17. If Q is of finite type, then $\mathcal{C}_{A,\nu}(kQ) = \mathcal{H}_{A,\nu}(kQ)$.

Let $\mathcal{K}$ be an infinite subset of the set of all finite fields so that $\{\,|k|\,\}_{k\in\mathcal{K}} \subset \mathbb{N}$ is infinite.

Let A be an integral domain that is generated by $\mathbb{Z}$ and elements ν_k such that $\nu_k^2 = |k|$ for each $k \in \mathcal{K}$. Let $\mathcal{C}_k := \mathcal{C}_{A,\nu_k}$ be the composition algebra for each $k \in \mathcal{K}$, and define the subalgebra

$$\mathcal{C} \subseteq \prod_{k\in\mathcal{K}} C_k$$

generated by the elements

$$\begin{aligned} t &= (\nu_k)_{k\in\mathcal{K}}, \\ t^{-1} &= (\nu_k^{-1})_{k\in\mathcal{K}}, \\ \text{and} \quad u_i &= ([S_i(k)])_{k\in\mathcal{K}} \qquad\qquad \text{for } i = 1, 2, \dots, n. \end{aligned}$$

Observe that $t^{\pm 1}$ lie in the center of C (since so is each ν_k) and if $p(t) = 0$ for some $p \in \mathbb{Z}[t]$, then $p(t) \equiv 0$ (because $p(t) = 0 \iff p(\nu_k) = 0 \iff p(|k|^{\frac{1}{2}}) = 0$ because there are infinitely many distinct $k \in \mathcal{K}$).

Thus, we may think of $\mathcal{C}$ as an $A[t, t^{-1}]$-algebra generated by the u_i, that is,

$$\mathcal{C}_Q := A[t, t^{-1}]\langle u_1, \dots, u_n\rangle.$$

Next, we define the **generic composition algebra** of Q,

$$\mathcal{C}_Q^* := \mathbb{Q}(t) \otimes_A \mathcal{C}_Q.$$

4.4 Quantum groups

The idea is that quantum groups are "quantized" universal enveloping algebras $U_t(\mathfrak{g})$ of $\mathfrak{g} = \mathfrak{g}_Q$, such that when $t = 1$, $U_t(\mathfrak{g}) = U(\mathfrak{g})$.

Like the classical case, we have that

$$U_t(\mathfrak{g}) = U_t(\mathfrak{n}^+) \otimes U_t(\mathfrak{h}) \otimes U_t(\mathfrak{n}^-).$$

Recall that

$$U(\mathfrak{n}^+) = {\mathbb{Q}\langle e_1, \ldots, e_n \rangle}/{\langle\langle (\mathrm{ad}\, e_i)^{1-c_{ij}}(e_j) = 0 \mid i \neq j \rangle\rangle}.$$

Notice that

$$(\mathrm{ad}\, e_i)^N e_j = \sum_{p=0}^{N} (-1)^p \binom{N}{p} e_i^p e_j e_i^{N-p}.$$

To "quantize" this algebra, we replace $n \in \mathbb{N}$ by the **quantum numbers**

$$[n]_t := \frac{t^n - t^{-n}}{t - t^{-1}}$$

As $t \to 1$, this approaches n in the limit. Similarly, we have **quantum binomial coefficients**

$$\binom{N}{p} \rightsquigarrow \begin{bmatrix} N \\ p \end{bmatrix}_t := \frac{[N]_t!}{[p]_t![N-p]_t!},$$

where

$$[N]_t! := [1]_t[2]_t \cdots [N]_t.$$

Therefore,

$$(\mathrm{ad}_t\, e_i)^N e_j = \sum_{p=0}^{N} (-1)^p \begin{bmatrix} N \\ p \end{bmatrix}_t e_i^p e_j e_i^{N-p}.$$

Then the **positive part of the quantized universal enveloping algebra** is

$$U_t(\mathfrak{n}^+) := {\mathbb{Q}(t)\langle e_1, \ldots, e_n \rangle}/{\langle\langle (\mathrm{ad}_t\, e_i)^{1-c_{ij}}(e_j) = 0 \mid i \neq j \rangle\rangle}.$$

Theorem 4.4.1 (Ringel,Green)**.** *The map* $u_i \mapsto e_i$ *gives a natural isomorphism of* $\mathbb{Q}(t)$*-algebras* $C_Q^* \cong U_t(\mathfrak{n}^+)$.

Exercise 4.4.2 (Continued from Exercise 4.3.15)**.** Prove this theorem for the Kronecker quiver $Q = (\bullet \longrightarrow \bullet)$.

Remark 4.4.3. For references for this section, see [Rin90, Rin96, Rin93, Gre95].

Quantum groups were introduced by Drinfel'd and Jimbo in 1986. They weren't taken too seriously until they appeared in Reshetikhin-Turaev invariants of 3-manifolds, which generalize the Jones polynomials. A question we might ask is whether or not this is connected to Hall algebras, since the quantum groups appear in both areas. Recent work has found Hall-algebra-like objects in invariants of manifolds, called **elliptic Hall algebras**.

4.5 Multilocular Categories

Fix a field k.

Definition 4.5.1. A k**-linear category** is an additive category $\mathbf{A}$ such that every hom-set has the structure of a k-vector space, and the composition map factors through the tensor product of hom-sets.

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{A}}(B,C)\times\mathrm{Hom}_{\mathbf{A}}(A,B) & \xrightarrow{\quad\circ\quad} & \mathrm{Hom}_{\mathbf{A}}(A,C) \\ & \searrow^{\otimes} \qquad \nearrow^{\circ\ (\text{k-linear map})} & \\ & \mathrm{Hom}_{\mathbf{A}}(B,C)\otimes_k\mathrm{Hom}_{\mathbf{A}}(A,B) & \end{array}$$

Example 4.5.2.

(a) Let A be a k-algebra. Then the category $\mathbf{Mod}(A)$ of (left) A-modules is k-linear. (In particular, for $A = kQ$.)

(b) If X is a projective algebraic variety with $\overline{k} = k$, and $X \subseteq \mathbb{P}^n_k$, then the category $\mathbf{Coh}(X)$ of coherent sheaves on X is k-linear. Similarly, the category $\mathbf{QCoh}(X)$ of quasi-coherent sheaves is k-linear.

Remark 4.5.3. Recall that any additive category has finite direct sums and direct products, and they coincide. Therefore, the notion of direct sums makes sense in any additive category. See section 4.1.1 of [HA1] for the precise definition.

Definition 4.5.4. A category $\mathbf{A}$ is called **multilocular** or **Krull-Schmidt** if

(a) each hom-set is a finite-dimensional k-vector space,

(b) every $X \in \mathrm{Ob}(\mathbf{A})$ can be decomposed into a finite direct sum of indecomposable objects; and

(c) the endomorphism ring $\mathrm{End}_{\mathbf{A}}(X)$ of an indecomposable object X is local.

Remark 4.5.5. Recall that Definition 4.5.4(b) implies that the direct decomposition is unique (up to permutation of factors).

Example 4.5.6.

(a) If A a finite-dimensional k-algebra and $\mathbf{A} = \mathbf{mod}(A)$ is the category of finite-dimensional A-modules, then $\mathbf{A}$ is multilocular.

(b) If X is a projective variety (e.g. an elliptic curve) over k and $\mathbf{A} = \mathbf{Coh}(X)$, then $\mathbf{A}$ is multilocular.

(c) Bounded derived categories of objects in (a) and (b) are also multilocular. $\mathcal{D}^b(\mathbf{mod}(A))$ and $\mathcal{D}^b(\mathbf{Coh}(X))$.

Proposition 4.5.7. *A multilocular category is determined by its full subcategory* $\mathbf{Ind}(\mathbf{A})$ *consisting of indecomposable objects in* $\mathbf{A}$ *and morphisms between them.*

Definition 4.5.8. Consider for two $U, V \in \mathrm{Ob}(\mathbf{Ind}(\mathbf{A}))$

$$\mathrm{rad}(U, V) = \left\{ f\colon U \to V \mid f \text{ is } \textit{not} \text{ invertible} \right\}.$$

Exercise 4.5.9. Check that $\mathrm{rad}(U, V)$ is a 2-sided ideal in $\mathrm{Mor}(\mathbf{Ind}(\mathbf{A}))$; that is, check that $g \circ f \in \mathrm{rad}(U, V)$ and $f \circ g \in \mathrm{rad}(U, V)$ for all $g \in \mathrm{Mor}(\mathbf{Ind}(\mathbf{A}))$ and $f \in \mathrm{rad}(U, V)$.

Definition 4.5.10. For any $U, V \in \mathrm{Ob}(\mathbf{Ind}(\mathbf{A}))$, define the **space of irreducibles**

$$\mathrm{Irr}(\mathbf{A}) := {}^{\mathrm{rad}(U,V)}/_{\mathrm{rad}(U,V)^2}.$$

Definition 4.5.11. Let $\mathbf{A}$ be a multilocular category. Define the **quiver of A**, denoted $\Gamma(A)$, as follows.

$$\begin{aligned} \text{Vertices of } \Gamma(A) &= \left\{ \text{ isomorphism classes of indecomposables in } \mathbf{A} \right\} \\ &= \left\{ \text{ isomorphism classes of objects in } \mathbf{Ind}(\mathbf{A}) \right\}. \end{aligned}$$

Arrows $[U] \to [V]$ in $\Gamma(\mathbf{A})$ are the elements of $\mathrm{Irr}(\mathbf{A})$; represented by arrows $U \to V$ that are not invertible, modulo those which are compositions of at least two.

Note that the number of arrows $[U] \to [V]$ in $\Gamma(\mathbf{A})$ is $\dim_k \mathrm{Irr}(U, V)$.

Part II

Differential Graded Algebras & Hochschild Homology

Chapter 5

Differential Graded Algebras

5.1 Differential Graded Algebras

Definition 5.1.1. By **graded vector space**, we mean a $\mathbb{Z}$-graded vector space. A (homological) **chain complex** is written

$$V_\bullet = \bigoplus_{n\in\mathbb{Z}} V_n,$$

and a (cohomological) **cochain complexx** is written

$$V^\bullet = \bigoplus_{n\in\mathbb{Z}} V^n.$$

The relation between $V_\bullet$ and $V^\bullet$ is given by inverting degrees: $V_n \mapsto V^{-n}$.

Definition 5.1.2 (Koszul Sign Convention)**.** Given maps $f, g\colon V_\bullet \to W_\bullet$, we have a bilinear map $f\otimes g\colon V_\bullet \otimes V_\bullet \to W_\bullet \otimes W_\bullet$. The **Koszul sign convention** says that for all $x, y \in V_\bullet$,

$$(f\otimes g)(x\otimes y) := (-1)^{|x||g|} f(x)\otimes g(y),$$

where $|x| = \deg(x)$ and $|g| = \deg(g)$.

Definition 5.1.3 (Shifts of degrees)**.** For a graded vector space V and any $m\in\mathbb{Z}$, we may define the m**-shifted vector space** by

$$V[m]_\bullet = \bigoplus_{n\in\mathbb{Z}} V[m]_n, \qquad V[m]_n = V_{n-m}$$
$$V[m]^\bullet = \bigoplus_{n\in\mathbb{Z}} V[m]^n, \qquad V[m]^n = V^{n+m}$$

The shift $V\mapsto V[m]$ defines a functor on $\mathbb{Z}$-graded vector spaces.

Definition 5.1.4. A (cochain) **differential graded algebra** (dg-algebra) is a k-algebra $A^\bullet$ with a decomposition as a graded vector space

$$A^\bullet = \bigoplus_{n\in\mathbb{Z}} A^n,$$

together with a differential $d\colon A^\bullet \to A[1]^\bullet$ satisfying

(a) d is an (odd) derivation: $d(ab) = (da)b + (-1)^{|a|}a(db)$ for all $a, b \in A$;

(b) $d^2 = 0$.

Remark 5.1.5. The chain version of a dg-algebra is $A_\bullet = \bigoplus_{n\in\mathbb{Z}} A_n$, with $|d| = -1$.

Definition 5.1.6. A **morphism of differential graded algebras** is a morphism of graded algebras $f\colon A^\bullet \to B^\bullet$ that commutes with the differential.

Definition 5.1.7. We write $\mathbf{dgAlg}_k$ for the category of differential graded k-algebras and morphisms between them.

Lemma 5.1.8. *Let $A^\bullet$ be a differential graded algebra. Then*

*(a) $Z^\bullet A := \{a \in A^\bullet \mid da = 0\}$ is a graded subalgebra of $A^\bullet$ (called the **cocycle subalgebra**);*

(b) $B^\bullet A := \{a \in A^\bullet \mid a = db$ for some $b \in A\}$ is a 2-sided graded ideal in $Z^\bullet A$.

Definition 5.1.9. The **cohomology algebra** of $A^\bullet$ is the graded algebra

$$H^\bullet A := {Z^\bullet A}/{B^\bullet A}.$$

Equivalently this is a dg-algebra with $d = 0$.

$A^\bullet \mapsto H^\bullet A$ defines a functor $H^\bullet\colon \mathbf{dgAlg}_k \longrightarrow \mathbf{grAlg}_k \hookrightarrow \mathbf{dgAlg}_k$

Remark 5.1.10. A natural question you might want to ask is: what extra structure needs to be added to $H^\bullet A$ to recover $A^\bullet$?

The answer is that $H^\bullet A$ carries the so-called A_∞**-structure** defined by a family of graded maps $m_n\colon H^\bullet A^{\otimes n} \to H^\bullet A$ of degree $2-n$ ($n \geq 1$) that allows one to recover $A^\bullet$ (up to A_∞-quasi-isomorphism). This result is known in the literature as Kadeishvili's Theorem [Kel99].

Example 5.1.11. The trivial example of dg-algebras is just ordinary algebras. You may consider an ordinary algebra A as a dg-algebra by putting it in degree zero.

$$A^\bullet = \bigoplus_{n\in\mathbb{Z}} A^n, \qquad A^n = \begin{cases} A & n = 0 \\ 0 & n \neq 0. \end{cases}$$

This construction defines a fully faithful functor $\mathbf{Alg}_k \hookrightarrow \mathbf{dgAlg}_k$

Why dg-algebras? $\mathbf{dgAlg}_k$ can be viewed as a "categorical closure" (natural generalization) of the category of finite-dimensional associative k-algebras.

Construction 5.1.12. Let A be a finite-dimensional associative k-algebra. Write

$$\begin{aligned} m\colon A \otimes A &\longrightarrow A \\ a \otimes b &\longmapsto ab \end{aligned}$$

for the multiplication map. Dualize this map: ($A^* := \mathrm{Hom}_k(A, k)$)

$$m^*\colon A^* \longrightarrow (A \otimes A)^* \cong A^* \otimes A^* \hookrightarrow T_k(A^*) := \bigoplus_{n \geq 0} (A^*)^{\otimes n}.$$

Note that we use the finite dimensional assumption to get $(A \otimes A)^* \cong A^* \otimes A^*$.

By the universal property of tensor algebras, there is a unique derivation $d\colon T_k(A^*) \to T_k(A^*)$ of degree 1 extending m^*. We give $T_k(A^*)$ the natural grading such that $|\xi| = 1$ for all $\xi \in A^*$.

$$\begin{array}{ccc} A^* & \xrightarrow{\quad m^* \quad} & T_k(A^*) \\ & \searrow & \nearrow{\scriptstyle d} \\ & T_k(A^*) & \end{array}$$

To construct d, use the Leibniz rule inductively. On degree 1, d is given by

$$d(\xi \otimes \eta) = m^*(\xi) \otimes \eta + (-1)^{|\xi|} \xi \otimes m^*(\eta).$$

This construction takes the algebra (A, m) to the dg-algebra $(T(A^*), d)$.

What does the associativity of m mean in terms of d? The associativity of m is the diagram

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{m \otimes 1} & A \otimes A \\ \downarrow{\scriptstyle 1 \otimes m} & & \downarrow{\scriptstyle m} \\ A \otimes A & \xrightarrow{\quad m \quad} & A. \end{array}$$

Lemma 5.1.13. *Given an algebra (A, m), construct as in Construction 5.1.12 the dg-algebra $(T_k(A^*), d)$. Then m is associative if and only if $d^2 = 0$.*

Exercise 5.1.14. Prove the previous lemma.

Remark 5.1.15. Construction 5.1.12 generalizes to arbitrary (not necessarily finite-dimensional) associative dg-algebras as a duality between dg-algebras and dg-coalgebras. This duality is an instance of Koszul duality.

We can also play this game in the opposite direction. Given a dg-algebra $(T_k V, d)$, define $m\colon V^* \otimes V^* \to V^*$ by $m = (d|_V)^*$.

Lemma 5.1.16. *Any free dg-algebra generated by a finite-dimensional space V of elements of degree 1 gives rise to a finite-dimensional associative algebra.*

Remark 5.1.17. Given Lemma 5.1.13 and Lemma 5.1.16, we can view free differential-graded algebras as a generalization of finite-dimensional associative algebras.

Exercise 5.1.18. Perform Construction 5.1.12 for (finite-dimensional) Lie algebras: extend the Lie bracket $[-,-]\colon \bigwedge^2(\mathfrak{g}) \to \mathfrak{g}$ to a map

$$[-,-]^*\colon \mathfrak{g}^* \longrightarrow \textstyle\bigwedge^2 \mathfrak{g}^* \hookrightarrow \textstyle\bigwedge^\bullet \mathfrak{g}^*$$

and get $d_{CE}\colon \bigwedge^\bullet \mathfrak{g}^* \to \bigwedge^\bullet \mathfrak{g}^*$. This gives a commutative cochain dg-algebra.

The relation is that $d_{CE}^2 = 0$ if and only if $[-,-]$ satisfies the Jacobi identity.

Definition 5.1.19. The complex $C^\bullet(\mathfrak{g}) = (\bigwedge^\bullet \mathfrak{g}^*, d_{CE})$ is called the **Chevalley-Eilenberg cochain complex**. It gives Lie algebra cohomology of $\mathfrak{g}$ with trivial coefficients:

$$H^\bullet(\mathfrak{g}, k) := H\left(C^\bullet(\mathfrak{g})\right).$$

Definition 5.1.20. Define the forgetful functor $(-)_\#\colon \mathbf{dgAlg}_k \to \mathbf{grAlg}_k$ that takes a dg-algebra (A, d) and forgets the differential d.

Definition 5.1.21.

(a) Let $\mathbf{dgAlg}_k^+$ be the full subcategory of $\mathbf{dgAlg}_k$ with objects the non-negatively graded dg-algebras, that is,

$$A_\# = \bigoplus_{p\in\mathbb{Z}} A_p \qquad A_p = 0\ \forall p < 0$$

If $A \in \mathbf{dgAlg}_k^+$, then A is called **non-negative**.

(b) $A \in \mathbf{dgAlg}_k^+$ is called **connected** if $A_0 = k$, i.e. $A_\# = k \oplus A_1 \oplus \dots$

(c) $A \in \mathbf{dgAlg}_k^+$ is called **semi-free** if $A_\#$ is a free algebra, i.e. $A_\# = T_k(V)$ for some non-negatively graded vector space $V = \bigoplus_{p\geq 0} V$.

(d) A morphism of dg-algebras $f\colon A \to B$ in $\mathbf{dgAlg}_k^+$ is called a **semi-free extension** of A if there is an isomorphism $\phi\colon B_\# \cong A_\# \sqcup_k T_k(V)$ in $\mathbf{grAlg}_k$ for some graded vector space V and we have a commutative diagram

$$\begin{array}{ccc} A_\# & \hookrightarrow & A_\# \sqcup_k T_k(V) \\ & {}_{f_\#}\searrow \quad \nearrow_{\phi}^{\cong} & \\ & B_\# & \end{array}$$

where $\sqcup_k$ is the coproduct (free product) in $\mathbf{dgAlg}_k$.

Remark 5.1.22. In particular, A is semi-free if and only if $k \hookrightarrow A$ is a semi-free extension of k. Notice also that A is semi-free and connected if and only if $A_{\#} \cong T_k(V)$, where $V = \bigoplus_{p \geq 1} V_p$.

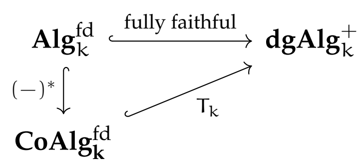
Remark 5.1.23. Previously, we showed that there is a bijection

$$\left\{ \begin{matrix} \text{finite-dimensional (non-unital)} \\ \text{associative } k\text{-algebras} \end{matrix} \right\} \Longleftrightarrow \left\{ \begin{matrix} \text{semi-free connected dg-algebras} \\ \text{generated by finitely many} \\ \text{elements in degree } 1 \end{matrix} \right\}$$

$$A \longmapsto T_k(A^*)$$

This is a sign of bad mathematics. We have some object and we keep adding adjectives to get to something interesting. Of course, some people need their adjectives like fish need water. We should ask why we have all the adjectives, and if there's some other (more natural) way to understand the equivalence.

This functor arises from Koszul duality (or the **bar-cobar construction**) between dg-coalgebras and dg-algebras. This gives a **Quillen equivalence**



Definition 5.1.24. Let $A \in \mathbf{grAlg}_k$. A linear map $d \colon A \to A$ is called a **derivation** of degree $r \in \mathbb{N}$ if

(a) $dA_i \subseteq A_{i+r}$ for all $i \in \mathbb{Z}$. We write $|d| = r$ for the **degree** of r.

(b) The **Leibniz rule:** $d(ab) = (da)b + (-1)^{r|a|}a(db)$.

If $r \equiv 0 \pmod 2$, then d is called an **even** derivation. Likewise, if $r \equiv 1 \pmod 2$, then d is called an **odd** derivation.

The next lemma says that any derivation (even or odd) on A is uniquely determined by its values on generators of A.

Lemma 5.1.25. *Let A be a graded algebra generated by S and $d_1, d_2 \colon A \to A$ derivations. If $d_1(s) = d_2(s)$ for all $s \in S$, then $d_1 = d_2$.*

Proof. Apply Leibniz rule iteratively. □

Corollary 5.1.26. *Let A be a graded algebra generated by S and $d \colon A \to A$ an odd derivation such that $d^2(s) = 0$ for all $s \in S$. Then $d^2 = 0$ in A.*

Proof. If d is an odd derivation, then d^2 is a derivation. Indeed,

$$\begin{aligned} d^2(ab) &= d\Big((da)b + (-1)^{|a|r}a(db)\Big) \\ &= (d^2a)b + \Big((-1)^{|da|r}(da)(db) + (-1)^{|a|r}(da)(db)\Big) + (-1)^{2|a|r}a(d^2b) \\ &= (d^2a)b + (-1)^{|a|\cdot 2r}a(d^2b) \end{aligned}$$

The middle two terms in the middle line cancel because d is an odd derivation. Then the result follows from Lemma 5.1.25. □

Lemma 5.1.27 (Künneth Formula)**.** *Let A be any dg-algebra, and B a connected dg-algebra ($B \in \mathbf{dgAlg}_k^+, B_0 = k$). Then*

$$H_*(A \sqcup_k B) \cong H_*(A) \sqcup_k H_*(B).$$

Exercise 5.1.28. Prove Lemma 5.1.27.

5.2 Algebraic de-Rham theory

Definition 5.2.1 (Algebraic de-Rham complex)**.** Let A be a commutative k-algebra. Define the A-module of **Kähler differentials** $\Omega^1_{\text{comm}}(A)$ generated by symbols da for each $a \in A$, satisfying the following relations:

(1) $d(\lambda a + \mu b) = \lambda(da) + \mu(db)$ for all $\lambda, \mu \in k$.

(2) $d(ab) = a(db) + b(da)$ for all $a, b \in A$.

Lemma 5.2.2. $\Omega^1_{\text{comm}}(A) \cong {}^{A \otimes_k A}\big/_{(ab \otimes c - a \otimes bc + ac \otimes b)_{a,b,c \in A}}$

Proof. Consider the A-module map

$$\begin{aligned} A \otimes A &\longrightarrow \Omega^1_{\text{comm}}(A) \\ a \otimes b &\longmapsto a(db) \end{aligned}$$

This map is surjective, and its kernel is spanned by elements of the form

$$ab \otimes c - a \otimes bc + ac \otimes b.$$

So it induces the required isomorphism. □

Definition 5.2.3. For a commutative dg-algebra and an A-module M, define the **space of derivations**

$$\text{Der}_k(A, M) = \{d \in \text{Hom}_k(A, M) \mid d(ab) = a(db) + b(da)\}.$$

If $f\colon M \to N$, then we get a map

$$\begin{aligned} f_*\colon \operatorname{Der}_k(A, M) &\longrightarrow \operatorname{Der}_k(A, N) \\ (A \xrightarrow{d} M) &\longmapsto \left(A \xrightarrow{d} M \xrightarrow{f} N\right) \end{aligned}$$

so $\operatorname{Der}_k(A, -)\colon A\text{-}\mathbf{Mod} \to k\text{-}\mathbf{Mod}$ defines a functor.

Example 5.2.4. If $M = A$, then $\operatorname{Der}_k(A, A) = \operatorname{Der}_k(A)$.

Proposition 5.2.5. *The functor* $\operatorname{Der}_k(A, -)\colon A\text{-}\mathbf{Mod} \to k\text{-}\mathbf{Mod}$ *is corepresented by* $\Omega^1_{\text{comm}}(A)$*, that is, there is a natural isomorphism*

$$\operatorname{Hom}_A(\Omega^1_{\text{comm}}(A), M) \cong \operatorname{Der}_k(A, M).$$

This proposition will be proved in the noncommutative case later. The proposition gives a universal property for $\Omega^1_{\text{comm}}(A)$, as follows.

Given any A-module M, and a derivation $\delta\colon A \to M$, then there is a unique A-module map $\phi\colon \Omega^1_{\text{comm}}(A) \to M$ such that the following commutes:

$$\begin{array}{ccc} A & \xrightarrow{\quad\delta\quad} & M \\ & {\scriptstyle d}\searrow \quad \nearrow{\scriptstyle \exists!\,\phi} & \\ & \Omega^1_{\text{comm}}(A) & \end{array}$$

In particular, if $M = \Omega^1_{\text{comm}}(A)$, then

$$\begin{aligned} \operatorname{Hom}_A(\Omega^1_{\text{comm}}(A), \Omega^1_{\text{comm}}(A)) &\longrightarrow \operatorname{Der}_k(A, \Omega^1_{\text{comm}}(A)) \\ \mathrm{id}_{\Omega^1_{\text{comm}}(A)} &\longmapsto (d\colon a \mapsto da) \end{aligned}$$

Definition 5.2.6. Define the **(commutative) de Rham algebra** of A as a graded commutative A-algebra $\Omega^\bullet_{\text{comm}}(A) := \bigwedge^\bullet \Omega^1_{\text{comm}}(A)$. In particular, we have

$$\begin{aligned} \Omega^0_{\text{comm}}(A) &= A \\ \Omega^1_{\text{comm}}(A) &= \text{Kähler differentials} \\ \Omega^2_{\text{comm}}(A) &= \bigwedge\nolimits^2 \Omega^1_{\text{comm}}(A) \\ &\vdots \end{aligned}$$

A typical element of this algebra looks like $a_0\, da_1 \wedge da_2 \wedge \ldots \wedge da_k$.

Remark 5.2.7 (Convention)**.** The phrase **graded commutative** means that

$$x \wedge y = (-1)^{|x||y|} y \wedge x$$

for any homogeneous elements x and y.

Definition 5.2.8. The **de Rham differential** d_{DR} is the canonical extension of the universal differential to the de Rham algebra.

$$\begin{array}{ccccc} A & \xrightarrow{d} & \Omega^1_{\text{comm}}(A) & \hookrightarrow & \Omega^\bullet_{\text{comm}}(A) \\ \big\downarrow & & & \overset{d_{DR}}{\nearrow} & \\ \Omega^\bullet_{\text{comm}}(A) & & & & \end{array}$$

Explicitly,

$$d_{DR}(a_0 da_1 \wedge \ldots \wedge da_n) := da_0 \wedge da_1 \wedge \ldots da_n.$$

One can check directly that this satisfies the Leibniz rule.

Definition 5.2.9. The **algebraic de Rham cohomology** is defined by

$$H^\bullet_{DR}(\text{Spec}(A); k) := H^\bullet\left(\Omega^\bullet_{\text{comm}}(A)\right).$$

Remark 5.2.10. If Spec(A) is smooth, then this is a "good" cohomology theory.

There are two important theorems, the first of which is the Grothendieck Comparison theorem.

Let $X = \text{Spec}(A)$ be a *smooth* affine variety over $\mathbb{C}$, where A is a finitely generated commutative $\mathbb{C}$-algebra. Embed $X = \text{Spec}(A) \hookrightarrow \mathbb{C}^N$, which has the standard classical topology. X has two topologies: the algebraic (Zariski) topology and the analytic (Euclidean) one. Then Grothendieck's theorem says that two different cohomology theories, one coming from each of the very different topologies, are the same.

Theorem 5.2.11 (Grothendieck)**.** *If X is a smooth complex affine variety, then*

$$H^\bullet_{top}(X; \mathbb{C}) \cong H^\bullet_{DR}(X; \mathbb{C}) \tag{5.1}$$

Remark 5.2.12.

(a) The isomorphism (5.1) is actually the composition of two different isomorphisms. First, a quasi-isomorphism $\Omega^\bullet_{\text{comm}}(A) \hookrightarrow \Omega^\bullet_{\text{smooth}}(X)$, called the **interpretation map**. Second, the classical de Rham theorem:

$$H^\bullet\left(\Omega^\bullet_{\text{smooth}}(X)\right) \cong H^\bullet_{top}(X; \mathbb{C}).$$

(b) The interpretation map is *not* defined by applying $\Omega^\bullet_{\text{comm}}(-)$ to $A \hookrightarrow C^\infty(X)$. That is, $\Omega^\bullet_{\text{comm}}(C^\infty(X)) \neq \Omega^\bullet_{\text{smooth}}(X)$. Indeed, if f, g are algebraically independent in A, then by definition of $\Omega^1_{\text{comm}}(A)$, df and dg are linearly independent over A in $\Omega^1_{\text{comm}}(A)$. For example, if we take $A = C^\infty(X)$, $f = x$ and $g = e^x$, then these are algebraically independent by $d(e^x) = e^x dx$ and dx are linearly dependent over $A = C^\infty(X)$.

Example 5.2.13. In the special case when $Y \subseteq \mathbb{C}^n$ is an affine hypersurface, $X = \mathbb{C}^N \setminus Y$, and $A = \mathcal{O}(X) \cong \mathbb{C}[y_1, \ldots, y_N][f^{-1}]$. Then

$$\Omega^\bullet_{\text{comm}}(A) = \mathbb{C}[y_1, \ldots, y_N, f^{-1}] \otimes \bigwedge^\bullet (dy_1, \ldots, dy_N)$$

A typical element here is

$$w = \frac{P(y_1, \ldots, y_N)}{f(y_1, \ldots, y_N)^k} dy_1 \wedge \ldots \wedge dy_k$$

Question 5.2.14. Find sharp bounds on poles of algebraic differential forms representing cohomology classes in $H^*_{\text{sing}}(X; \mathbb{C})$. This is answered in [ABG73].

Question 5.2.15. What is the relation (if any) between the topological cohomology $H^\bullet_{\text{top}}(X; \mathbb{C})$ and the algebraic de-Rham cohomology $H^\bullet_{\text{DR}}(X; \mathbb{C})$ if X is singular? It turns out that the naïve answer is incorrect: $H^\bullet_{\text{DR}}(X; \mathbb{C})$ is the wrong theory to think about in the singular case.

We must modify the definition of the algebraic de Rham complex using the **Hodge filtration**. Recall that if I is an ideal in a commutative algebra B, then we may define the I-adic filtration $\{F_i\}_{i \geq 0}$ by

$$\begin{array}{ccccccc} F_0 & \supseteq & F_1 & \supseteq & F_2 & \supseteq & \ldots \\ \| & & \| & & \| & & \\ B & \supseteq & I & \supseteq & I^2 & \supseteq & \ldots \end{array}$$

The Hodge filtration is an extension of the I-adic filtration to B.

If $X = \text{Spec}(A)$ is not smooth, then choose a closed embedding $i \colon X \hookrightarrow Y$, where $Y = \text{Spec}(B)$ is smooth. Equivalently, choose $i^* \colon B \twoheadrightarrow A$, where B is regular (e.g. $B = k[x_1, \ldots, x_N]$). Take $I = \ker(i^*)$.

Definition 5.2.16. Consider $\Omega^\bullet_{\text{comm}}(B)$ and define the **Hodge filtration** on $\Omega^\bullet_{\text{comm}}(B)$ by

$$F^n \Omega^j_{\text{comm}}(B) := \begin{cases} \Omega^j_{\text{comm}}(B) & j \geq n \\ I^{n-j} \Omega^j_{\text{comm}}(B) & j < n \end{cases}$$

$$\begin{array}{ll} F^0 \Omega^\bullet_{\text{comm}}(B) : & \Omega^0_{\text{comm}}(B) \oplus \Omega^1_{\text{comm}}(B) \oplus \Omega^2_{\text{comm}}(B) \oplus \ldots = \Omega^\bullet_{\text{comm}}(B) \\ F^1 \Omega^\bullet_{\text{comm}}(B) : & I\Omega^0_{\text{comm}}(B) \oplus \Omega^1_{\text{comm}}(B) \oplus \Omega^2_{\text{comm}}(B) \oplus \ldots \\ F^2 \Omega^\bullet_{\text{comm}}(B) : & I^2\Omega^0_{\text{comm}}(B) \oplus I\Omega^1_{\text{comm}}(B) \oplus \Omega^2_{\text{comm}}(B) \oplus \ldots \\ & \vdots \end{array}$$

Definition 5.2.17. Assume now that $\dim(X) = n$ and define the **crystalline cohomology**

$$H^\bullet_{\text{cryst}}(A, n) := \left(\Omega^\bullet_{\text{comm}}(B) / F^{n+1} \Omega^\bullet_{\text{comm}}(B), d_{\text{DR}} \right).$$

Definition 5.2.18. The **stable crystalline cohomology** of A is

$$H^\bullet_{\text{cryst}}(A,\infty) := \left(\varprojlim_n \Omega^\bullet_{\text{comm}}(B)\big/F^{n+1}\Omega^\bullet_{\text{comm}}(B), d_{\text{DR}} \right).$$

Theorem 5.2.19 (Grothendieck,Hartshorne)**.**

(a) The crystalline cohomology $H^\bullet(A,n)$ is independent of the choice of B; it is an invariant of $X = \operatorname{Spec}(A)$.

(b) $H_{\text{top}}(X;\mathbb{C}) \cong H^\bullet_{\text{cryst}}(A,\infty)$

Remark 5.2.20. Theorem 5.2.19 appeared in a paper [Har75] of Hartshorne explaining ideas of Grothenieck, so the attribution is a bit unclear.

Assume that $X = \operatorname{Spec}(A)$ is smooth, and consider $i\colon X \hookrightarrow Y$, where Y is smooth, $i^*\colon B \twoheadrightarrow A$, where A, B are both regular.

Proposition 5.2.21. *For all $i \geq 0$, we have*

$$H^i_{\text{cryst}}(A,n) \cong \begin{cases} H^i(\Omega^\bullet_{\text{comm}}(A)) & i < n \\ \Omega^i_{\text{comm}}(A)\big/d(\Omega^{i-1}_{\text{comm}}(A)) & i = n \\ 0 & i > n. \end{cases}$$

Exercise 5.2.22. Prove Proposition 5.2.21 in the case when $A = k[x_1,\dots,x_n]$, $B = k[x_1,\dots,x_N]$ with $N > n$, $X = \mathbb{A}^n$, $Y = \mathbb{A}^N$, and i is the affine space embedding $i\colon \mathbb{A}^n \hookrightarrow \mathbb{A}^N$. The map $i^*\colon B \twoheadrightarrow A$ is

$$i^*(x_j) = \begin{cases} x_j & j \leq n \\ 0 & j > n \end{cases}$$

with $I = (x_{n+1},\dots,x_N) \subseteq B$.

In general, Proposition 5.2.21 follows from the Hochschild-Kostant-Rosenberg Theorem (Theorem 6.1.5, below).

Chapter 6

Hochschild Homology

6.1 Hochschild Homology

Definition 6.1.1. Let A be an associative algebra (not necessarily commutative) and M any A-bimodule. Define the **Hochschild complex** $C_\bullet(A, M)$ by

$$C_n(A, M) := M \otimes A^{\otimes n}$$

with differential $b_n \colon C_n(A, M) \to C_{n-1}(A, M)$ given by

$$\begin{aligned} b_n(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n \\ + (-1)^n a_n a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1} \end{aligned}$$

for all $a_0 \in M$, $a_1, \ldots, a_n \in A$.

Remark 6.1.2. Equivalently, we may define

$$b = \sum_{i=0}^{n} (-1)^i d_i$$

where $d_i \colon C_n \to C_{n-1}$ are defined by

$$\begin{aligned} d_i \colon a_0 \otimes \cdots \otimes a_n &\longmapsto a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n, \qquad i = 0, \ldots, n-1 \\ d_n \colon a_0 \otimes \cdots \otimes a_n &\longmapsto a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}, \qquad i = n. \end{aligned}$$

Lemma 6.1.3. *The map* b *in the Hochschild complex is a differential; that is,* $b^2 = 0$.

Definition 6.1.4. Let $C_\bullet(A) := C_\bullet(A, A)$, with $M = {}_AA_A$, and define the **Hochschild Homology**

$$HH_\bullet(A) := H_\bullet(C_\bullet(A)).$$

Theorem 6.1.5 (Hochschild-Kostant-Rosenberg)**.** *If A be a regular commutative algebra defined over a field k of characteristic zero. Then there is a natural isomorphism of graded algebras*

$$HH_\bullet(A) \cong \Omega^\bullet_{\mathrm{comm}}(A).$$

In particular, this implies that $HH_q(A) = 0$ for all $q > n$.

Remark 6.1.6. Notice that this is not the same as saying that Hochschild homology is the same as de Rham cohomology. This isomorphism is as graded algebras, not dg-algebras; differential forms have a differential d_{DR} that is absent in Hochschild homology. This means that there should be another (cohomological) differential on Hochschild homology going in the other direction; this is the **Connes differential** B.

We will postpone the proof of Theorem 6.1.5 until Section 6.5 in order to develop some of the ingredients.

6.2 Tor interpretation of Hochschild homology

This section defines the Tor interpretation of Hochschild homology.

Definition 6.2.1. Let A be any associative k-algebra. Define $A\langle\varepsilon\rangle_\bullet := A \sqcup_k k\langle\varepsilon\rangle$ with $|a| = 0$, $|\varepsilon| = 1$. We have

$$\begin{aligned} A\langle\varepsilon\rangle_0 &= A \\ A\langle\varepsilon\rangle_1 &= A\varepsilon A \\ A\langle\varepsilon\rangle_2 &= A\varepsilon A\varepsilon A \\ &\vdots \end{aligned}$$

Define a differential d on A by $da = 0$ for all $a \in A$ and $d\varepsilon = 1_A$. Then $d^2 = 0$ because $d^2 = 0$ on all generators, so $(A\langle\varepsilon\rangle, d)$ is a chain dg-algebra.

Question 6.2.2. What is $H_\bullet\,(A\langle\varepsilon\rangle, d)$? Well, the class of the identity 1_A induces the identity in homology. But we have $d\varepsilon = 1_A$, so $[1_A]$ is a boundary and therefore $[1_A] = [0_A]$. Hence, $H_\bullet(A\langle\varepsilon\rangle, d) = 0$.

Remark 6.2.3. We can identify $A\langle\varepsilon\rangle_n \cong A^{\otimes(n+1)}$ via

$$a_0\varepsilon a_1\varepsilon\cdots\varepsilon a_n \longmapsto a_0 \otimes a_1 \otimes \cdots \otimes a_n. \tag{6.1}$$

This defines a chain complex called the (acyclic) **bar complex**.

Typically, the element $a_0 \otimes a_1 \otimes \cdots \otimes a_n$ is written $(a_0 \mid a_1 \mid \ldots \mid a_n)$ in the context of the bar complex.

Remark 6.2.4. Under the identification (6.1), we have

$$\begin{aligned} d(a_0 \varepsilon a_1) &= da_0 \cdot \varepsilon a_1 + (-1)^{|a_0|} a_0\, d(\varepsilon a_1) \\ &= a_0 d(\varepsilon a_1) \\ &= a_0 (d\varepsilon) a_1 + (-1)^{|\varepsilon|} a_0 \varepsilon\, da_1 \\ &= a_0 a_1 \end{aligned}$$

and moreover, we can write this dg-algebra as a complex

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & A\langle\varepsilon\rangle_n & \xrightarrow{d} & A\langle\varepsilon\rangle_{n-1} & \longrightarrow & \cdots & & & & & & \\ & & \downarrow\cong & & \downarrow\cong & & & & & & & & \\ \cdots & \longrightarrow & A^{\otimes(n+1)} & \xrightarrow{b'} & A^{\otimes n} & \longrightarrow & \cdots & \longrightarrow & A^{\otimes 2} & \xrightarrow{\mu} & A & \longrightarrow & 0 \end{array}$$

where

$$b'(a_0 \otimes \cdots \otimes a_n) := \sum_{i=0}^{n-1} (-1)^i (a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n.$$

Notice that $(b')^2 = 0$ and moreover, $b' \colon A \otimes A \to A$ is just multiplication.

Definition 6.2.5. Let A be an associative k-algebra. Define the **bar complex** $B_\bullet A$ of A as the complex $B_n A := A^{\otimes (n+2)}$ with the differential $b'_n \colon B_n A \to B_{n-1} A$ as defined above.

Proposition 6.2.6. *The 2-sided bar construction gives a quasi-isomorphism* $\mu \colon B_\bullet A \twoheadrightarrow A$*, which is a free* A*-bimodule resolution of* ${}_A A_A$*.*

$$\begin{array}{c} \left[\cdots \longrightarrow A^{\otimes(n+1)} \xrightarrow{b'_{n+1}} A^{\otimes n} \xrightarrow{b'_n} \cdots \longrightarrow A^{\otimes 3} \longrightarrow A^{\otimes 2} \longrightarrow 0\right] \\ \downarrow \mu \\ \left[\cdots \longrightarrow 0 \longrightarrow A \longrightarrow 0\right] \end{array}$$

Definition 6.2.7. We call $A^e := A \otimes A^{op}$ the **enveloping algebra** of A.

Remark 6.2.8. There are isomorphisms of categories between left A^e-modules, right A^e-modules, and (A, A)-bimodules.

$$A^e\text{-}\mathbf{Mod} \cong \mathbf{Bimod}(A, A) \cong \mathbf{Mod}\text{-}A^e$$

Given an (A, A)-bimodule M, the action of A^e on the left/right is given as follows for $m \in M$:

$$m \cdot (b \otimes a^\circ) = a \cdot m \cdot b = (a \otimes b^\circ) \cdot m,$$

where the superscript $\circ$ denotes an element $a \in A$ considered as an element a° of A^{op}.

Theorem 6.2.9. *For any associative* k*-algebra* A*, we have a natural isomorphism*

$$HH_\bullet(A, M) \cong \mathrm{Tor}_\bullet^{A^e}(A, M)$$

where $A^e = A \otimes A^{op}$.

This is known as the Tor interpretation of Hochschild homology.

Proof. Recall the bar construction $B_\bullet A = (B_n A, b')$, where $B_n A = A^{\otimes(n+2)}$

$$\begin{array}{ccc} B_n A & \xrightarrow{\quad b' \quad} & B_{n-1}A \\ a_0 \otimes \cdots \otimes a_{n+1} & \longmapsto & \displaystyle\sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} \end{array}$$

We have from Proposition 6.2.6 a quasi-isomorphism between $B_\bullet A$ and the chain complex that is just A in dimension zero. This is the standard (or bar) resolution of A; it is a resolution of bimodules. Since $B_n A = A \otimes_k A^{\otimes n} \otimes_k A$ is the free (A, A)-bimodule on the vector space $A^{\otimes n}$, then $B_n A$ provides a free resolution of A as bimodules.

So now we may use this to compute Tor. We have an isomorphism of (A, A)-bimodules as follows.

$$\begin{array}{ccc} B_n A = A \otimes A^{\otimes n} \otimes A & \xrightarrow{\quad \cong \quad} & A^e \otimes_{A^e} A^{\otimes n} \\ a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1} & \longmapsto & (a_0 \otimes a_{n+1}^\circ) \otimes_{A^e} (a_1 \otimes \cdots \otimes a_n) \end{array}$$

This gives another isomorphism of (A, A)-bimodules:

$$\begin{array}{ccc} M \otimes_{A^e} B_n A & \xrightarrow{\quad \cong \quad} & M \otimes A^{\otimes n} =: C(A, M) \\ m \otimes_{A^e} (a_0 \otimes \cdots \otimes a_{n+1}) & \longmapsto & a_{n+1} m a_0 \otimes a_1 \otimes \cdots \otimes a_n \end{array}$$

(This isomorphism factors through $M \otimes_{A^e} A^e \otimes A^{\otimes n}$.)

Let's compute the image of the differential b' on $C_n(A, M)$.

$$\begin{array}{ccc} C_n(A, M) & \xrightarrow{\ \cong\ } & M \otimes_{A^e} B_n A \\ \Big\downarrow b' & & \Big\downarrow 1_M \otimes b' \\ C_{n-1}(A, M) & \xleftarrow{\ \cong\ } & M \otimes_{A^e} B_{n-1} A \end{array}$$

Let's chase an element of $C_n(A, M)$ around this diagram.

$$
\begin{aligned}
(m, a_1, \dots, a_n) &\longmapsto m \otimes_{A^e} (1 \otimes a_1 \otimes \dots \otimes a_n \otimes 1) \\
&\overset{1_M \otimes b'}{\longmapsto} m \otimes_{A^e} \Big((a_1 \otimes a_2 \otimes \dots \otimes a_n \otimes 1) \\
&\qquad + \sum_{i=1}^{n-1} (-1)^i (1 \otimes a_1 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n \otimes 1) \\
&\qquad + (-1)^n (1 \otimes a_1 \otimes \dots \otimes a_n) \Big) \\
&\longmapsto b(m \otimes a_1 \otimes a_2 \otimes \dots \otimes a_n)
\end{aligned}
$$

But notice that

$$
\begin{aligned}
b(m \otimes a_1 \otimes \dots \otimes a_n) = m a_1 \otimes a_2 \otimes \dots \otimes a_n \\
+ \sum_{i=1}^{n-1} (-1)^i (m \otimes a_1 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n) \\
+ (-1)^n (a_n m \otimes a_1 \otimes \dots \otimes a_{n-1}).
\end{aligned}
$$

This is exactly the Hochschild differential. Therefore, we have that

$$
(M \otimes_{A^e} B_\bullet A, 1_M \otimes b') \cong (C_\bullet(A, M), b)
$$

is an isomorphism of chain complexes. Hence,

$$
\mathrm{Tor}_\bullet^{A^e}(A, M) \cong H_\bullet(M \otimes_{A^e} B_\bullet A) \cong HH_\bullet(A, M) \qquad \square
$$

Remark 6.2.10. Notice that the definition of b' on $B_\bullet A$ makes sense even for non-unital algebras. This leads us to the following generalization of the definition of unital algebra.

Definition 6.2.11 (Wodzicki)**.** A non-unital algebra is called **homologically unital** (H-unital) if $B_\bullet A$ is acyclic.

6.3 Koszul Complexes

Let R be a commutative ring, and let E be an R-module. Let $x \colon E \to R$ be a linear form. Then consider

$$
\textstyle\bigwedge_R^\bullet E = \bigoplus_{p=0}^{\infty} \bigwedge_R^p E = R \oplus E \oplus \bigwedge_R^2 E \oplus \cdots
$$

Then x extends uniquely to a derivation of degree (-1).

$$\begin{array}{ccccc} E & \xrightarrow{x} & R & \hookrightarrow & \bigwedge_R^\bullet E \\ & \searrow & & \nearrow_{\delta_x} & \\ & & \bigwedge_R^\bullet E & & \end{array}$$

Explicitly, let $e_0 \wedge e_1 \wedge \ldots \wedge e_p \in \bigwedge_R^\bullet E$. Then

$$\delta_x(e_0 \wedge \ldots \wedge e_p) = \sum_{i=0}^{p} (-1)^i x(e_i) e_0 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_p$$

One can show by that $\delta_x^2 \equiv 0$ since $\delta(r) = 0$ and $\delta(e) = x(e)$ for all $e \in E$.

Definition 6.3.1. The **Koszul complex** of the pair (E, x) is the commutative (chain) dg-algebra $(\bigwedge^\bullet{}_R E, \delta_x)$.

Example 6.3.2. Consider a sequence $x = (x_1, \ldots, x_m)$ in R and take $E = R^{\oplus m}$ and define the functional

$$\begin{array}{ccc} E & \xrightarrow{x\cdot -} & R \\ \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix} & \longmapsto & \sum_{i=1}^{m} x_i r_i \end{array}$$

The corresponding Koszul complex will be written $K_\bullet(x) = K_\bullet(R, x) := (\bigwedge^\bullet{}_R(R^m), \delta_x)$. This is a commutative (chain) dg-algebra.

A different way to describe $K_\bullet(x)$ is as follows:

$$K_\bullet(x) = R[\xi_1, \ldots \xi_m \mid \delta\xi_i = x_i] = \frac{R\langle \xi_1, \ldots, \xi_n \rangle}{\langle\langle \xi_i \xi_j + \xi_j \xi_i \rangle\rangle}, \ \ \delta\xi_i = x_i$$

This isomorphism is given by $r \mapsto r$ and

$$\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \mapsto \xi_i.$$

Definition 6.3.3. A sequence $x = (x_1, \ldots, x_m)$ is called **regular** if x_{i+1} is not a zerodivisor in ${}^{R}/_{(x_1, x_2, \cdots, x_i)}$.

Definition 6.3.4. Let $R = k[X_1, \dots, X_N]$ be a polynomial algebra over a field k. If $(P_1, \dots, P_m)$ is a regular sequence, then $A = {}^{R}\!/\!_{(P_1,\dots,P_m)}$ is called a **complete intersection**.

Proposition 6.3.5 (Complete Intersection Criterion)**.** *A sequence $x = (x_1, \dots, x_m)$ is regular in R if and only if the corresponding Koszul complex $K_\bullet(x)$ is acyclic in positive degree. More precisely,*

$$H_n(K_\bullet(x)) = \begin{cases} 0 & n > 0 \\ {}^{R}\!/\!_{(x_1, x_2, \cdots, x_n)} & n = 0. \end{cases}$$

Proof. We only prove the forward direction. Assume that $x = (x_1, \dots, x_m)$ is regular in R. Proof by induction on m.

Take $m = 1$. Then the corresponding complex is

$$K_\bullet(x_1): \quad 0 \longrightarrow R \xrightarrow{x_1 \cdot} R \longrightarrow 0,$$

with the rightmost R in degree zero. We know that x_1 is regular, which implies that multiplication by x_1 is injective. Hence, $H_1(K_\bullet(x_1)) = 0$, and

$$H_0(K_\bullet(x_1)) = \operatorname{coker}(x_1) = {}^{R}\!/\!_{(x_1)}.$$

Now suppose that the claim is true for $m-1$, that is, for any sequence $(x_1, \dots, x_{m-1})$. Let's add x_m to this sequence to get $x = (x_1, \dots, x_m)$. We have a short exact sequence of complexes (horizontally), where the top row is degree 1 and the bottom row is degree zero.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & R & \longrightarrow & R & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle x_m \cdot} & & \downarrow & & \\
0 & \longrightarrow & R & \longrightarrow & R & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Rewriting, the short exact sequence is the following:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_0 & \longrightarrow & K_\bullet(x_m) & \longrightarrow & K_1 & \longrightarrow & 0 \\
 & & \shortparallel & & & & \shortparallel & & \\
 & & R & & & & R[1] & &
\end{array} \tag{6.2}$$

where R is the complex consisting of R in dimension zero. (Recall our convention Definition 5.1.3 where $K_\bullet[n]_i = K_{i-n}$ and $K^*[n]^i = K^{i+n}$.)

Let $L = K_\bullet(x_1, \ldots, x_{m-1})$ and tensor Eq. (6.2) with L.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & L \otimes_R K_\bullet(x_m) & \longrightarrow & L[1] & \longrightarrow & 0 \\ & & & & \wr\| & & & & \\ & & & & K_\bullet(x_1, \ldots, x_m) & & & & \end{array}$$

Now take the associated long exact sequence in homology:

$$\cdots \longrightarrow H_{n+1}(L) \longrightarrow H_{n+1}(K_\bullet(x)) \longrightarrow H_{n+1}(L[1]) \longrightarrow$$
$$\hookrightarrow H_n(L) \longrightarrow H_n(K_\bullet(x)) \longrightarrow H_n(L[1]) \longrightarrow$$
$$\hookrightarrow H_{n-1}(L) \longrightarrow H_{n-1}(K_\bullet(x)) \longrightarrow H_{n-1}(L[1]) \longrightarrow \cdots$$

Notice that $H_j(L[1]) = H_{j-1}(L)$.

This implies in particular that the following sequence is short exact:

$$0 \to \operatorname{coker}\left(H_n(L) \xrightarrow{\partial} H_n(L)\right) \to H_n(K_\bullet(x)) \to \ker\left(H_{n-1}(L) \xrightarrow{\partial} H_{n-1}(L)\right) \to 0$$

It is easy to check (exercise!) that the connecting homomorphism ∂ is induced on homology by $x_m \cdot : L \to L$

If $n > 1$, then $H_n(L) = 0$, $H_{n-1}(L) = 0$, so we have $H_n(K_\bullet(x)) = 0$.

If $n = 1$, then by induction we get

$$H_1(K_\bullet(x)) \cong \ker\left(H_0(L) \xrightarrow{\partial} H_0(L)\right) = \ker\left({}^{R}/_{(x_1, \ldots, x_{m-1})} \xrightarrow{x_m} {}^{R}/_{(x_1, \ldots, x_{m-1})}\right)$$

This is zero because x_m is regular.

If $n = 0$, then we have

$$H_0(K_\bullet(x)) \cong \operatorname{coker}\left(H_n(L) \xrightarrow{\partial} H_n(L)\right) \cong {}^{R}/_{(x_1, \ldots, x_m)}. \qquad \square$$

Corollary 6.3.6. *Let R be a commutative ring and $I \subset R$ an ideal generated by a regular sequence, say $I = (x_1, \ldots, x_m)$. Then there is a natural isomorphism of R-modules*

$$\operatorname{Tor}^R_\bullet\left({}^{R}/_{I}, {}^{R}/_{I}\right) \cong \textstyle\bigwedge^\bullet_{R/I}\left({}^{I}/_{I^2}\right) \tag{6.3}$$

Proof. Note that by Proposition 6.3.5, $K_\bullet(R, x)$ gives a free R-module resolution to ${}^{R}/_{I}$. Hence, we have a quasi-isomorphism

$$K_\bullet(R, x) \xrightarrow{\sim} {}^{R}/_{I}.$$

Therefore,

$$\begin{aligned} K_n(x) \otimes_R {}^R/_I &\cong {\textstyle\bigwedge}^n_R (R^m) \otimes_R {}^R/_I \\ &\cong {\textstyle\bigwedge}^n_{R/I} \left(\left({}^R/_I \right)^{\oplus m} \right) \\ &\cong {\textstyle\bigwedge}^n_{R/I} \left({}^{R^{\oplus m}}/_{I^{\oplus m}} \right) \\ &\cong {\textstyle\bigwedge}^n_{R/I} \left({}^I/_{I^2} \right) \end{aligned}$$

The last isomorphism comes from the following diagram

$$\begin{array}{ccccccc} I^{\oplus m} & \hookrightarrow & R^{\oplus m} & \xrightarrow{x\cdot} & I & \longrightarrow & {}^I/_{I^2} \\ & & & \searrow & & \nearrow_{\cong} & \\ & & & & {}^{R^{\oplus m}}/_{I^{\oplus m}} & & \end{array}$$

Note that $x_i \in I$ implies that $\delta_x \otimes \mathrm{id}_{R/I} = 0$ on $\bigwedge^\bullet_{R/I}(I/I^2)$. Therefore,

$$\mathrm{Tor}^R_\bullet \left({}^R/_I, {}^R/_I \right) \cong H_\bullet \left(K_\bullet(x) \otimes_R {}^R/_I \right) \cong {\textstyle\bigwedge}^\bullet_{R/I} \left({}^I/_{I^2} \right) \qquad \square$$

Remark 6.3.7. In fact, more is true: $\mathrm{Tor}^R_\bullet \left({}^R/_I, {}^R/_I \right)$ has a natural structure of a graded commutative algebra, and the canonical isomorphism

$${}^I/_{I^2} \cong \mathrm{Tor}^R_1 \left({}^R/_I, {}^R/_I \right)$$

extends to an isomorphism of graded algebras

$${\textstyle\bigwedge}^\bullet_{R/I} \left({}^I/_{I^2} \right) \cong \mathrm{Tor}^R_\bullet \left({}^R/_I, {}^R/_I \right)$$

6.4 (Formal) Smoothness

6.4.1 Grothendieck's notion of smoothness

Let $\mathbf{Alg}_k$ be the category of all associative k-algebras for a commutative ring k. Let $\mathbf{C} \subset \mathbf{Alg}_k$ be a full subcategory, e.g. $\mathbf{C} = \mathbf{CommAlg}_k$ or $\mathbf{C} = \mathbf{Alg}_k$.

Question 6.4.1. How do we define smooth objects in $\mathbf{C}$?

Let's enlarge $\mathbf{C}$ by embedding it into the category of functors $\widehat{\mathbf{C}} = \mathbf{Fun}(\mathbf{C}, \mathbf{Set})$ via the **Yoneda embedding:**

$$\begin{aligned} \mathbf{C} &\xrightarrow{\ h\ } \widehat{\mathbf{C}} \\ A &\longmapsto h^A = \mathrm{Hom}(A, -) \\ \left(A \xrightarrow{f} B \right) &\longmapsto f^* = (- \circ f) \end{aligned}$$

Lemma 6.4.2 (Yoneda)**.** h *is full and faithful.*

Definition 6.4.3 (Grothendieck)**.** (a) A functor $F \in \mathrm{Ob}(\widehat{\mathbf{C}})$ is called **smooth** if for all pairs (C, I) with $C \in \mathrm{Ob}\,\mathbf{C}$ and $I \lhd C$ a 2-sided nilpotent ideal (i.e. $I^N = 0$ for some $N > 1$),

$$F(p) \colon F(C) \twoheadrightarrow F\left({}^{C}/_{I}\right)$$

is a surjective map of sets, where $p \colon C \to {}^{C}/_{I}$ is the projection.

(b) $A \in \mathrm{Ob}(\mathbf{C})$ is called **formally smooth** if h^A is smooth.

Remark 6.4.4. Definition 6.4.3(b) is the same as the following: for all pairs (C, I) with $C \in \mathrm{Ob}\,\mathbf{C}$ and $I \lhd C$ a two-sided nilpotent ideal, and for any $\phi \colon A \to {}^{C}/_{I}$, there exists $\widetilde{\phi}$ such that the following diagram commutes:

$$\begin{array}{ccc} & & C \\ & \overset{\widetilde{\phi}}{\nearrow} & \downarrow \\ A & \xrightarrow[\phi]{} & {}^{C}/_{I} \end{array}$$

Remark 6.4.5. The property of $A \in \mathrm{Ob}(\mathbf{C})$ being smooth depends on the category $\mathbf{C}$ in which A lives. If (say) $\mathbf{C} \subsetneq \mathbf{C}'$, and $A \in \mathrm{Ob}(\mathbf{C})$, it may happen that A is smooth in $\mathbf{C}$ but not in $\mathbf{C}'$.

Remark 6.4.6. Free algebras in $\mathbf{C}$ are always smooth: assume that $U \colon \mathbf{C} \to \mathbf{Set}$ is the forgetful functor. Then this has a left adjoint $k\langle - \rangle \colon \mathbf{Set} \to \mathbf{C}$, $X \mapsto k\langle X \rangle$.

For example, if $C = \mathbf{Alg}_k$, then $k\langle X \rangle$ is the free k-algebra based on X. If $\mathbf{C} = \mathbf{CommAlg}_k$, then $k\langle X \rangle = k[X]$. Note that

$$\mathrm{Hom}_{\mathbf{C}}(k\langle X \rangle, C) \longrightarrow \mathrm{Hom}_{\mathbf{C}}(k\langle X \rangle, {}^{C}/_{I})$$

corresponds under the adjunction $k\langle - \rangle \dashv U$ to

$$\mathrm{Hom}_{\mathbf{Set}}(X, C) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Hom}_{\mathbf{Set}}(X, {}^{C}/_{I})$$

for any ideal $I \lhd C$, because sets are free (they have no relations).

The moral is that smooth algebras behave like free algebras with respect to nilpotent extensions.

Example 6.4.7. Let k be a field, and consider $\mathbf{C} = \mathbf{Alg}_k$.

(a) Any free algebra $k\langle x_1, \ldots, x_n \rangle$ is smooth.

(b) If Q is any quiver, then kQ is smooth.

(c) In fact, all smooth algebras are hereditary, but not conversely. The **Weyl algebra**

$$A = A_1(k) = {}^{k\langle x, y\rangle}/_{(xy - yx = 1)}$$

is hereditary but not smooth.

(d) If Γ is discrete, then $k\Gamma$ is smooth if and only if Γ is virtually free (contains a free subgroup of finite index).

(e) If A and B are smooth, then their coproduct $A \sqcup_k B$ is smooth as well.

(f) If A is smooth, and M is a projective A-bimodule, then the tensor algebra $T_A M$ is smooth.

(g) Let A be a commutative algebra viewed as an object in $\mathbf{Alg}_k$. Then A is smooth in $\mathbf{Alg}_k$ if $X = \operatorname{Spec}(A)$ is a smooth affine curve (more precisely, A is smooth in $\mathbf{CommAlg}_k$ and $\operatorname{gldim}(A) \leq 1$).

(h) $A = k[x, y]$ is *not* smooth in $\mathbf{Alg}_k$.

Proposition 6.4.8. *If A is smooth in $\mathbf{Alg}_k$, then for any vector space V, $k\left[\operatorname{Rep}_V(A)\right]$ is smooth in $\mathbf{CommAlg}_k$.*

6.4.2 Quillen's notion of smoothness

Let k be a commutative ring and let A be a commutative k-algebra with multiplication $\mu\colon A \otimes_k A \to A$. Let $I = \ker(\mu)$.

Definition 6.4.9 (Quillen)**.** A is called **smooth** over k if

(a) A is **flat** as a k-module (i.e. $- \otimes_k A$ is an exact functor).

(b) for any maximal ideal $\mathfrak{m}$ of A, $\mu^{-1}(\mathfrak{m})$ is a maximal ideal of $A \otimes_k A$, and $I_{\mu^{-1}(\mathfrak{m})}$ is generated by a regular sequence in $(A \otimes_k A)_{\mu^{-1}(\mathfrak{m})}$.

Remark 6.4.10. Sometimes, one assumes that A is **pseudo-flat** (or **stably flat**) instead of flat. Flatness can be formulated as $\operatorname{Tor}_n^k(A, M) = 0$ for all $n > 0$, and **stably flat** is $\operatorname{Tor}_n^k(A, A) = 0$ for all $n > 0$.

Proposition 6.4.11. *Let k be a Noetherian ring and let A be a k-algebra which is (stably) flat over k and (essentially) of finite type over k. Then the following are equivalent:*

(a) A is smooth in the sense of Definition 6.4.9.

(b) $I = \ker(\mu)$ is locally a complete intersection.

(c) ***(Jacobian Criterion)*** *If* $f\colon k[X_1,\dots,X_n] \twoheadrightarrow A$ *is a finite presentation of* A *then* $(\ker f)_{\mathfrak{q}}$ *is generated by* $P_1,\dots,P_N \in k[X_1,\dots,X_n]$ *for all primes* $\mathfrak{p} \in \operatorname{Spec}(A)$ *such that* $\mathfrak{q} = f^{-1}(\mathfrak{p})$*, and moreover* $dP_1,\dots,dP_N$ *are linearly independent in* $\Omega^1_{\mathrm{comm}}(k[X_1,\dots,X_n]) \otimes_{k[X_1,\dots,X_n]} A_{\mathfrak{p}}$.

(d) *A is formally smooth in* **CommAlg**$_k$ *in the sense of Definition 6.4.3.*

Remark 6.4.12. The last condition gives a categorical characterization of smoothness: it says that in a category **C** of algebras, the smooth objects are those which behave like free objects with respect to nilpotent extensions.

Example 6.4.13. If k is an algebraically closed field, then (c) says that the algebra of regular functions $A = \mathcal{O}(X)$ on a nonsingular affine variety over k is formally smooth. Therefore $k[x,y]$ is smooth in **CommAlg**$_k$.

6.5 Proof of Hochschild-Kostant-Rosenberg

The strategy of this proof is to construct a map $\varepsilon_n\colon \Omega^n_{\mathrm{comm}}(A) \to HH_n(A)$, and another map $\pi_n\colon HH_n(A) \to \Omega^n_{\mathrm{comm}}(A)$ such that $\pi_n \circ \varepsilon_n$ is a scalar multiple of the identity on $\Omega^n_{\mathrm{comm}}(A)$. This exhibits $\Omega^n_{\mathrm{comm}}(A)$ as a direct summand of $HH_n(A)$.

We then apply Theorem 6.2.9 and Corollary 6.3.6 to show that this is an isomorphism.

6.5.1 The antisymmetrizer map

Let S_n be the symmetric group on n letters. Define the action of S_n on $C_n(A)$ by

$$\sigma \cdot (a_0 \otimes \cdots \otimes a_n) = a_0 \otimes a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(n)}$$

This gives $C_n(A)$ the structure of a $k[S_n]$-module. Define the **antisymmetrizer**

$$\varepsilon_n := \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\sigma \in k[S_n]$$

Notice that ε_n is not quite a Young symmetrizer; it's off by a factor of $n!$. We have that $\varepsilon_n^2 = n!\varepsilon_n$ in $k[S_n]$.

Next, define the map

$$\begin{aligned} A \otimes \textstyle\bigwedge^n_k(A) &\xrightarrow{\quad \varepsilon_n \quad} C_n(A) \\ a_0 \otimes a_1 \wedge a_2 \wedge \cdots \wedge a_n &\longmapsto \varepsilon_n \cdot (a_0 \otimes a_1 \otimes \cdots \otimes a_n) \end{aligned}$$

or equivalently,

$$a_0 \otimes a_1 \wedge a_2 \wedge \cdots \wedge a_n \longmapsto \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) a_0 \otimes a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(n)}.$$

This map is well-defined because ε_n is the antisymmetrizer and the exterior power is antisymmetric.

Recall from Definition 5.1.19 the **Chevalley-Eilenberg differential** δ of Lie algebra cohomology. Explicitly, thinking of A as a Lie algebra with commutator $[a, b] = ab - ba$, the Chevalley-Eilenberg differential $\delta_n \colon A \otimes \bigwedge_k^n (A) \to A \otimes \bigwedge_k^{n-1}(A)$ is defined by

$$\begin{aligned} \delta_n(a_0 \otimes a_1 \wedge \cdots \wedge a_n) := &\sum_{i=1}^{n} (-1)^i [a_0, a_i](a_0 \otimes (a_1 \wedge \ldots \wedge \widehat{a_i} \wedge \ldots \wedge a_n) \\ &+ \sum_{1 \le i < j \le n} (-1)^{i+j-1} a_0 \otimes [a_i, a_j] \wedge a_1 \wedge \ldots \wedge \widehat{a_i} \wedge \ldots \wedge \widehat{a_j} \wedge \ldots \wedge a_n \end{aligned}$$

Lemma 6.5.1. *For any associative k-algebra, the following diagram commutes:*

$$\begin{array}{ccc} A \otimes \bigwedge_k^n(A) & \xrightarrow{\varepsilon_n} & C_n(A) \\ \downarrow{\scriptstyle \delta_n} & & \downarrow{\scriptstyle b} \\ A \otimes \bigwedge_k^{n-1}(A) & \xrightarrow{\varepsilon_{n-1}} & C_{n-1}(A) \end{array}$$

where δ is the Chevalley-Eilenberg differential for A considered as a Lie algebra.

Proof. By induction on n. (Doable by explicit calculation). □

Corollary 6.5.2. *If A is commutative, then $\delta_n \equiv 0$. Therefore, $b_n \circ \varepsilon_n = 0$ for all n, so $\mathrm{im}(\varepsilon_n) \subseteq \ker(b_n)$. Therefore, ε_n induces a map*

$$\overline{\varepsilon}_n \colon A \otimes \bigwedge_k^n A \twoheadrightarrow \ker(b_n) \to \varepsilon_n Z_n(C_\bullet(A)) \twoheadrightarrow HH_n(A).$$

This is nothing more than the map induced by ε_n on homology.

Lemma 6.5.3. *If A is commutative, then $\overline{\varepsilon}_n$ factors as*

$$\begin{array}{ccc} A \otimes \bigwedge_k^n A & \xrightarrow{\quad \overline{\varepsilon}_n \quad} & HH_n(A) \\ & {\scriptstyle p_n} \searrow \quad \nearrow {\scriptstyle \varepsilon_n} & \\ & \Omega^n_{\mathrm{comm}}(A) & \end{array} \tag{6.4}$$

Where $p_n \colon a_0 \otimes (a_1 \wedge \ldots \wedge a_n) \mapsto a_0 da_1 \wedge \ldots \wedge da_n$.

Proof Sketch. Recall that $\Omega^n_{\text{comm}}(A)$ is the n-th exterior algebra of $\Omega^1_{\text{comm}}(A)$, that is $\Omega^n_{\text{comm}}(A) := \bigwedge^n_A \left(\Omega^1_{\text{comm}}(A)\right)$. In $\Omega^1_{\text{comm}}(A)$, we have relations

$$-d(a_1 a_2) + a_1\, da_2 + a_2\, da_1 = 0$$

for $a_1, a_2 \in A$. These imply a relation in $\Omega^n_{\text{comm}}(A)$ of the form

$$\begin{aligned} 0 = &- a_0 d(a_1 a_2) \wedge da_3 \wedge \ldots \wedge da_n \\ &+ a_0 a_1\, da_2 \wedge da_3 \wedge \ldots \wedge da_n + a_0 a_2\, da_1 \wedge da_3 \wedge \ldots \wedge da_n \end{aligned}$$

The right hand side of this relation is in the image of p_n, as $p_n(x)$ where

$$\begin{aligned} x = &-a_0 \otimes ((a_1 a_2) \wedge a_3 \wedge \ldots a_n) \\ &+ a_0 a_1 \otimes (a_2 \wedge a_3 \wedge \ldots \wedge a_n) + a_0 a_2 \otimes (a_1 \wedge a_3 \wedge \ldots \wedge a_n) \end{aligned}$$

We only checked this for indices $i = 1, j = 2$, but in principle this works for any pair of indices i and j.

It suffices to check that $\overline{\varepsilon}_n(x) \subseteq \operatorname{im}(b_{n+1})$. Indeed

$$\overline{\varepsilon}_n(x) = -b_{n+1} \left(\sum_{\substack{\sigma \in S_n \\ \sigma(1) < \sigma(2)}} \operatorname{sgn}(\sigma)\sigma \cdot (a_0 \otimes a_1 \otimes \cdots \otimes a_n) \right)$$

The proof of this is then by induction. □

Definition 6.5.4. The map $\varepsilon_n \colon \Omega^n_{\text{comm}}(A) \to HH_n(A)$ from (6.4) is called the **HKR map**.

For the second step, define the projection map

$$\begin{aligned} \pi_n \colon C_n(A) &\longrightarrow \Omega^n_{\text{comm}}(A) \\ a_0 \otimes \cdots \otimes a_n &\longmapsto a_0\, da_1 \wedge \cdots \wedge da_n \end{aligned}$$

Lemma 6.5.5. $\pi_n \circ b_{n+1} = 0$

Proof. By direct calculation. Easy. □

Therefore, we get a well-defined map $\pi_n \colon HH_n(A) \to \Omega^n_{\text{comm}}(A)$. Indeed, $\pi_n|_{\operatorname{im}(b_{n+1})} = 0$, so π_n induces a map

$$HH_n(A) = {}^{\ker(b_n)}\!/_{\operatorname{im}(b_{n+1})} \longrightarrow \Omega^n_{\text{comm}}(A).$$

Lemma 6.5.6. *The composition*

$$\Omega^n_{\text{comm}}(A) \xrightarrow{\varepsilon_n} HH_n(A) \xrightarrow{\pi_n} \Omega^n_{\text{comm}}(A)$$

is equal to $n!\, \mathrm{id}_{\Omega^n_{\text{comm}}(A)}$.

Proof.

$$a_0\, da_1 \wedge \ldots \wedge da_n \xmapsto{\varepsilon_n} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)(a_0 \otimes a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(n)})$$

$$\xmapsto{\pi_n} \sum_{\sigma \in S_n} a_0\,(da_1 \wedge \ldots \wedge da_n) = n! a_0\, da_1 \wedge \ldots \wedge da_n$$

□

To summarize, if A is any commutative algebra A, then $\pi_n \circ \varepsilon_n = n! \mathrm{id}_{\Omega^n_{\mathrm{comm}}(A)}$. Hence, $\Omega^n_{\mathrm{comm}}(A)$ is a direct summand of $HH_n(A)$.

6.5.2 The case $n = 1$

Recall that we had a map $\varepsilon_n \colon \Omega^n_{\mathrm{comm}}(A) \to HH_n(A)$ given by

$$a_0\, da_1 \wedge \ldots \wedge da_n \mapsto \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)(a_0, a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(n)}).$$

We also saw that $\pi_n \circ \varepsilon_n = n!\, \mathrm{id}_{\Omega^n_{\mathrm{comm}}(A)}$, so for any commutative algebra A, this map is split injective.

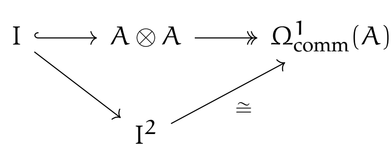
Let's look at the case $n = 1$.

Lemma 6.5.7. *For any commutative unital algebra A, the map ε_1 is an isomorphism of A-modules $HH_1(A) \cong \Omega^1_{\mathrm{comm}}(A)$.*

Proof Sketch. Recall that $\Omega^1_{\mathrm{comm}}(A) \cong {}^{A \otimes A}\!/_{(a \otimes bc - ab \otimes c - ac \otimes b)}$. Consider the multiplication map $\mu \colon A \otimes A \to A$ and let $I = \ker(\mu)$. This is the two sided ideal generated by $\langle\langle 1 \otimes a - a \otimes 1 \rangle\rangle \subset A \otimes A$. We have $A \cong A \otimes A / I$, so we can identify

$$\Omega^1_{\mathrm{comm}}(A) \cong {}^{I}\!/_{I^2}.$$

This isomorphism comes from the diagram



We will prove that $HH_1(A) \cong I/I^2$ homologically. Consider the short exact sequence of (A, A)-bimodules (note that $A^{op} = A$ if A is commutative, so $A^e = A \otimes_k A$)

$$0 \longrightarrow I \longrightarrow A^e \longrightarrow A \longrightarrow 0 \qquad (6.5)$$

and apply $A \otimes_{A^e} (-)$ to get the long exact sequence

$$0 \longrightarrow \operatorname{Tor}_1^{A^e}(A, A) \longrightarrow A \otimes_{A^e} I \longrightarrow A \longrightarrow A \otimes_{A^e} A \longrightarrow 0$$

Since $\operatorname{Tor}_1^{A^e}(A, A) = HH_1(A)$ and $A \otimes_{A^e} A = HH_0(A) = {}^{A}/_{[A,A]} = A$, we get the isomorphism

$$HH_1(A) \cong A \otimes_{A^e} I.$$

On the other hand, tensor Eq. (6.5) with $- \otimes_{A^e} I$ to get the long exact sequence

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & I \otimes_{A^e} I & \longrightarrow & I & \longrightarrow & A \otimes_{A^e} I & \longrightarrow & 0 \\ & & x \otimes y & \longmapsto & xy & & & & \end{array}$$

This shows that $A \otimes_{A^e} I \cong I/I^2$.

Putting it all together, we get an isomorphism

$$HH_1(A) \cong A \otimes_{A^e} I \cong {}^{I}/_{I^2} \cong \Omega^1_{\text{comm}}(A) \qquad \square$$

Remark 6.5.8. $\mu^*\colon \operatorname{Spec}(A) \to \operatorname{Spec}(A \otimes A) \cong \operatorname{Spec}(A) \times \operatorname{Spec}(A)$ is the diagonal embedding.

6.5.3 The general case

Lemma 6.5.9 (Local to global principle)**.** *If $f\colon M \to N$ is a homomorphism of A-modules over a commutative algebra A, then f is an isomorphism if and only if $f_{\mathfrak{m}}\colon M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is an isomorphism for any maximal ideal $\mathfrak{m}$ of A.*

Proof. In the forward direction, this is clear because localization is functorial.

Conversely, assume that $f_{\mathfrak{m}}$ is an isomorphism for all maximal ideals $\mathfrak{m}$ of A. If f is (say) not injective, then $\ker(f) \neq 0$, so there is some nonzero $x \in \ker(f)$. The annihilator of this element is nonzero. $0 \neq \operatorname{Ann}(x) \subset A$. Then we have that $\operatorname{Ann}(x) \subseteq \mathfrak{m}$, which implies that $\ker(f_{\mathfrak{m}})$ is nonzero. This is a contradiction because $f_{\mathfrak{m}}$ is an isomorphism. $\square$

Here k is a field. Assume that A is a smooth commutative k-algebra. Let $\mathfrak{m} \subset A$ be a maximal ideal, and let $\mu\colon A \otimes A \to A$ denote multiplication. For $R = A \otimes A$ and $I = \ker(\mu)$, we have:

$$HH_n(A) \cong \operatorname{Tor}_n^R\left({}^{R}/_{I}, {}^{R}/_{I}\right)$$
$$\Omega^n_{\text{comm}}(A) \cong \bigwedge\nolimits^n_{R/I}\left({}^{I}/_{I^2}\right)$$

Now let $R = (A \otimes A)_{\mu^{-1}(\mathfrak{m})}$ and $I = (\ker \mu)_{\mu^{-1}(\mathfrak{m})}$ so that $R/I \cong A_\mathfrak{m}$. Then by Corollary 6.3.6 and Theorem 6.2.9, we have isomorphisms for all maximal ideals $\mathfrak{m}$ of A:

$$\Omega^n_{\text{comm}}(A)_\mathfrak{m} \cong \bigwedge\nolimits^n_{R/I}\left(I/I^2\right) \cong \mathrm{Tor}^R_n\left(R/I, R/I\right) \cong \mathrm{Tor}^{A \otimes A}(A, A)_\mathfrak{m} = HH_n(A)_\mathfrak{m}$$

Then by Lemma 6.5.9, we conclude

$$\Omega^n_{\text{comm}}(A) \cong HH_n(A).$$

6.6 Noncommutative Differential Forms

We will follow the approach of Cuntz-Quillen in this section.

Definition 6.6.1. Let A be an associative unital k-algebra. Define

$$\Omega^n A := A \otimes \overline{A}^{\otimes n},$$

where $\overline{A} = A/k \cdot 1_A$. We write elements of $\Omega^n(A)$ as

$$(a_0, a_1, \ldots, a_n) := a_0 \otimes a_1 \otimes \cdots \otimes a_n$$

with $a_0 \in A$ and $a_1, \ldots, a_n \in \overline{A}$.

Definition 6.6.2. Put

$$\Omega^\bullet(A) := \bigoplus_{n \geq 0} \Omega^n(A)$$

and define $d\colon \Omega^\bullet A \to \Omega^{\bullet+1}(A)$ by

$$d(a_0, a_1, \ldots, a_n) = (1, a_0, a_1, \ldots, a_n) \tag{6.6}$$

Note that $d^2 = 0$. Also define

$$(a_0, \ldots, a_n) \cdot (a_{n+1}, a_{n+2}, \ldots, a_k) = \sum_{i=1}^{n} (-1)^{n-i} (a_0, \ldots, a_i a_{i+1}, \ldots, a_k) \tag{6.7}$$

Theorem 6.6.3.

(a) Formulas (6.6) and (6.7) define a DG (cochain) algebra structure on $\Omega^\bullet(A)$ which is the unique one satisfying

$$a_0 \cdot da_1 \cdot \cdots \cdot da_n = (a_0, a_1, \ldots, a_n)$$

(b) *(Universal Property) Given any DG-algebra $\Gamma^\bullet = \bigoplus_{n\geq 0}\Gamma^n$ and any algebra homomorphism $u\colon A \to \Gamma^0$, there is a unique map of algebras $u_*\colon \Omega^\bullet(A) \to \Gamma$ such that the following commutes*

$$\begin{array}{ccc} \Omega^\bullet(A) & \xrightarrow{\quad u_* \quad} & \Gamma \\ & \nwarrow \quad \nearrow_{u} & \\ & A & \end{array}$$

Proof of Theorem 6.6.3(a). First, we will prove uniqueness. Suppose we have any DG algebra (B, d) that contains A as a subalgebra in degree 0, with $(B^\bullet \supseteq A)$. Then the following formulas in B hold:

$$d(a_0\, da_1\, da_2 \cdots da_n) = da_0\, da_1 \,\cdots da_n \tag{6.8}$$

$$(a_0\, da_1 \cdots da_n)\cdot(a_{n+1}\, da_{n+2}\cdots da_k) = \\ (-1)^n a_0a_1\, da_2\cdots da_k + \sum_{i=1}^{n}(-1)^{n-i}a_0\, da_1\cdots d(a_ia_{i+1})\cdots da_k \tag{6.9}$$

(6.8) is immediate from the Leibniz rule and $d^2 = 0$, and (6.9) follows by induction on n.

So assuming that $\Omega^\bullet A$ is a DG-algebra, by applying the above information, we see that (6.6) and (6.7) define a DG structure satisfying

$$a_0 \cdot da_1 \cdots da_n = (a_0, \dots, a_n).$$

For existence, we deduce (6.6) and (6.7) in the following way. Apply Construction 6.6.4 (below) to $(\Omega^\bullet A, d)$ and define a DG-algebra

$$\mathcal{E}^\bullet = (\underline{\mathrm{End}}^\bullet(\Omega^\bullet A), d_{\mathrm{Hom}}).$$

This is the collection of all k-linear endomorphisms of $\Omega^\bullet A$ of all degrees. Define the left-multiplication operator $\ell\colon A \to \mathcal{E}^\bullet$ by $a \mapsto \ell_a$, where ℓ_a is the linear map

$$\ell_a\colon (a_0, a_1, \dots, a_n) \longmapsto (aa_0, a_1, \dots, a_n).$$

We may then extend ℓ to a map $\ell_*\colon \Omega^\bullet A \to \mathcal{E}^\bullet$ by

$$\ell_*(a_0, a_1, \dots, a_n) \longmapsto \ell_a \cdot d_{\mathrm{Hom}}(\ell_{a_1})\cdots d_{\mathrm{Hom}}(\ell_{a_n})$$

Notice that $\mathrm{im}(\ell_*) \subseteq \mathcal{E}^\bullet$ is the DG subalgebra generated as a DG-algebra by $\ell(A) \subseteq \mathcal{E}^\bullet$. (Strictly speaking, we should check (6.8) and (6.9) for $\ell(A) \subseteq \mathcal{E}^\bullet$.)

Next, we define

$$\begin{aligned} \mathrm{ev}\colon \mathcal{E}^\bullet &\longrightarrow \Omega^\bullet A \\ f &\longmapsto f(1) \end{aligned}$$

Notice that for all $i = 0, \dots, n$,

$$\begin{aligned} [d, \ell_{a_i}](1, a_{i+1}, \dots, a_n) &= (d\ell_{a_i} - \ell_{a_i} d)(1, a_{i+1}, \dots, a_k) \\ &= d(a_i, a_{i+1}, \dots, a_n) - 0 \\ &= (1, a_i, a_{i+1}, \dots, a_n) \end{aligned}$$

This implies that

$$\mathrm{ev}\,(\ell_*(a_0, a_1, \dots, a_n)) = (a_0, a_1, \dots, a_n).$$

Hence, we have shown that ev is a retraction (left-inverse) for ℓ_*. Therefore, $\ell_*\colon \Omega^\bullet(A) \to \mathcal{E}^\bullet$ is injective. So we identify $\Omega^\bullet(A)$ with its image under ℓ_* inside of $\mathcal{E}^\bullet$. Using this isomorphism, we transport the graded algebra structure of $\mathcal{E}^\bullet$ to $\Omega^\bullet(A)$.

So we need to check that the differential d_{Hom} transports to $d_{\Omega A}$. To that end, consider the diagram below (where $\mathrm{im}(\ell_*)^n$ denotes the elements of $\mathrm{im}(\ell_*) \subseteq \mathcal{E}^\bullet$ of degree n).

$$\begin{array}{ccc} \Omega^n(A) & \xrightarrow{\ \ell_*\ } & \mathrm{im}(\ell_*)^n \\ \downarrow & & \downarrow d_{\mathrm{Hom}} \\ \Omega^{n+1}(A) & \xrightarrow{\ \ell_*\ } & \mathrm{im}(\ell_*)^{n+1} \end{array} \qquad \begin{array}{ccc} (a_0, \dots, a_n) & \longmapsto & \ell_{a_0}[d, \ell_{a_1}] \cdots [d, \ell_{a_n}] \\ \downarrow & & \downarrow d_{\mathrm{Hom}} \\ (1, a_0, \dots, a_n) & \longmapsto & [d, \ell_{a_0}][d, \ell_{a_1}] \cdots [d, \ell_{a_n}] \end{array}$$

□

Proof of Theorem 6.6.3(b). Given any algebra map $u\colon A \to \Gamma^\bullet$, define

$$\begin{aligned} \Omega^\bullet(A) &\xrightarrow{\qquad u_* \qquad} \Gamma \\ (a_0, \dots, a_n) &\longmapsto (ua_0) d_\Gamma(ua_1) \cdots d_\Gamma(ua_n) \end{aligned}$$

This is the required DG algebra homomorphism by formulas (6.8) and (6.9). □

Construction 6.6.4 (Morphism Complex)**.** Consider two cochain complexes of k-modules $M^\bullet$ and $N^\bullet$. Define

$$\underline{\mathrm{Hom}}^\bullet(K^\bullet, N^\bullet) := \left[\cdots \to \underline{\mathrm{Hom}}^n(K^\bullet, N^\bullet) \xrightarrow{d_{\mathrm{Hom}}} \underline{\mathrm{Hom}}^{n+1}(K^\bullet, N^\bullet) \to \cdots \right]$$

where $\underline{\mathrm{Hom}}^n(K^\bullet, N^\bullet)$ is the collection of linear maps $f^\bullet\colon K^\bullet \to N^\bullet$ of degree n

$$\underline{\mathrm{Hom}}^n(K^\bullet, N^\bullet) := \prod_{i \in \mathbb{Z}} \mathrm{Hom}_k(K^i, N^{i+n})$$

and $d^n_{\mathrm{Hom}}\colon \underline{\mathrm{Hom}}^n \to \underline{\mathrm{Hom}}^{n+1}$ is given by

$$d^n_{\mathrm{Hom}}(f) = [d,f] := d_N \circ f - (-1)^n f \circ d_K$$

Notice that $d_N^2 = d_K^2 = 0$ so $d^2_{\mathrm{Hom}} = 0$ as well.

If $K^\bullet = N^\bullet$, then $\underline{\mathrm{End}}^\bullet(K^\bullet) := \underline{\mathrm{Hom}}^\bullet(K^\bullet, K^\bullet)$ is naturally a DG-algebra with composition as the product.

This construction defines an internal Hom in the category of cochain complexes.

Definition 6.6.5. Given any associative, unital k-algebra A, $\Omega^\bullet(A)$ is called the **DG-envelope** of A.

Corollary 6.6.6 (Corollary to Theorem 6.6.3)**.** *The functor Ω is left-adjoint to the forgetful functor $U\colon \mathbf{dgAlg}_k \to \mathbf{Alg}_k$, given by $U\colon \Gamma^\bullet \mapsto \Gamma^0$.*

Notice that $\Omega^1(A) = A \otimes \overline{A}$ is naturally an (A,A)-bimodule. Elements of $\Omega^1(A)$ are written $a_0\, da_1 = (a_0, a_1)$. What is the bimodule structure? The left A-module structure is:

$$\begin{aligned} A \otimes \Omega^1(A) &\longrightarrow \Omega^1(A) \\ a \otimes a_0\, da_1 &\longmapsto aa0\, da_1 \\ a \otimes (a_0, a_1) &\longmapsto (aa_0, a_1) \end{aligned}$$

The right A-module structure is:

$$\begin{aligned} \Omega^1(A) \otimes A &\longrightarrow \Omega^1(A) \\ a_0\, da_1 \otimes a &\longmapsto a_0 \cdot da_1 \cdot a \overset{\text{Leibniz}}{=\!=\!=} a_0\, d(a_1 a) - a_0 a_1\, da \\ (a_0, a_1) \otimes a &\longmapsto (a_0, a_1 a) - (a_0 a_1, a) \end{aligned}$$

Exercise 6.6.7. Define Ω^1 in the category of Lie algebras.

Proposition 6.6.8. *There is a natural isomorphism of graded algebras*

$$\Omega^\bullet(A) \cong T_A(\Omega^1(A)).$$

Proof. Recall the universal property of tensor algebras: if A is an algebra, M an (A,A)-bimodule, then for all algebra maps $u_0\colon A \to B$ and (A,A)-bimodule maps $u_1\colon M \to B$, there is a unique algebra map $u\colon T_A M \to B$ such that $u|_A = u_0$ and $u|_M = u_1$.

In our case, $B = \Omega^\bullet(A)$, $u_0\colon A \hookrightarrow \Omega^\bullet(A)$, $u_1\colon \Omega^1(A) \hookrightarrow \Omega^\bullet(A)$. So we get a $u\colon T_A(\Omega^1(A)) \to \Omega^\bullet(A)$.

This is an isomorphism because

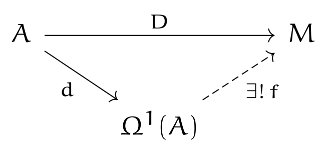
$$\begin{aligned}\Omega^1(A) \otimes_A \Omega^n(A) &= (A \otimes \overline{A}) \otimes_A (A \otimes \overline{A}^{\otimes n}) \\ &\cong A \otimes \overline{A} \otimes \overline{A}^{\otimes n} \\ &= A \otimes \overline{A}^{\otimes (n+1)} = \Omega^{n+1}(A)\end{aligned}$$

□

Remark 6.6.9. This should be compared to the commutative case, where we have $\Omega^\bullet_{DR}(A) = \bigwedge^\bullet{}_A \Omega^1_{\text{comm}}(A)$. So we might think of $\Omega^1(A)$ as the space of noncommutative Kähler differentials on A.

Definition 6.6.10. Given a derivation $D\colon A \to M$, and a bimodule map $f\colon M \to N$, then the **derivation induced from** D **by** f is the derivation $f_* D = f \circ D\colon A \to N$.

Lemma 6.6.11. *The derivation* $d\colon A \to \Omega^1(A)$ *is the universal derivation on* A *in the sense that any derivation* $D\colon A \to M$ *is induced from* d *by a unique bimodule map* $f\colon \Omega^1(A) \to M$*;* $D = f_* d$.



In other words, this lemma says that the (A, A)-bimodule $\Omega^1(A)$ represents the functor $\text{Der}_k(A, -)\colon$ $\textbf{Bimod}(A) \to \textbf{Vect}_k$; there is a natural isomorphism

$$\text{Der}_k(A, M) \cong \text{Hom}_{A^e}(\Omega^1(A), M).$$

Proof. Given any algebra A and bimodule M, define the **semidirect product** $A \ltimes M = A \oplus M$ with the multiplication

$$(a_1, m_1) \cdot (a_2, m_2) = (a_1 a_2, a_1 m_2 + m_1 a_2).$$

Notice that $M^2 = 0$.

Given a derivation $D\colon A \to M$, we can make $A \rtimes M$ into a DG-algebra by setting $da = D(a)$ for all $a \in A$ and $dm = 0$ for all $m \in M$. So we have an algebra map $u\colon A \to A \rtimes M$ given by $a \mapsto (a, 0)$. So by the universal property of DG-envelopes, there is a unique map

$$u_*\colon \Omega^\bullet(A) \to A \rtimes M.$$

Hence, there is a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{D} & M \\ \Vert & & \uparrow u_*|_{\Omega^1(A)} \\ A & \xrightarrow{d} & \Omega^1(A) \end{array}$$

□

Lemma 6.6.12. *There is a short exact sequence of bimodules*

$$0 \longrightarrow \Omega^1(A) \xrightarrow{j} A \otimes A \xrightarrow{\mu} A \longrightarrow 0$$

where $j(a_0\, da_1) = (a_0 a_1, 1) - (a_0, a_1)$ *and* $\mu(a \otimes b) = ab$.

Thus, $\Omega^1(A) \cong \ker(\mu)$ as an (A, A)-bimodule. If we identify $\Omega^1(A)$ as $\ker(\mu)$, then the natural isomorphism $\mathrm{Der}_k(A, M) \cong \mathrm{Hom}_{A^e}(\Omega^1(A), M)$ is easy to describe. Given an (A, A)-bimodule map $\Theta\colon \Omega^1(A) \to M$, the corresponding derivation is

$$\begin{aligned} D\colon A &\longrightarrow M \\ a &\longmapsto \Theta(a \otimes 1 - 1 \otimes a) \end{aligned}$$

Notice that for $M = A \otimes A$, j corresponds to the derivation $\Delta_A\colon A \to A \otimes A$ given by

$$\begin{aligned} \Delta_A\colon A &\longrightarrow A \otimes A \\ a &\longmapsto a \otimes 1 - 1 \otimes a. \end{aligned}$$

Δ_A is called the **canonical double derivation**.

Remark 6.6.13. Notice that $A \otimes A$ has two commuting (A, A)-bimodule structures, making it into a bi-bimodule:

$$x \cdot (a \otimes b) \cdot y = xa \otimes by \qquad \text{(outer)}$$

$$x \cdot (a \otimes b) \cdot y = ay \otimes xb \qquad \text{(inner)}$$

If we write $A^e \cong A \otimes A$, then $A \otimes A$ is an (A^e, A^e)-bimodule.

Here, $\mathrm{Der}_k(A, A \otimes A)$ is a bimodule with respect to the inner bimodule structure. Define

$$\Pi_\lambda(A) := T_A(\mathrm{Der}(A, A \otimes A)) \big/ (\Delta_A - \lambda)$$

for some $\lambda \in A$.

Exercise 6.6.14. If $A = kQ$ is the path algebra of a quiver, and $\lambda = \sum_{i \in Q_0} \lambda_i e_i \in A$ with $\lambda \in k$, then $\Pi_\lambda(Q)$ is what we called the deformed preprojective algebra of Q,

$$k\overline{Q} \Big/ \left(\sum_{a \in Q_1} [a, a^*] = \sum_{i \in Q_0} \lambda_i e_i \right)$$

Here $\overline{Q}$ denotes the **doubled quiver:** for each arrow $a \in Q_1$, add another arrow a^* in the opposite direction.

Exercise 6.6.15. If $A = \mathcal{O}(X)$ for a smooth affine curve X, then $\Pi_0(X) \cong \mathcal{O}(T^*X)$ and $\Pi_1(X) \cong \mathcal{D}(X)$, where $\mathcal{D}(X)$ is the ring of differential operators on X.

Connecting the two exercises, the intuition is that deformed preprojective algebras are the differential operators on quiver algebras.

Chapter 7

Higher Hochschild Homology

7.1 Some Homotopy Theory

We recall some definitions of homotopy theory here. For references, see [HA2, GJ09, May92].

Question 7.1.1. If X is a topological space, we define the classical singular homology $H_\bullet(X, A)$ with coefficients in an abelian group A. Can we define a homology theory of a space with coefficients in commutative algebras?

Recall that we may model topological spaces with simplicial sets, which are functors $X\colon \Delta^{op} \to \mathbf{Set}$, where Δ is the simplicial category.

Definition 7.1.2. Δ is the **simplicial category** with

$$\mathrm{Ob}(\Delta) = \left\{ [n] = \{0 < 1 < \ldots < n\} \right\}$$

$$\mathrm{Mor}(\Delta) = \left\{ f\colon [n] \to [m] \mid i \leq j \implies f(i) \leq f(j) \right\}$$

The simplicial category is generated by morphisms

$$\delta^i\colon [n] \to [n+1] \qquad 0 \leq i \leq n, \ (n \geq 0),$$

$$\sigma^j\colon [n+1] \to [n] \qquad 0 \leq i \leq n, \ (n \geq 1).$$

Informally, δ^i is the map that omits i in the image, and σ^j is the map that takes the value j twice.

We write $d_i = X(d^i)$ and $s_j = X(\sigma^j)$ for their images under the functor $X\colon \Delta^{op} \to \mathbf{C}$.

Definition 7.1.3. Let $\mathbf{C}$ be any category. The **category of simplicial objects in C** is $\mathbf{Fun}(\Delta^{op}, \mathbf{C}) = \mathbf{C}^{\Delta^{op}}$.

Similarly, the **category of cosimplicial objects in C** is $\mathbf{Fun}(\Delta, \mathbf{C}) = \mathbf{C}^{\Delta}$.

If $\mathbf{C} = \mathbf{Set}$, then $\mathbf{C}^{\Delta^{op}} = \mathbf{sSet}$ is called simplicial sets.

Example 7.1.4. An example of a cosimplicial space is the geometric simplicies

$$\Delta^k = \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i=0}^{k} x_i \leq 1 \right\}$$

The functor $\Delta^\bullet \colon \Delta \to \mathbf{Top}$ defines a cosimplicial space, that is, a cosimplicial object in $\mathbf{Top}$, sending a set $[n]$ to Δ^n and a map $f\colon [n] \to [m]$ to $f_* \colon \Delta^n \to \Delta^m$ given by $f_*(e_i) = e_{f(i)}$, where e_j is the j-th standard basis vector of $\mathbb{R}^k$

$$\begin{array}{ccc} \Delta & \xrightarrow{\Delta^\bullet} & \mathbf{Top} \\ [n] & \longmapsto & \Delta^n \\ \left([n] \xrightarrow{f} [m]\right) & \longmapsto & \left(\Delta^n \xrightarrow{f_*} \Delta^m\right) \end{array}$$

Definition 7.1.5. The **geometric realization** of a simplicial set $X_\bullet$ is

$$|X| = \coprod_{n \geq 0} X_n \times \Delta^n / \sim$$

where $(x, f_* y) \sim (f^* x, y)$ for any $f\colon [n] \to [m]$ in Δ.

The map $f\colon [n] \to [m]$ gives a map $f_*\colon \Delta^n \to \Delta^m$ between geometric simplicies, and a map $X(f) = f^* \colon X_m \to X_n$ between sets.

Lemma 7.1.6. *The geometric realization functor* $|-|\colon \mathbf{sSet} \to \mathbf{Top}$ *is left adjoint to the functor* $\mathcal{S}\colon \mathbf{Top} \to \mathbf{sSet}$ *sending a topological space* X *to the simplicial set of* n*-simplicies inside* X*;*

$$\begin{array}{ccc} |-|\colon \mathbf{sSet} & \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} & \mathbf{Top} \colon \mathcal{S} \\ X & \longmapsto & |X| \\ \mathrm{Hom}_{\mathbf{Top}}(\Delta^n, X) & \longleftarrow\!\shortmid & X \end{array}$$

Theorem 7.1.7. *Let* $\mathbf{C}$ *be a cocomplete locally small category. Then the category* $\mathbf{C}^{\Delta}$ *of cosimplicial objects in* $\mathbf{C}$ *is equivalent to the category of simplicial adjunctions*

$$L\colon \mathbf{sSet} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{C} \colon R$$

L is called the **realization** of a simplicial set in the category $\mathbf{C}$.

Proof sketch. Given $\Delta^\bullet \colon \mathbb{\Delta} \to \mathbf{C}$ a cosimplicial object in $\mathbf{C}$, we will construct L and R.

Consider the Yoneda embedding $Y\colon \mathbb{\Delta} \hookrightarrow \mathbf{sSet}$. Let $L = \mathrm{Lan}_Y(\Delta^\bullet)$ be the left Kan extension of $\Delta^\bullet$ along Y; this exists because $\mathbf{C}$ is cocomplete and locally small.

$$\begin{array}{ccc} \mathbb{\Delta} & \xrightarrow{\Delta^\bullet} & \mathbf{C} \\ {\scriptstyle Y}\big\downarrow & \nearrow_{L} & \\ \mathbf{sSet} & & \end{array}$$

Define $R\colon \mathbf{C} \to \mathbf{sSet}$ by

$$\begin{aligned} R(c)\colon \Delta &\longrightarrow \mathbf{Set} \\ [n] &\longmapsto \mathrm{Hom}_{\mathbf{C}}(\Delta^n, c) \end{aligned}$$

This is right-adjoint to L by construction. □

Example 7.1.8. We have already seen one example of this – the geometric realization.

Example 7.1.9. Let $\mathbf{C} = \mathbf{Cat}$, and define the cosimplicial object $\mathbb{\Delta} \to \mathbf{Cat}$ that sends $[n]$ to the category $\vec{n}$ with presentation

$$\vec{n} = (0 \to 1 \to 2 \to \cdots \to n)$$

The right adjoint (as in Theorem 7.1.7) $\mathcal{N}\colon \mathbf{Cat} \to \mathbf{sSet}$ is called the **nerve** of a category, and sends a category $\mathbf{C}$ to the simplicial set $\mathcal{N}\mathbf{C}$ with

$$\mathcal{N}\mathbf{C}_n = \mathrm{Hom}_{\mathbf{Cat}}(\vec{n}, \mathbf{C}) = \left\{ c_0 \to c_1 \to \cdots \to c_n \;\middle|\; c_i \in \mathrm{Ob}(\mathbf{C}) \right\}$$

Example 7.1.10. Let G be a discrete group. Then G can be thought of as a category with one object whose morphisms are elements of G. To distinguish this category from the group G, we write $\underline{G}$ for the category.

Then $\mathcal{N}\underline{G} = BG$ is the classifying space of G.

Example 7.1.11. Let G be an algebraic group over k, i.e. a representable functor $G\colon \mathbf{CommAlg}_k \to \mathbf{Gr}$, $A \mapsto G(A)$ (for example, $G = GL_n(-)$). Let $\mathcal{O}(G)$ be the corresponding representation

$$\mathrm{Hom}_{\mathbf{CommAlg}_k}(\mathcal{O}(G), A) \cong G(A)$$

Let $BG_\bullet$ be the classifying space of G; we have

$$BG_n = G^n = G \times G \times \cdots \times G.$$

This classifying space is a simplicial scheme

$$\begin{aligned} BG_\bullet\colon \mathbf{CommAlg}_k &\longrightarrow \mathbf{sSet} \\ A &\longmapsto B(G(A)) \end{aligned}$$

Then $\mathcal{O}(BG_\bullet)$ is a cosimplicial commutative algebra, sending $[n] \in \mathrm{Ob}(\Delta)$ to the commutative algebra $\mathcal{O}(G \times G \times \cdots \times G) = \mathcal{O}(G)^{\otimes n}$.

By Theorem 7.1.7, we get a simplicial adjunction $L\colon \mathbf{sSet} \rightleftarrows \mathbf{CommAlg}_k\colon R$, $L \dashv R$. This yields the following theorem.

Definition 7.1.12. A simplicial set X is called reduced if X_0 has cardinality 1. The category of all such simplicial sets is denoted $\mathbf{sSet}_0$.

Theorem 7.1.13. *The realization functor corresponding to $\mathcal{O}(BG_\bullet) \in \mathbf{CommAlg}_k$ restricted to $\mathbf{sSet}_0$ is given by*

$$\begin{aligned} \mathbf{sSet}_0 &\longrightarrow \mathbf{CommAlg}_k \\ X &\longmapsto \mathcal{O}\left(\mathrm{Rep}_G(\pi_1(|X|, *))\right) \end{aligned}$$

7.2 PROPS

There are several ways of thinking about algebras: using monads, operads, or PROPs. We will talk about PROPs.

Definition 7.2.1. A **symmetric monoidal category** is a category $\mathbf{S}$ equipped with a bifunctor $\boxtimes\colon \mathbf{S} \times \mathbf{S} \to \mathbf{S}$, together with a distinguished object $\mathbb{1}$ and natural isomorphisms

$$\begin{aligned} \alpha_{X,Y,Z}&\colon (X \boxtimes Y) \to \boxtimes Z \\ \lambda_X&\colon \mathbb{1} \boxtimes X \to X \\ \rho_X&\colon X \boxtimes \mathbb{1} \to X \\ \tau_{X,Y}&\colon X \boxtimes Y \to Y \boxtimes X \end{aligned}$$

A symmetric monoidal category is called **strict** if α, λ, ρ are identities.

Definition 7.2.2. A **PROP** (short for products and permutations category) is a strict symmetric monoidal category $\mathbf{S}$ with $\mathrm{Ob}(\mathbf{S}) = \mathbb{N}$, and $n \boxtimes m = n + m$.

Example 7.2.3. Consider $\mathcal{F}$ the full subcategory of finite sets with objects $[n] = \{1, 2, \ldots, n\}$ and $0 = \emptyset$, and tensor product given by disjoint union.

Example 7.2.4. Let $\mathcal{G}$ be the prop of finitely generated free groups, with objects F_n, the free group $\langle n \rangle$ on the set $\underline{n} = \{1, 2, \dots, n\}$. Tensor product is given by the free products $\langle n \rangle \boxtimes \langle m \rangle = \langle n + m \rangle$. Morphisms are group homomorphisms.

Definition 7.2.5. A k-algebra over a PROP $\mathcal{P}$ is the category of strong monoidal functors

$$\mathbf{Alg}_k(S) = \mathbf{Fun}^{\otimes}(\mathcal{P}, (\mathbf{Vect}_k, \otimes_k))$$

Lemma 7.2.6. *The* k*-algebra over the prop* $\mathcal{F}$ *is the category of commutative* k*-algebras.*

Proof. Define the functor $\mathbf{CommAlg}_k \to \mathbf{Alg}_k(\mathcal{F})$ that sends a commutative algebra A to the functor $\underline{A} \colon \mathcal{F} \to \mathbf{Vect}_k$; this functor $\underline{A}$ sends $\underline{n}$ to $A^{\otimes n}$ and $f \colon \underline{n} \to \underline{m}$ to the function

$$\begin{array}{ccc} A^{\otimes n} & \longrightarrow & A^{\otimes m} \\ a_1 \otimes \cdots \otimes a_n & \longmapsto & b_1 \otimes \cdots \otimes b_m \end{array}$$

where

$$b_i = \prod_{j \in f^{-1}(i)} a_j$$

for $i = 1, \dots, m$.

In the other direction, define a functor $\mathbf{Alg}_k(\mathcal{F}) \to \mathbf{CommAlg}_k$ that sends a strong monoidal functor T to the commutative k-algebra $T(\underline{1})$; here multiplication $\mu \colon A^{\otimes 2} \to A$ is given by the image under T of the morphism $\underline{2} \to \underline{1}$, and the unit $\eta \colon k \to A$is given by the image under T of the morphism $\underline{0} \to \underline{1}$.

Check that these are inverse functors giving an isomorphism of categories. □

Proposition 7.2.7. *The* k*-algebra over the prop* $\mathcal{G}$ *is the category of commutative Hopf* k*-algebras.*

Definition 7.2.8. A **co-group object** in **C** is an object $A \in \mathrm{Ob}(\mathbf{C})$ such that for any $B \in \mathrm{Ob}(\mathbf{C})$ $\mathrm{Hom}_C(A, B)$ is a group (in the ordinary sense) and for any $f \colon B \to B'$, the morphism $f_* \colon \mathrm{Hom}_{\mathbf{C}}(A, B) \to \mathrm{Hom}_{\mathbf{C}}(A, B')$ is a morphism of groups.

That is, $\mathrm{Hom}_{\mathbf{C}}(A, -) \colon C \to \mathbf{Set}$ factors through the category of groups.

Remark 7.2.9. I. Berstein, for which the Berstein seminar at Cornell is named, is famous for studying cogroups in the category of spaces – called "co-H-spaces."

Example 7.2.10.

(a) Cogroups in **Set** or **Top** are trivial

(b) Cogroups in the opposite category of commutative algebras are cocommutative Hopf algebras.

(c) (Kan) Cogroups in the category of groups are precisely the free groups.

(d) Reduced suspensions are cogroups in the homotopy category of pointed topological spaces. But these are not all of the cogroups (due to I. Berstein).

Theorem 7.2.11. *Let $\mathcal{G}$ be the PROP of finitely generated free groups, and* **C** *a monoidal category. Then the category of strong monoidal functors from $\mathcal{G}$ to* **C** *is isomorphic to the category of cogroup objects in* **C**.

7.3 Higher Hochschild Homology

These ideas were developed by Loday and Pirashvili in 2002.

Let A be a commutative k-algebra. Recall that A corresponds to the strong monoidal functor $\underline{A}\colon \mathcal{F} \to \mathbf{Vect}_k$ sending $\underline{n}$ to $A^{\otimes n}$. We may extend $\underline{A}$ to a functor $\widetilde{\underline{A}}\colon \mathbf{Set} \to \mathbf{Vect}_k$ by

$$\widetilde{\underline{A}}(X) = \operatorname{colim}_{\underline{n}\to X} \underline{A}(\underline{n})$$

Let $X_\bullet\colon \Delta^{op} \to \mathbf{Set}$ be a simplicial set. Then we define a simplicial vector space by composing with $\widetilde{\underline{A}}$.

$$V_\bullet = \widetilde{\underline{A}} \circ X_\bullet\colon \Delta^{op} \to \mathbf{Vect}_k$$

Theorem 7.3.1 (Dold-Kan Correspondence)**.** *Let* **A** *be any abelian category. Then there is an equivalence of categories* N *between chain complexes in* **A** *and simplicial objects in* **A**,

$$N\colon \mathbf{sA} \simeq \mathbf{Com}(\mathbf{A}).$$

So to our simplicial vector space $V_\bullet$, we apply the Dold-Kan correspondence functor N to get a chain complex $NV_\bullet$.

Definition 7.3.2. The higher hochschild homology of X with coefficients in A is

$$HH_*(X, A) := H_\bullet(NV_\bullet)$$

In the above definition, we conflate a topological space X with a simplicial set X.

Example 7.3.3. Let $X = \mathbb{S}^1_\bullet$ be the simplicial circle. Then the higher hochschild homology in this case is classical hochschild homology.

$$HH_\bullet(\mathbb{S}^1, A) \cong HH_\bullet(A)$$

This leads to the following generalization of the HKR theorem (Theorem 6.1.5).

Theorem 7.3.4 (Pirashvili 2002)**.** *Assume* A *is a smooth* k*-algebra. Let* $X = \mathbb{S}^d_\bullet$ *be the simplicial* n*-sphere with* $d \geq 1$*. Then there is a natural isomorphism of graded commutative algebras*

$$\mathrm{HH}_\bullet(\mathbb{S}^d, A) \cong \mathrm{Sym}_A\left(\Omega^1_{\mathrm{comm}}[d]\right).$$

We may consider $\mathrm{HH}_\bullet(-, A)$ as a homology theory of spaces with coefficients in commutative algebras. This leads to the question of whether or not this is a stable homotopy invariant of X, that is, if for two spaces X and Y we have $\Sigma X \cong \Sigma Y$, is $\mathrm{HH}_\bullet(X, A) \cong \mathrm{HH}_\bullet(Y, A)$?

The answer to this question is yes, but only so long as A is smooth. In general, the answer is no [DT16].

7.4 Homotopy Theory of Simplicial groups

Let **Group** be the category of groups, and **sGroup** the category of simplicial groups $\Gamma\colon \Delta^{op} \to$ **Group** with

$$\begin{aligned} d_i\colon \Gamma_n &\to \Gamma_{n-1} \quad \text{face maps} \\ s_j\colon \Gamma_n &\to \Gamma_{n+1} \quad \text{degeneracy maps} \end{aligned}$$

Definition 7.4.1. The **Moore complex** of a simplicial group is the complex

$$(N\Gamma)_* := \left[\ \cdots \longrightarrow N\Gamma_{n-1} \xrightarrow{d_0} N\Gamma_n \longrightarrow \cdots\ \right]$$

where

$$(N\Gamma)_n := \bigcap_{i=1}^{n} \ker\left(d_i\colon \Gamma_n \to \Gamma_{n-1}\right)$$

We have that $d_0 \circ d_0 = 0$, so this is indeed a complex.

This defines a complex of not-necessarily abelian groups. We further define

$$\pi_*(\Gamma) := {}^{\ker(d_0)}\!\big/_{\mathrm{im}(d_0)}$$

In general these are not groups, but spaces of left-cosets with an action of the group. But in many interesting cases, $\pi_*(\Gamma)$ are groups.

Let's consider an example where this was used.

Example 7.4.2. Let M be a manifold (or more generally, any topological space). The **configuration space of** n**-ordered points in** M is

$$\mathrm{Conf}_n(M) = \left\{ (x_0, \ldots, x_{n-1}) \in M \times \cdots \times M \,\middle|\, x_i \neq x_j \text{ when } i \neq j \right\}$$

There are maps

$$\begin{aligned} p_i\colon \operatorname{Conf}_{n+1}(M) &\longrightarrow \operatorname{Conf}_n(M) \\ (x_0,\dots,x_n) &\longmapsto (x_0,\dots,\widehat{x_i},\dots,x_n) \end{aligned}$$

From these maps, we have induced maps on the fundamental groups:

$$d_i := (p_i)_*\colon \pi_1(\operatorname{Conf}_{n+1}(M)) \longrightarrow \pi_1(\operatorname{Conf}_n(M))$$

We define a semi-simplicial group (it doesn't have degeneracy maps)

$$\operatorname{Conf}(M)^{\pi_1} = \left\{\pi_1(\operatorname{Conf}_{n+1}(M)), d_i\right\}_{n\geq 0}$$

Although this is only semi-simplicial, we may still define it's Moore complex because the Moore complex doesn't consider degeneracy map.

Then we can ask about $\pi_*(\operatorname{Conf}(M)^{\pi_1})$. The following theorem provides a partial answer for the 2-sphere.

Theorem 7.4.3 (Berrick, Cohen, Wong, Wu 2008)**.** *If* $M = S^2$*, then*

$$\pi_n(\operatorname{Conf}(S^2)^{\pi_1}) \cong \pi_n(S^2)$$

for all $n \geq 4$.

Definition 7.4.4. A **simplicial group** $\Gamma_* = \{\Gamma_n\}_{n\geq 0}$ is **semi-free** if there is a subset $B_n \subseteq \Gamma_n$ for each n, such that

(a) Γ_n is free on B_n for all $n \geq 0$, and

(b) $B := \bigcup_{n\geq 0} B_n$ is closed under degeneracy.

The second condition here simply states that $s_j(B_n) \subseteq B_{n+1}$ for all $n \geq 0$ and all j. Write $\overline{B}_n$ for the set of non-degenerate generators of Γ_*,

$$\overline{B_n} := B_n \setminus \bigcup_{j=0}^{n-1} s_j(B_{n-1}).$$

Recall that $\mathbf{sSet}_0 \subset \mathbf{sSet}$ is the full subcategory of reduced simplicial sets, i.e., those $X \in \mathbf{sSet}_0$ such that $X_0 = \{*\}$.

Definition 7.4.5. The **Kan loop group construction** assigns to each reduced simplicial set X a semifree simplicial group $\mathbb{G}X$, where $(\mathbb{G}X)_n$ has the presentation:

$$(\mathbb{G}X)_n := \langle X_{n+1} \mid s_0(x) = 1\ \forall\, x \in X_n\rangle.$$

This construction is a functor $\mathbb{G}\colon \mathbf{sSet}_0 \to \mathbf{sGroup}$.

We write $B_n = X_{n+1} \setminus s_0(X_n)$, and the inclusion $B_n \hookrightarrow X_n$ induces an isomorphism between $(\mathbb{G}X)_n$ and the free group on B_n.

The face and degeneracy maps of $\mathbb{G}X$ are as follows.

- $s_j^{\mathbb{G}X}$ is induced by s_j^X;
- $d_i^{\mathbb{G}X}$ is induced by d_{i+1}^X for all $i > 0$;
- $d_0^{\mathbb{G}X}\colon (\mathbb{G}X)_n \to (\mathbb{G}X)_{n-1}$ is given by

$$d_0^{\mathbb{G}X}(x) := d_1(x) \cdot d_0(x)^{-1}.$$

Definition 7.4.6. The **Kan construction** is a pair of adjoint functors

$$\mathbb{G}\colon \mathbf{sSet}_0 \rightleftarrows \mathbf{sGroup}\colon \overline{W} \quad (\bot)$$

$\overline{W}$ is called the **(simplicial) classifying space** functor which is determined by $\overline{W}\Gamma = B\Gamma$, where Γ is a discrete simplicial group.

Theorem 7.4.7 (Kan 1958)**.** *The adjunction $\mathbb{G} \dashv \overline{W}$ is a Quillen equivalence, and moreover $\mathbb{G}$ preserves cofibrations, $\overline{W}$ preserves fibrations, and both $\mathbb{G}$ and $\overline{W}$ preserve weak equivalences.*

To say that an adjunction is a Quillen equivalence is to say that induces an equivalence on the homotopy categories.

Recall that $\mathbf{Ho}(\mathbf{sSet}) \simeq \mathbf{Ho}(\mathbf{Top})$ via the adjunction $|-| \dashv \mathcal{S}$. This descends to an equivalence $\mathbf{Ho}(\mathbf{sSet}_0) \simeq \mathbf{Ho}(\mathbf{Top}_{*,0})$ between the homotopy category of pointed simplicial sets and that of pointed, connected spaces, given by the adjunction $|-| \dashv E\mathcal{S}$. Here $E\mathcal{S}$ is the **Eilenberg singular complex functor**

$$E\mathcal{S}(X) := \left\{ f \in \mathrm{Hom}_{\mathbf{Top}}(\Delta^n, X) \;\middle|\; f \text{ takes all vertices to the basepoint} \right\}$$

The conclusion of all this is that we may use simplicial groups to model (pointed connected) spaces.

7.5 Representation Homology

The material in this section is due to Berest, Ramadoss, and Yeung [BRY17].

Recall the prop $\mathcal{G}$ of finitely generated free groups. The objects of $\mathcal{G}$ are the free groups of order n, and morphisms are group homomorphisms. We saw that

$$\mathbf{Alg}_k(\mathcal{G}) := \mathbf{Fun}^{\otimes}(\mathcal{G}, \mathbf{Vect}_k) \cong \mathbf{CommHopfAlg}_k.$$

We also defined an affine algebraic group scheme as a representable functor $G\colon \mathbf{CommAlg}_k \to \mathbf{Group}$, sending A to $G(A) = \mathrm{Hom}_{\mathbf{CommAlg}_k}(\mathcal{O}(G), A)$, where $\mathcal{O}(G)$ is the coordinate ring of G. We see that $\mathcal{O}(G)$ is a commutative Hopf algebra with coproduct dual to the multiplication $G \times G \to G$.

Example 7.5.1. The general linear group is an affine algebraic group scheme $G = GL_n\colon \mathbf{CommAlg}_k \to \mathbf{Group}$, sending $A \mapsto GL_n(A)$. Then we have $\mathcal{O}(G) = k[x_{ij}, \det(x_{ij})^{-1}]$.

Given a commutative Hopf algebra H, consider the functor $\underline{H}\colon \mathcal{G} \to \mathbf{Vect}_k$ that sends the free group F_n to $H^{\otimes n}$. We may include $\mathcal{G}$ into the category $\mathbf{FGr}$ of free groups (Γ, S) such that $\Gamma = \langle S \rangle$. Morphisms in $\mathbf{FGr}$ are all group homomorphisms (so may not necessarily preserve the generating set).

Take the left Kan extension $\widetilde{\underline{H}} = \mathrm{Lan}_I(\underline{H})$.

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\underline{H}} & \mathbf{CommAlg}_k \\ {\scriptstyle I}\downarrow & \nearrow_{\widetilde{\underline{H}}} & \\ \mathbf{FGr} & & \end{array}$$

Definition 7.5.2 ([BRY17])**.** The **representation homology** of a reduced simplicial set X with coefficients in $\mathcal{H} = \mathcal{O}(G)$ is

$$HR_*(X, G) := H_*\left(N\widetilde{\underline{H}}(\mathbb{G}X)\right) = \pi_*\widetilde{\underline{H}}(\mathbb{G}X)$$

$$\Delta^{op} \xrightarrow{\mathbb{G}X} \mathbf{FGr} \xrightarrow{\widetilde{\underline{H}}} \mathbf{CommAlg}_k$$

Fact 7.5.3. *We have that*

$$HR_0(X, G) \cong k\left(\mathrm{Rep}_G(\pi_1|X|, *\right)$$

We can relate the representation homology back to Hochschild homology.

Theorem 7.5.4 ([BRY17])**.** *For any simplicial set $X \in$ **sSet**,*

$$HR_*(\Sigma(X_+), G) \cong HH_*(X, \mathcal{O}(G))$$

Here, $X_+ = X \sqcup \{*\}$ is X with an artificially added basepoint. The simplicial set X_+ is defined so that $|X|_+ = |X_+|$.

Chapter 8

Quillen Homology

8.1 Model Categories

Definition 8.1.1 (See [HA2]). A **model category C** is a category with three distinguished classes of morphisms:

(1) **weak equivalences** (WE), with arrows decorated with a tilde ($\xrightarrow{\sim}$)

(2) **fibrations** (Fib), with arrows decorated with two heads ($\twoheadrightarrow$)

(3) **cofibrations** (Cof), with arrows decorated by a hook ($\hookrightarrow$)

these classes satisfy five axioms, (MC1) - (MC5).

(MC1) **C** has all finite limits and colimits. In particular, **C** has initial and terminal objects.

(MC2) Let $f\colon A \to B$ and $g\colon B \to C$. If any two of f, g and gf are weak equivalences, then so is the third.

(MC3) All three classes WE, Fib and Cof are closed under taking retracts.

(MC4) Consider the diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle f}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle g} \\ B & \longrightarrow & Y \end{array}$$

with $f \in$ Cof and $g \in$ Fib. If either f or g is in addition a weak equivalence, then there exists $h\colon B \to Y$ making the diagram commute.

(MC5) Any morphism $X \xrightarrow{f} A$ may be factored in two ways: either as a fibration followed by an **acyclic cofibration** (weak equivalence that is also a cofibration)

$$X \hookrightarrow QA \xrightarrow{\sim} A,$$

or as an **acyclic fibration** (weak equivalence that is also a fibration), followed by a cofibration

$$X \overset{\sim}{\hookrightarrow} RX \twoheadrightarrow A.$$

Here, QA is a **cofibrant object** in **C** (i.e. $\emptyset \twoheadrightarrow QA$ is a cobibration) and RX is a **fibrant object** in **C** (i.e. $RX \hookrightarrow *$ is a fibration).

Remark 8.1.2. In practice, a model category **C** is usually **fibrant** or **cofibrant** – all objects are either fibrant or cofibrant.

Definition 8.1.3. The **homotopy category** of a model category **C** is the category in which we formally invert all weak equivalences; $\mathbf{Ho}(\mathbf{C}) := \mathbf{C}[\mathrm{WE}^{-1}]$. This comes with a canonical map $\gamma_{\mathbf{C}} \colon C \to \mathbf{Ho}(C)$.

There is a canonical functor $\gamma \colon \mathbf{C} \to \mathbf{Ho}(\mathbf{C})$ called the **localization functor**.

Remark 8.1.4. There is another approach to homotopy theory using homotopy categories instead of model categories, which only has one axiom. But it's very hard to use in practice; see [DHKS04].

Example 8.1.5.

(a) Let A be an associative k-algebra. Let $\mathbf{C} = \mathbf{Com}(A)$ be the category of chain complexes of A-modules. There are two model structures on **C**: the projective and injective ones. The injective structure is dual to the projective one, so we only describe the projective model structure.

In the projective model structure:

- the fibrations are morphisms $f_\bullet \colon M_\bullet \to N_\bullet$ such that f_n is surjective for all n.
- the cofibrations are $f_\bullet \colon M_\bullet \to N_\bullet$ such that f_n is injective for all n, and $\operatorname{coker}(f_n \colon M_n \to N_n)$ is a projective A-module.
- weak equivalences are quasi-isomorphisms of complexes.

Here the zero object (both initial and terminal) is the zero complex. $\mathbf{Com}(A)$ is fibrant: $K_\bullet \twoheadrightarrow 0$ is always surjective. An object $P_\bullet$ is fibrant if P_n is a projective A-module.

(b) $\mathbf{dgAlg}_k$, $\mathbf{dgLie}_k$, $\mathbf{dgCommAlg}_k$ are all model categories, when the characteristic of k is zero.

- The weak equivalences are quasi-isomorphisms of DG-algebras.
- the fibrations are degreewise surjective homomorphisms of DG-algebras.
- the cofibrations are (retracts of) semi-free-extensions of DG-algebras: a map $f\colon A \to B$ is a semi-free-extension if there is an isomorphism of the underlying graded algebras between B and the coproduct of A and a free algebra. In the case of $\mathbf{dgAlg}_k$, for example, we get $B_\# \cong A_\# \sqcup_k T_k V$.

(c) If $\mathbf{C}$ (= $\mathbf{Set}, \mathbf{Group}, \mathbf{Alg}_k, \mathbf{Cat}$) is a category, then simplicial objects in $\mathbf{C}$ forms a model category $\mathbf{sC} = \mathbf{Fun}(\Delta^{op}, \mathbf{C})$, with the model structure as follows.

- weak equivalences are **weak homotopy equivalences**: $f\colon X_\bullet \to Y_\bullet$ that induce homotopy equivalences on the geometric realizations: $f_*\colon \pi_*(|X_\bullet|, x_0) \cong \pi_*(|Y_\bullet|, f(x_0))$.
- cofibrations are degreewise injective maps $f\colon X_\bullet \to Y_\bullet$.
- fibrations are determined by the weak equivalences and cofibrations

Remark 8.1.6. The homotopy category of the category $\mathbf{Com}(A)$ is equal to the derived category of complexes; $\mathbf{Ho}(\mathbf{Com}(A)) = \mathcal{D}(\mathbf{Com}(A))$. Therefore, we think of $\mathbf{Ho}(\mathbf{dgAlg}_k)$ as the nonlinear analogue of the derived category.

Similarly, we have $\mathbf{Ho}(\mathbf{sSet}) \cong \mathbf{Ho}(\mathbf{Top})$, and for any commutative ring k, $\mathbf{Ho}(\mathbf{sAlg}_k) \cong \mathbf{Ho}(\mathbf{dgAlg}_k)$.

Given a functor $F\colon \mathbf{C} \to \mathbf{D}$ between two model categories, how can we get a functor between their homotopy categories? F doesn't induce a functor $\mathbf{Ho}(\mathbf{C}) \to \mathbf{Ho}(\mathbf{D})$ unless F preserves weak equivalences. The idea, however, is to approximate the non-existent induced functor using Kan extensions.

Definition 8.1.7. Given a functor $F\colon \mathbf{C} \to \mathbf{D}$ between two model categories $\mathbf{C}, \mathbf{D}$, the **left-derived functor** of F is the right Kan extension of F along $\gamma_{\mathbf{C}}\colon \mathbf{C} \to \mathbf{Ho}(\mathbf{C})$.

$$\begin{array}{ccccc} \mathbf{C} & \xrightarrow{F} & \mathbf{D} & \xrightarrow{\gamma_{\mathbf{D}}} & \mathbf{Ho}(\mathbf{D}) \\ \downarrow{\scriptstyle \gamma_{\mathbf{C}}} & & & \nearrow{\scriptstyle \mathbb{L}F = \mathrm{Ran}_{\gamma_{\mathbf{C}}}(\gamma_{\mathbf{D}} \circ F)} & \\ \mathbf{Ho}(\mathbf{C}) & & & & \end{array}$$

Definition 8.1.8. The **left-derived functor** is a pair $(\mathbb{L}F, t)$ of $\mathbb{L}F\mathrm{colon}\, \mathbf{Ho}(\mathbf{C}) \to \mathbf{Ho}(\mathbf{D})$ with a natural transformation $t\colon \mathbb{L}F \circ \gamma_C \Rightarrow \gamma_D \circ F$ such that for any other pair (G, s) with $G\colon \mathbf{Ho}(\mathbf{C}) \to \mathbf{Ho}(\mathbf{D})$ and $s\colon G \circ \gamma_C \Rightarrow \gamma_D \circ F$, there is a unique $\widetilde{s}\colon \mathbb{L}F \circ \gamma_C \Rightarrow G \circ \gamma_C$ such that $t = s \circ \widetilde{s}$.

There is similarly a definition of a right-derived functor. This is the left Kan extension, instead of the right one.

Theorem 8.1.9 (Adjunction theorem)**.** *Given a pair of adjoint functors $F \dashv G$ between model categories* **C** *and* **D** *such that F preserves cofibrations, or equivalently G preserves fibrations, then $\mathbb{L}F \dashv \mathbb{R}G$.*

8.2 Quillen Homology and Cyclic Formalism

Recall that for a unital associative k-algebra A we defined the universal DG-algebra of noncommutative differential forms $(\Omega^\bullet A, d)$, and showed that $\Omega^\bullet(A) = T_A(\Omega^1(A))$.

If we try to define the de Rham cohomology naively, as the cohomology of $\Omega^\bullet(A)$, we only get trivial things.

Proposition 8.2.1. *For any associative k-algebra A*

$$H^n(\Omega^\bullet(A), d) = \begin{cases} k & n = 0 \\ 0 & n \geq 1 \end{cases}$$

Proof. For $n \geq 1$, there is an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & d\Omega^{n-1}(A) & \xrightarrow{S} & \Omega^n(A) & \xrightarrow{d} & d\Omega^n & \longrightarrow & 0 \\ & & \omega & \longmapsto & 1 \otimes \omega & & & & \end{array}$$

□

So how do we extract from $\Omega^\bullet(A)$ some interesting homological information? There are two different answers to this question: the Quillen homology (a simple way of looking at homology) or the cyclic formalism of mixed complexes (due to Connes, Quillen, Kassel, Tsyzan, . . .). Quillen homology appears in Chapter V of [HA2].

Quillen homology can be defined in any model category, and recovers the usual homology for many algebraic structures.

Definition 8.2.2. If **C** is any category, we may define an **abelian group object** $A \in \mathrm{Ob}(\mathbf{C})$ such that the functor $\mathrm{Hom}_{\mathbf{C}}(-, A)\colon \mathbf{C} \to \mathbf{Set}$, represented by A factors through the category of abelian groups.

This says that for all $B \in \mathrm{Ob}(\mathbf{C})$, $\mathrm{Hom}_{\mathbf{C}}(B, A)$ is an abelian group and for any $f\colon B \to B' \in \mathrm{Mor}(\mathbf{C})$,

$$f^*\colon \mathrm{Hom}_{\mathbf{C}}(B', A) \to \mathrm{Hom}_{\mathbf{C}}(B, A)$$

is a homomorphism of abelian groups.

Proposition 8.2.3. *Let* **C** *be a Cartesian category (i.e.* **C** *has all finite products and a terminal object* $E \in \mathrm{Ob}(\mathbf{C})$*). Then* $A \in \mathrm{Ob}(\mathbf{C})$ *is an abelian object if* **C** *if and only if there are morphisms*

$$\begin{array}{rl} m : A \times A \to A & \textit{(product)} \\ i : A \to A & \textit{(inverse)} \\ e : E \to A & \textit{(unit)} \end{array}$$

such that the following diagrams commute.

$$\begin{array}{ccc} A \times A \times A & \xrightarrow{(m,\mathrm{id})} & A \times A \\ \downarrow m & & \downarrow m \\ A \times A & \xrightarrow{m} & A \end{array} \qquad \text{(associativity)}$$

$$\begin{array}{ccccc} A \times E & \xrightarrow{(\mathrm{id},!)} & A \times A & \xleftarrow{(!,\mathrm{id})} & E \times A \\ & \searrow^{\cong} & \downarrow m & \swarrow_{\cong} & \\ & & A & & \end{array} \qquad \text{(unit axioms)}$$

$$\begin{array}{ccccc} A & \xrightarrow{(\mathrm{id},\mathrm{id})} & A \times A & & \\ {\scriptstyle (\mathrm{id},\mathrm{id})}\downarrow & & \searrow^{\mathrm{id}} & \searrow^{\mathrm{id} \times i} & \\ A \times A & & & & A \times A \\ & \searrow_{i \times \mathrm{id}} & & & \downarrow m \\ & & A \times A & \xrightarrow[m]{} & A \end{array} \qquad \text{(inverse axioms)}$$

$$\begin{array}{ccc} A \times A & \xrightarrow{(p_2,p_1)} & A \times A \\ & \searrow^{m} \quad \swarrow_{m} & \\ & A & \end{array} \qquad \text{(commutativity)}$$

Exercise 8.2.4. Prove the previous proposition.

Definition 8.2.5. If **C** is a category, we denote by $\mathbf{C}_{\mathrm{ab}}$ the category of abelian objects in **C**. This comes with the inclusion functor $i\colon C_{\mathrm{ab}} \to \mathbf{C}$ which is faithful (but *not* in general full).

Example 8.2.6.

(a) If $\mathbf{C} = \mathbf{Set}$, then $\mathbf{C}_{\mathrm{ab}} = \mathbf{Ab} = \mathbb{Z}\text{-}\mathbf{Mod}$.

(b) If $\mathbf{C} = \mathbf{Group}$, then $\mathbf{C}_{\mathrm{ab}} = \mathbf{Ab}$. Here $i\colon \mathbf{Ab} \to \mathbf{Group}$ is fully faithful.

(c) If $\mathbf{C} = \mathbf{sSet}$ then $\mathbf{C}_{\mathrm{ab}} = \mathbf{sAb}$.

(d) If $\mathbf{C} = \mathbf{Lie}_k$, then $\mathbf{C}_{ab}$ is the category of abelian Lie algebras, i.e. Lie algebras with trivial bracket. Indeed, if $\mathfrak{a}$ is an abelian Lie algebra, then for any Lie algebra $\mathfrak{g}$,

$$\mathrm{Hom}_{\mathbf{Lie}}(\mathfrak{g}, \mathfrak{a}) \cong \mathrm{Hom}_{\mathbf{Lie}}\left({}^{\mathfrak{g}}/_{[\mathfrak{g},\mathfrak{g}]}, \mathfrak{a} \right) \cong \mathrm{Hom}_k\left({}^{\mathfrak{g}}/_{[\mathfrak{g},\mathfrak{g}]}, \mathfrak{a} \right)$$

(e) If $\mathbf{C} = \mathbf{Alg}_k$ (or $\mathbf{dgAlg}_k$ or $\mathbf{sAlg}_k$), then $\mathbf{C}_{ab} = \{0\}$; there is only one abelian object in this category. Similarly for commutative algebras.

Let $\mathbf{C}$ be a model category and let $\mathbf{C}_{ab}$ be the category of abelian objects. Assume that $i\colon \mathbf{C}_{ab} \to \mathbf{C}$ has a left adjoint, namely

$$\mathbf{C} \underset{i}{\overset{\mathrm{Ab}}{\rightleftarrows}} \mathbf{C}_{ab} \quad (\perp)$$

Assume that $\mathbf{C}_{ab}$ also has a model structure and that the adjunction $\mathrm{Ab} \dashv i$ is a Quillen adjunction. Then we also have an adjunction

$$\mathbf{Ho}(\mathbf{C}) \underset{\mathbb{R}i}{\overset{\mathbb{L}\,\mathrm{Ab}}{\rightleftarrows}} \mathbf{Ho}(\mathbf{C}_{ab}) \quad (\perp)$$

Definition 8.2.7. The **Quillen homology** in $\mathbf{C}$ is (the homology theory assoicated to) the derived abelianization functor $\mathbb{L}\,\mathrm{Ab}\colon \mathbf{Ho}(\mathbf{C}) \to \mathbf{Ho}(\mathbf{C}_{ab})$.

In practice, to compute the Quillen homology, we use the fact that for all objects X of $\mathbf{C}$, $\mathbb{L}\,\mathrm{Ab}(X) \cong \mathrm{Ab}(QX)$, where $QX \xrightarrow{\sim} X$ is the cofibrant resolution of X in $\mathbf{C}$.

Remark 8.2.8. If $\mathbf{C}$ is an "algebraic" category, then $\mathbf{C}_{ab}$ is an *abelian* category with a (small) projective generator $P \in \mathrm{Ob}(\mathbf{C}_{ab})$ such that $\mathbf{C}_{ab} \cong \mathrm{End}(P)$-$\mathbf{Mod}$. We denote $\mathbb{Z}(\mathbf{C}) := \mathrm{End}(P)$, and call it the ring associated to $\mathbf{C}$.

8.2.1 Quillen Homology of Simplicial Sets

Example 8.2.9. If $\mathbf{C} = \mathbf{sSet}$, then $\mathbf{C}_{ab} = \mathbf{sAb}$. Given a simplicial set $X_\bullet$, the abelianization is given by the simplicial abelian group $\mathbb{Z}X_\bullet$; which at level n is the free abelian group $\mathbb{Z}X_n$ generated by the set X_n.

Since $\mathbf{sSet}$ is cofibrant, then we may take $QX_\bullet = X_\bullet$ for all simplicial sets $X_\bullet$, that is,

$$\mathbb{L}\,\mathrm{Ab}(X_\bullet) = \mathrm{Ab}(X_\bullet) = \mathbb{Z}X_\bullet.$$

Recall that $|-|\colon \mathbf{sSet} \to \mathbf{Top}$ has a right adjoint the singular complex functor $S\colon \mathbf{Top} \to \mathbf{sSet}$, given by $S(Y)_n = \mathrm{Hom}_{\mathbf{Top}}(\Delta^n, Y)$

Then

$$\pi_*(\mathbb{L}\,\mathrm{Ab}(S(Y))) = \pi_*(\mathrm{Ab}(S(Y))) = H_*\left(N(\mathbb{Z}S(Y))\right) = H_*(Y,\mathbb{Z}),$$

where $N(\mathbb{Z}S(Y))$ is the Dold-Kan complex of the simplicial abelian group $\mathbb{Z}S(Y)$.

Remark 8.2.10. In this case, **sAb** (and **sGroup**) is a fibrant model category by a theorem of Moore (every simplicial group is a Kan complex in **sSet**, and therefore fibrant). Hence, $\mathbb{R}i(A_\bullet) = i(RA_\bullet) = i(A_\bullet)$ for any simplicial abelian group A.

Example 8.2.11. Let $\mathbf{C} = \mathbf{s\,Group}$ and then $\mathbf{C}_{ab} = \mathbf{sAb}$. Then for a simplicial group $\Gamma_\bullet$, the abelianization functor takes the abelianization at each level:

$$\mathrm{Ab}(\Gamma_\bullet)_n = (\Gamma_n)_{ab} = {}^{\Gamma_n}/_{[\Gamma_n,\Gamma_n]}$$

If G is a group, let's think of G as a discrete simplicial group. Then

$$\mathbb{L}\,\mathrm{Ab}(G) \cong \mathrm{Ab}(QG)$$

where $QG \xrightarrow{\sim} G$ is the cofibrant replacement of G. We may assume that $\Gamma_\bullet = QG$ is a semi-free simplicial abelian group.

Theorem 8.2.12 (Quillen). *For all $n \geq 0$, there are natural isomorphisms*

$$\pi_n \mathbb{L}\,\mathrm{Ab}(G) \cong H_{n+1}(G;\mathbb{Z})$$

Sketch. Recall that $H_\bullet(G;\mathbb{Z}) := H_\bullet(BG)$, where BG is the classifying space of G. Let's apply the functor $B\colon$ **Group** $\to$ **sSet** to $\Gamma_\bullet = QG_\bullet$. This gives a bisimplicial (free) abelian group

$$\Gamma = \mathbb{Z}(B_*\Gamma_*) = \{\mathbb{Z}(B_p\Gamma_q)\}_{p,q\geq 0}.$$

There are two filtrations comping from the two different simplicial degrees, giving rise to two different spectral sequences. The E^1 page of the first sequence is the following:

$${}^{I}E^1_{p,q} = H_q(B\Gamma_p) \cong H_q(\Gamma_p,\mathbb{Z}) = \begin{cases} \mathbb{Z} & q = 0 \\ \Gamma_p/[\Gamma_p,\Gamma_p] & q = 1 \\ 0 & q > 1 \end{cases}$$

$${}^{I}E^2_{p,q} = \begin{cases} \pi_p \mathbb{L}\,\mathrm{Ab}(\Gamma) & q = 1,\ \text{any } p \\ Z & p = q = 0 \\ 0 & \text{otherwise} \end{cases}$$

This sequence converges to

$$ {}^{I}H_n(\mathrm{Tot}(\mathbb{Z}B_\bullet\Gamma_\bullet)) = \begin{cases} \mathbb{Z} & n = 0 \\ \pi_{n-1}\mathbb{L}\,\mathrm{Ab}(\Gamma) & n \geq 1 \end{cases} $$

The second spectral sequence has first page:

$$ {}^{II}E^1_{p,q} = \begin{cases} \mathbb{Z}[BG] & p = 0 \\ 0 & p \neq 0. \end{cases} $$

Then this converges to

$$ {}^{II}H_n(\mathrm{Tot}(\mathbb{Z}B_\bullet\Gamma_\bullet) = H_n(BG) $$

for all n. Therefore, comparing the two expressions for the homology of the total complex, we obtain the desired conclusion.

$$ \pi_n\mathbb{L}\,\mathrm{Ab}(\Gamma) \cong H_{n+1}(G;\mathbb{Z}) $$ □

8.2.2 Quillen Homology of Algebras

Example 8.2.13. Let k be a commutative ring and consider the category $\mathbf{Alg}_k$ of k-algebras. Then there is only one abelian group object: $(\mathbf{Alg}_k)_{\mathrm{ab}} = \{0\}$. Fix $A \in \mathrm{Ob}(\mathbf{Alg}_k)$ and consider $\mathbf{C} := {}^{\mathbf{Alg}_k}/_A$, whose objects are algebra maps $\phi\colon B \to A$ with codomain A and morphisms are commutative triangles

$$ \begin{array}{ccc} B & \xrightarrow{\quad f \quad} & B' \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle\phi'} & \\ & A & \end{array} $$

Given $M \in \mathbf{Bimod}(A)$, recall we defined the k-algebra $A \ltimes M = A \oplus M$ with

$$ (a_1, m_1) \cdot (a_2, m_2) = (a_1 a_2, a_1 m_2 + m_1 a_2) $$

This comes with a projection $\pi\colon A \ltimes M \twoheadrightarrow A$, making π an object of $\mathbf{C}$.

Moreover, this construction is functorial in M, giving a functor

$$ \begin{aligned} A \ltimes (-)\colon \mathbf{Bimod}(A) &\longrightarrow {}^{\mathbf{Alg}_k}/_A \\ M &\longmapsto A \ltimes M \end{aligned} $$

Lemma 8.2.14. *This functor is fully faithful, with essential image being* $\mathbf{C}_{\mathrm{ab}}$.

Proposition 8.2.15.

(a) The functor $A \ltimes (-)$ *is fully faithful, with essential image being* $\mathbf{C}_{\mathrm{ab}}$

(b) $A \ltimes (-)$ *has left adjoint*

$$\begin{aligned} \Omega^1_k\left({}^{(-)}/_{A}\right) : {}^{\mathbf{Alg}_k}/_{A} &\longrightarrow \mathbf{Bimod}(A) \\ \left(B \xrightarrow{\phi} A\right) &\longmapsto A \otimes_B \Omega^1(B) \otimes_B A \end{aligned}$$

Proof.

(a) There are inverse functors given by

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Bimod}(A)}(M, M') &\xrightarrow{\cong} \mathrm{Hom}_{\mathbf{Alg}_k/A}(A \ltimes M, A \ltimes M') \\ f &\longmapsto (\mathrm{id}, f) \\ g|_M &\longleftarrow g \end{aligned}$$

Therefore $A \ltimes (-)$ is fully faithful.

Let $f \in \mathrm{Hom}_{\mathbf{Alg}_k/A}(B/A, A \ltimes M/A)$. Then $f\colon B \to A \ltimes M$ is an algebra map such that

$$\begin{array}{ccc} B & \xrightarrow{\quad f \quad} & A \ltimes M \\ & {\scriptstyle \phi}\searrow \qquad \swarrow{\scriptstyle \pi} & \\ & A & \end{array}$$

commutes.

(b) If we write $f = (f_0, D)$ with $f_0\colon B \to A$ and $D\colon B \to M$, then this diagram tells us that $f_0 = \phi$, and f is an algebra homomorphism if and only if D is a derivation $D\colon B \to {}_{\phi}M_{\phi}$. Since f_0 is determined by ϕ, then f is determined by the choice of D.

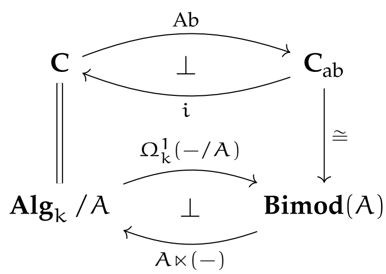
Hence,

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Alg}_k/A}\left({}^{B}/_{A}, {}^{A \ltimes M}/_{A}\right) &\cong \mathrm{Der}_k(B, {}_{\phi}M_{\phi}) \\ &\cong \mathrm{Hom}_{B^e}\left(\Omega^1_k(B), \mathrm{Hom}_{A^e}\left({}_{A^e}A^e_{B^e}, M\right)\right) \\ &\cong \mathrm{Hom}_{A^e}\left(A^e \otimes_{B^e} \Omega^1_k(B), M\right) \\ &= \mathrm{Hom}_{\mathbf{Bimod}(A)}\left(A \otimes_B \Omega^1(B) \otimes_B A, M\right) \end{aligned}$$

The second line follows from Lemma 6.6.11. All of these isomorphisms are natural, so this shows that $A \ltimes (-)$ is right adjoint to $\Omega^1_k(-/A)$.

□

Hence we have the setup for Quillen homology for algebras.

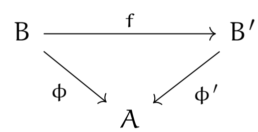


Definition 8.2.16. A right **DG-module** M over a DG-algebra A is a DG-vector space $(M^\bullet, d_M)$ which is a graded A-module:

$$d_M(m \cdot a) = d_M(m) \cdot a + (-1)^{|m|} m \cdot d_A(a).$$

Let $\mathbf{dgAlg}_k$ be the category of DG-algebras over k, and $\mathbf{dgBimod}(A)$ the category of DG-bimodules over a DG-algebra A.

Remark 8.2.17. If $\mathbf{C}$ is a model category, and $A \in \mathrm{Ob}(\mathbf{C})$ is fixed, then $\mathbf{C}/A$ has a model structure where



is a weak equivalence, fibration, or cofibration in $\mathbf{C}/A$ if and only if f is a weak equivalence, fibration, or cofibration in $\mathbf{C}$.

Consider the category $\mathbf{Mor}(\mathbf{C}) = C^{\{\bullet \to \bullet\}}$ with objects all arrows in $\mathbf{C}$ and morphisms commutative squares. This also has a models structure coming from $\mathbf{C}$.

In general, if $\mathbf{I}$ is a small (or even finite) category, then $\mathbf{C}^{\mathbf{I}} = \mathbf{Fun}(\mathbf{I}, \mathbf{C})$ does *not* inherit a model structure from $\mathbf{C}$. But if $\mathbf{I} = \{\bullet \leftarrow \bullet \to \bullet\}$, $\mathbf{I} = \{\bullet \to \bullet \leftarrow \bullet\}$, or $\mathbf{I} = \Delta$ or $\mathbf{I} = \Delta^{op}$, then $\mathbf{C}^{\mathbf{I}}$ does have a model structure.

Lemma 8.2.18. *The adjunction*

$$\Omega^1_k(-/A) \colon \mathbf{dgAlg}_k/A \rightleftarrows \mathbf{dg\,Bimod}(A) \colon A \ltimes (-)$$

is a Quillen adjunction.

Proof. By the basic axioms of model categories, the following are equivalent:

(1) F preserves cofibrations and G preserves fibrations,

(2) F preserves cofibrations and acyclic cofibrations,

(3) G preserves fibrations and acyclic fibrations.

The third one is clear in our case. Therefore,

$$\mathbb{L}\Omega^1_k\left({}^{-}/_{A}\right) : \mathbf{Ho}\left({}^{\mathbf{Alg}_k}/_{A}\right) \rightleftarrows \mathbf{Ho}(\mathbf{dg\,Bimod}(A)) : A \ltimes M$$

□

How do we compute $\mathbb{L}\Omega^1_k\left({}^{-}/_{A}\right)$?

$$\mathbb{L}\Omega^1_k\left({}^{-}/_{A}\right) \cong A \otimes_{QB} \Omega^1_k(QB) \otimes_{QB} A$$

where QB is a fibrant resolution of B.

$$\begin{array}{ccc} QB & \xrightarrow{\sim} & B \\ & {\scriptstyle \phi_P} \searrow \quad \swarrow {\scriptstyle \phi} & \\ & A & \end{array}$$

Definition 8.2.19. The **noncommutative cotangent complex** of A is

$$\mathbb{L}^{NC}_{k/A} := \mathbb{L}\Omega^1_k\left({}^{A}/_{A}\right)$$

where ${}^{A}/_{A} = \mathrm{id}_A : A \to A$.

In particular,

$$\mathbb{L}_{k\backslash A} = A \otimes_{QA} \Omega^1_k(QA) \otimes_{QA} A \in \mathbf{Ho}(\mathbf{dgBimod}_A) = \mathcal{D}\left(\mathbf{dgBimod}(\mathbf{A})\right)$$

Example 8.2.20. If $A = k[x, y]$, then $QA = k\langle x, y, t \mid dt = [x, y]\rangle$. Then

$$\mathbb{L}_{k\backslash k[x,y]} = k[x, y] \otimes_{k\langle x,y,t\rangle} \otimes\, \Omega^1(k\langle x, y, t\rangle) \otimes_{k\langle x,y,t\rangle} \otimes\, k[x, y]$$

What kind of homology theory does this give?

Theorem 8.2.21. *For an ordinary* $\phi : B \to A$ *in* $\mathbf{Alg}_k$*,*

$$H_n\left(\mathbb{L}\Omega^1_k\left({}^{B}/_{A}\right)\right) = \begin{cases} \Omega^1_k\left({}^{B}/_{A}\right) & n = 0 \\ \mathrm{Tor}^B_{n+1}(A, A) & n \geq 1 \end{cases}$$

This is called **André-Quillen Homology**.

8.3 André-Quillen Homology

Recall from Remark 8.2.8 that if $\mathbf{C}$ is an "algebraic" category, then $\mathbf{C}_{ab}$ is an *abelian* category with a (small) projective generator $P \in \mathrm{Ob}(\mathbf{C}_{ab})$ such that $\mathbf{C}_{ab} \cong \mathrm{End}(P)\text{-}\mathbf{Mod}$. We denote $\mathbb{Z}(\mathbf{C}) := \mathrm{End}(P)$, and call it the ring associated to $\mathbf{C}$.

Example 8.3.1.

(a) $\mathbf{C} = \mathbf{Group}$, then $\mathbf{C}_{ab} \cong \mathrm{Ab} = \mathbb{Z}\text{-}\mathbf{Mod}$, and $\mathbb{Z}(\mathbf{C}) = \mathbb{Z}$.

(b) If $\mathbf{C} = \mathbf{LieAlg}_k$, then $\mathbf{C}_{ab} \cong \mathbf{AbLieAlg_k} \cong \mathbf{Vect}_k = k\text{-}\mathbf{Mod}$. Therefore, $\mathbb{Z}(\mathbf{LieAlg}_k) = k$.

(c) If $\mathbf{C} = \mathbf{Alg}_k$, then $\mathbf{C}_{ab} = \{0\}$, so $\mathbb{Z}(\mathbf{Alg}_k) = 0$.

(d) Fix $A \in \mathrm{Ob}(\mathbf{Alg}_k)$ and let $C = \mathbf{Alg}_k\,/A$ be the category of ring homomorphisms whose codomain is A. Then $\mathbf{C}_{ab} \cong \mathbf{Bimod}(A) \cong A^e\text{-}\mathbf{Mod}$. Therefore, $\mathbb{Z}(\mathbf{Alg}_k\,/A) = A \otimes A^{op} = A^e$.

Recall the adjunction

$$\mathbf{Alg}_k/_A \underset{A \ltimes (-)}{\overset{\Omega^1_k(-/A)}{\rightleftarrows}} \mathbf{Bimod}(A) \tag{8.1}$$

where the left adjoint is defined by

$$\Omega^1_k\left({}^{(-)}/_A\right) : \mathbf{Alg}_k/_A \longrightarrow \mathbf{Bimod}(A)$$
$$\left(B \xrightarrow{\phi} A\right) \longmapsto A \otimes_B \Omega^1(B) \otimes_B A$$

Let $\mathbf{C}$ be a model category, and consider the adjunction $\mathrm{Ab}\colon \mathbf{C} \leftrightarrows \mathbf{C}_{ab} : i$. This is a Quillen adjunction, which implies that we may approximate these functors by derived functors and get an adjunction on the level of homotopy categories:

$$\mathbf{Ho}(\mathbf{C}) \underset{\mathbb{R}i}{\overset{\mathbb{L}\,\mathrm{Ab}}{\rightleftarrows}} \mathbf{Ho}(\mathbf{C}_{ab})$$

Now replace in Eq. (8.1) the categories by their DG-analogues. Then we get an adjunction

$$\mathbf{dgAlg}_k/_A \underset{A \ltimes (-)}{\overset{\Omega^1_k(-/A)}{\rightleftarrows}} \mathbf{dg\,Bimod}(A)$$

This gives an adjunction on the level of homotopy categories by the above discussion.

$$\mathbf{Ho}\left({}^{\mathbf{dgAlg}_k}/_A\right) \underset{A\ltimes(-)}{\overset{\mathbb{L}\Omega^1_k(-/A)}{\rightleftarrows}} \perp \ \mathbf{Ho}(\mathbf{dg\,Bimod}(A))$$

where

$$\mathbb{L}\Omega^1_k\left({}^B/_A\right) \cong \Omega^1_k\left({}^{QAB}/_A\right) = A\otimes_{QB}\Omega^1_k(QB)\otimes_{QB}A$$

for some cofibrant replacement QB of B. We write

$${}^{QB}/_A := QB \overset{\sim}{\twoheadrightarrow} B \xrightarrow{\phi} A.$$

Theorem 8.3.2. *If* ${}^B/_A \in \operatorname{Ob}(\mathbf{Alg}_k/A) \subset \operatorname{Ob}(\mathbf{dgAlg}_k/A)$, *then*

$$\mathbb{L}_*\omega^1_K\left({}^B/_A\right) := H_*\left(A\otimes_{QB}\Omega^1_k(QB)\otimes_{QB}A\right) \cong \begin{cases}\Omega^1_k({}^B/_A) & n=0\\ \operatorname{Tor}^B_{n+1}(A,A) & n\geq 1.\end{cases}$$

Proof. Let Q := QB be fixed. Consider its bar complex

$$\mathcal{B}Q := \left[\cdots\to Q\otimes Q^{\otimes 2}\otimes Q \xrightarrow{b'} Q\otimes Q\otimes Q \xrightarrow{b'} Q\otimes Q\to 0\right] \underset{\sim}{\xrightarrow{\mu}} Q$$

Recall that $\Omega^1_k Q \cong \ker(Q\otimes Q\xrightarrow{\mu} Q)$. Therefore,

$$(\tau_{\geq 1}\mathcal{B}Q)[-1] \overset{\sim}{\twoheadrightarrow} \Omega^1_k Q$$

where for all $n\geq 0$,

$$(\tau_{\geq 1}\mathcal{B}Q)[-1]_n = (\tau_{\geq 1}\mathcal{B}_Q)_{n+1} := Q\otimes Q^{\otimes n}\otimes Q.$$

By applying the functor $A\otimes_Q(-)\otimes_Q A$ to this quasi-isomorphism, we get the resolution of A-bimodules

$$A\otimes_Q(\tau_{\geq 1}\mathcal{B}Q)[-1]\otimes_Q A \underset{\text{q-iso}}{\xrightarrow{\sim}} \Omega^1_k\left({}^{QB}/_A\right)$$

The left hand side is the complex

$$\left[\cdots\to A\otimes Q^{\otimes 2}\otimes A\to A\otimes Q\otimes A\to 0\right]$$

If π: QB $\to$ B is the acyclic fibration between B and its fibrant replacement, apply it to the above complex as follows:

$$\begin{array}{c}\left[\cdots \longrightarrow A\otimes Q^{\otimes 2}\otimes A \longrightarrow A\otimes Q\otimes A \longrightarrow 0\right]\\ \qquad\downarrow{\scriptstyle 1\otimes\pi\otimes\pi\otimes 1} \qquad\qquad \downarrow{\scriptstyle 1\otimes\pi\otimes 1}\\ \left[\cdots \longrightarrow A\otimes B^{\otimes 2}\otimes A \longrightarrow A\otimes B\otimes A \longrightarrow 0\right]\end{array}$$

The vertical arrows together define a quasi-isomorphism. This second line is more concisely expressed by the following:

$$A \otimes (\tau_{\geq 1}\mathcal{B}B)[-1] \otimes A$$

Taking the homology of this complex gives us what we want. □

Corollary 8.3.3. *In particular, if* $^A/_A$ *is the object* $A \xrightarrow{\mathrm{id}_A} A$*, then by the theorem,*

$$\mathbb{L}^{NC}_{k\backslash A} := \mathbb{L}Q^q_k\left({^A/_A}\right) \simeq \Omega^1_k(A)$$

for any $A \in \mathrm{Ob}(\mathbf{Alg}_k)$.

Definition 8.3.4. We call $\mathbb{L}^{NC}_{k\backslash A}$ the **noncommutative cotangent complex** of A.

Fix a commutative ring k and let A be a commutative k-algebra. Let $\mathbf{C} = \mathbf{CommAlg}_k/_A$ be the category of commutative algebras over A. Then $\mathbf{C}_{ab} \cong \mathbf{Mod}\text{-}A$. Here, we have an adjunction

$$\Omega^1_{\mathrm{comm}}(-/A) \dashv A \ltimes (-).$$

The left-adjoint is given by

$$(B \xrightarrow{\phi} A) \longmapsto A \otimes_B \Omega^1_{\mathrm{comm}}(B).$$

The right-adjoint is given by the k-algebra $A \ltimes M$ with multiplication

$$(a_1, m_1) \cdot (a_2, m_2) = (a_1 a_2, a_1 m_2 + a_2 m_1).$$

Now replace $\mathbf{CommAlg}_k$ by $\mathbf{dgCommAlg}_k$ or $\mathbf{sCommAlg}_k$ (although we must choose simplicial commutative algebras when the characteristic of k is not zero). Then $\mathbf{Ho}(\mathbf{dg\,Mod}\text{-}A) \cong \mathcal{D}(\mathbf{Mod}\text{-}A)$ is the usual derived category, and we have an adjunction of derived functors

$$\mathbb{L}\Omega^1_{\mathrm{comm}}(-/A) \dashv A \ltimes (-)$$

Definition 8.3.5 (Grothendieck, Illusie)**.** The **cotangent complex** of an affine scheme $\mathrm{Spec}(A)$ is

$$\mathbb{L}_{k\backslash A} := \mathbb{L}\Omega^1_{\mathrm{comm}}(\mathrm{id}_A) \in \mathcal{D}(\mathbf{Mod}\text{-}A)$$

Definition 8.3.6 (André,Quillen)**.**

(a) The **André-Quillen homology** of A is

$$\mathcal{D}_q(k\backslash A) := H_q(\mathbb{L}_{k\backslash A}) = H_q\left(A \otimes_{QA} \Omega^1_{\mathrm{comm}}(QA)\right)$$

(b) If M is an A-module, then the **André-Quillen** homology of M is

$$\mathcal{D}_q(k\backslash A, M) := H_q\left(\mathbb{L}_{k\backslash A} \otimes_A M\right)$$

Theorem 8.3.7 (Quillen)**.** *Suppose A is a finitely generated k-algebra. Then A is (formally) smooth if and only if $\mathbb{L}_{k\backslash A} \simeq \Omega^1_{\mathrm{comm}}(A)$ ($\iff \mathcal{D}_q(k\backslash A) = 0$ for all $q \geq 1$) and $\Omega^1_{\mathrm{comm}}(A)$ is a projective A-module.*

Remark 8.3.8. The theorem also characterizes formally smooth algebras in $\mathbf{Alg}_k$ because $\Omega^1_k(A)$ is a projective (A, A)-bimodule if and only if A is formally smooth.

8.4 Automorphic Sets and Quandles

For references for this section, see [Nel11, Bri88]. To make things confusing, we will follow the definitions from [Bri88] but use the notation of [Nel11].

Definition 8.4.1. A non-empty set R is called **automorphic** (or a **rack**) if there is a binary operation on R

$$\begin{aligned} \triangleright\colon R \times R &\longrightarrow R \\ (x, y) &\longmapsto x \triangleleft y \end{aligned}$$

such that $\ell_x = x \triangleright -\colon R \to R$ is an automorphism of $(R, \triangleright)$.

Equivalently, we can restate this as follows:

(R1) $\ell_x\colon R \to R$ is a bijection for all $x \in R$, and

(R2) $\ell_x(y \triangleright z) = \ell_x(y) \triangleright \ell_x(z)$ for all $x, y, z \in R$.

This second condition says that the operation $\triangleright$ is self-distributive:

$$x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z).$$

Given a rack $(R, \triangleright)$, define a new operation

$$\begin{aligned} \triangleright^{-1}\colon R \times R &\longrightarrow R \\ (x, y) &\longmapsto \ell_x^{-1}(y) = x \triangleright^{-1} y. \end{aligned}$$

Definition 8.4.2. Thus, a rack is a tuple $(R, \triangleright, \triangleright^{-1})$ where $\triangleright, \triangleright^{-1}\colon R \times R \to R$ such that

(R1′) $x \triangleright (x \triangleright^{-1} y) = x \triangleright^{-1} (x \triangleright y) = y$ for all $x, y \in R$.

(R2′) $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$.

We may also rewrite self-distributivity as

$$\ell_x \circ \ell_y = \ell_{\ell_x(y)} \circ \ell_x,$$

which is occasionally useful.

Example 8.4.3. Let G be a group, and define

$$\begin{aligned} \triangleright\colon G \times G &\longrightarrow G \\ (x, y) &\longmapsto xyx^{-1} \end{aligned}$$

This is called a **conjugacy rack**.

Remark 8.4.4. Let **Rack** be the category of racks. Then we have an adjoint pair

$$\mathbf{Rack} \underset{\text{Conj}}{\overset{\text{Gr}}{\rightleftarrows}} \mathbf{Group} \qquad (\perp)$$

$$(R, \triangleright) \longmapsto \mathrm{Gr}(R) := \langle R \mid xyx^{-1} = x \triangleleft y \rangle$$

Example 8.4.5. An alternative way to look at racks is **crossed G-sets** or **augmented racks**: consider quadruples (X, G, ρ, Φ) where

- X is a nonempty set,
- G is a group,
- $\rho\colon G \to \mathrm{Aut}(X)$ is a left action of G on X, and
- $\Phi\colon X \to G$ is an **anchor map**,

satisfying $\Phi(g \cdot x) = g\Phi(x)g^{-1}$. Every such quadruple gives the data of a rack via

$$\begin{aligned} \triangleright\colon X \times X &\longrightarrow X \\ (x, y) &\longmapsto \Phi(x) \cdot y \end{aligned}$$

$$\begin{aligned} \triangleright^{-1}\colon X \times X &\longrightarrow X \\ (x, y) &\longmapsto \Phi(x)^{-1} \cdot y \end{aligned}$$

We should verify that this satisfies the rack axioms (R1′) and (R2′). This is easy to check.

Notice that Φ is a rack homomorphism $X \to \Phi$, where G has the conjugacy rack structure. Φ is commonly called the **augmentation**.

Example 8.4.6. Conversely, given a rack $(R, \triangleright)$, define $\mathrm{Aut}(R)$ as the automorphism group of $(R, \triangleright)$ and let $\mathrm{IAut}(R)$ be the **inner automorphisms** of R.

$$\mathrm{IAut}(R) := \langle \ell_x \in \mathrm{Aut}(R) \mid x \in R \rangle \subseteq \mathrm{Aut}(R).$$

Define also

$$C(R) := \langle \phi \in \mathrm{Aut}(R) \mid \phi \circ \ell_x = \ell_x \circ \phi \, \forall \, x \in R \rangle$$

Then this defines a crossed G-set $(R, G = \mathrm{IAut}(R), \rho, \phi)$ where ρ is the natural action of G on R and $\phi(x) = \ell_x$ for all $x \in R$.

Theorem 8.4.7. *These two constructions in Example 8.4.5 and Example 8.4.6 define an equivalence of categories* **Rack** $\simeq$ $\mathbf{C}^\circ$*, where* $\mathbf{C}$ *is the category of crossed* G*-sets and* $\mathbf{C}^\circ$ *is the full subcategory with objects* (X, G, ρ, ϕ) *such that* ρ *is faithful and* $\mathrm{im}(\phi)$ *generates* G*.*

Definition 8.4.8. The **free rack** on a nonempty set X is a rack $\mathbb{R}\langle X \rangle$ with a set map $j \colon X \to \mathbb{R}\langle X \rangle$ satisfying the universal property

$$\begin{array}{ccc} \mathbb{R}\langle X \rangle & \overset{\exists !}{\dashrightarrow} & R \\ j \uparrow & \nearrow & \\ X & & \end{array}$$

A model for a free rack is defined as a crossed G-set

$$\mathbb{R}\langle X \rangle = \mathbb{F}\langle X \rangle \times X$$

where $\mathbb{F}\langle X \rangle$ is the free group on X and the action is

$$\begin{aligned} \mathbb{F}\langle X \rangle \times \mathbb{R}\langle X \rangle &\longrightarrow \mathbb{R}\langle X \rangle \\ \gamma, (\alpha, x) &\longmapsto (\gamma\alpha, x) \end{aligned}$$

with

$$\begin{aligned} \Phi \colon \mathbb{R}\langle X \rangle &\longrightarrow \mathbb{F}\langle X \rangle \\ (\alpha, x) &\longmapsto \alpha x \alpha^{-1}. \end{aligned}$$

Example 8.4.9. Consider $\mathbb{R}\{x\} = \mathbb{F}\langle x \rangle \times \{x\} \cong \mathbb{F}\{x\} \cong \mathbb{Z}$. An element $(x^n, x) \in \mathbb{F}\langle x \rangle \times \{x\}$ corresponds to $n \in \mathbb{Z}$. Then the rack operation is given by

$$(x^n, x) \triangleright (x^m, x) = (x^{m+1}, x)$$

Hence, $\mathbb{Z}$ is a rack with operation $\triangleright \colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, $n \triangleright m = m + 1$. This does not depend on n.

Definition 8.4.10. The **torsion** of a rack $(R, \triangleright)$ is a map

$$\begin{aligned} \chi \colon R &\longrightarrow R \\ x &\longmapsto x \triangleright x \end{aligned}$$

Fact 8.4.11. $\chi \in C(R)$, *meaning* $\chi(x \triangleright y) = \chi(x) \triangleright \chi(y)$ *for all* $x, y \in R$.

Remark 8.4.12. χ defines an automorphism of the identity functor of **Rack**, and the center of this category $\mathbb{Z}(\textbf{Rack})$ is generated by χ.

Definition 8.4.13. A **Quandle** is a rack with trivial torsion ($\chi = \mathrm{id}_R$).

Proposition 8.4.14. *Let*

$$\mathbb{Q}\langle X \rangle := \mathbb{R}\langle X \rangle \big/ \langle (\alpha, x) \sim (\alpha, x) \triangleright (\alpha, x)$$

The equivalence relation $\sim$ *is compatible with* $\triangleright$ *on* $\mathbb{R}\langle X \rangle$

$$(a, x) \sim (a', x') \text{ and } (b, y) \sim (b, y') \implies (a, x) \triangleright (b, y) \sim (a', x') \triangleright (b', y')$$

and the quotient set $\mathbb{Q}\langle X \rangle$ *becomes a quandle which is the* ***free quandle on*** X.

Proposition 8.4.15. $\mathbb{Q}\langle X \rangle$ *is the union of conjugacy classes of elements* $x \in F\langle X \rangle$.

Example 8.4.16. Let V be a finite-dimensional $\mathbb{R}$-vector space and $(-,-)$ a nondegenerate symmetric bilinear form on V. Then for each nonzero $\alpha \in V$, define reflection s_α through the hyperplane perpindicular to α

$$\begin{aligned} s_\alpha \colon V &\longrightarrow V \\ x &\longmapsto x - \tfrac{2(\alpha, x)}{(\alpha, \alpha)} \alpha \end{aligned}$$

A (non-crystallographic) **root system** in V is $\mathcal{R} \subseteq V \setminus \{0\}$ such that

(1) $|\mathcal{R}| < \infty$

(2) $s_\alpha(\beta) \in \mathcal{R}$ for all $\alpha, \beta \in \mathcal{R}$.

The **coxeter group** of $\mathcal{R} \subseteq V$ is $W(\mathcal{R}) = \langle s_\alpha \mid \alpha \in \mathcal{R} \rangle$. It is always finite when $\mathcal{R}$ is a root system.

A root system is a finite automorphic set with

$$\begin{aligned} \triangleright \colon \mathcal{R} \times \mathcal{R} &\longrightarrow \mathcal{R} \\ (\alpha, \beta) &\longmapsto s_\alpha(\beta) \end{aligned}$$

In this case, $X = \mathcal{R}$, $G = W(\mathcal{R})$, $\Phi \colon \alpha \mapsto s_\alpha$.

Definition 8.4.17. A **knot** in $\mathbb{R}^3$ (or S^3) is a smooth embedding $S^1 \hookrightarrow \mathbb{R}^3$ (or S^3).

A **diagram** D_K of a knot K is a projection of a knot onto a plane in $\mathbb{R}^3$ which is **regular** in the sense that each point in D_K has at most 2-preimages; these should be transverse in the diagram.

Points with two preimages are called **crossings**; let $\mathrm{Cross}(D_K)$ be the set of crossings of the diagram.

Call the connected components of $D_K \setminus \mathrm{Cross}(D_K)$ **arcs**; the set of these is $\mathrm{Arc}(D_K)$.

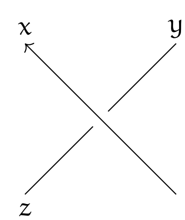
Definition 8.4.18. The **knot group** $\pi(K)$ of a knot K is the fundamental group of the knot compliment: $\pi(K) := \pi_1(\mathbb{R}^3 \setminus K)$.

Joke 8.4.19 (Life Advice)**.** We are like cars. In order to go fast, we must first warm-up.

Definition 8.4.20 (Joyce 1980)**.** The **knot quandle** QK of the knot K in $\mathbb{R}^3$ with regular diagram D_K is defined by the coequalizer

$$\mathrm{coeq}\left(\mathbb{Q}\langle \mathrm{Cross}(D_K)\rangle \overset{f}{\underset{g}{\rightrightarrows}} \mathbb{Q}\langle \mathrm{Arc}(D_K)\rangle\right)$$

where the top map f takes a crossing c to $x \rhd y$ and the bottom map g sends a crossing c to z.



A crossing c.

Proposition 8.4.21 (Joyce)**.** $\mathrm{Gr}(QK) \cong \pi(K)$*, where* $\pi(K)$ *is the knot group of* K*, and* Gr *is as in Remark 8.4.4.*

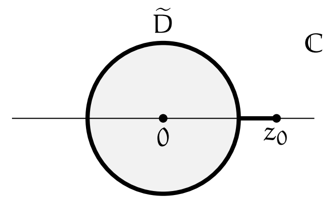
Proof. Gr maps the presentation of QK to the classical Wirtinger presentation $z = xyx^{-1}$. □

8.4.1 Topological Interpretation

Let $\mathbf{C}$ be a category of triples (X, Y, x_0) where X is a topological space, $Y \subseteq X$ is a subspace, and $* \in X \setminus Y$. A morphism $f\colon (X, Y, x_0) \to (X', Y', x_0')$ is a map $f\colon X \to X'$ such that $f^{-1}(Y') = Y$ and $f(x_0) = x_0'$.

Define $(\widetilde{D}, \{0\}, z_0) \in \mathrm{Ob}(\mathbf{C})$ as follows:

- $z_0 \in \mathbb{R}$ with $z_0 > 1$.
- $\widetilde{D}$ is the union of the unit disk in the complex plane $\mathbb{C}$ with the line connecting 0 and z_0. This set is called a **noose**.



Define for every $(X, Y, *)$ the homotopy classes of maps into $(X, Y, *)$ from the noose $(\widetilde{D}, \{0\}, z_0)$.

$$\Delta(X, Y, *) := [(\widetilde{D}, \{0\}, z_0), (X, Y, *)]_{\mathbf{Ho}(\mathbf{C})}$$

There is a natural action of $\pi_1(X \setminus Y, *)$ on $\Delta(X, Y, *)$ given as follows. If $[f] \in \Delta(X, Y, *)$ is represented by $f\colon \widetilde{D} \to X$, then for $\gamma \in \pi_1(X \setminus Y, *)$, $\gamma(f)\colon \widetilde{D} \to X$ is defined by mapping $I \subseteq [1, z_0]$ by γ and the rest by restriction and reparameterization of f.

The anchor map $\varphi\colon \Delta(X, Y, *) \to \pi_1(X \setminus Y, *)$ is given by

$$\varphi\colon f \mapsto f|_{\partial \widetilde{D}}.$$

Then $(\Delta(X, Y, *), \pi_1(X \setminus Y, *), \varphi)$ is a crossed set. Then the associated quandle

$$(\Delta(X, Y, *), \triangleright)$$

is called the **fundamental quandle** of $(X \setminus Y, *)$.

Now take a knot $K \subseteq \mathbb{R}^3$ and apply this construction. We get a map

$$\Phi\colon \Delta(\mathbb{R}^3, K, *) \xrightarrow{\ \varphi\ } \pi_1(\mathbb{R}^3 \setminus K, *) \xrightarrow{\ \mathrm{ab}\ } H_1(\mathbb{R}^3 \setminus K; \mathbb{Z}) \cong \mathbb{Z}$$

The latter map is the abelianization map, taking a meridian m of the knot compliment to a generator $\overline{m}$ of H_1.

Proposition 8.4.22 (Joyce)**.** $QK \cong \Phi^{-1}(\overline{m})$ *is a subquandle of the fundamental quandle of* K.

Theorem 8.4.23 (Joyce)**.** QK *is a complete knot invariant.*

Remark 8.4.24. QK is a complete knot invariant, whereas $\pi(K)$ is not!

8.4.2 Quillen Homology of Racks

We want to construct an abalianization functor for racks and quandles, and compute $\Omega(QK)$.

Definition 8.4.25. The **Alexander module** M_K is a $\mathbb{Z}[t^{\pm 1}]$-module associated to the knot K.

Theorem 8.4.26 (Joyce)**.** $\Omega(QK)$ *is isomorphic to the classical Alexander module of* K*.*

Definition 8.4.27. An abelian rack is an abelian object in **Rack**, or equivalently a rack object in **Ab**.

Explicitly, M is an abelian object in **Rack** if and only if

(1) M has the structure of an abelian group $(M, +, 0)$; and

(2) $\triangleright\colon M \times M \to M$ is a homomorphism of abelian groups.

The latter of these two conditions says that

$$\triangleright((m,p) + (n,q)) = \triangleright(m,p) + \triangleright(n,q)$$

for all $m, n, q, p \in M$, or equivalently

$$(m+n) \triangleright (p+q) = (m \triangleright p) + (n \triangleright q).$$

In particular, we have an automorphism of M

$$\begin{aligned} \alpha\colon M &\longrightarrow M \\ x &\longmapsto 0 \triangleright x \end{aligned}$$

and an endomorphism of M

$$\begin{aligned} \varepsilon\colon M &\longrightarrow M \\ x &\longmapsto x \triangleright 0 \end{aligned}$$

Therefore,

$$x \triangleright y = (x+0) \triangleright (0+y) = x \triangleright 0 + 0 \triangleright y = \varepsilon(x) + \alpha(y).$$

Hence, α, ε completely determine the rack operation $\triangleright$. Moreover, α and ε commute:

$$\varepsilon(\alpha(y)) = \varepsilon(0 \triangleright y) = (0 \triangleright y) \triangleright 0 = (0 \triangleright y) \triangleright (0 \triangleright 0) = 0 \triangleright (y \triangleright 0) = \alpha(\varepsilon(y))$$

Theorem 8.4.28. *The category of abelian racks is isomorphic to the category of modules over the ring*

$$A = {}^{\mathbb{Z}[t^{\pm 1}, e]}\big/_{\langle e^2 - e(1-t)\rangle}.$$

Proof. It is easy to show that every A-module has a rack structure

$$\begin{aligned} \triangleright \colon M &\longrightarrow M \\ (x, y) &\longmapsto ex + ty. \end{aligned}$$

Let's check – it's clear that $\triangleright$ is a homomorphism of abelian groups, but we need to check

$$\ell_x(y \triangleright z) = \ell_x(y) \triangleright \ell_x(z)$$

The left hand side is

$$\begin{aligned} \ell_x(y \triangleright z) &= \ell_x(ey + tz) \\ &= ex + t(ey + tz) \\ &= ex + tey + t^2 z \end{aligned}$$

The right hand side is

$$\begin{aligned} \ell_x(y) \triangleright \ell_x(z) &= (ex + ty) \triangleright (ex + tz) \\ &= e(ex + ty) + t(ex + tz) \\ &= (e^2 + te)x + ety + t^2 z \end{aligned}$$

Since $e^2 - e + te = 0$, these coincide.

Conversely, Let M be an abalian rack. Note that for all $x \in M$,

$$\begin{aligned} \varepsilon(x) = x \triangleright 0 &= x \triangleright (0 \triangleright 0) \\ &= (x \triangleright 0) \triangleright (x \triangleright 0) \\ &= \varepsilon(x) \triangleright \varepsilon(x) \\ &= (\varepsilon(x) \triangleright 0) + (0 \triangleright \varepsilon(x)) \\ &= \varepsilon^2(x) + \alpha\varepsilon(x) \end{aligned}$$

Therefore,

$$\varepsilon^2 + \alpha\varepsilon - \varepsilon \sim 0$$

in an abelian rack M. Then as before, we have $\varepsilon\alpha = \alpha\varepsilon$ and $\varepsilon^2 - (1-\alpha)\varepsilon = 0$. Hence, M is an A-module. □

Corollary 8.4.29. *The category of abelian quandles is equivalent to the category of modules over* $\mathbb{Z}[t^{\pm 1}]$.

Proof. Recall that Quandles are racks $(R, \triangleright)$ such that $x \triangleright x = x$. By Theorem 8.4.28,

$$\mathbf{Quandle}_{\mathrm{ab}} \subseteq \mathbf{Rack}_{\mathrm{ab}} \cong A\text{-}\mathbf{Mod}\,.$$

So any abelian quandle M is an A-module, where A is as in Theorem 8.4.28. Moreover, for all $x \in M$,

$$x = x \triangleright x = ex + tx \implies (e + t - 1)x = 0.$$

Now define $f \colon A \to \mathbb{Z}[t^{\pm 1}]$ by

$$\begin{aligned} A &\xrightarrow{\;f\;} \mathbb{Z}[t^{\pm 1}] \\ t &\longmapsto t \\ e &\longmapsto e. \end{aligned}$$

Then $f_* \colon \mathbb{Z}[t^{\pm 1}]\text{-}\mathbf{Mod} \hookrightarrow A\text{-}\mathbf{Mod}$ induces the inclusion of $\mathbf{Quandle}_{\mathrm{ab}}$ into $\mathbf{Rack}_{\mathrm{ab}}$. □

Corollary 8.4.30. *The inclusion functor* $i \colon \mathbf{Rack}_{\mathrm{ab}} \hookrightarrow \mathbf{Rack}$ *has left-adjoint (abelianization) given by*

$$\begin{aligned} \Omega \colon \mathbf{Rack} &\longrightarrow \mathbf{Rack}_{\mathrm{ab}} \cong A\text{-}\mathbf{Mod} \\ (R, \triangleright) &\longmapsto {\mathbb{Z}R}/{(x \triangleright y = ex + ty)} \end{aligned}$$

Similarly, for quandles, we have

$$\begin{aligned} \Omega \colon \mathbf{Quandle} &\longrightarrow \mathbf{Quandle}_{\mathrm{ab}} \cong \mathbb{Z}[t^{\pm 1}]\text{-}\mathbf{Mod} \\ (Q, \triangleright) &\longmapsto {\mathbb{Z}Q}/{(x \triangleright y = (1-t)x + ty)} \end{aligned}$$

Now we want to apply this to knots. Recall that for a knot K in $\mathbb{R}^3$ or S^3, we defined the knot quandle QK as a coequalizer in Definition 8.4.20.

$$\operatorname{coeq}\left(Q\langle \mathrm{Cross}(D_K) \rangle \underset{g}{\overset{f}{\rightrightarrows}} Q\langle \mathrm{Arc}(D_K) \rangle \right)$$

It is convenient to define the knot quandle this way because we want to apply Ω to it, and Ω is a left-adjoint functor so preserves colimits.

Proposition 8.4.31. *For any knot* K*, with diagram* D_K*,*

$$\Omega(QK) \cong \operatorname{coker}\left(\begin{aligned} \mathbb{Z}[t^{\pm 1}]\langle \mathrm{Cross}(D_K) \rangle &\longrightarrow \mathbb{Z}[t^{\pm 1}]\langle \mathrm{Arc}(D_K) \rangle \\ c &\longmapsto (1-t)x + ty - z \end{aligned} \right)$$

Proof. Since Ω is a left adjoint, then Ω preserves colimits. In an abelian category, the coequalizer of f and g is the cokernel of the difference $f - g$. $\square$

Example 8.4.32. Let K be the trefoil knot. Then

$$\Omega(QK) = \operatorname{coker}\left(\mathbb{Z}[t^{\pm 1}]^{\oplus 3} \xrightarrow{\begin{pmatrix} 1-t & -1 & t \\ t & 1-t & -1 \\ -1 & t & 1-t \end{pmatrix}} \mathbb{Z}[t^{\pm 1}]^{\oplus 3} \right)$$
$$\cong \mathbb{Z}[t^{\pm 1}] / (t^2 - t + 1) \oplus \mathbb{Z}[t^{\pm 1}]$$

Remark 8.4.33. It's interesting to compare this with groups. In that case, the functor Ω: **Group** $\to$ **Ab** is the usual abelianization functor. Applying this to the knot group $\pi K \cong \mathrm{Gr}(QK)$ gives $\Omega(\pi K) \cong \mathbb{Z}$. This doesn't depend on the knot! So the knot quandle QK is a stronger invariant.

Let K be a knot and let $X = S^3 \setminus K$. Then the commutator subgroup $\pi_1(X, *)^{(1)}$ of $\pi_1(X, *)$ defines a covering space $\widetilde{X}$ of $\pi_1(X, *)$ called the **infinite cyclic cover**.

$$\pi_1(S^3 \setminus K, *)^{(1)} \to \pi_1(S^3 \setminus K, *) \to \mathbb{Z}$$

$\mathbb{Z}$ acts on $\widetilde{X}$ by covering transformations; write $\mathbb{Z} = \{t^n\}_{n \in \mathbb{Z}}$. This makes $H_1(\widetilde{X}; \mathbb{Z})$ into a module over $\mathbb{Z}[t^{\pm 1}]$.

Definition 8.4.34. The **Alexander module** A_K of a knot K is $A_K = H_1(\widetilde{X}; \mathbb{Z})$, thought of as a $\mathbb{Z}[t^{\pm 1}]$-module.

For any knot

$$\mathcal{A}_K \cong \mathbb{Z}[t^{\pm 1}] / (\Delta_K(t)) \oplus \mathbb{Z}[t^{\pm 1}],$$

where $\Delta_K(t)$ is the **Alexander polynomial** of K. The annihilator of the torsion part of $\mathcal{A}_K$, is the module generated by this polynomial.

Theorem 8.4.35 (Joyce)**.** $\Omega(QK)$ *is isomorphic to the classical Alexander module* $\mathcal{A}_K$ *of a knot* K*.*

Can we construct $\mathcal{A}_K$ from groups instead of quandles? The answer is yes, if we work relatively. Work in the category **Group** $/\mathbb{Z}$ of groups over $\mathbb{Z}$.

Exercise 8.4.36. Show that

$$\Omega^1\left({}^{-}/_{\mathbb{Z}}\right) : {}^{\mathbf{Group}}/_{\mathbb{Z}} \leftrightarrows \left({}^{\mathbf{Group}}/_{\mathbb{Z}}\right)_{\mathrm{ab}} \cong \mathbb{Z}[t^{\pm 1}]\text{-}\mathbf{Mod} : (\mathbb{Z} \ltimes -)$$

is an adjunction, and moreover

$$\Omega^1\left({}^{\pi K}/_{\mathbb{Z}}\right) \cong \mathcal{A}_K.$$

8.4.3 Rack (co)homology

Given a rack $(X, \triangleright)$, we define a chain complex

$$CR_*(X) = \left[\cdots \to CR_n(X) \xrightarrow{\delta} CR_{n-1}(X) \to \cdots\right]$$

where

$$CR_n(X) = \mathbb{Z}[X^{n+1}] = \mathbb{Z}[\underbrace{X \times X \times \cdots \times X}_{n+1}]$$

and $\delta\colon CR_n(X) \to CR_{n-1}(X)$ is defined by $\delta = \delta^0 - \delta^1$, where

$$\delta^i = \sum_{j=1}^{n} (-1)^j \delta^i_j$$

$$\delta^0_j(x_1, \dots, x_n) = (x_1, \dots, \widehat{x_j}, \dots, x_n)$$

$$\delta^1_j(x_1, \dots, x_n) = (x_1, \dots, x_{j-1}, x_j \triangleright x_{j+1}, x_j \triangleright x_{j+2}, \dots, x_j \triangleright x_n)$$

$$\vdots$$

Brute force calculation shows that $\delta^2 = 0$.

Example 8.4.37. $\delta\colon CR_1(X) \to CR_0(X)$ is given by $\delta(x, y) = x - x \triangleright y$

Definition 8.4.38. For an abelian group A, we define the **Rack homology**

$$HR_*(X; A) := H_*(CR_*(X) \otimes_{\mathbb{Z}} A)$$

and the **Rack cohomology**

$$HR^*(X; A) := H^*(\mathrm{Hom}(CR_*(X), A))$$

Example 8.4.39. $HR_0(X) \cong \mathbb{Z}$ and $HR_1(X) = \mathbb{Z}[\text{orbits of } X]$.

Theorem 8.4.40. *Rack Homology is Quillen Homology (up to degree shifting).*

This says that there is an adjunction

$$\mathbb{L}\Omega\colon \mathbf{Ho}(\mathbf{sRack}) \leftrightarrows \mathbf{s}(A\text{-}\mathbf{Mod})_{ab} \cong \mathcal{D}(\mathbf{Ch}_{\geq 0}(A\text{-}\mathbf{Mod})),$$

and

$$\pi_*(\mathbb{L}\Omega) \otimes \mathbb{Z} \cong HR_{*+1}(-; \mathbb{Z}).$$

Remark 8.4.41. For quandles, $\mathbb{L}\Omega\colon \mathbf{Ho}(\mathbf{sQuandle}) \to \mathcal{D}(\mathbb{Z}[t^{\pm 1}] - \mathbf{Mod})$ is given by

$$\mathbb{L}\Omega(QK) \cong \mathrm{cone}\left(\mathbb{Z}[t^{\pm 1}]\langle \mathrm{Cross} \rangle \longrightarrow \mathbb{Z}[t^{\pm 1}]\langle \mathrm{Arc} \rangle\right)$$

$$c \longmapsto (1-t)x + ty - z$$

Part III

Differential Graded Categories

Chapter 9

Differential Graded Categories

Throughout this section, we make the following assumptions and use the following notation.

- k is a commutative ring (e.g. a field or $\mathbb{Z}$).
- **Mod**(k) is the category of k-modules.
- We will work with both cohomologically and homologically graded modules: in the first case, we write $V^\bullet = \bigoplus_{p\in\mathbb{Z}} V^p$ and in the second case, we write $V_\bullet = \bigoplus_{p\in\mathbb{Z}} V_p$, following the standard convention.
- The degree shift functor is defined by $V^\bullet[1] = \bigoplus_{p\in\mathbb{Z}} V^{p+1}$; this means that $V^\bullet[1]^p = V^{p+1}$ for cohomologically graded modules, and $V_\bullet[1]_p = V_{p-1}$ in the homological case.
- If $f\colon V \to W$ is of degree n, then $f(V^p) \subseteq W^{p+n}$ for all $p \in \mathbb{Z}$. This is equivalent to $f\colon V \to W[n]$ being degree zero.
- If V, W are graded k-modules, then $V \otimes W = \bigoplus_{n\in\mathbb{Z}} (V \otimes W)^n$ and

$$(V \otimes W)^n = \bigoplus_{p+q=n} V^p \otimes W^q$$

- We follow the **Koszul sign rule**: if $f\colon V \to V'$ and $g\colon W \to W'$, then

$$(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w).$$

- For a category **C**, write $\mathbf{C}(X,Y) := \mathrm{Hom}_{\mathbf{C}}(X,Y)$ for $X, Y \in \mathrm{Ob}(\mathbf{C})$.

- We often call complexes of k-modules **differential graded (DG) k-modules**. If (V, d_V) is a DG k-module, then $V[1]$ has differential $d_{V[1]} = -d_V$.
- Similarly $(V, d_V) \otimes (W, d_W) = (V \otimes W, d_{V \otimes W})$ where

$$d_{V \otimes W} = d_V \otimes \mathrm{id}_W + \mathrm{id}_V \otimes d_W$$

  with the Koszul sign rule in mind.

We begin with the following definition.

Definition 9.0.1. A k**-category A** is a category enriched in $\mathbf{Mod}(k)$.

Explicitly, **A** consists of the following data:

(1) a set of objects $\mathrm{Ob}(\mathbf{A})$;

(2) for every pair $X, Y \in \mathrm{Ob}(\mathbf{A})$, a k-module $\mathbf{A}(X, Y)$;

(3) for every triple $X, Y, Z \in \mathrm{Ob}(\mathbf{A})$, a k-bilinear map

$$\begin{array}{ccc} \mathbf{A}(Y, Z) \times \mathbf{A}(X, Y) & \longrightarrow & \mathbf{A}(X, Z) \\ (f, g) & \longmapsto & f \circ g \end{array}$$

satisfying the usual associativity condition;

(4) for each $X \in \mathrm{Ob}(\mathbf{A})$, $1_X \in \mathbf{A}(X, X)$ satisfying the unit condition with respect to composition.

Notice that k-bilinearity of the composition map in (3) above means that it factors through the tensor product of k-complexes:

$$\begin{array}{ccc} \mathbf{A}(Y, Z) \times \mathbf{A}(X, Y) & \xrightarrow{\quad\circ\quad} & \mathbf{A}(X, Z) \\ & {\scriptstyle \otimes} \searrow \qquad \nearrow {\scriptstyle \circ} & \\ & \mathbf{A}(Y, Z) \otimes \mathbf{A}(X, Y) & \end{array}$$

We will distinguish between "k-categories" and "k-linear categories."

Definition 9.0.2. A k**-linear category** is a k-category **A** that has all finite direct sums.

Example 9.0.3. An associative unital k-algebra is a k-category with one object, but it is *not* k-linear.

Example 9.0.4. Let A be a k-algebra and let $e_1, \ldots, e_n$ be a complete set of orthogonal idempotents:

- $e_i^2 = e_i$ for all $i = 1, 2, \dots, n$;
- $e_i e_j = 0$ for $i \neq j$;
- $\sum_{i=1}^n e_i = 1_A$.

Then define a category $\underline{A}$ with objects $\{e_1, \dots, e_n\}$ and $\underline{A}(e_i, e_j) = e_j A e_i$ as a k-category. Composition is given by the product in A. This is occasionally called the **Pierce decomposition** of A.

As an example, consider a finite quiver $Q = (Q_0, Q_1)$ where $\{e_i\}_{i \in Q_0}$ are vertex idempotents. Then for $A = kQ$, $\underline{A} = \underline{kQ}$ is the **path category** of Q.

Example 9.0.5. If B is a k-algebra, then $\mathbf{Mod}(B)$ is a k-linear category.

Definition 9.0.6. A **DG-category A** is a category enriched in the category of complexes of k-modules.

Explicitly, **A** consists of the data:

(1) A set of objects $\mathrm{Ob}(\mathbf{A})$.

(2) For all $X, Y \in \mathrm{Ob}(\mathbf{A})$, a complex of morphisms $\mathbf{A}(X, Y) \in \mathbf{Com}(k)$.

(3) The composition of morphisms is a morphisms of complexes and factors through the tensor product of complexes

$$\mathbf{A}(Y, Z) \otimes \mathbf{A}(X, Y) \xrightarrow{\circ} \mathbf{A}(X, Z),$$

satisfying the usual associativity condition.

(4) For all $X \in \mathrm{Ob}(\mathbf{A})$, $1_X \in \mathbf{A}(X, X)$.

Remark 9.0.7. We call a DG-category **A homological** (resp. **cohomological**) if its morphism complexes $\mathbf{A}(X, Y)$ are homologically (resp. cohomologically) graded. We will deal with both homological and cohomological DG-categories.

Example 9.0.8. A differential graded k-category with a single object $*$ is a unital associative differential graded k-algebra **A**. Indeed, by Definition 9.0.6(3), the following must commute:

$$\begin{array}{ccc} \mathbf{A}(*,*) \otimes \mathbf{A}(*,*) & \xrightarrow{\circ} & \mathbf{A}(*,*) \\ \downarrow d_{A \otimes A} & & \downarrow d_A \\ \mathbf{A}(*,*) \otimes \mathbf{A}(*,*)[1] & \xrightarrow{\circ} & \mathbf{A}(*,*)[1] \end{array}$$

Chasing the diagram around counterclockwise, we get

$$\begin{aligned} \circ d_{A \otimes A}(f \otimes g) &= \circ(d_A \otimes \mathrm{id}_A + \mathrm{id}_A \otimes d_A)(f \otimes g) \\ &= \circ(d_A f \otimes g + (-1)^{|d_A||f|} f \otimes d_A g) \\ &= (df)g + (-1)^{|f|} f\, dg \end{aligned}$$

Chasing the diagram around clockwise, however, we get $d_A(fg)$. Hence we recover the Leibniz rule:

$$d_A(fg) = (df)g + (-1)^{|f|} f\, dg$$

Fix a k-algebra B. Define

$$\mathbf{C}(B) := \mathbf{Com}(\mathbf{Mod}(B))$$

the category of complexes of right B-modules. This is not a DG-category, but just a k-category. We can define a DG-category by enlarging homs as follows.

Definition 9.0.9. The **morphism complex** of $X, Y \in \mathrm{Ob}(\mathbf{C}(B))$ is the complex

$$\underline{\mathrm{Hom}}_B^\bullet(X, Y) := \bigoplus_{n\in\mathbb{Z}} \underline{\mathrm{Hom}}_B^n(X, Y)$$

where

$$\underline{\mathrm{Hom}}_B^n(X, Y) = \prod_{p\in\mathbb{Z}} \mathrm{Hom}_B(X^p, Y^{p+n}).$$

This is precisely the set of morphisms of graded B-modules of degree n. The differential $d_{\mathrm{Hom}} \colon \mathrm{Hom}_B^\bullet(X, Y) \to \mathrm{Hom}_B^{\bullet+1}(X, Y)$ is defined by

$$d_{\mathrm{Hom}}(f) := [d, f] = d_Y \circ f - (-1)^{|f|} f \circ d_X \colon \underline{\mathrm{Hom}}_B^n(X, Y) \to \underline{\mathrm{Hom}}_B^{n+1}(X, Y)$$

Since $d_Y^2 = 0 = d_X^2$, we have $d_{\mathrm{Hom}}^2 = 0$.

Definition 9.0.10. $\mathbf{C}_{DG}(B)$ is the DG category of complexes of B-modules. The objects of $\mathbf{C}_{DG}(B)$ are the objects of $\mathbf{C}(B)$, that is, complexes of B-modules, and

$$\mathbf{C}_{DG}(B)(X, Y) := \underline{\mathrm{Hom}}_B^\bullet(X, Y)$$

What's the relation between $\mathbf{C}_{DG}(B)$ and $\mathbf{C}(B)$? Consider

$$\underline{\mathrm{Hom}}_B^\bullet(X, Y) = \left[\cdots \to \underline{\mathrm{Hom}}^{-1} \xrightarrow{d_{\mathrm{Hom}}^{-1}} \underline{\mathrm{Hom}}^0 \xrightarrow{d_{\mathrm{Hom}}^0} \underline{\mathrm{Hom}}^1 \to \cdots \right]$$

Define

$$Z^0(\underline{\mathrm{Hom}}^\bullet(X, Y)) := \ker(d_{\mathrm{Hom}}^0)$$

this is the set of $f^\bullet \colon X^\bullet \to Y^\bullet$ of degree zero such that $d_Y f = f d_X$; the usual morphisms of complexes $\mathbf{C}(B)(X, Y)$.

Now consider the homology in degree 0:

$$H^0\left(\underline{\mathrm{Hom}}_B^\bullet(X, Y)\right) = {\ker d_{\mathrm{Hom}}^0}\big/{\mathrm{im}\, d_{\mathrm{Hom}}^{-1}}$$

Here $\operatorname{im}(d_{\text{Hom}}^{-1})$ is the set of those $f^\bullet \colon X^\bullet \to Y^\bullet$ such that

$$f = d_{\text{Hom}}^{-1}(h) = d_Y \circ h + h \circ d_X$$

for $h \in \underline{\text{Hom}}_B^\bullet(X,Y)$. More concisely stated, this is the null-homotopic morphisms $f^\bullet \colon X^\bullet \to Y^\bullet$ in $\mathbf{C}(B)$.

Now let $\mathcal{H}(B)$ be the category of complexes of B-modules up to homotopy. Then

$$\mathcal{H}(B)(X,Y) = \mathbf{C}(B)(X,Y)/\sim$$

where $\sim$ is homotopy equivalence, and

$$H^0\left(\underline{\text{Hom}}_B^\bullet(X,Y)\right) = \mathcal{H}(B)(X,Y).$$

Thus, we see that $\mathbf{C}(B)(X,Y) = Z^0\left[\underline{\text{Hom}}^\bullet(X,Y)\right]$ and $\mathcal{H}(B)(X,Y) = H^0\left[\underline{\text{Hom}}^\bullet(X,Y)\right]$.

Remark 9.0.11.

(a) $f \sim g \implies H^i(f) = H^i(g)$ for all $i \in \mathbb{Z}$.

(b) We say that $f \colon X \to Y$ is a **homotopy equivalence** if there is some $g \colon Y \to X$ such that $fg \sim 1_Y$ and $gf \sim 1_X$.

From both (a) and (b) we conclude that if f is a homotopy equivalence, then f is a quasi-isomorphism. But the converse is not true!

Example 9.0.12. Consider $f \colon X \to 0$ where

$$X = \left[0 \to k[t] \xrightarrow{t} k[t] \xrightarrow{t=0} k \to 0\right]$$

Then f is a quasi-isomorphism but *not* a homotopy equivalence.

Definition 9.0.13. Let $\mathbf{A}$ be a DG k-category. Define:

(a) The **opposite category** $\mathbf{A}^{\text{op}}$ where $\mathbf{A}^{\text{op}}(X,Y) := \mathbf{A}(Y,X)$.

(b) The category $\mathbf{Z}^0(\mathbf{A})$ has the same objects as $\mathbf{A}$ and morphisms

$$\mathbf{Z}^0(\mathbf{A})(X,Y) := Z^0\left(\mathbf{A}(X,Y)\right).$$

(c) The category $\mathbf{H}^0(\mathbf{A})$ has the same objects as $\mathbf{A}$ and morphisms

$$\mathbf{H}^0(\mathbf{A})(X,Y) := H^0(\mathbf{A}(X,Y)).$$

Example 9.0.14. If $\mathbf{A} = \mathbf{C}_{DG}(B)$, then $Z^0(\mathbf{A}) = \mathbf{C}(B)$ and $\mathbf{H}^0(\mathbf{A}) = \mathcal{H}(B)$.

9.1 DG functors

Definition 9.1.1. A functor $F\colon \mathbf{A} \to \mathbf{B}$ between DG-categories is called a **DG-functor** if for all $X, Y \in \mathrm{Ob}(\mathbf{A})$,

$$F(X,Y)\colon \mathbf{A}(X,Y) \to \mathbf{B}(FX, FY)$$

is a morphism of complexes of k-modules compatible with composition.

Example 9.1.2. If **A** and **B** are DG-k-algebras, that is DG-categories with a single object, then $F\colon A \to B$ is a DG-functor if and only if F is a homomorphism of DG-algebras.

Definition 9.1.3. Given two DG-functors $F, G\colon \mathbf{A} \to \mathbf{B}$, a **DG natural transformation** $\alpha\colon F \to G$ is given by

$$\alpha = \left\{ (\alpha_X\colon F(X) \to G(X)) \in Z^0\left(\mathbf{B}(FX, GX)\right) \right\}_{X \in \mathrm{Ob}(\mathbf{A})}$$

such that the following diagram commutes for all $f\colon X \to Y$:

$$\begin{array}{ccc} F(X) & \xrightarrow{\alpha_X} & G(X) \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\alpha_Y} & G(Y) \end{array}$$

Definition 9.1.4. $\mathbf{Fun}(\mathbf{A}, \mathbf{B})$ is the k-category of all DG-functors $\mathbf{A} \to \mathbf{B}$ with morphisms being DG natural transformations.

We enrich the k-category $\mathbf{Fun}(\mathbf{A}, \mathbf{B})$ into a DG-category $\mathbf{Fun}_{DG}(\mathbf{A}, \mathbf{B})$ in the same way as we enrich $C(\mathbf{B})$ to $C_{DG}(\mathbf{B})$.

Define for two fixed DG-functors $F, G\colon \mathbf{A} \to \mathbf{B}$ the **complex of graded morphisms** as follows.

$$\mathcal{H}\mathrm{om}^\bullet(F, G) := \bigoplus_{n \in \mathbb{Z}} \mathcal{H}\mathrm{om}^n(F, G)$$

where $\mathcal{H}\mathrm{om}^n(F, G)$ is the k-module of all DG-natural transformations α with $\alpha_X \in \mathbf{B}(FX, GX)^n$ for all $X \in \mathrm{Ob}(\mathbf{A})$.

The differential

$$d_{\mathcal{H}\mathrm{om}}\colon \mathcal{H}\mathrm{om}^n \to \mathcal{H}\mathrm{om}^{n+1}$$

is induced from the one on $\mathbf{B}(FX, GX)$.

Definition 9.1.5. The category $\mathbf{Fun}_{DG}(\mathbf{A}, \mathbf{B})$ has the same objects as the k-category $\mathbf{Fun}(\mathbf{A}, \mathbf{B})$, namely the DG-functors $\mathbf{A} \to \mathbf{B}$, but the morphisms are now

$$\mathbf{Fun}_{DG}(\mathbf{A}, \mathbf{B})(F, G) := \mathcal{H}\mathrm{om}^\bullet(F, G).$$

Remark 9.1.6. Note that $\mathbf{Fun}(\mathbf{A}, \mathbf{B}) = Z^0\left(\mathbf{Fun}_{DG}(\mathbf{A}, \mathbf{B})\right)$.

9.2 The DG category of small DG categories

Definition 9.2.1. The category $\mathbf{dgCat}_k$ is the category of small DG k-categories, whose objects are small DG k-categories and whose morphisms are DG-functors.

Our goal is to study the homotopy theory of $\mathbf{dgCat}_k$. Here are some properties of $\mathbf{dgCat}_k$:

- $\mathbf{dgCat}_k$ has an initial object, which is the empty DG-category, and a terminal object $*$, which is the one object category with $\mathrm{End}(*) = 0$.
- $\mathbf{dgCat}_k$ is monoidal. If $\mathbf{A}, \mathbf{B}$ are DG-categories, then their **tensor product** $\mathbf{A} \boxtimes \mathbf{B}$ is the category whose objects are pairs $(X, Y) \in \mathrm{Ob}(\mathbf{A}) \times \mathrm{Ob}(\mathbf{B})$, and whose morphisms are

$$(\mathbf{A} \boxtimes \mathbf{B})((X, Y), (X', Y')) = \mathbf{A}(X, X') \otimes \mathbf{B}(Y, Y')$$

  The unit for this tensor product is k, considered as a one-object DG-category.
- $\mathbf{dgCat}_k$ has an internal Hom, given by $\mathbf{Fun}_{DG}(\mathbf{A}, \mathbf{B})$.

Proposition 9.2.2. *For any* $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathrm{Ob}(\mathbf{dgCat}_k)$, *we have*

$$\mathrm{Hom}_{\mathbf{dgCat}_k}(\mathbf{A} \otimes \mathbf{B}, \mathbf{C}) \cong \mathrm{Hom}_{\mathbf{dgCat}_k}(\mathbf{A}, \mathbf{Fun}_{DG}(\mathbf{B}, \mathbf{C})) .$$

Theorem 9.2.3. $\mathbf{dgCat}_k$ *is a symmetric closed monoidal category, with tensor product* $\boxtimes$ *and internal hom* $\mathbf{Fun}_{DG}(\mathbf{A}, -)$.

9.3 DG-modules

Let $\mathbf{A} \in \mathrm{Ob}(\mathbf{dgCat}_k)$ be a small DG-category.

Definition 9.3.1. A (right) DG $\mathbf{A}$-module M is a DG-functor $M \colon \mathbf{A}^{op} \to C_{DG}(k)$.

For notation, we write:

$$C(\mathbf{A}) := \mathbf{Fun}(\mathbf{A}^{op}, C_{DG}(k)) \qquad \text{a k-category}$$
$$C_{DG}(\mathbf{A}) := \mathbf{Fun}_{DG}(\mathbf{A}^{op}, C_{DG}(k)) \qquad \text{a DG k-category}$$

Note that $C(\mathbf{A}) = Z^0(C_{DG}(\mathbf{A}))$.

By Proposition 9.2.2, giving a DG $\mathbf{A}$-module is equivalent to giving

- A complex $M(X)$ of k-modules for each $X \in \mathrm{Ob}(\mathbf{A})$
- for each pair of objects $X, Y \in \mathrm{Ob}(\mathbf{A})$, a morphism of complexes

$$M(Y) \otimes \mathbf{A}(X, Y) \to M(X)$$

  defining a right-action of $\mathbf{A}(X, Y)$ on $M(Y)$; this must satisfy the usual associativity and unit conditions.

9.3.1 Examples of DG-modules

Example 9.3.2. Let $\mathbf{A} = A$ be a DG k-algebra, which is a DG category with one object $*$. Then a right module M is a functor $M\colon A^{op} \to C_{DG}(k)$ sending the single object $*$ to some chain complex $M(*)$ of k-modules. The DG-algebra homomorphism $A^{op} = \mathrm{End}(*) \to \underline{\mathrm{End}}^\bullet(M(*))$ corresponds to a right-action of A on M. Hence M is a usual right DG-module over A.

If A is an ordinary k-algebra, then a DG-module M over A is a complex of A-modules.

Many classical constructions in homological algebra can be interpreted in terms of DG-modules.

Example 9.3.3. Let $A = \bigwedge_k(V)$, where $V = kx$ is a 1-dimensional k-vector space generated by x of degree -1. Then $A = k \oplus kx$, where the first summand is in degree zero and the second is in degree -1. The differential is zero $d_A = 0$.

A DG-module over A is called a (cohomological) **mixed complex**. It is given by a graded k-module $M = \bigoplus_{p\in\mathbb{Z}} M^p$ with two endomorphisms

$$\begin{aligned} B\colon M &\longrightarrow M \\ m &\longmapsto (-1)^{|m|} m\cdot x \end{aligned} \qquad\qquad \begin{aligned} b\colon M &\longrightarrow M \\ m &\longmapsto d_M(m). \end{aligned}$$

These endomorphisms have degrees $|B| = -1$ and $|b| = +1$, with $B^2 = 0$ and $b^2 = 0$ and $Bb + bB = 0$.

Dually, a (homological) **mixed complex** is given by $\left(M = \bigoplus_{p\in\mathbb{Z}} M_p, B, b\right)$ where B is degree $+1$, b is degree -1, with relations $B^2 = 0$, $b^2 = 0$ and $Bb + bB = 0$.

Example 9.3.4. Let A be an associative unital k-algebra and consider the reduced Hochschild complex

$$C_n(A) = C_n(A, A) = A \otimes \overline{A}^{\otimes n}$$

where $\overline{A} = {}^{A}/_{k\cdot 1}$. We have a Hochschild differential

$$b\colon C_n(A) \to C_{n-1}(A)$$

where

$$b_n(a_0, \dots, a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0, \dots, a_i a_{i+1}, \dots a_n) + (-1)^n (a_n a_0, \dots, a_{n-1}).$$

But there is also another differential in the other direction,

$$B\colon C_n(A) \to C_{n+1}(A),$$

where

$$B(a_0, a_1, \dots, a_n) = \sum_{i=0}^{n} (-1)^{ni}(1, a_i, a_{i+1}, \dots, a_n, a_0, a_1, \dots, a_{i-1}).$$

One can check by brutal calculation that $b^2 = 0$, $B^2 = 0$, and $bB + Bb = 0$.

Definition 9.3.5. The differential B is called the **Connes differential** of Hochschild homology.

Fact 9.3.6 (Rinehart 1963, Connes 1985)**.** *If A is a smooth commutative k-algebra, then the Hochschild-Kostant-Rosenberg Theorem says that $HH_n(A) \cong \Omega^n(A)$. In fact, the following diagram commutes*

$$\begin{array}{ccc} HH_n(A) & \xrightarrow{B} & HH_{n+1}(A) \\ \cong \downarrow \text{HKR} & & \cong \downarrow \text{HKR} \\ \Omega^n(A) & \xrightarrow{d_{DR}} & \Omega^{n+1}(A) \end{array}$$

where the map B is induced by the Connes differential $B\colon C(A) \to C(A)$.

Thus, the Connes differential B can be viewed as an abstract version of the de Rham differential. Associated to a mixed complex, is **cyclic homology** theory. In many cases, this turns out to be an extremely useful variation of Hochschild theory.

Definition 9.3.7. Let $M = (C_\bullet(A), b, B)$ be a homological mixed complex. Let u be a formal variable of degree $|u| = -2$.

- Define the **negative cyclic complex** $CC^-(A) := (C_\bullet(A)[[u]], b + uB)$. Consider an element of $CC^-(A)$ as a formal power series

$$\sum_{n \geq 0} x_n u^n = (x_0, x_1, \dots, x_n, \dots) \in C_\bullet(A)^\infty$$

- Define the **periodic cyclic complex** $CC^{per}(A) := (C_\bullet(A)((u)), b + uB)$, where $C_\bullet(A)((u))$ is the ring of formal Laurent series in $C_\bullet(A)$.

- Define the **cyclic complex**

$$CC(A) = {CC^{per}(A)}\big/{CC^-(A)} = {C(A)((u))}\big/{uC_\bullet(A)[[u]]}$$

with differential induced by $b + uB$.

There is a short exact sequence of complexes

$$0 \longrightarrow CC^-(A) \longrightarrow CC^{per}(A) \longrightarrow CC(A) \longrightarrow 0$$

Definition 9.3.8. The **cyclic homology** $HC_\bullet$ is the homology of the cyclic complex.

Now back to examples of DG-modules.

Example 9.3.9 (Yoneda DG-modules)**.** Let $\mathbf{A}$ be a DG-category. Consider the functor

$$\begin{aligned} \widehat{(-)}\colon \mathbf{A} &\longrightarrow C_{DG}(\mathbf{A}) \\ X &\longmapsto \widehat{X} = \mathbf{A}(-,X) \end{aligned}$$

where $\mathbf{A}(-,X)\colon Y \mapsto \mathbf{A}(Y,X) \in \mathrm{Ob}(C_{DG}(\mathbf{A}))$. $\widehat{X}$ is called a **DG-module represented by the object** X. By (an enriched version of) the Yoneda lemma, we have

$$\mathcal{H}om(\widehat{X}, M) \cong M(X)$$

for any DG-module M.

The Yoneda functor $\widehat{(-)}$ embeds $\mathbf{A}$ into the DG-category of DG $\mathbf{A}$-modules, which allows one to perform various operations on objects of $\mathbf{A}$ by regarding them as DG-modules.

Example 9.3.10 (Morphism Cone)**.** Consider $X, Y \in \mathrm{Ob}(\mathbf{A})$. Then as in the previous example, we have $\widehat{X}, \widehat{Y} \in \mathrm{Ob}(C_{DG}(\mathbf{A}))$. Given $f\colon X \to Y$ a morphism in $Z^0\mathbf{A}(X,Y)$, we get a morphism $\widehat{f}\colon \widehat{X} \to \widehat{Y}$ of DG-modules. Then we define the **morphism cone** of f to be the complex

$$\mathrm{cone}(f) := \left(\widehat{Y} \oplus \widehat{X}[1], d_{\mathrm{cone}}\right)$$

where

$$d_{\mathrm{cone}} = \begin{bmatrix} d_{\widehat{Y}} & \widehat{f} \\ 0 & -d_{\widehat{X}} \end{bmatrix}.$$

Chapter 10

Model Categories Interlude

10.1 Cofibrantly generated model categories

We now introduce a very important class of DG-functors.

Definition 10.1.1. Let $F: \mathbf{A} \to \mathbf{B}$ be a DG functor. Then F is called a **quasi-equivalence** if

(QE1) For all $X, Y \in \mathrm{Ob}(\mathbf{A})$, F induces a quasi-isomorphism between $\mathbf{A}(X, Y)$ and $\mathbf{B}(FX, FY)$ in $C(k)$.

(QE2) The induced functor $H^0(F): H^0(A) \to H^0(B)$ is an equivalence of k-categories.

Our goal is to demonstrate that there is a model category structure on $\mathbf{dgCat}_k$ such that the weak equivalences are the DG-functors $F: \mathbf{A} \to \mathbf{B}$ that are quasi-equivalences. To do this, we must define cofibrations and fibrations satisfying Quillen's axioms.

Remark 10.1.2. To define a (closed) model category, we only need to specify two out of three classes of morphisms: either weak equivalences and the fibrations or the weak equivalences and the cofibrations; the dual class (either cofibrations or fibrations) is forced by the axioms (see [HA2]). We will only define the cofibrations and weak equivalences.

In fact, we can do better. We may choose small subsets $\mathcal{I} \subseteq \mathrm{Cof}$ and $\mathcal{J} \subseteq \mathrm{Cof} \cap \mathrm{WE}$ which **generate** the model structure in a natural way. This is a **cofibrantly generated model category**.

Definition 10.1.3. A model category $\mathbf{C}$ is called **cofibrantly generated** if there are two sets of morphisms:

(1) the **generating cofibrations** $\mathcal{I} \subseteq \mathrm{Cof}$ and

(2) the **generating acyclic cofibrations** $\mathcal{J} \subseteq \text{Cof} \cap \text{WE}$

subject to the following.

(a) The acyclic fibrations are precisely those morphisms which have the right-lifting property with respect to all morphisms in $\mathcal{I}$, i.e. $\text{Fib} \cap WE = \text{RLP}(\mathcal{I})$.

(b) Fibrations are precisely those morphisms which have the right-lifting-property with respect to $\mathcal{J}$, i.e. $\text{Fib} = \text{RLP}(\mathcal{J})$.

(c) The domain of every $f \in \mathcal{I}$ is **small** with respect to Cof.

(d) The domain of every $f \in \mathcal{J}$ is **small** with respect to $\text{Cof} \cap \text{WE}$.

(See Definition 10.2.1 for the definition of small objects.)

Example 10.1.4. The category $\mathbf{C} = \mathbf{Com}_{\geq 0}(R)$ of non-negatively graded chain complexes over a k-algebra R has a model structure where

- The weak equivalences are quasi-isomorphisms;
- the fibrations are morphisms of complexes that are objectwise surjective in positive degree;
- the cofibrations are morphisms of complexes that are objectwise injective whose cokernels are projective R-modules.

In this case, $\mathcal{J}$ is the set of generating acyclic cofibrations

$$\mathcal{J} = \{0 \to D(n)\}_{n \geq 1},$$

where $D(n)$ is the n**-disk**; the chain complex

$$D(n) = \left[\; 0 \longrightarrow R \xrightarrow{\text{id}} R \longrightarrow 0 \;\right]$$

with R in degrees n and $n-1$ and zeros elsewhere.

$\mathcal{I}$ is the set of generating cofibrations, given by

$$\mathcal{I} = \left\{S(n-1) \hookrightarrow D(n)\right\}_{n \geq 0} \cup \{0 \to S(0)\}$$

where $S(n-1)$ is the complex with R in degree $(n-1)$ and zeroes elsewhere.

Remark 10.1.5. The complex $D(n)$ as above represents the functor τ_n that picks out the n-th term of a complex

$$\begin{aligned} \tau_n \colon \mathbf{Com}_{\geq 0}(R) &\longrightarrow \mathbf{Mod}(R) \\ A_\bullet &\longmapsto A_n. \end{aligned}$$

Example 10.1.6. In the category of topological spaces with the usual model structure,

$$\mathcal{J} = \left\{ D_n \overset{i_0}{\hookrightarrow} D_n \times [0,1] \right\}_{n \geq 0}$$
$$\mathcal{I} = \left\{ S^{n-1} = \partial D_n \hookrightarrow D_n \right\}_{n \geq 0}.$$

Then the class of fibrations consists of those maps which satisfy the right-lifting-property with respect to $\mathcal{J}$; this is exactly the definition of a Serre fibration (the fibrations in the usual model structure on **Top**).

$$\begin{array}{ccc} D_n & \xrightarrow{\tilde{f}_0} & E \\ {\scriptstyle i_0}\downarrow & \overset{\exists\, \tilde{f}_t}{\nearrow} & \downarrow \\ D_n \times [0,1] & \xrightarrow[f_t]{} & B \end{array}$$

Example 10.1.7. Let $\mathbf{C} = \mathbf{sSet}$. This is a bit different from the previous example, because $\mathbf{C}$ is a cofibrant model category. Then

$$\mathcal{J} = \left\{ \Lambda_k[n] \to \Delta[n] \right\}_{\substack{n \geq 1 \\ 0 \leq k \leq n}}$$
$$\mathcal{I} = \left\{ \partial\Delta[n] \to \Delta[n] \right\}_{n \geq 0}$$

Recall that $\Delta[n]$ is the standard n-simplex

$$\Delta[n] = \mathrm{Hom}_{\Delta}(-,[n]) \colon \Delta^{op} \to \mathbf{Set}$$
$$\Delta[n]_k = \left\{ (j_0, j_1, \dots, j_k) \colon 0 \leq j_0 \leq \dots \leq j_k \leq n \right\}$$

Note that the geometric realization of $\Delta[n]$ is Δ^n, the n-dimensional geometric simplex. We define

$$\partial\Delta[n] := \bigcup_{0 \leq i \leq n} d^i \Delta[n-1]$$

The geometric realization of $\partial\Delta[n]$ is the $(n-1)$-sphere S^{n-1}.

The k**-th horn** is

$$\Lambda_k[n] := \bigcup_{\substack{0 \leq i \leq n \\ i \neq k}} d^i \Delta[n-1]$$

the geometric realization of the k-th horn is the boundary of the geometric n-simplex without the k-th face. $|\Lambda_k[n]| = \partial\Delta^n \setminus (k\text{-th face})$.

10.2 Quillen's Small Object Argument

Let **C** be a cocomplete category, and let $\mathcal{F} \subseteq \mathrm{Mor}(\mathbf{C})$ be a class of morphisms.

Definition 10.2.1. $A \in \mathrm{Ob}(\mathbf{C})$ is (sequentially) **small with respect to** $\mathcal{F}$ if the functor $\mathbf{C}(A, -) = \mathrm{Hom}_{\mathbf{C}}(A, -)$ commutes with sequential colimits of morphisms in $\mathcal{F}$.

That is, given a sequence

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} X_n \xrightarrow{f_{n+1}} X_{n+1} \longrightarrow \cdots$$

with $f_i \in \mathcal{F}$. Taking the colimit of this sequence, we obtain maps of sets $\mathbf{C}(A, X_n) \to \mathbf{C}(A, \mathrm{colim}_{n X_n})$. Taking the colimit of the sets $\mathbf{C}(A, X_n)$, we get a canonical map

$$\mathrm{colim}_n\, (\mathbf{C}(A, X_n)) \to \mathbf{C}(A, \mathrm{colim}_n\, X_n).$$

We say that A is small with respect to $\mathcal{F}$ if this canonical map is a bijection of sets.

Definition 10.2.2. If $\mathcal{F} = \mathrm{Mor}(\mathbf{C})$, and A is sequentially small with respect to $\mathcal{F}$, then A is simply called a **small object** of **C**.

Remark 10.2.3. This generalizes to any arbitrary ordinal λ, which gives a notion of λ-small objects.

Remark 10.2.4. Sometimes small objects are called **compact objects**.

Example 10.2.5.

(a) In $\mathbf{C} = \mathbf{Set}$, a set A is small if and only if A is a finite set.

(b) In $\mathbf{C} = \mathbf{Mod}(R)$, an R-module A is small if and only if A is a **coherent** R-module, i.e. it has a finite presentation: $A = \mathrm{coker}\,(R^m \to R^n)$.

(c) If $\mathbf{C} = \mathbf{QCoh}(X)$ for X a quasi-projective variety, then a quasi-coherent sheaf $\mathcal{F}$ is small if and only if it is actually a coherent sheaf.

(d) In $\mathbf{C} = \mathbf{sSet}$, a simplicial set X is small if and only if X has only finitely many nondegenerate simplicies. This is called a **finite simplicial set**.

(e) In $\mathbf{C} = \mathbf{Com}(R)$ is the category of chain complexes over a ring R, a chain complex $A_\bullet$ is small if and only if $A_\bullet$ is a bounded complex and each term is coherent (the cokernel of a map between free R-modules). Such a complex is called **perfect**.

(f) Let $\mathbf{C} = \mathbf{Top}$ and let $\mathcal{F}$ be the class of closed inclusions of topological spaces. A space X is small with respect to $\mathcal{F}$ if and only if X is compact.

Theorem 10.2.6 (Quillen's Small Object Argument)**.** *Let* **C** *be a cofibrantly generated model category with generating sets of cofibrations* $\mathcal{I} \subset \mathrm{Cof}$ *and* $\mathcal{J} \subset \mathrm{Cof} \cap \mathrm{WE}$*. Then we may factor any* $f\colon X \to Y$ *in two ways*

(a) $f\colon X \xhookrightarrow{i} Q \xrightarrow[\sim]{p} Y$

(b) $f\colon X \xhookrightarrow[\sim]{j} Z \xrightarrow{q} Y$

So that both factorizations (i, p) *and* (j, q) *are natural (functorial) in* f*.*

Remark 10.2.7. This strengthens the axiom (MC5) of model categories, which doesn't guarantee naturality in the choice of factorizations.

Proof of Theorem 10.2.6. We will construct a natural factorization (j, q) with j an acyclic cofibration and q a fibration. The construction of (i, p) is similar.

First, set $Z_0 := X$, $q_0 = f\colon Z_0 \to Y$ and consider the set of diagrams

$$U_0 = \left\{ \begin{array}{ccc} A & \xrightarrow{\alpha_0} & Z_0 \\ \big\downarrow i & & \big\downarrow q_0 \\ B & \xrightarrow{\beta_0} & Y \end{array} \;\middle|\; i \in I, A \text{ sequentially small} \right\}$$

This yields the pushout diagram

$$\begin{array}{ccc} \coprod A & \xrightarrow{\sum \alpha_0} & Z_0 \\ {\scriptstyle \coprod i}\big\downarrow & \searrow & \big\downarrow j_1 \\ \coprod B & \longrightarrow & Z_1 \end{array}$$

where

$$Z_1 = \operatorname{colim}\left(\coprod B \xleftarrow{\coprod i} \coprod A \xrightarrow{\sum \alpha_0} Z_0 \right).$$

This also gives us a map $q_1\colon Z_1 \to Y$ from the universal property of pushouts, with $\sum \beta_0\colon \coprod B \to Y$ and $q_0\colon Z_0 \to Y$.

Because cofibrations are preserved under pushouts and coproducts, j_1 is a cofibration. Therefore, we may repeat the process with the set of all diagrams

$$U_1 = \left\{ \begin{array}{ccc} A & \xrightarrow{\alpha_1} & Z_1 \\ \big\downarrow i & & \big\downarrow q_1 \\ B & \xrightarrow{\beta_1} & Y \end{array} \;\middle|\; i \in I, A \text{ sequentially small} \right\}$$

Yielding as before Z_2 and $q_2\colon Z_2 \to Y$.

By induction, we get a sequence

$$X = Z_0 \xhookrightarrow{j_1} Z_1 \xhookrightarrow{j_2} Z_2 \xhookrightarrow{j_3} \cdots \hookrightarrow Z_n \xhookrightarrow{j_n} \cdots$$

each of which has a map $q_i \colon Z_i \to Y$. Now define $Z := \operatorname{colim}_n(Z_n)$; this comes with natural maps $\widetilde{j}_n \colon Z_n \to Z$ and $q \colon Z \to Y$.

We factor $f \colon X \to Y$ as

$$X = Z_0 \xrightarrow{\widetilde{j}_0} Z \xrightarrow{q} Y.$$

Since each j_n is a cofibration, $\widetilde{j}_0 \colon X = Z_0 \to Z$ is a cofibration as well.

Now it only remains to prove that $q \colon Z \to Y$ is an acyclic fibration. To do this, it suffices to show that q has the right-lifting property with respect to all morphisms in $\mathcal{I}$.

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & Z \\ {\scriptstyle i}\downarrow & \overset{\exists\,\widetilde{\beta}}{\nearrow} & \downarrow{\scriptstyle q} \\ B & \xrightarrow[\beta]{} & Y \end{array}$$

Since A is small, we know

$$\operatorname{colim}_n \mathbf{C}(A, Z_n) \cong \mathbf{C}(A, Z).$$

Surjectivity of this map gives factorizations

$$\begin{array}{ccc} & & Z_n \\ & \overset{\exists\,\widetilde{\alpha}}{\nearrow} & \downarrow{\scriptstyle \widetilde{j}_n} \\ A & \xrightarrow{\alpha} & Z \end{array}$$

Then by construction of Z_{n+1}, there is a diagram

$$\begin{array}{ccc} A & \xrightarrow{\widetilde{\alpha}} & Z_n \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j_{n+1}} \\ B & \longrightarrow & Z_{n+1} \end{array}$$

Composing the bottom map with the inclusion $Z_{n+1} \to Z$ gives the required $\widetilde{\beta}$.

Hence, q has the right-lifting-property with respect to $\mathcal{I}$, and is therefore an acyclic fibration. □

Remark 10.2.8. We may produce an abstract version of the small object argument in any complete and cocomplete category $\mathbf{C}$.

- Let $\mathcal{F} \subseteq \mathrm{Mor}(\mathbf{C})$ be a class of morphisms.
- Let $\mathcal{F}_{\text{inj}}$ be the set of all morphisms with the right-lifting property with respect to $\mathcal{F}$, i.e. $\mathcal{F}_{\text{inj}} = \mathrm{RLP}(\mathcal{F})$.
- Let $\mathcal{F}_{\text{cof}}$ be the set of all morphisms with the left-lifting-property with respect to $\mathcal{F}_{\text{inj}}$, i.e. $\mathcal{F}_{\text{cof}} = \mathrm{LLP}(\mathcal{F}_{\text{inj}})$.
- Let $\mathcal{F}_{\text{cell}}$ be the set of all $f\colon A \to B$ such that there exists a chain

$$Z_0 \overset{j_1}{\hookrightarrow} Z_1 \overset{j_2}{\hookrightarrow} Z_2 \overset{j_3}{\hookrightarrow} \cdots \hookrightarrow Z_n \overset{j_n}{\hookrightarrow} \cdots$$

  with $A \cong Z_0$ and $\mathrm{colim}_n(Z_n) \cong B$ and each j_n is a pushout of some map in $\mathcal{F}$ and the induced map $Z_0 \to \mathrm{colim}_n(Z_n)$ is isomorphic to f in the morphism category $\mathbf{Mor}(\mathbf{C})$.

Theorem 10.2.9. *Assume in addition that the domain of every $f \in \mathcal{F}$ is small with respect to $\mathcal{F}$. Then there is a functor*

$$\begin{aligned} \mathbf{Mor}(\mathbf{C}) &\longrightarrow \mathcal{F}_{\text{cell}} \times \mathcal{F}_{\text{inj}} \\ f &\longmapsto (q(f), j(f)), \end{aligned}$$

where $\mathbf{Mor}(\mathbf{C})$ is the category of morphisms and natural squares, and $\mathcal{F}_{\text{cell}}$ and $\mathcal{F}_{\text{inj}}$ are considered subcategories of $\mathbf{C}$.

Example 10.2.10. If $\mathbf{C}$ is a cofibrantly generated model category, and $\mathcal{F} = \mathcal{I}$ or $\mathcal{F} = \mathcal{J}$, then this is just the small object argument.

10.3 Applications of the Small Object Argument

10.3.1 Promoting Model Structures

One of the main applications of the small object argument is the following.

Construction 10.3.1. Let $F\colon \mathbf{C} \leftrightarrows \mathbf{D}\colon G$ be a pair of adjoint functors. Suppose that $\mathbf{C}$ is a cofibrantly generated model category, with $\mathcal{I}$ a set of generating cofibrations and $\mathcal{J}$ a set of generating acyclic cofibrations. We will define a model structure on $\mathbf{D}$ from the one on $\mathbf{C}$.

For $f\colon X \to Y$ in $\mathbf{D}$, say that

- f is a weak equivalence in $\mathbf{D}$ if and only if Gf is a weak equivalence in $\mathbf{C}$;

$$f \in \mathrm{WE}_{\mathbf{D}} \iff Gf \in \mathrm{WE}_{\mathbf{C}}$$

- f is a fibration in $\mathbf{D}$ if and only if Gf is a fibration in $\mathbf{C}$;

$$f \in \mathrm{Fib}_{\mathbf{D}} \iff Gf \in \mathrm{Fib}_{\mathbf{C}}$$

- f is a cofibration in $\mathbf{D}$ if and only if f has the left lifting property with respect to all acyclic fibrations in $\mathbf{D}$.

$$\mathrm{Cof}_{\mathbf{D}} := \mathrm{LLP}\left(\mathrm{WE}_{\mathbf{D}} \cap \mathrm{Fib}_{\mathbf{D}}\right)$$

Theorem 10.3.2. *Suppose that*

(a) $G\colon \mathbf{D} \to \mathbf{C}$ *commutes with all sequential colimits, and*

(b) If a map $f\colon X \to Y$ *in* $\mathbf{D}$ *is both a cofibration and has the left lifting property with respect to all fibrations in* $\mathbf{D}$*, then it is a weak equivalence in* $\mathbf{D}$*.*

$$\mathrm{Cof}_{\mathbf{D}} \cap \mathrm{LLP}(\mathrm{Fib}_{\mathbf{D}}) \subseteq \mathrm{WE}_{\mathbf{D}}$$

Then $\mathbf{D}$ *is a cofibrantly generated model category, with*

$$\mathcal{I}_{\mathbf{D}} := \{Fi\colon i \in \mathcal{I}\}$$
$$\mathcal{J}_{\mathbf{D}} := \{Fj\colon j \in \mathcal{J}\}$$

Moreover, the adjunction $F \dashv G$ *is a Quillen pair between these model categories.*

The proof of this theorem is essentially repeated application of the small object argument.

Remark 10.3.3. It is usually hard to check condition Theorem 10.3.2(b) directly. A useful criterion (again due to Quillen) is the following proposition.

Proposition 10.3.4. *If* $\mathbf{D}$ *satisfies all the conditions of the above construction except for (b), and*

(a) there is a functorial fibrant replacement functor $X \mapsto RX$ *in* $\mathbf{D}$*: i.e.*

$$X \overset{\sim}{\hookrightarrow} RX \twoheadrightarrow *$$

(b) Every $B \in \mathrm{Ob}(\mathbf{D})$ *has a natural path object, i.e.*

$$\begin{array}{ccc} & & P(B) \\ & \overset{\sim}{\nearrow} & \downarrow \\ B & \xrightarrow{\Delta} & B \times B \end{array}$$

Then Theorem 10.3.2(b) holds in $\mathbf{D}$*.*

Remark 10.3.5. Note that Proposition 10.3.4(a) holds when $\mathbf{D}$ is fibrant, and Proposition 10.3.4(b) holds when $\mathbf{D}$ is a simplicial model category.

Example 10.3.6. Let k be a commutative ring, and let $\mathbf{C} = \mathbf{C}^+(k)$ be the category of chain complexes over k. Then

- the weak equivalences are the quasi-isomorphisms
- the fibrations are maps $f \colon X_\bullet \to Y_\bullet$ such that f_n is surjective in positive degree
- the cofibrations are maps $f \colon X_\bullet \to Y_\bullet$ such that $\operatorname{coker}(f_n)$ is projective for all $n \geq 0$.

This is the standard (projective) model structure on $\mathbf{C}(k)$.

We have an adjunction

$$\mathbf{C}^+(k) \underset{\mathrm{Sym}_k}{\overset{U}{\rightleftarrows}} \mathbf{dgAlg}_k^+ \quad (\top)$$

where U is the forgetful functor and Sym_k is the symmetric algebra functor.

If k is a field of characteristic zero, then teh aboe theorem applies and gives the standard (projective) model structure on $\mathbf{dgAlg}_k^+$.

But, if k has positive characteristic, say characteristic two, the theorem does *not* apply. Indeed, if the conclusion of the theorem were true, then the left adjoint $F = \mathrm{Sym}_k$ (being Quillen) would perserve all weak equivalences between cofibrant objects. As k is a field, $\mathbf{C}^+(k)$ is a cofibrant model category (every k-module is projective). Take the "n-disk" in $\mathbf{C}^+(k)$,

$$D(n) = \left[0 \to k \xrightarrow{\mathrm{id}} k \to 0\right]$$

with k in degrees n and $n-1$.

Then $0 \hookrightarrow D(n)$ is a weak equivalence in $\mathbf{C}^+(k)$, but

$$\begin{array}{ccc} \mathrm{Sym}_k(0) & \hookrightarrow & \mathrm{Sym}_k(D(n)) \\ \downarrow\cong & & \downarrow\cong \\ k & \hookrightarrow & k[x, y \mid dx = y, dy = 0] \end{array}$$

is *not* a quasi-isomorphism because x^2 is a cycle of degree $2n$ which is not a boundary. (In fact, if n is even, then $dx^2 = dx\,x + (-1)^n x\,dx = 2x\,dx = 2xy = 0$. But there is no z such that $dz = x^2$.)

Another example where the theorem fails is the free k-algebra/group of units adjunction between simplicial groups and simplicial k-algebras

$$\mathbf{sGr} \underset{\mathrm{k}[-]}{\overset{(-)^\times}{\rightleftarrows}} \mathbf{sAlg_k} \quad (\top)$$

The problem here is that $\mathrm{k}[-]$ does not map cofibrations to cofibrations.

10.3.2 Recognition Theorem

The next important theorem allows one to identify model structures in practice.

Theorem 10.3.7 (Recognition Theorem). *Let* $\mathbf{C}$ *be a complete and cocomplete category, and let* W *be a class of morphisms. Let* $\mathcal{I}, \mathcal{J} \subseteq \mathrm{Mor}(\mathbf{C})$ *be two sets of morphisms such that:*

(a) *W satisfies the two-out-of-three property and is stable under retracts*

(b) *for all $f \in \mathcal{I}$, the domain of f is small with respect to $\mathcal{I}_{\mathrm{cell}}$.*

(c) *for all $f \in \mathcal{J}$, the domain of f is small with respect to $\mathcal{J}_{\mathrm{cell}}$.*

(d) $\mathcal{J}_{\mathrm{cell}} \subseteq W \cap \mathcal{I}_{\mathrm{inj}}$.

(e) $\mathcal{I}_{\mathrm{inj}} \subseteq W \cap \mathcal{J}_{\mathrm{inj}}$.

(f) $W \cap \mathrm{I}_{\mathrm{cof}} \subseteq \mathrm{J}_{\mathrm{cof}}$ *or* $W \cap \mathcal{J}_{\mathrm{inj}} \subseteq \mathcal{I}_{\mathrm{inj}}$.

Then $\mathcal{C}$ *has a unique cofibrantly generated model structure with class of weak equivalences* W*.*

Chapter 11

The category of DG-categories

11.1 Model Structure on $\mathbf{dgCat}_k$

There are really two model structures on $\mathbf{dgCat}_k$ – the Dwyer-Kan model structure, and the Morita model structure due to Töen. The latter can be obtained from the former by a localizing procedure (so-called **Bousfield localization**). We will discuss only the Dwyer-Kan model structure where the weak equivalences are the quasi-equivalences.

Let $\mathbf{dgCat}_k$ be the category of small DG-categories over a commutative ring k. Throughout this section, we use the homological convention. For all $X, Y \in \mathrm{Ob}(\mathbf{A})$, $\mathbf{A}(X, Y)$ is equipped with a differential of degree (-1).

Recall:

Definition 11.1.1. A functor $F\colon \mathbf{A} \to \mathbf{B}$ is a **quasi-equivalence** if

(a) for all $X, Y \in \mathrm{Ob}(\mathbf{A})$, $F\colon \mathbf{A}(X, Y) \to \mathbf{B}(FX, FY)$ is a quasi-isomorphism of complexes

(b) $\mathbf{H}_0 F\colon \mathbf{H}_0(\mathbf{A}) \to \mathbf{H}_0(\mathbf{B})$ is an equivalence of categories.

The goal is to construct a cofibrantly generated model structure on $\mathbf{dgCat}_k$ in which the weak equivalences are the quasi-equivalences.

Construction 11.1.2 (Drinfeld DG-Category). Define a DG-category $\mathbf{K}$ as follows. It has two objects:

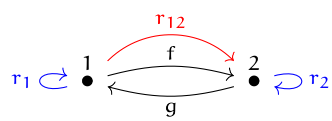
$$\mathrm{Ob}(\mathbf{K}) = \{1, 2\}$$

and $\mathrm{Mor}(\mathbf{K})$ is generated by $f \in \mathbf{K}(1,2)_0$, $g \in \mathbf{K}(2,1)_0$, $r_1 \in \mathbf{K}(1,1)_1$, $r_2 \in \mathbf{K}(2,2)_1$ and $r_{12} \in \mathbf{K}(1,2)_2$.

The differential is defined by

$$\begin{aligned} df &= dg = 0 \\ dr_1 &= g \circ f - \mathrm{id}_1 \\ dr_2 &= f \circ g - \mathrm{id}_2 \\ dr_{12} &= f \circ r_1 - r_2 \circ f \end{aligned}$$

Pictorially, this looks like



where the black arrows sit in degree zero, the blue arrows in degree one, and the red arrows in degree two.

Definition 11.1.3. Let $\mathbf{A}$ be any DG-category.

(a) a morphism $s \in \mathbf{Z}_0(\mathbf{A})$ is called a **homotopy equivalence** if it becomes invertible in $\mathbf{H}_0(\mathbf{A})$

(b) an object $X \in \mathrm{Ob}(\mathbf{A})$ is called **contractible** if the DG-algebra $\mathbf{A}(X, X)$ is acyclic, or equivalently, if there is $h \in \mathbf{A}(X, X)_1$ such that $dh = \mathrm{id}_X$. Such an h is called a **contraction** of X.

Lemma 11.1.4 (Drinfeld)**.** *For any DG-category $\mathbf{B}$, there is a bijection between the set of DG functors $\mathbf{K} \to \mathbf{B}$ and the set of pairs (s, h) where $s \in \mathrm{Mor}(\mathbf{Z}_0(\mathbf{B}))$ and h is a contraction of $\mathrm{cone}(\widehat{s}) \in C_{DG}(\mathbf{B})$.*

In the lemma, $\widehat{s}$ denotes the image of s under the Yoneda embedding $\mathbf{A} \to C_{DG}(\mathbf{A})$ and $\mathrm{cone}(-)$ is as in Example 9.3.10.

Proof. The DG-category $\mathbf{K}$ is defined by generators and relations. Hence, the datum of a DG-functor $F\colon \mathbf{K} \to \mathbf{B}$ corresponds to the data of objects $1 \mapsto X$, $2 \mapsto Y$, and morphisms in $\mathbf{B}$:

- the image of f, a morphism $s \in \mathbf{B}(X, Y)_0$,
- the image of g, a morphism $p \in \mathbf{B}(Y, X)_0$,
- the image of r_1, a morphism $r_X \in \mathbf{B}(X, X)_1$,
- the image of r_2, a morphism $r_Y \in \mathbf{B}(Y, Y)_1$,
- the image of r_{12}, a morphism $r_{XY} \in \mathbf{B}(X, Y)_2$.

These must satisfy the relations

$$\begin{aligned} ds &= 0 \\ dp &= 0 \\ dr_X &= p \circ s - 1_X \\ dr_Y &= s \circ p - 1_Y \\ dr_{XY} &= s \circ r_X - r_X \circ s. \end{aligned} \tag{11.1}$$

On the other hand, by definition, a morphism $s \in Z_0(\mathbf{B})$ is a morphism $s \in B(X, Y)_0$ such that $ds = 0$.

A degree-one morphism $h \in \mathbf{C}_{DG}(\mathrm{cone}(\widehat{s}), \mathrm{cone}(\widehat{s}))_1$ is given by a matrix

$$\begin{pmatrix} \widehat{r}_Y & \widehat{r}_{XY} \\ \widehat{p} & \widehat{r}_X \end{pmatrix}$$

with $p \in \mathbf{B}(Y, X)_0$, $r_X \in \mathbf{B}(X, X)_1$ and $r_Y \in \mathbf{B}(Y, Y)_1$. This presentation of h is because $\mathrm{cone}(\widehat{s}) := \widehat{Y} \oplus \widehat{X}[1]$, and

$$d_{\mathrm{cone}(\widehat{s})} = \begin{pmatrix} d_{\widehat{Y}} & \widehat{s} \\ 0 & -d_{\widehat{X}} \end{pmatrix}$$

so a morphism $\mathrm{cone}(\widehat{s}) \to \mathrm{cone}(\widehat{s})$ of degree 1 fits into the diagram below.

$$\begin{array}{ccccccc} & & \cdots \longrightarrow & Y_{n+1} \oplus X_n & \xrightarrow{d_{\mathrm{cone}}} & Y_n \oplus X_{n-1} & \longrightarrow \cdots \\ & & \swarrow^{h} & & \swarrow^{h} & & \\ Y_{n+2} \oplus X_{n+1} & \xrightarrow{d_{\mathrm{cone}}} & Y_{n+1} \oplus X_n & \longrightarrow \cdots & & & \end{array}$$

Such an h is a contraction if and only if its differential equals $1_{\mathrm{cone}(\widehat{s})}$, that is,

$$\begin{pmatrix} \widehat{r}_X & \widehat{r}_{XY} \\ \widehat{p} & \widehat{r}_X \end{pmatrix} \begin{pmatrix} d_{\widehat{Y}} & \widehat{s} \\ 0 & -d_{\widehat{X}} \end{pmatrix} - (-1)^{-1} \begin{pmatrix} d_{\widehat{Y}} & \widehat{s} \\ 0 & -d_{\widehat{X}} \end{pmatrix} \begin{pmatrix} \widehat{r}_Y & \widehat{r}_{XY} \\ \widehat{p} & \widehat{r}_X \end{pmatrix} = \begin{pmatrix} \widehat{1}_X & 0 \\ 0 & \widehat{1}_X \end{pmatrix}$$

By performing the matrix multiplications and taking into account the fully faithfulness of the Yoneda DG-functor $\widehat{(-)}$, we recover exactly the relations (11.1). □

Before we continue, let us set some definitions and notation.

Definition 11.1.5. Let $\underline{k}$ be the DG-category with one object, denoted 3, and morphisms $\underline{k}(3, 3) = k$ in degree zero.

Note that for any $\mathbf{B} \in \mathbf{dgCat}_k$, the class of DG-functors $F\colon \underline{k} \to B$ is in bijection with objects of $\mathbf{B}$.

For $n \in \mathbb{Z}$, we recall that S_n denotes the chain complex $k[n] \in \mathbf{C}(k)$:

$$S_n = \left[0 \to k \to 0\right]$$

with k in degree n.

Definition 11.1.6. Let $\mathbf{S}(n)$ be the DG-category with two objects $\{4, 5\}$ and morphisms

$$\begin{aligned} \mathbf{S}(n)(4,4) &= \mathbf{S}(n)(5,5) = k \\ \mathbf{S}(n)(5,4) &= 0 \\ \mathbf{S}(n)(4,5) &= \mathbf{S}_n. \end{aligned}$$

The composition is given by multiplication in k.

For $n \in \mathbb{Z}$, recall that the n-disk in $\mathbf{C}(k)$ is given by the mapping cone on S_{n-1}:

$$D_n := \operatorname{cone}(\mathrm{id}_{S_{n-1}}) = \left[0 \to k \xrightarrow{\mathrm{id}} k \to 0\right]$$

Definition 11.1.7. Now, define $\mathbf{D}(n)$ to be the DG-category with two objects $\{6, 7\}$ and morphisms

$$\begin{aligned} \mathbf{D}(n)(6,6) &= \mathbf{D}(n)(7,7) = k \\ \mathbf{D}(n)(7,6) &= 0 \\ \mathbf{D}(n)(6,7) &= D_n \end{aligned}$$

Definition 11.1.8. Finally, for $n \in \mathbb{Z}$, define the DG-functor $i(n) \colon \mathbf{S}(n-1) \to \mathbf{D}(n)$ on objects by $4 \mapsto 6$ and $5 \mapsto 7$, and on morphisms by

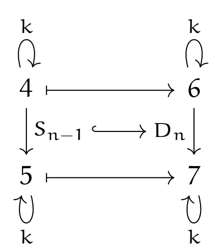
$$\begin{aligned} \mathbf{S}(n-1)(4,4) = k &\xrightarrow{\mathrm{id}} k = \mathbf{D}(n)(6,6) \\ \mathbf{S}(n-1)(5,5) = k &\xrightarrow{\mathrm{id}} k = \mathbf{D}(n)(7,7) \\ \mathbf{S}(n-1)(5,4) = 0 &\to 0 = \mathbf{D}(n)(7,6) \\ \mathbf{S}(n-1)(4,5) = S_{n-1} &\hookrightarrow D_n = \mathbf{D}(n)(6,7) \end{aligned}$$

where the map $S_{n-1} \hookrightarrow D_n$ is defined by

$$\begin{array}{ccccccc} 0 & \longrightarrow & k & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \\ 0 & \longrightarrow & k & \longrightarrow & k & \longrightarrow & 0 \end{array}$$

This functor is summarized in the following diagram.

$$\mathbf{S}(n-1) \xrightarrow{i(n)} \mathbf{D}(n)$$



Denote by $\mathcal{I}$ the set of DG-functors

$$\mathcal{I} = \{i(n)\}_{n\in\mathbb{Z}} \cup \{\emptyset \to \underline{k}\}$$

Definition 11.1.9. Let $\mathbf{B}_0$ be the DG-category with two objects $\{8, 9\}$ and

$$\mathbf{B}_0(8,8) = \mathbf{B}_0(9,9) = k\mathbf{B}_0(8,9) \qquad\qquad = \mathbf{B}_0(9,8) = 0$$

Let $j(n)\colon \mathbf{B}_0 \to \mathbf{D}(n)$ be the DG-functor sending $8 \mapsto 6$ and $9 \mapsto 7$.

Denote by $\mathcal{J}$ the set of DG-functors

$$\mathcal{J} = \{j(n)\}_{n\in\mathbb{Z}} \cup \left\{\underline{k} \xrightarrow[3\mapsto 1]{} \mathbf{K}\right\}$$

Theorem 11.1.10 (Tabuada 2005)**.** *Let $\mathbf{C} = \mathbf{dgCat}_k$, and let W denote the set of quasi-equivalences of DG-categories (Definition 11.1.1). Let $\mathcal{I}$ and $\mathcal{J}$ be as above. Then $(\mathbf{C}, W, \mathcal{I}, \mathcal{J})$ satisfies the hypotheses of the Recognition Theorem (Theorem 10.3.7). Hence, $\mathbf{dgCat}_k$ admits a cofibrantly generated model structure with weak equivalences being quasi-equivalences and* $\mathrm{Fib} = \mathrm{RLP}(\mathcal{J})$*.*

Fibrations in $\mathbf{dgCat}_k$ can actually be described explicitly.

Proposition 11.1.11. *A DG-functor $F\colon \mathbf{A} \to \mathbf{B}$ is a fibration if and only if*

(F1) For all $X, Y \in \mathrm{Ob}(\mathbf{A})$, the morphism in $\mathbf{C}(k)$

$$F(X,Y)\colon \mathbf{A}(X,Y) \to \mathbf{B}(FX,FY)$$

is degree-wise surjective.

(F2) Given $X \in \mathrm{Ob}(\mathbf{A})$ and a homotopy equivalence $v\colon F(X) \to Y$ in $\mathbf{B}$, there is a homotopy equivalence $u\colon X \to X'$ in $\mathbf{A}$ such that $F(u) = v$.

Corollary 11.1.12. $\mathbf{dgCat}_k$ *is a* ***fibrant*** *model category, that is, every DG-category is a fibrant object.*

Cofibrations and cofibrant objects in $\mathbf{dgCat}_k$ are harder to describe, although there is a necessary condition that is easy to verify.

Proposition 11.1.13 (Toën 2007)**.** *If $\mathbf{A}$ is a cofibrant DG-category in $\mathbf{dgCat}_K$, then for all $X, Y \in \mathrm{Ob}(\mathbf{A})$, the complex $\mathbf{A}(X,X)$ is cofibrant in $\mathbf{C}(k)$, i.e., consists of projective k-modules.*

This is the start of the theory of **noncommutative motives** in derived algebraic geometry.

Chapter 12

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