MATH 7350: SIMPLICIAL METHODS IN ALGEBRA AND GEOMETRY

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0.2 Overview

In Part I, we will investigate the classical version of the Dold–Kan correspondence.

Theorem 0.2.1 (Dold–Kan)**.** *For any ring* R*, there is a natural equivalence of categories between the category of simplicial* R*-modules and nonnegatively graded chain complexes of* R*-modules:*

$$\mathbf{sMod}(R) \simeq \mathbf{Ch}_+(R).$$

The Dold–Kan correspondence provides a link between homological algebra and homotopical algebra. There is a dual version of the Dold–Kan correspondence providing an equivalence of categories of cosimplicial modules and nonnegatively graded cochain complexes.

$$\mathbf{cMod}(R) \simeq \mathbf{Ch}^+(R).$$

Variations on the Dold–Kan correspondence that we will discuss are a "differential form" version due to Karoubi and a "duplicial" categorical version due to Dwyer–Kan. For the duplicial version, we replace the simplex category $\mathbb{\Delta}$ by a category $\mathbb{K}$.

Another important variation of Dold–Kan is a monoidal version: let $R = k$ be a commutative ring, and consider the category $\mathbf{sMod}(k)$ of simplicial k-modules. Monoidal objects in $\mathbf{sMod}(k)$ are the simplicial k-algebras $\mathbf{sAlg}(k)$, while monoidal objects in $\mathbf{Ch}_+(k)$ are the differential graded k-algebras, $\mathbf{dgAlg}^+(k)$. We might ask: is there an equivalence of these categories? The answer is no, but there is a Quillen equivalence between the two categories.

Theorem 0.2.2 (Schwede–Shipley 2003)**.** *If* k *is a commutative ring, there is a Quillen equivalence of model categories*

$$\mathbf{sAlg}(k) \simeq_Q \mathbf{dgAlg}^+(k).$$

There is another monoidal version of the Dold–Kan correspondence in the cosimplicial case which we will call Belinson's theorem. It comes from number theory, and isn't too well known to homotopy theorists.

Theorem 0.2.3 (Belinson)**.** *Let* k *be a commutative ring. There is an equivalence of categories between the categories of cosimplicial commutative* k*-algebras and nonnegatively graded, commutative, differential graded algebras, after we restrict to the small objects in both categories.*

$$\mathbf{CommAlg}(k)^{\text{small}} \simeq \mathbf{Comm\,dgAlg}^+(k)^{\text{small}}$$

In the case of commutative DGA's, the small objects are easy to describe: they are those commutative differential graded k-algebras that are generated in degree zero and degree one.

There is also a categorified version of the Dold–Kan correspondence.

Theorem 0.2.4 (Categorical Dold–Kan)**.** *There is a Quillen equivalence between small categories enriched in simplicial modules and differential graded categories over a commutative ring* k *(i.e. small additive categories enriched over* $\mathbf{Ch}_+(k)$*).*

$$\mathbf{sMod}(k)\text{-}\mathbf{Cat} \simeq_Q \mathbf{dgCat}^+(k).$$

The category $\mathbf{dgCat}^+(k)$ *is given the Dwyer–Kan model structure, in which the weak equivalences are the quasi–equivalences.*

An application of this categorical version is found in rational homotopy theory.

Definition 0.2.5. A map $f\colon X \to Y$ of simply connected spaces is a **rational homotopy equivalence** if the induced map

$$\pi_n(f)\colon \pi_n(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to \pi_n(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism.

The **rational homotopy type** of a simply connected space X is the list of its rationalized homotopy groups $\pi_n(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Theorem 0.2.6 (Quillen 1969)**.** *Rational homotopy types of simply connected spaces are in one-to-one correspondence with homotopy types of reduced differential graded Lie algebras over* $\mathbb{Q}$*.*

Theorem 0.2.7 (Sullivan 1971)**.** *Rational homotopy types of simply connected spaces are in one-to-one correspondence with homotopy types of commutative differential graded* $\mathbb{Q}$*-algebras.*

Remark 0.2.8. The previous theorem works with any field of characteristic zero.

Remark 0.2.9. The correspondence between $\mathcal{C}omm\,\mathbf{dgAlg}^+(k)$ and reduced differential graded Lie algebras over $\mathbb{Q}$ is an instance of Koszul duality.

How do we translate from a space to a commutative dg $\mathbb{Q}$-algebra? For manifolds, this is clear: we may take the algebra of forms over the manifold.

Example 0.2.10. If X is a smooth real manifold, we can obtain a commutative differential graded $\mathbb{R}$-algebra $\Omega^\bullet(X) \in H_0\left(\mathcal{C}omm\,\mathbf{dgAlg}^+(k)\right)$.

In general, however, which algebraic objects can be taken as algebraic models for general spaces? An answer (Katzakov–Pantev–Toën 2008, Pridham 2008, Moriya 2012) is to look at closed tensor dg-categories

In part II, we will begin with simplicial groups. The main theorem here is Kan's loop group theorem. Classically, there is a Quillen equivalence between connected, pointed topological spaces and connected simplicial sets. Kan's loop group functor $\mathbb{G}\colon \mathbf{sSet}_0 \to \mathbf{sGr}$ gives a Quillen equivalence between connected simplicial sets and simplicial groups.

Theorem 0.2.11 (Kan)**.** *There is a Quillen equivalence between connected simplicial sets and simplicial groups.*

$$\mathbf{sSet}_0 \simeq_Q \mathbf{sGr}$$

This theorem generalizes the classifying space of a group Γ, which is a connected, pointed topological space $B\Gamma$. We will spend time investigating whether or not group-theoretic constructions (or property of groups) extends (up to homotopy) to spaces. For example, singular homology of spaces is an extension of abelianization of groups (due to Kan). More recently, people have been investigating the homotopy analog of inclusions of normal subgroups, called homotopy normal maps.

Chapter 1

Simplicial and cosimplicial sets

1.1 The (co)simplicial category

Definition 1.1.1. The **simplicial category** $\mathbb{\Delta}$ is the category whose objects are the finite totally ordered sets $[n] = \{0 \leq 1 \leq 2 \leq \ldots \leq n\}$, $n \geq 0$, and the morphisms are order-preserving maps

$$\mathrm{Hom}_{\mathbb{\Delta}}([m],[n]) = \left\{ f \in \mathrm{Hom}_{\mathbf{Sets}}([m],[n]) \;\middle|\; f(i) \leq f(j) \text{ in } [n] \text{ if } i \leq j \text{ in } [m] \right\}.$$

Remark 1.1.2 (Trivial observations)**.**

(a) The category $\mathbb{\Delta}$ has no interesting automorphisms: $\mathrm{Aut}_{\mathbb{\Delta}}([n]) = \{\mathrm{id}\}$ for all $n \geq 0$.

(b) $\mathbb{\Delta}$ has a terminal objects $[0]$ but no initial object.

There are two special classes of morphisms in $\mathbb{\Delta}$.

Definition 1.1.3.

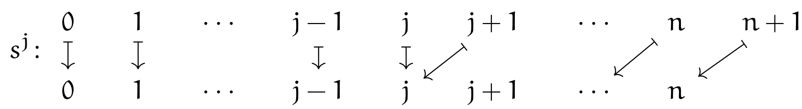
(a) The **coface maps** $d^i \colon [n-1] \hookrightarrow [n]$ for $0 \leq i \leq n$, $n \geq 1$ are defined by the property that d^i is injective and i does not appear in the image of d^i.

$$d^i(k) = \begin{cases} k & (k < i) \\ k+1 & (i \leq k) \end{cases}$$

$$\begin{array}{ccccccccccc} & 0 & 1 & \cdots & i-1 & i & i+1 & \cdots & n-1 & & \\ d^i\colon & \downarrow & \downarrow & & \downarrow & & \searrow & & \searrow & & \searrow \\ & 0 & 1 & \cdots & i-1 & i & i+1 & \cdots & n-1 & & n \end{array}$$

(b) The **codegeneracy maps** $s^j \colon [n+1] \twoheadrightarrow [n]$ for $0 \le j \le n$, $n \ge 1$ are defined by the property that s^j is surjective and takes the value j twice.

$$s^j(k) = \begin{cases} k & (k \le j) \\ k-1 & (k > j) \end{cases}$$



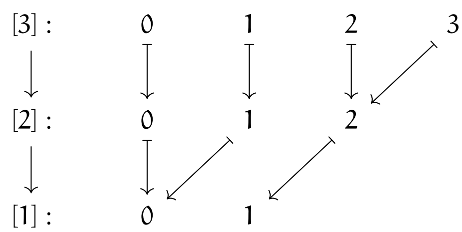
Lemma 1.1.4. *Every* $f \in \mathrm{Hom}_{\Delta}([n],[m])$ *can be decomposed (in a unique way) into the composition*

$$f = d^{i_1} d^{i_2} \cdots d^{i_r} s^{j_1} \cdots s^{j_s}$$

where $m = n + r - s$, $i_1 < i_2 < \ldots < i_r$ *and* $j_1 < j_2 < \ldots < j_s$.

Proof. See for example [**?**, Lemma 22]. □

Example 1.1.5. Consider the morphism $f \colon [3] \to [1]$ given by $f(0) = f(1) = 0$ and $f(2) = f(3) = 1$. We may write $f = s^0 \circ s^2$.



Lemma 1.1.6. *The morphisms* d^i *and* s^j *satisfy the* **cosimplicial identities***:*

$$\begin{aligned} d^j d^i &= d^i d^{j-1} \qquad (i < j) \\ s^j s^i &= s^i s^{i+1} \qquad (i \le j) \\ s^j d^i &= \begin{cases} d^i s^{j-1} & (i < j) \\ \mathrm{id} & (i = j) \text{ or } (i = j+1) \\ d^{i-1} s^j & (i > j+1) \end{cases} \end{aligned} \tag{1.1.1}$$

Remark 1.1.7. Δ can be defined abstractly as the category with objects $\{[n]\}_{n \ge 0}$ and morphisms generated by the cofaces d^i and codegeneracies s^j subject to the cosimplicial identities.

Definition 1.1.8. Let $\mathcal{C}$ be a category.

(a) A **cosimplicial object** in $\mathcal{C}$ is a functor $X^* \colon \Delta \to \mathcal{C}$, written $[n] \mapsto X^n = X^*([n])$.

(b) A **simplicial object** in $\mathcal{C}$ is a functor $X_* \colon \Delta^{op} \to \mathcal{C}$, written $[n] \mapsto X_n = X_*([n])$.

The category of simplicial objects in $\mathcal{C}$ is denoted

$$C = \mathcal{C}^{\Delta^{op}} = \mathbf{Fun}(\Delta^{op}, \mathcal{C}) = C_{\Delta}$$

and the category of cosimplicial objects in $\mathcal{C}$ is denoted

$$C = \mathcal{C}^{\Delta} = \mathbf{Fun}(\Delta, \mathcal{C}).$$

In both cases, the morphisms are natural transformations of functors.

Example 1.1.9. If $\mathcal{C} = \mathbf{Sets}$, then $C = \mathbf{sSet}$ is the category of simplicial sets.

Definition 1.1.10. The **simplicial category** is the dual Δ^{op} of the cosimplicial category

The dual category Δ^{op} has the presentation $\mathrm{Ob}(\Delta^{op}) = \mathrm{Ob}(\Delta) = \{[n]\}_{n \geq 0}$. Morphisms in Δ^{op} are generated by the **face maps**

$$d_i \colon [n] \to [n-1], \qquad 0 \leq i \leq n, n \geq 1,$$

and the degeneracy maps

$$s_j \colon [n] \to [n+1], \qquad 0 \leq j \leq n, n \geq 0.$$

The **simplicial relations** are as follows:

$$\begin{aligned} d_i d_j &= d_{j-1} d_i \qquad (i < j) \\ s_i s_j &= s_{j+1} s_i \qquad (i \leq j) \\ d_i s_j &= \begin{cases} s_{j-1} d_i & (i < j) \\ \mathrm{id} & (i = j) \text{ or } (i = j+1) \\ s_j d_{i-1} & (i > j+1) \end{cases} \end{aligned} \tag{1.1.2}$$

Remark 1.1.11. It is convenient to think of simplicial objects in $\mathcal{C}$ as right "modules" over Δ.

$$X_* = \left[\; X_0 \;\; \begin{matrix} \xleftarrow{d_0} \\ \xrightarrow{s_0} \\ \xleftarrow[d_1]{} \end{matrix} \;\; X_1 \;\; \begin{matrix} \xleftarrow{d_0} \\ \xrightarrow{s_0} \\ \xleftarrow{d_1} \\ \xrightarrow{s_1} \\ \xleftarrow[d_2]{} \end{matrix} \;\; X_2 \dashrightarrow \cdots \; \right]$$

Likewise, a cosimplicial object in $\mathcal{C}$ is a left "module" over Δ.

We will later encounter the importance of cosimplicial objects when we prove the following:

Theorem 1.1.12. *Let $\mathcal{D}$ be a cocomplete, locally small category. Then the category of cosimplicial objects in $\mathcal{D}$ is equivalent to the category of simplicial adjunctions*

$$\mathbf{sSet} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}, \quad L \dashv R$$

with left adjoint defined on **sSet**.

Definition 1.1.13 (Some more terminology)**.** Let $\mathcal{C} = \mathbf{Sets}$.

(a) If X_* is a simplicial set, then X_n is the set of n**-simplicies**.

(b) An n-simplex $x \in X_n$ is called **degenerate** if $x \in \operatorname{im}(s_j)$ for some j. Denote the set of degenerate n-simplicies by

$$X_n^{\text{dej}} := \bigcup_{j=0}^{n-1} s_j(X_{n-1})$$

Exercise 1.1.14. Using the simplicial identities, we can express

$$X_n^{\text{dej}} = \bigcup_{\substack{f\colon [n]\twoheadrightarrow[k] \\ f\neq \text{id}}} X(f)(X_k) \cong \operatorname{colim}_{\substack{f\colon [n]\twoheadrightarrow[k] \\ f\neq \text{id}}} X(f)(X_k) \tag{1.1.3}$$

when such colimits exist in $\mathcal{C}$.

Remark 1.1.15. Formula (1.1.3) makes sense for more general categories, so we can define an object of degenerate simplicies for any simplicial object in a category that has such colimits.

1.2 Examples

1.2.1 Geometric examples

The geometric examples of simplicial sets provide the basis for simplicial homotopy theory.

Example 1.2.1 (Geometric simplicies)**.** The **geometric** n**-simplex** is the topological space Δ^n defined by

$$\Delta^n = \left\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i=1}^{n} x_i = 1,\ x_i \geq 0 \right\} \subseteq \mathbb{R}^{n+1}.$$

Alternatively, this is the convex hull of the standard unit vectors $e_i \in \mathbb{R}^{n+1}$. There is a natural functor $\mathbb{\Delta} \to \textbf{Top}$ given on objects by $[n] \mapsto \Delta^n$ and on morphisms $f\colon [n] \to [m]$ by the restriction to Δ^n of the map $\mathbb{R}^{n+1} \to \mathbb{R}^{m+1}$ sending e_i to $e_{f(i)}$.

$$\begin{array}{ccc} \Delta^n & \xrightarrow{\Delta^*(f)} & \Delta^m \\ \cup\!| & & \cup\!| \\ \mathbb{R}^n & & \mathbb{R}^m \\ \Psi & & \Psi \\ e_i & \longmapsto & e_{f(i)} \end{array}$$

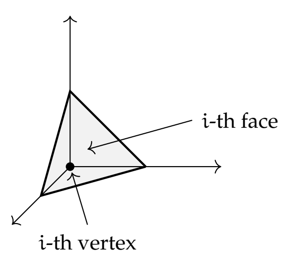
The functor $\Delta^*\colon \mathbb{\Delta} \to \textbf{Top}$ is a cosimplicial space. Let's compute the coface and codegeneracy maps on Δ^*. Recall that

$$d^i(e_k) = \begin{cases} e_k & (k < i) \\ e_{k+1} & (k \geq i) \end{cases}$$

Therefore, $d^i\colon \Delta^{n-1} \to \Delta^n$ is given by

$$\begin{aligned} d^i(x_0, x_1, \dots, x_{n-1}) &= d^i\left(\sum_{k=0}^{n-1} x_k e_k\right) \\ &= \sum_{k=0}^{n-1} x_k d^i(e_k) \\ &= \sum_{k=0}^{i-1} x_k e_k + \sum_{k=i+1}^{n} x_{k-1} e_k = (x_0, x_1, \dots, x_{i-1}, 0, x_i, \dots, x_{n-1}) \end{aligned}$$

Geometrically, in Δ^n we define the i-th $(n-1)$-dimensional face to be the one opposite to e_i. Then the map $d^i\colon \Delta^{n-1} \hookrightarrow \Delta^n$ is just the inclusion of the i-th face.



Dually, $s^j\colon \Delta^{n+1} \to \Delta^n$ is given by

$$s^j(x_0, x_1, \dots, x_n) = (x_0, x_1, \dots, x_{j-1}, x_j + x_{j+1}, \dots, x_{n+1})$$

Geometrically, s^j collapses the edge between the j-th and $(j+1)$-st vertices in Δ^{n+1} to a point.

Definition 1.2.2. Given any nonempty subset $\sigma \subseteq [n] = \{0, 1, \dots, n\}$, we define Δ_σ to be the convex hull of $\{e_i \mid i \in \sigma\} \subseteq \Delta^n$. This is called the **$\sigma$-face** of Δ^n.

Given this definition, we have a bijection between nonempty subsets of $[n]$ and faces in Δ^n

Definition 1.2.3. A **finite polyhedron** is a topological space X homeomorphic to a union of faces in Δ^n

$$X = \bigcup_{i=1}^{r} \Delta_{\sigma_i} \subseteq \Delta^n$$

The choice of this homeomorphism is a **triangulation** of X.

Definition 1.2.4. A **simplicial complex** X on a set V (the **vertex set** of X) is a collection of non-empty finite subsets in V closed under taking subsets: for all $\sigma \in X$, if τ is a nonempty subset of σ, then $\tau \in X$.

Remark 1.2.5. We do not assume that V is finite and that $V = \bigcup_{\sigma \in X} \sigma$.

To associate to (X, V) a space, we consider the $\mathbb{R}$-vector space $\mathbb{V} = \mathrm{Span}_{\mathbb{R}}(V)$ and define for each $\sigma \in X$, $\Delta_\sigma \subseteq \mathbb{V}$ as the convex hull of $\sigma \subset V \subseteq \mathbb{V}$. Δ_σ is a topological space equipped with a topology induced from $\mathbb{V}$.

Definition 1.2.6. The **geometric realization** $|X|$ of a simplicial complex is

$$|X| = \bigcup_{\sigma \in X} \Delta_\sigma \subseteq \mathbb{V}.$$

Definition 1.2.7. A **polyhedron** is a topological space homeomorphic to $|X|$ for some simplicial complex.

Remark 1.2.8. Simplicial complexes have bad functorial properties: if $Y \subseteq X$ is a simplicial subcomplex, the quotient X/Y is not necessarily a simplicial complex.

Simplicial sets can be viewed as a generalization of simplicial complexes. To an (ordered) simplicial complex (X, V) (where V is given an order), we can define the simplicial set $SS_*(X)$ as follows:

$$SS_n(X) = \left\{ (v_0, v_1, \dots, v_n) \in V^{n+1} \;\middle|\; \{v_0, v_1, \dots, v_n\} \subset X \right\}.$$

For $f \colon [n] \to [m]$ in Δ, we define

$$\begin{aligned} SS_m(X) &\xrightarrow{SS(f)} SS_n(X) \\ (v_0, v_1, \dots, v_m) &\longmapsto (v_{f(0)}, v_{f(1)}, \dots, v_{f(n)}) \end{aligned}$$

Explicitly,

$$\begin{aligned} SS_n(X) &\xrightarrow{d_i} SS_{n-1}(X) \\ (v_0, v_1, \dots, v_n) &\longmapsto (v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n) \end{aligned}$$

$$\begin{array}{ccc} SS_n(X) & \xrightarrow{\qquad s^j \qquad} & SS_{n+1}(X) \\ (v_0, v_1, \ldots, v_m) & \longmapsto & (v_0, \ldots, v_j, v_j, \ldots, v_n) \end{array}$$

Exercise 1.2.9. Show that:

(a) Show that X can be recovered from $SS_*(X)$; more precisely, the set of nondegenerate simplicies in $SS_*(X)$ are in bijection with X.

(b) For any ordered simplicial complex X, its geometric realization is homeomorphic to the geometric realization of the simplicial set $SS_*(X)$ (see REFERENCE GEOMETRIC REALIZATION OF SIMPLICIAL SETS DEFINITION)

$$|X| \cong |SS_*(X)|$$

Definition 1.2.10. Given a topological space X, we define the **singular simplicial set** $S_*(X)$ by

$$S_n(X) = \mathrm{Hom}_{\mathbf{Top}}(\Delta^n, X)$$

for $n \geq 0$. This becomes a functor $S_*\colon \mathbf{Top} \to \mathbf{sSet}$ defined on morphisms $f\colon [n] \to [m] \in \mathrm{Mor}(\mathbb{\Delta})$ by

$$\begin{array}{ccc} S_m(X) & \xrightarrow{S_*(f)} & S_n(X) \\ \phi & \longmapsto & \phi \circ \mathbb{\Delta}(f) \end{array}$$

Definition 1.2.11. A **discrete** or **constant simplicial set** is a simplicial set in image of the embedding $\mathbf{Set} \hookrightarrow \mathbf{sSet}$. More explicitly, for any set X, we can define a simplicial set whose n-simplicies are X for all n, and all faces and degeneracies are identity maps.

Definition 1.2.12. For any locally small category $\mathcal{C}$, we define the category $\widehat{\mathcal{C}}$ of **presheaves on** $\mathcal{C}$ or $\mathcal{C}$**-sets** by,

$$\widehat{\mathcal{C}} := \mathbf{Fun}(\mathcal{C}^{op}, \mathbf{Set}).$$

There is a canonical categorical equivalence

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\qquad h \qquad} & \widehat{\mathcal{C}} \\ c & \longmapsto & h_c = \mathrm{Hom}_{\mathcal{C}}(-, c) \\ (c_1 \xrightarrow{f} c_2) & \longmapsto & h_f = f \circ - \end{array}$$

Lemma 1.2.13 (Yoneda)**.** *The functor h is fully faithful: more generally, for any $c \in \mathrm{Ob}(\mathcal{C})$, and any $X \in \mathrm{Ob}(\widehat{\mathcal{C}})$,*

$$\mathrm{Hom}_{\widehat{\mathcal{C}}}(h_c, X) \xrightarrow{\sim} X(c).$$

Explicitly, the equivalence is given by

$$\phi := \left\{\phi_d \mid h_c(d) \to X(d)\right\}_{d \in \mathrm{Ob}(\mathcal{C})} \mapsto \phi_c(\mathrm{id}_c) \in X(c).$$

Definition 1.2.14. Presheaves on $\mathcal{C}$ isomorphic to h_c for $c \in \mathrm{Ob}(\mathcal{C})$ are called **representable** presheaves.

Recall that $\mathbf{sSet} = \mathbf{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{Set})$. In the new notation, a simplicial set is a presheaf on $\mathbb{\Delta}$, that is, $\mathbf{sSet} = \widehat{\mathbb{\Delta}}$.

Definition 1.2.15. A **standard simplex** $\Delta[n]_*$ is the simplicial set of the form $h_{[n]}$ for $n \geq 0$ fixed.

The simplicial set $\Delta[n]_*$ is the functor

$$\begin{aligned} \mathbb{\Delta}^{\mathrm{op}} &\xrightarrow{h_{[n]} = \Delta[n]_*} \mathbf{Set} \\ [k] &\longmapsto \Delta[n]_k = \mathrm{Hom}_{\mathbb{\Delta}}([k],[n]) \end{aligned}$$

Explicitly, there is a bijection

$$\Delta[n]_k \cong \left\{(j_0, j_1, \ldots, j_k) \mid 0 \leq j_0 \leq j_1 \leq \ldots \leq j_k \leq n\right\}.$$

Consider now the generating maps of $\mathbb{\Delta}$:

$$\begin{aligned} d^i &\colon [k-1] \to [k] & 0 \leq i \leq k,\ k \geq 1 \\ s^\ell &\colon [k+1] \to [k] & 0 \leq \ell \leq k, k \geq 0 \end{aligned}$$

Under $h_{[n]}$, the maps d^i become:

$$\begin{aligned} \mathbb{\Delta}[n]_k &\xrightarrow{d_i = h_{d^i}} \mathbb{\Delta}[n]_{k-1} \\ (j_0, j_1, \ldots, j_k) &\longmapsto (j_0, \ldots, \widehat{j_i}, \ldots, j_k). \end{aligned}$$

and similarly, the maps s^ℓ become:

$$\begin{aligned} \mathbb{\Delta}[n]_k &\xrightarrow{s_\ell = h_{s^\ell}} \mathbb{\Delta}[n]_{k+1} \\ (j_0, j_1, \ldots, j_k) &\longmapsto (j_0, \ldots, j_\ell, j_\ell, \ldots, j_k). \end{aligned}$$

By inspection, if we think of the geometric n-simplex Δ^n as an abstract simplicial complex, then $\Delta[n]_*$ is the associated simplicial set $\Delta[n]_* = SS_*(\Delta^n)$.

Lemma 1.2.16. $\Delta[n]_*$ *corepresents the functor*

$$\begin{aligned} \mathbf{sSet} &\longrightarrow \mathbf{Set} \\ X_* &\longmapsto X_n \end{aligned}$$

Proof. It immediate from the Yoneda lemma (this is a proof by changing notation):

$$\mathrm{Hom}_{\mathbf{sSet}}(\Delta[n]_*, X_*) = \mathrm{Hom}_{\widehat{\mathbb{\Delta}}}(h_{[n]}, X) \cong X([n]) = X_n \qquad \square$$

This lemma allows us to associate a unique simplicial map $\widehat{x}\colon \Delta[n]_* \to X_*$ to each simplex $x \in X_n$ such that $\widehat{x}(\mathrm{id}_{[n]}) = x$. Frequently, we will just omit the hat and think of these simplicies as morphisms of simplicial sets.

The Yoneda functor in this case is a functor

$$\begin{aligned} \mathbb{\Delta} &\hookrightarrow \widehat{\mathbb{\Delta}} = \mathbf{sSet} \\ [n] &\longmapsto \Delta[n]_* = h_{[n]} \end{aligned}$$

This assignment $[n] \mapsto \Delta[n]_*$ defines a cosimplicial simplicial set $\Delta[\bullet]$.

The category csSet of cosimplicial simplicial sets can be curried:

$$\mathrm{csSet} = \mathbf{Fun}(\mathbb{\Delta}, \mathbf{sSet}) = \mathbf{Fun}(\mathbb{\Delta}, \mathbf{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{Set})) = \mathbf{Fun}(\mathbb{\Delta} \times \mathbb{\Delta}^{\mathrm{op}}, \mathbf{Set}).$$

We think of objects in csSet as (graded) bimodules over $\mathbb{\Delta}$, i.e. as functors $\mathbb{\Delta} \times \mathbb{\Delta}^{\mathrm{op}} \to \mathbf{Set}$.

This allows us to define two very important simplicial subsets in $\Delta[n]_*$: the boundaries and the horns.

Definition 1.2.17. $\partial\Delta[n]_*$ is the **boundary** of $\Delta[n]_*$:

$$\partial\Delta[n]_* := \bigcup_{i=0}^{n} d^i(\mathbb{\Delta}[n-1]_*)$$

where $d^i\colon \Delta[n-1]_* \to \Delta[n]_*$ is the morphism of simplicial sets.

The boundary $\partial\Delta[n]_*$ is the smallest subcomplex of $\Delta[n]_*$ generated by its $(n-1)$-dimensional faces.

Example 1.2.18. There is a bijection

$$\Delta[0]_k \cong \{\underbrace{(0,0,\ldots,0)}_{(k+1)}\}_{k\geq 0},$$

so there is only one non-degenerate simplex $(0) \in \Delta[0]_0$.

Likewise, there is a bijection

$$\Delta[1]_k \cong \left\{(j_0, j_1, \ldots, j_k) \,\middle|\, j_0 \leq j_1 \leq \ldots \leq j_k \leq 1\right\} = \left\{(\underbrace{0,0,\ldots,0}_{i},\underbrace{1,\ldots,1}_{k+1-i}) \,\middle|\, 0 \leq i \leq k+1\right\}$$

Some non-degenerate simplicies are $(0) \in \Delta[1]_0$ and $\{(0,1)\} \subseteq \Delta[1]_1$.

These morphisms d^0 and d^1 of simplicial sets are given by

$$\begin{array}{ccc} \Delta[0]_* & \longrightarrow & \Delta[1]_* \\ (0,\dots,0) & \longmapsto & (0,\dots,0) \end{array}$$

$$\begin{array}{ccc} \Delta[0]_* & \longrightarrow & \Delta[1]_* \\ (0,\dots,0) & \longmapsto & (1,\dots,1) \end{array}$$

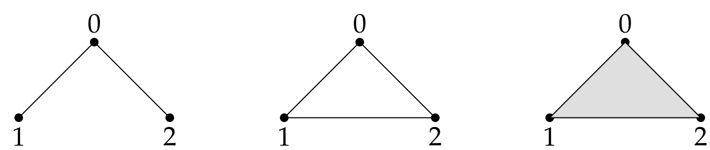
Then

$$\partial\Delta[1]_* = d^0(\Delta[0]_*) \cup d^1(\Delta[0]_*) = \{(0,\dots,0),(1,\dots,1)\}$$

Definition 1.2.19. Fix $k \geq 0$. The **k-th horn** $\Lambda_k[n]_*$ is the simplicial set

$$\Lambda_k[n]_* := \bigcup_{\substack{0\leq i\leq n \\ i\neq k}} d^i(\Delta[n-1]_*) \subseteq \partial\Delta[n]_* \subseteq \Delta[n]_*$$

Geometrically, the k-th horn is the cone with vertex k.



$$\Lambda_0[2] \quad \hookrightarrow \quad \partial\Delta[2]_* \quad \hookrightarrow \quad \Delta[2]_*$$

Simplicial spheres S^n

Definition 1.2.20. The **simplicial n-sphere** S^n is defined as the quotient simplicial set

$$S^n_* := {}^{\Delta[n]_*}\big/_{\partial\Delta[n]_*}.$$

Equivalently, this is the pushout

$$\begin{array}{ccc} \partial\Delta[n]_* & \hookrightarrow & \Delta[n]_* \\ \downarrow & & \downarrow \\ * & \longrightarrow & S^n_* \end{array}$$

Example 1.2.21. The **simplicial circle** S^1_* is built as follows.

$$\Delta[1]_k = \{(\underbrace{0,\dots,0}_{i},\underbrace{1,\dots,1}_{k+1-i}\}$$

$$\partial\Delta[1]_k = \{(\underbrace{0,\ldots,0}_{k+1}), (\underbrace{1,\ldots,1}_{k+1})\}$$

$$S^1_k = \Delta[1]_k / \partial\Delta[1]_k \cong \{*, (\underbrace{0,\ldots,0}_{i}, \underbrace{1,\ldots,1}_{k+1-i} \mid i = 1,\ldots,k\}$$

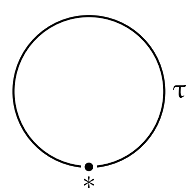
Where $* = (0,\ldots,0) \sim (1,\ldots,1)$. To be explicit,

$$\begin{aligned}
S^1_0 &= \{*\} \\
S^1_1 &= \{*, \tau = (0,1)\} \\
S^1_2 &= \{s_1(t), s_0(\tau), s_0^2(*)\} \\
&\vdots \\
S^1_k &= \{s_{k-1}s_{k-2}\cdots\widehat{s_i}\cdots s_0(\tau) \mid 0 \le i < k\} \cup \{s_0^k(*)\}
\end{aligned}$$

Note that in particular, $*$ and τ are the only nondegenerate simplicies, and that by the relations $s_1 s_0(*) = s_0^2(*)$. The maps d_i and s_j are uniquely determined by the simplicial relations (1.1.2).

The geometric realization of the simplicial circle is the topological space S^1.

Remark 1.2.22. This can be seen as the simplest or smallest simplicial model of S^1 built from the standard cell decomposition consisting of a single 0-cell $*$ and a single 1-cell τ:



Remark 1.2.23. We may identify S^1_k with the cyclic group $\mathbb{Z}/(k+1)\mathbb{Z}$ by $s_0^k(*) \mapsto 0$ and $s_{k-1}s_{k-2}\cdots\widehat{s_i}\cdots s_0(t) \mapsto i \pmod{k+1}$. However, the face and degeneracy maps are not group homomorphisms, so this is *not* a simplicial group. Nevertheless, it is an example of a **crossed simplicial group**, which we will encounter later.

$$S^1_* = \left[\; 0 \; \underset{d_1}{\overset{d_0}{\leftleftarrows}} \!\!\xrightarrow{s_0}\; \mathbb{Z}/2\mathbb{Z} \; \underset{d_2}{\overset{d_0}{\leftleftarrows}}\!\!\overset{s_0,\,s_1}{\rightrightarrows}\!\!\xleftarrow{d_1} \; \mathbb{Z}/3\mathbb{Z} \dashrightarrow \cdots \;\right]$$

1.3 The cyclic category

To formalize the observation from Remark 1.2.23, we introduce a new category.

Definition 1.3.1. The **cyclic category** (or **Connes' category**) $\Delta\mathbf{C}$ has objects $\{[n]\}_{n\geq 0}$ and morphisms generated by

$$\begin{aligned} d^i &\colon [n-1] \to [n] \\ s^j &\colon [n] \to [n+1] \\ \tau_n &\colon [n] \to [n] \end{aligned}$$

called **cofaces** d^i, **codegeneracies** s^j and **cyclic maps** τ_n. These satisfy three types of relations:

(a) d^i and s^j satisfy the cosimplicial relations (1.1.1) as in Δ,

(b) cyclic relations between τ_n, d^i and s^j:

$$\begin{aligned} \tau_n d^i &= d^{i-1}\tau_n \qquad (1 \leq i \leq n) \\ \tau_n d^0 &= d^n \\ \tau_n s^j &= s^{j-1}\tau_{n+1} \qquad (1 \leq j \leq n) \\ \tau_n s^0 &= s^n \tau_{n+1}^2, \end{aligned}$$

(c) $\tau_n^{n+1} = \mathrm{id}_{[n]}$ for all $n \geq 0$.

Remark 1.3.2.

(a) Although $\Delta\mathbf{C}$ has the "same" objects as Δ, we do not regard $[n]$ as sets because morphisms cannot be viewed as set maps. For example, $[0]$ is not terminal in $\Delta\mathbf{C}$. In fact, $\Delta\mathbf{C}$ has neither an initial object nor a terminal object.

(b) Interestingly, $\Delta\mathbf{C}^{\mathrm{op}} \cong \Delta\mathbf{C}$, yet $\Delta^{\mathrm{op}} \not\cong \Delta$. This is known as **Connes' Duality**: cyclic and cocyclic objects are the same.

(c) The two relations $\tau_n d^0 = d^n$ and $\tau_n s^0 = s^n \tau_{n+1}^2$ are redundant and can be omitted. Indeed,

$$d^n = \tau_n^{n_1} d_n = (\tau_n)^n \tau_n d^n = \tau_n^n (d^{n-1}\tau_{n-1}) = \cdots = \tau_n d^0 \tau_{n-1}^n = \tau_n d^0$$

The structure of $\Delta\mathbf{C}$ described by the the following:

Theorem 1.3.3. *The cyclic category $\Delta\mathbf{C}$ contains the simplicial category Δ as a subcategory (but not full) and*

(a) $\mathrm{Aut}_{\Delta\mathbf{C}}([n]) \cong \mathbb{Z}/_{(n+1)\mathbb{Z}}$ *for all* $n \geq 0$,

(b) every morphism $f \in \mathrm{Hom}_{\Delta\mathbf{C}}([n],[m])$ *can be uniquely factored as* $f = \phi\gamma$ *where* $\gamma \in \mathrm{Aut}_{\Delta\mathbf{C}}([n])$ *and* $\phi \in \Delta$.

We will later prove this in general for any crossed simplicial groups.

Definition 1.3.4. A **cyclic object** in $\mathcal{C}$ is a functor $X\colon \mathbf{\Delta C}^{\mathrm{op}} \to \mathcal{C}$.

In $\mathbf{\Delta C}^{\mathrm{op}}$, the objects are $\{[n]\}_{n\geq 0}$ and morphisms are

$$\begin{aligned} d_i &= (d^i)^{\mathrm{op}}\colon [n] \to [n-1] && (0 \leq i \leq n, n \geq 1) \\ s_j &= (s^j)^{\mathrm{op}}\colon [n] \to [n+1] && (0 \leq j \leq n, n \geq 0) \\ t_n &= \tau_n{}^{\mathrm{op}}\colon [n] \to [n] && (n \geq 0) \end{aligned}$$

with relations dual to those in Definition 1.3.1.

Example 1.3.5. Define a functor $C_*\colon \mathbf{\Delta C}^{\mathrm{op}} \to \mathbf{Set}$ on objects by

$$\begin{aligned} C_*\colon \mathbf{\Delta C}^{\mathrm{op}} &\longrightarrow \mathbf{Set} \\ [n] &\longrightarrow C_n = \mathrm{Aut}_{\mathbf{\Delta C}^{\mathrm{op}}}([n]) \cong \mathbb{Z}/_{(n+1)\mathbb{Z}} \end{aligned}$$

and on morphisms as follows. Recall that any $f\colon [n] \to [m]$ in $\mathbf{\Delta C}$ can be uniquely factored as $f = \phi \circ \gamma$, where $\gamma \in \mathrm{Aut}_{\mathbf{\Delta C}}([n])$ and $\phi \in \mathrm{Hom}_{\mathbf{\Delta}}([n],[m])$. Take any $g \in \mathrm{Aut}_{\mathbf{\Delta C}}([m])$ and any $a \in \mathrm{Hom}_{\mathbf{\Delta C}}([n],[m])$ and consider $f = ga$.

By unique factorization, there is a unique $\phi = g^*(a) \in \mathrm{Hom}_{\mathbf{\Delta}}([n],[m])$ and a unique $\gamma = a_*(g) \in \mathrm{Aut}_{\mathbf{\Delta C}}([n])$ such that

$$g \circ a = g^*(a) \circ a_*(g).$$

Thus, for a fixed $g \in \mathrm{Aut}_{\mathbf{\Delta C}}([m],[n])$ we defined a map

$$g^*\colon \mathrm{Hom}_{\mathbf{\Delta C}}([n],[m]) \to \mathrm{Hom}_{\mathbf{\Delta}}([n],[m])$$

and for a fixed $a \in \mathrm{Hom}_{\mathbf{\Delta}}([n],[m])$, we have

$$a_*\colon \mathrm{Aut}_{\mathbf{\Delta C}}([m]) \to \mathrm{Aut}_{\mathbf{\Delta C}}([n]).$$

such that $g \circ a = g^*(a) \circ a_*(g)$.

Dually in $\mathbf{\Delta C}^{\mathrm{op}}$, for a fixed $g \in \mathrm{Aut}_{\mathbf{\Delta C}^{\mathrm{op}}}([m])$ we have

$$g^*\colon \mathrm{Hom}_{\mathbf{\Delta C}^{\mathrm{op}}}([m],[n]) \to \mathrm{Hom}_{\mathbf{\Delta}^{\mathrm{op}}}([m],[n])$$

and for a fixed $a \in \mathrm{Hom}_{\mathbf{\Delta}^{\mathrm{op}}}([m],[n])$, we have

$$a_*\colon \mathrm{Aut}_{\mathbf{\Delta C}^{\mathrm{op}}}([m]) \to \mathrm{Aut}_{\mathbf{\Delta C}^{\mathrm{op}}}([n]).$$

such that $a \circ g = a_*(g) \circ g^*(a)$.

Since composition in $\mathbf{\Delta C}^{\mathrm{op}}$ is associative, we have

$$(a' \circ a)_*(g) = a'_*(a_*(g))$$

for any $a' \in \mathrm{Hom}_{\Delta\mathbf{C}^{op}}([n],[m])$ in $\Delta\mathbf{C}^{op}$.

Thus, we can define the functor C_* on morphisms by

$$\begin{aligned} C_*\colon \Delta\mathbf{C}^{op} &\longrightarrow \mathbf{Set} \\ \big(a\colon [m]\to[n]\big) &\longmapsto (a_*\colon C_m \to C_n) \end{aligned}$$

Explicitly, by the formula

$$a \circ g = a_*(g) \circ g^*(a),$$

we have

$$d_i \circ t_n = (d_i)_*(t_n) \circ t_n^*(d_i).$$

On the other hand, we have the relation

$$d_i \circ t_n = t_{n-1} \circ d_{i-1}$$

so by uniqueness we have $d_i^C(t_n) = (d_i)_*(t_n) = t_{n-1}$. Likewise, by other relations in $\Delta\mathbf{C}^{op}$, we have

$$\begin{aligned} d_i^C(t_n) &= (d_i)_*(t_n) = t_{n-1} \\ d_0^C(t_n) &= \mathrm{id} \\ s_j^C(t_n) &= t_{n+1} \\ s_0^C(t_n) &= t_{n+1}^2 \end{aligned}$$

Lemma 1.3.6. *We have the isomorphism of simplicial sets*

$$S^1_* \cong C_*$$

given on simplicies by

$$\begin{aligned} s_0^n(*) &\mapsto (t_n)^0 = 1, \\ s_{n-1}\cdots\widehat{s_i}\cdots s_0(\tau) &\mapsto t_n^{i+1} \end{aligned}$$

1.4 Categorical applications

Recall that a category is **small** if its objects form a proper set.

Definition 1.4.1. Associated to any small category $\mathcal{C}$ is the **nerve** of $\mathcal{C}$, denoted

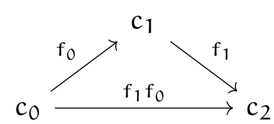
$\mathcal{N}_*\mathcal{C}$ or $B_*\mathcal{C}$, defined by

$$\begin{aligned}
\mathcal{N}_0\mathcal{C} &= \mathrm{Ob}(\mathcal{C}) \\
\mathcal{N}_1\mathcal{C} &= \mathrm{Mor}(\mathcal{C}) \\
\mathcal{N}_2\mathcal{C} &= \left\{ c_0 \xrightarrow{f_0} c_1 \xrightarrow{f_1} c_2 \right\} \\
&\vdots \\
\mathcal{N}_n\mathcal{C} &= \left\{ c_0 \xrightarrow{f_0} c_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} c_n \right\}
\end{aligned}$$

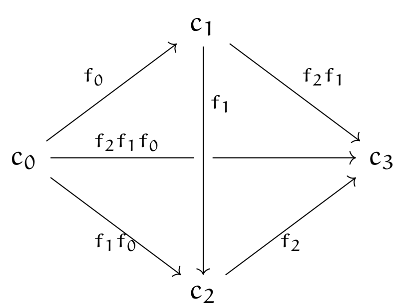
We also write the objects of $\mathcal{N}_n\mathcal{C}$ as n-tuples of morphisms in $\mathcal{C}$ with the understanding that they must be composable:

$$\mathcal{N}_n\mathcal{C} = \{(f_{n-1}, f_{n-2}, \ldots, f_1, f_0) \mid f_i \circ f_{i-1} \text{ exists}\}$$

The n-simplicies of $\mathcal{N}_*\mathcal{C}$ are the n-tuples of composable morphisms in $\mathcal{C}$. A 2-simplex in $\mathcal{C}$ is given by a pair of composable morphisms, which we visualize as follows:



And a 3-simplex is a triple of composable morphisms, which forms a tetrahedron:



These pictures immediately suggests the definition of the face and degeneracy maps for the nerve of $\mathcal{C}$:

$$d_i\left(c_0 \xrightarrow{f_0} c_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} c_n\right) = \left(c_0 \xrightarrow{f_1} \cdots c_{i-1} \xrightarrow{f_i \circ f_{i-1}} c_{i+1} \xrightarrow{f_{i+1}} \cdots \to c_n\right).$$

$$s_j\left(c_0 \xrightarrow{f_0} c_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} c_n\right) = \left(c_0 \to c_1 \to \cdots \to c_j \xrightarrow{\mathrm{id}_{c_j}} c_j \to \cdots \to c_n\right)$$

Another way to view this construction is the following. Let $s, t\colon \mathrm{Mor}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C})$ be the source and target maps, and let $i\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{C})$ be the identity morphism map. The composition map is $\circ\colon \operatorname{Mor}(\mathcal{C}) \times_{\operatorname{Ob}(\mathcal{C})} \operatorname{Mor}(\mathcal{C}) \to \operatorname{Mor}(\mathcal{C})$, where $\operatorname{Mor}(\mathcal{C}) \times_{\operatorname{Ob}(\mathcal{C})} \operatorname{Mor}(\mathcal{C})$ is the fiber product

$$\begin{array}{ccc} \operatorname{Mor}(\mathcal{C}) \times_{\operatorname{Ob}(\mathcal{C})} \operatorname{Mor}(\mathcal{C}) & \xrightarrow{p_1} & \operatorname{Mor}(\mathcal{C}) \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle t} \\ \operatorname{Mor}(\mathcal{C}) & \xrightarrow{\quad s \quad} & \operatorname{Ob}(\mathcal{C}). \end{array}$$

Notice that the structure of $\mathcal{C}$ gives us a diagram

$$\operatorname{Ob}(\mathcal{C}) \underset{\overleftarrow{\ t\ }}{\overset{\overleftarrow{\ s\ }}{\xrightarrow{\ i\ }}} \operatorname{Mor}(\mathcal{C}) \quad \begin{matrix} \xleftarrow{p_2} \\ \xrightarrow{i_2} \\ \xleftarrow{\circ} \\ \xrightarrow{i_1} \\ \xleftarrow{p_1} \end{matrix} \quad \operatorname{Mor}(\mathcal{C}) \times_{\operatorname{Ob}(\mathcal{C})} \operatorname{Mor}(\mathcal{C})$$

The nerve construction can be viewed as an extension of the category described by the diagram above to a full simplicial set. The n-simplicies of the nerve $\mathcal{N}_n\mathcal{C}$ are the iterated fiber products

$$\mathcal{N}_n\mathcal{C} = \underbrace{\operatorname{Mor}(\mathcal{C}) \times_{\operatorname{Ob}(\mathcal{C})} \operatorname{Mor}(\mathcal{C}) \times_{\operatorname{Ob}(\mathcal{C})} \cdots \times_{\operatorname{Ob}(\mathcal{C})} \operatorname{Mor}(\mathcal{C})}_{n}.$$

The maps s_j and d_i are the structure maps of the iterated fiber products.

Question 1.4.2. Clearly, $\mathcal{N}_*\mathcal{C}$ (as a simplicial set up to isomorphism) determines the category $\mathcal{C}$ (up to isomorphism). What kind of simplicial set do we obtain in this way?

The answer to this question is that we obtain an **∞-category** or a **quasicategory**. More about this later.

Example 1.4.3. Let X be a topological space, and let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open covering. Define a category $\mathcal{C} = X_{\mathcal{U}}$ with

$$\operatorname{Ob}(\mathcal{C}) = \left\{(x, U_i) \mid x \in U_i \in \mathcal{U}\right\} = \coprod_{i\in I} U_i$$

$$\operatorname{Hom}_{\mathcal{C}}\left((x, U_i), (y, U_j)\right) = \begin{cases} \emptyset & \text{if } x \neq y \\ \{x \to y\} & \text{if } x = y \text{ in } U_i \cap U_j \end{cases}$$

With this definition,

$$\operatorname{Mor}(\mathcal{C}) = \coprod_{(i,j)\in I^2} U_i \cap U_j.$$

The nerve of $X_{\mathcal{U}}$ is called the **Čech nerve** of the covering $\mathcal{U}$. In fact, this is a simplicial topological space.

It is a classical fact that if $\mathcal{U}$ is contractible (all U_i are contractible and finite intersections of the U_i are either empty or contractible) and "good," then the geometric realization of the Čech nerve determines X up to homotopy:

$$|\mathcal{N}_*\mathcal{C}| \simeq X.$$

This allows one to define homotopy types for various objects (e.g. étale homotopy types).

Question 1.4.4. Let $X = \mathrm{Spec}(A)$ be an affine algebraic variety over $\mathbb{C}$. We can consider its homotopy type defined in terms of coverings. There is a dual way: we can assign a homotopy type to A by viewing it as an object in $\mathbf{sCommAlg}_{\mathbb{C}}$. What's the relation between these two?

Example 1.4.5. Let G be a discrete group. There are two ways to view G as a category:

(a) The category $\underline{G}$ has $\mathrm{Ob}(\underline{G}) = \{*\}$ and $\mathrm{Mor}(\underline{G}) = G$. The **nerve of** G is $\mathcal{N}_*\underline{G}$ for which we will write B_*G. In this case, $B_nG = G^n$ for $n \geq 0$ and the faces $d_i \colon G^n \to G^{n-1}$ is given by

$$d_i(g_1, \ldots, g_n) = \begin{cases} (g_2, g_3, \ldots, g_n) & (i = 0) \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_n) & (1 \leq i \leq n) \\ (g_1, \ldots, g_{n-1}) & (i = n). \end{cases}$$

The degeneracies $s_j \colon G^n \to G^{n+1}$ are

$$s_j(g_1, \ldots, g_n) = (g_1, \ldots, g_j, 1, g_i, \ldots, g_n)$$

(b) The category $\underline{EG}$ has $\mathrm{Ob}(\underline{EG}) = G$ and morphisms

$$\mathrm{Hom}_{\underline{EG}}(g_1, g_2) = \{h \in G \mid hg_1 = g_2\} = \{h = g_2 g_1^{-1}\}.$$

The nerve $E_*G = \mathcal{N}(\underline{E}G)$ has $E_nG = G^{n+1}$. The face maps are

$$\begin{array}{ccc} E_nG & \xrightarrow{\quad d_i \quad} & E_{n-1}G \\ (g_0, \ldots, g_n) & \longmapsto & (g_0, \ldots, \widehat{g_i}, \ldots, g_n) \end{array}$$

and the degeneracies are

$$\begin{array}{ccc} E_nG & \xrightarrow{\quad s_j \quad} & E_{n+1}G \\ (g_0, \ldots, g_n) & \longmapsto & (g_1, \ldots, g_{j-1}, g_j, g_j, g_{j+1}, \ldots, g_n) \end{array}$$

Remark 1.4.6. B_*G is not a simplicial group, because d_i is not always group homomorphism. However, if G is abelian, then B_*G is a simplicial abelian group.

Remark 1.4.7. Note that we do not use the group operation to define E_*G, so in fact E_*X makes sense for any set or space X with $E_nX = X^{n+1}$ and s_j, d_i as above.

However, there are relations between E_*G and B_*G.

(1) There is a natural projection of simplicial sets $p_* \colon E_*G \to B_*G$ given by

$$(g_0, g_1, \dots, g_n) \mapsto (g_0g_1^{-1}, g_1g_2^{-1}, \dots, g_{n-1}g_n^{-1}).$$

(2) There is a right action of G on E_*G given by

$$\begin{array}{rcl} E_*G \times G & \longrightarrow & E_*G \\ (g_0, \dots, g_n), g & \longmapsto & (g_0g, g_1g, \dots, g_ng) \end{array}$$

such that the following diagram commutes:

$$\begin{array}{ccc} E_*G & \xrightarrow{p_*} & B_*G \\ \big\downarrow & \nearrow_{\cong} & \\ E_*G/G & & \end{array}$$

This gives an example of a **simplicial principal** G**-bundle**.

(3) Note that E_*G is a simplicial group which acts on the left on B_*G,

$$\begin{array}{rcl} E_*G \times B_*G & \longrightarrow & B_*G \\ (g_0, \dots, g_n) \cdot (h_1, \dots, h_n) & \longmapsto & (g_0h_1g_1^{-1}, g_1h_2g_2^{-1}, \dots, g_{n-1}h_ng_n^{-1}) \end{array}$$

Remark 1.4.8. B_*G is actually a cyclic set; the functor extends to the cyclic category:

$$\begin{array}{ccc} \Delta^{op} & \xrightarrow{B_*G} & \textbf{Set} \\ \big\downarrow & \nearrow & \\ \Delta\mathbf{C}^{op} & & \end{array}$$

More generally, fix $z \in Z(G) \subseteq G$, define $t_n \colon G^n \to G^n$ by

$$(g_1, \dots, g_n) \mapsto (z(g_1g_2\cdots g_n)^{-1}, g_1, \dots, g_{n-1})$$

Then

$$t_n^{n+1}(g_1, \ldots, g_n) = (zg_1z^{-1}, \ldots, zg_nz^{-1}) = (g_1, \ldots, g_n).$$

By since z is central, then $t_n^{n+1} = \mathrm{id}_{G^n}$.

This gives a cyclic set $B_*(G, z)$ called the **twisted nerve**. If $z = 1$, then this is the usual nerve.

Example 1.4.9 (Simplicial Borel construction)**.** Let X be a set and let G act on X, notated by $(g, x) \mapsto g \cdot x$. Then let $\mathcal{C} = C \ltimes X$ be the **action groupoid** of this action. The objects of $\mathcal{C}$ are just elements of X, and

$$\mathrm{Hom}_{G \ltimes X}(x, y) = \{g \in G \mid g \cdot x = y\}.$$

We write $B_*(G \ltimes X) = \mathcal{N}_*(G \ltimes X)$ for the nerve of this category. In particular, we have

$$\begin{aligned} B_0(G \ltimes X) &= X \\ B_1(G \ltimes X) &= G \times X \\ &\vdots \\ B_n(G \ltimes X) &= G^n \times X \end{aligned}$$

The face maps $d_i \colon B_n(G \ltimes X) \to B_{n-1}(G \ltimes X)$ are given by

$$d_i(g_1, \ldots, g_n, x) = \begin{cases} (g_2, g_3, \ldots, g_n, x) & (i = 0) \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_n, x) & (1 \le i \le n-1) \\ (g_1, \ldots, g_{n-1}, g_n \cdot x) & (i = n) \end{cases}$$

and the degeneracies are given by inserting identities:

$$\begin{aligned} s_j \colon B_n(G \ltimes X) &\longrightarrow B_{n+1}(G \ltimes X) \\ (g_1, \ldots, g_n, x) &\longmapsto (g_1, \ldots, 1, \ldots, g_n, x) \end{aligned}$$

Notice two things: if X is just a point, then $B_*(G \ltimes X) \cong B_*G$. If $X = G$ and G acts on itself by multiplication, then $B_*(G \ltimes G) \cong E_*G$. In general,

$$B_*(G \ltimes X) \cong E_*G \times_G X$$

The n-simplicies of this construction are given by $(E_nG \times X)/\sim$, where $\sim$ is the relation on n-simplicies given by

$$(g_1, \ldots, g_n; x) \sim (g_1, \ldots, g_{n-1}, 1; g_n \cdot x).$$

This is known as the **simplicial Borel construction**.

Exercise 1.4.10 (Homotopy Normal Maps)**.** How do you you extend the notion of inclusion of a normal subgroup $N \lhd G$ to a **normal map** $N \xrightarrow{f} G$? Give a definition and evidence for its correctness.

If we think of G as $\underline{G}$, the category with one object, then the action of G on a set X is just a functor $\underline{G} \to$ **Sets**.

Definition 1.4.11. Given any small diagram $X\colon \mathcal{C} \to$ **Sets** (or $X\colon \mathcal{C} \to$ **Top**), define the **Bousfield–Kan category**, alternatively denoted $\mathcal{C}_X$ or $\mathcal{C} \ltimes X$ or more commonly $\mathcal{C} \int X$, with objects

$$\mathrm{Ob}(\mathcal{C} \textstyle\int X) = \left\{ (c, x) \;\middle|\; c \in \mathrm{Ob}(\mathcal{C}), x \in X(c) \right\}$$

and morphisms

$$\mathrm{Hom}_{\mathcal{C} \int X}\left((c,x),(c',x')\right) = \left\{ f \in \mathrm{Hom}_{\mathcal{C}}(c,c') \;\middle|\; X(f)(x) = x' \right\}$$

Definition 1.4.12. Let $X\colon \mathcal{C} \to$ **Sets** be a diagram of sets with small domain. The **Bousfield–Kan construction** $B_*(\mathcal{C} \int X)$ of X is the nerve of the Bousfield–Kan category $\mathcal{C} \int X$

$$B_*(\mathcal{C} \textstyle\int X) := \mathcal{N}_*(\mathcal{C} \textstyle\int X).$$

Example 1.4.13. (a) Notice that if $X\colon C \to$ **Set** sends every object to a point, then $B_*(\mathcal{C} \int X) = B_* C$.

(b) If $X\colon \underline{G} \to$ **Set** for some group G, then $B_*(\underline{G} \int X) = B_*(G \ltimes X)$ is the Borel construction.

(c) Take $\mathcal{C} = \{0 \to 1\}$. Then $X\colon \mathcal{C} \to$ **Top** is just a map $f\colon X_0 \to X_1$ of spaces and $\mathcal{C} \int X$ is given by

$$\mathrm{Ob}(\mathcal{C} \textstyle\int X) = \left\{ (0, x_0) \;\middle|\; x_0 \in X_0 \right\} \cup \left\{ (1, x_1) \;\middle|\; x_1 \in X_1 \right\} = X_0 \sqcup X_1$$

$$\mathrm{Hom}_{\mathcal{C} \int X}\left((i,x_i),(j,x_j)\right) = \begin{cases} \mathrm{id}_x & \text{if } i = j \text{ and } x_i = x_j \\ f & \text{if } i = 0, j = 1 \text{ and } f(x_0) = x_1 \\ \emptyset & \text{otherwise} \end{cases}$$

With some consideration, one observes that

$$\mathrm{Mor}(\mathcal{C} \textstyle\int X) \cong X_0 \sqcup X_0 \sqcup X_1$$

where the first factor of X_0 corresponds to $\{\mathrm{id}_{x_0} \mid x_0 \in X_0\}$, the last factor X_1 corresponds to $\{\mathrm{id}_{x_1} \mid x_1 \in X_1\}$, and the middle factor is

$$X_0 \cong \bigsqcup_{x \in X_0} \mathrm{Hom}_{\mathcal{C} \int X}(x, f(x)).$$

The Bousfield–Kan construction corresponding to this construction has n-simplicies for $n \in \{0, 1\}$

$$\mathcal{N}_0(\mathcal{C} \int X) = X_0 \sqcup X_1$$
$$\mathcal{N}_1(\mathcal{C} \int X) = X_0 \sqcup X_0 \sqcup X_1$$

The only nondegenerate simplicies are those in degrees zero and one: a copy of $X_0 \sqcup X_1$ in degree zero and a copy of X_0 in degree one.

This will later imply that the realization of the Bousfield–Kan construction in this case is the mapping cylinder: $|\mathcal{N}_*(\mathcal{C} \int X)| \cong \mathrm{Cyl}(f)$.

Exercise 1.4.14. Show that the Bousfield–Kan construction corresponding to the diagram $X \colon \mathcal{C} \to \mathbf{Sets}$ where $\mathcal{C} = \{1 \leftarrow 0 \rightarrow 2\}$ corresponds to the mapping torus.

Let **Cat** be the category of all small categories, into which sets embed as discrete categories. It is then natural to consider diagrams of categories:

$$X \colon \mathcal{C} \to \mathbf{Cat}\,.$$

The Grothendieck construction is the generalization of the Bousfield–Kan construction sets are replaced by categories.

Definition 1.4.15. Let $X \colon \mathcal{C} \to \mathbf{Cat}$ be a diagram of categories with small domain. The **Grothendieck construction** $\mathcal{C} \int X$ (or $\mathcal{C} \ltimes X$) is the category with objects

$$\mathrm{Ob}(\mathcal{C} \int X) = \{(c, x) \mid c \in \mathrm{Ob}(\mathcal{C}), x \in \mathrm{Ob}(\mathcal{X}(c))\}$$

$$\mathrm{Hom}_{\mathcal{C} \int X}\left((c, x), (c', x')\right) = \left\{(f, \phi) \,\middle|\, f \in \mathrm{Hom}_{\mathcal{C}}(c, c'), \phi \in \mathrm{Hom}_{X(c')}(X(f)(x), x')\right\}$$

Composition of (f, ϕ) with (f', ϕ') is given by the formula:

$$(c, x) \xrightarrow{(f,\phi)} (c', x') \xrightarrow{(f',\phi')} (c'', x'').$$
$$(f', \phi') \circ (f, \phi) = (f' \circ f, \phi' \circ X(f')(\phi))$$

The composition $f' \circ f$ is in $\mathcal{C}$, and the composition $\phi' \circ X(f')(\phi)$ is in $X(c'')$.

Exercise 1.4.16. Check that composition in the Grothendieck construction $\mathcal{C} \int X$ is well-defined.

Definition 1.4.17. If $\xi \colon X \implies X'$ is a natural transformation of diagrams, we define a functor

$$\mathcal{C} \int \xi \colon \mathcal{C} \int X \to \mathcal{C} \int X'$$

on objects by

$$(c, x) \mapsto (c, \xi_c(x))$$

and on morphisms by

$$(f, \phi) \mapsto (f, \xi_{c'}(\phi)).$$

This then defines a functor

$$\mathcal{C} \int - \colon \mathbf{Fun}(\mathcal{C}, \mathbf{Cat}) \to \mathbf{Cat}\,.$$

Theorem 1.4.18 (Thomason)**.** *Let* $X\colon \mathcal{X} \to \mathbf{Cat}$ *be a diagram of categories. Then the homotopy colimit of the diagram in* **Top**

$$\mathcal{C} \xrightarrow{X} \mathbf{Cat} \xrightarrow{\mathcal{N}} \mathbf{sSet} \xrightarrow{|-|} \mathbf{Top}$$

is weakly equivalent to the realization of the nerve of the Grothendieck construction:

$$\operatorname{hocolim}_{\mathcal{C}} |\mathcal{N} \circ X| \simeq |\mathcal{N}(\mathcal{C} \textstyle\int X)|$$

This theorem tells us that $N_*(\mathcal{C} \int X)$ is a simplicial model for the homotopy colimit of the diagram $|\mathcal{N} \circ X|$. This theorem can be restated and proved in terms of simplicial sets.

1.5 ∞-categories

Question 1.5.1. Can we characterize the image of the nerve functor $\mathcal{N}\colon$ **Cat** $\to$ **sSet**? Given a simplicial set, is there a small category $\mathcal{C}$ such that $X_* = \mathcal{N}_*\mathcal{C}$.

Definition 1.5.2. If $\mathcal{C}$ is a category and $M \subseteq \mathrm{Mor}(\mathcal{C})$ is a class of morphisms in $\mathcal{C}$, then we say that a morphism $f\colon A \to B$ has the **right-lifting property** with respect to M ($f \in \mathrm{RLP}(M)$) if for every commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{h} & A \\ \big\downarrow g & \overset{\widetilde{k}}{\nearrow} & \big\downarrow f \\ D & \xrightarrow{k} & B \end{array}$$

with $g \in M$, there exists a **lift** $\widetilde{k}\colon C \to A$ such that $\widetilde{k}g = h$ and $f\widetilde{k} = k$.

Dually, $f\colon A \to B$ has the **left-lifting property** with respect to M if for every commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{k} & C \\ \big\downarrow f & \overset{\widetilde{h}}{\nearrow} & \big\downarrow g \\ B & \xrightarrow{h} & D \end{array}$$

with $g \in M$, there exists a **lift** $\widetilde{h}\colon B \to C$ such that $g\widetilde{h} = h$ and $\widetilde{h}f = k$.

Definition 1.5.3. A horn $\Lambda_k[n]$ is an **inner horn** if $0 < k < n$. A horn $\Lambda_k[n]$ is an **outer horn** if $k = 0$ or $k = n$.

Consider the inner horn inclusions

$$\left\{ i_n^k \colon \Lambda_k[n] \hookrightarrow \Delta[n] \right\}_{\substack{0<k<n \\ n\geq 2}} \subseteq \mathrm{Mor}(\mathbf{sSet}).$$

Definition 1.5.4 (Joyal)**.** A (small) **quasi-category** is a simplicial set X such that the canonical projection $X \to *$ has the right-lifting property with respect to the inner horn inclusions.

$$\begin{array}{ccc} \Lambda_k[n] & \longrightarrow & X \\ {\scriptstyle i_n^k}\big\downarrow & \overset{\tilde{p}_n}{\nearrow} & \big\downarrow{\scriptstyle p_X} \\ \Delta[n] & \xrightarrow{p_n} & * \end{array}$$

Let $\Lambda_k^n(X) = \mathrm{Hom}_{\mathbb{\Delta}}(\Lambda_k[n], X)$. The inner horn inclusions $i_n^k \colon \Lambda_k[n] \hookrightarrow \Delta[n]$ gives a map

$$\begin{array}{ccc} (i_n^k)^* \colon \mathrm{Hom}_{\mathbb{\Delta}}(\Delta[n], X) & \longrightarrow & \mathrm{Hom}_{\mathbb{\Delta}}(\Lambda_k[n], X) \\ \big\downarrow{\scriptstyle\cong} & & {\scriptstyle \mathrm{def}}\big\| \\ X_n & \longrightarrow & \Lambda_k^n \end{array}$$

The definition of a quasi-category can be restated as follows:

Definition 1.5.5 (Definition 1.5.4, revised)**.** A simplicial set X is called a **quasi-cateogry** if all maps $(i_n^k)^* \colon X_n \to \Lambda_k^n(X)$ are surjective for all $n \geq 0$ and all $0 < k < n$.

Informally, we say that each inner horn in X can be filled to an n-simplex.

A similar statement gives a characterization of the nerve, and also demonstrates how quasi-categories are a generalization of categories.

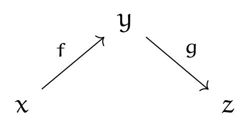
Theorem 1.5.6 (Characterization of the nerve)**.** *A simplicial set X is isomorphic to the nerve of some (necessarily unique up to isomorphism) small category $\mathcal{C}$ if and only if X is a quasi-category and each map $(i_n^k)^* \colon X_n \to \Lambda_k^n(X)$ is a bijection for $n \geq 2$, $0 < k < n$.*

Remark 1.5.7. The condition in Definition 1.5.5 is typically called the **weak Kan condition**, and quasi-categories are classically called **weak Kan complexes**. The usual (strong) **Kan condition** is the condition that all maps $(i_n^k)^* \colon X_n \to \Lambda_k^n(X)$ are surjective for all $n \geq 0$ and all $0 \leqq k < n$. Note that the only difference is that the surjectivity now holds for $k = 0$.

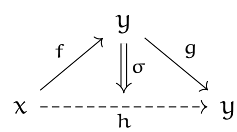
We will later encounter many examples of simplicial sets satisfying the strong Kan condition (and therefore the weak one as well): for example, the singular complex of a topological space and simplicial groups.

What does a quasi-category look like? Here's how you think about it:

- X_0 is the set of objects in a quasi-category X;
- X_1 is the usual set of morphisms in the quasi-category X;
- The source and target maps are $d_0, d_1 : X_1 \to X_0$. Now, given $f, g \in X_1$, which are composable, i.e. $d_1(f) = d_0(g)$, we have a horn $\sigma_0 \in \Lambda^2_1(X)$:



which by the weak Kan condition with $n = 2$ determines a 2-simplex $\sigma \in X_2$ "filling" σ_0 so that the composition $h = g \circ f$ is determined by $d_1(\sigma)$.



This composite h is only well-defined up to the weak Kan conditions for $n > 2$.

It turns out that the basic constructions of category theory can be extended to quasi-categories (due to Lurie [**?**, **?**]). Lurie's motivation was to give the singular simplicial set of a topological space a category-like structure. From the point of view of homotopy theory, ∞-categories are equivalent to topological spaces.

A theorem of Joyal demonstrates that quasi-categories model $(\infty, 1)$**-categories**, which are higher categories with all k-morphisms invertible for $k > 1$.

Theorem 1.5.8 (Joyal)**.** *On the full subcategory of quasi-categories in* **sSet** *there is a model structure.*

1.6 Some categorical constructions

1.6.1 Remarks on limits colimits

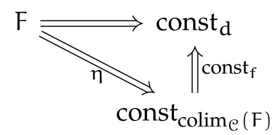
Unless otherwise stated, all categories $\mathcal{C}$ will be **locally small**, i.e. for fixed $X, Y \in \mathrm{Ob}(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a set.

Limits and colimits may be characterized by either a local universal property or a global universal property.

Definition 1.6.1. Let $\mathcal{C}, \mathcal{D}$ be categories and let $d \in \mathrm{Ob}(\mathcal{C})$ define a **constant functor** at d to be the functor $\mathrm{const}_d : \mathcal{C} \to \mathcal{D}$ defined by $c \mapsto d$ for all $c \in \mathrm{Ob}(\mathcal{C})$ and $f \mapsto \mathrm{id}_d$ for all $f \in \mathrm{Mor}(\mathcal{C})$.

Sending objects of $\mathcal{C}$ to their constant functors defines a functor $\mathrm{const}\colon \mathcal{C} \to \mathbf{Fun}(\mathcal{C}, \mathcal{D})$.

Definition 1.6.2 (Local universal property)**.** Given $F\colon \mathcal{C} \to \mathcal{D}$, the colimit $\mathrm{colim}_{\mathcal{C}}(F)$ satisfies the **local universal property** that the natural transformation $\eta\colon F \implies \mathrm{const}_{\mathrm{colim}_{\mathcal{C}}(F)}$ factors through all other morphism from F to constants in $\mathcal{D}$:

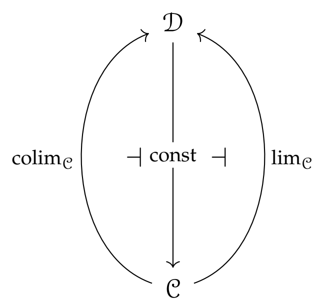


for some unique $f\colon\ \mathrm{colim}_{\mathcal{C}}(F) \to \mathrm{const}_d$.

Definition 1.6.3 (Global universal property)**.** The colimit and limit functors

$$\mathrm{colim}_{\mathcal{C}}\colon \mathcal{D}^{\mathcal{C}} \to \mathcal{D}$$
$$\lim_{\mathcal{C}}\colon \mathcal{D}^{\mathcal{C}} \to \mathcal{D}$$

are left and right adjoints to the functor $\mathrm{const}\colon \mathcal{D} \to \mathbf{Fun}(\mathcal{C}, \mathcal{D}) = \mathcal{D}^{\mathcal{C}}$.



Remark 1.6.4. For ordinary limits and colimits, these two definitions are essentially equivalent, but once we pass to homotopy limits and colimits, there is an important (and often confusing) difference between the two approaches.

Example 1.6.5.

(a) Let $\mathcal{C}$ be a discrete category with set of objects $\mathrm{Ob}(\mathcal{C}) = I$. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is a collection of objects indexed by $i \in I$: $F = \{X_i\}_{i \in I}$. The colimit of this diagram is a **coproduct** and the limit of this diagram is a **product**:

$$\mathrm{colim}_{\mathcal{C}}(F) = \coprod_{i \in I} X_i,$$
$$\lim_{\mathcal{C}}(F) = \prod_{i \in I} X_i.$$

(b) Let $\mathcal{C}$ be the category

$$0 \rightrightarrows 1$$

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is a diagram of the shape

$$X_0 \underset{g}{\overset{f}{\rightrightarrows}} X_1$$

A colimit for this diagram is an object Y and two morphisms $\phi_0\colon X_0 \to Y$ and $\phi_1\colon X_1 \to Y$ such that the diagram below commutes:

$$\begin{array}{ccc} X_0 & \underset{g}{\overset{f}{\rightrightarrows}} & X_1 \\ & {\scriptstyle\phi_0}\searrow \quad \swarrow{\scriptstyle\phi_1} & \\ & Y & \end{array}$$

This amounts to a **coequalizer** for this diagram.

Exercise 1.6.6. If $\mathcal{D}$ is additive,

$$\operatorname{coeq}\left(X_0 \underset{g}{\overset{f}{\rightrightarrows}} X_1 \right) = \operatorname{coker}(f-g)$$

Remark 1.6.7. Any colimit or limit can be constructed using coproducts and coequalizers. This is *not* true for homotopy colimits.

Definition 1.6.8. A category $\mathcal{D}$ is called **cocomplete** if it has colimits for all small diagrams $F\colon \mathcal{C} \to \mathcal{D}$. It is called **complete** if it has limits for all small diagrams.

1.6.2 Ends and coends

Let $\mathcal{C}$ be a small category and let $\mathcal{D}$ be a locally small, cocomplete category consider the bifunctor

$$S\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$$

Definition 1.6.9. The **coend** of S is defined by

$$\int^{c\in\mathcal{C}} S(c,c) := \operatorname{coeq}\left(\coprod_{f\colon c_0 \to c_1 \in \mathrm{Mor}(\mathcal{C})} S(c_1,c_0) \underset{f_*}{\overset{f^*}{\rightrightarrows}} \coprod_{c\in\mathrm{Ob}(\mathcal{C})} S(c,c) \right)$$

Where

$$f^* = i_{c_1} \circ S(1,f)\colon S(c_1,c_0) \to S(c_1,c_1) \hookrightarrow \coprod_{c\in\mathrm{Ob}(\mathcal{C})} S(c,c)$$

and

$$f_* = i_{c_0} \circ S(f,1)\colon S(c_1,c_0) \to S(c_0,c_0) \hookrightarrow \coprod_{c\in\mathrm{Ob}(\mathcal{C})} S(c,c)$$

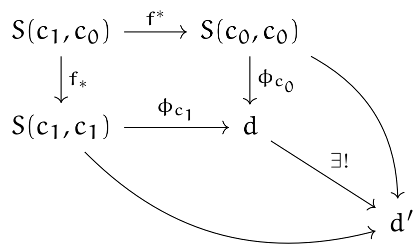
The universal property of the coend is as follows. The coend

$$d := \int^{c \in \mathcal{C}} S(c, c)$$

is an object in $\mathcal{D}$ together with morphisms

$$\phi_c \colon S(c, c) \to d,$$

one for each $c \in \mathrm{Ob}(\mathcal{C})$ such that for all $f \colon c_0 \to c_1$ in $\mathrm{Mor}(\mathcal{C})$, and any other $d' \in \mathrm{Ob}(\mathcal{D})$ with morphisms $\psi_c \colon S(c, c) \to d'$, the following diagram commutes:



and the coend is universal with respect to this property.

Example 1.6.10. Given $F \colon \mathcal{C} \to \mathcal{D}$, define a bifunctor

$$S \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$$

by $S(c', c) = F(c)$ and $S(f', f) = F(f)$. Then

$$\int^{c \in \mathcal{C}} S(c, c) \cong \mathrm{colim}_{\mathcal{C}}\, F$$

Example 1.6.11. Let R be a ring, not necessarily commutative. Denote by **Mod**-R and R-**Mod** the categories of right- and left- R-modules. Let $\mathcal{C}$ be any small category. Given two functors $F \colon \mathcal{C}^{\mathrm{op}} \to$ **Mod**-R and $G \colon \mathcal{C} \to R$-**Mod**, define a bifunctor

$$F \boxtimes_R G \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Ab}$$

by $(c, c') \mapsto F(c') \otimes_R G(c)$ and $(f, f') \mapsto F(f) \otimes_R G(f')$.

Then define the **functor tensor product**

$$F \otimes_{\mathcal{C},R} G := \int^{c \in \mathcal{C}} F(c) \boxtimes_R G(c)$$

Explicitly, we have

$$F \otimes_{\mathcal{C},R} G \cong \bigoplus_{c \in \mathcal{C}} F(c) \otimes_R G(c) \Big/ \Big\langle F(f)x' \otimes_R y - x' \otimes_R G(f)y \Big\rangle_{\substack{f \colon c \to c' \in \mathrm{Mor}(\mathcal{C}) \\ x \in F(c'), y \in G(c)}}$$

Exercise 1.6.12. Fix $c \in \mathrm{Ob}(\mathcal{C})$ and consider the functor $R^{op}[h_c] \colon \mathcal{C}^{op} \to \mathbf{Mod}\text{-}R$ which takes $c' \in \mathrm{Ob}(\mathcal{C})$ to the free right R-module generated by $\mathrm{Hom}_{\mathcal{C}}(c', c)$,

$$R^{op}\left[\mathrm{Hom}_{\mathcal{C}}(-, c)\right].$$

Similarly, define the functor $R[h^c] \colon \mathcal{C} \to R\text{-}\mathbf{Mod}$ which takes $c' \in \mathrm{Ob}(\mathcal{C})$ to the free left R-module generated by $\mathrm{Hom}_{\mathcal{C}}(c, c')$.

Prove that for all functors F, G

$$R^{op}[h_c] \otimes_{\mathcal{C},R} G \cong G(c)$$

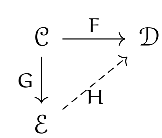
$$F \otimes_{\mathcal{C},R} R[h^c] \cong F(c)$$

Exercise 1.6.13. If $\underline{R} = \mathrm{const}_{R_R} \colon \mathcal{C}^{op} \to \mathbf{Mod}\text{-}R$, then prove that

$$\underline{R} \otimes_{\mathcal{C},R} G \cong \mathrm{colim}_{C}(G)$$

1.6.3 Kan extensions

Suppose we are given $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{C} \to \mathcal{E}$ where $\mathcal{C}$ is small and $\mathcal{D}$ is locally small and cocomplete. We want to extend F through G by a functor H such that the following diagram commutes.



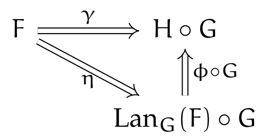
This is often impossible, but we can do the next best thing: obtain a universal natural transformation either $F \Rightarrow HG$ (left Kan extension) or $F \Leftarrow HG$ (right Kan extension).

Remark 1.6.14. You may have heard the slogan that "all concepts are Kan extensions." This statement may be exaggerated, but we may approximate it from the left and from the right.

Example 1.6.15. If $\mathcal{C}, \mathcal{D}$, and $\mathcal{E}$ are discrete categories, then it may happen that there are distinct objects $c, c' \in \mathrm{Ob}(\mathcal{C})$ such that $F(c) \neq F(c')$ but $G(c) = G(c')$ in $\mathcal{E}$. Then for any $H \colon \mathcal{E} \to \mathcal{D}$, $HG(c) = HG(c')$ yet $F(c) \neq F(c')$. In such an example, the diagram may never commute.

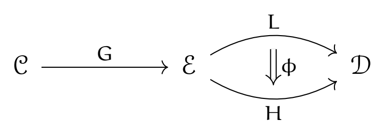
Definition 1.6.16. A **left Kan extension** of $F \colon \mathcal{C} \to \mathcal{D}$ along $G \colon \mathcal{C} \to \mathcal{E}$ is a functor $\mathrm{Lan}_G(F) \colon \mathcal{D} \to \mathcal{E}$ together with a natural transformation $\eta \colon F \implies \mathrm{Lan}_G(F) \circ G$ which is universal among all pairs $(H \colon \mathcal{E} \to \mathcal{D}, \gamma \colon F \implies H \circ G)$

in the sense that there is a unique $\phi\colon \mathrm{Lan}_G(F) \implies H$ such that the following diagram of natural transformations commutes:



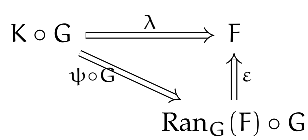
Recall that given functors $F\colon \mathcal{C} \to \mathcal{E}$, $L, H\colon \mathcal{E} \to \mathcal{D}$ and a natural transformation $\phi\colon L \to H$, then $\phi \circ G$ is the natural transformation $\phi_G\colon LG \implies HG$ given on the object $c \in \mathrm{Ob}(\mathcal{C})$ by

$$(\phi \circ G)_c = \phi_{G(c)}\colon LG(c) \to HG(c).$$



Dually, we have right Kan extensions.

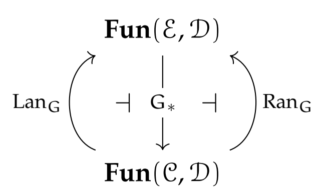
Definition 1.6.17. A **right Kan extension** of $F\colon \mathcal{C} \to \mathcal{D}$ along $G\colon \mathcal{C} \to \mathcal{E}$ is a functor $\mathrm{Ran}_G(F)\colon \mathcal{E} \to \mathcal{D}$ together with $\varepsilon\colon \mathrm{Ran}_G(F) \circ G \implies F$ which is universal in the sense that for all pairs $(K\colon \mathcal{E} \to \mathcal{D}, \lambda\colon K \circ G \implies F)$, there is a unique $\psi\colon K \implies \mathrm{Ran}_G(F)$ such that the following diagram of natural transformations commutes:



There is a global definition of Kan extensions as well. Fix $G\colon \mathcal{C} \to \mathcal{E}$ and consider the restriction functor

$$G_* = - \circ G\colon \mathbf{Fun}(\mathcal{E}, \mathcal{D}) \to \mathbf{Fun}(\mathcal{C}, \mathcal{D})$$

The left and right Kan extensions are left and right adjoints of this restriction.



The global definition of adjunctions then tells us that

$$\mathrm{Hom}_{\mathbf{Fun}(\mathcal{E},\mathcal{D})}(\mathrm{Lan}_G(F), H) \cong \mathrm{Hom}_{\mathbf{Fun}(\mathcal{C},\mathcal{D})}(F, H \circ G).$$

Example 1.6.18. Let G be a finite group and let H be a subgroup. Think of H and G as categories with a single object. There is an inclusion of subgroups

$$i\colon \underline{H} \hookrightarrow \underline{G}.$$

A linear representation of H in $\mathbf{Vect}_k$ is given by a homomorphism $\rho\colon H \to \mathrm{Aut}_k(V)$ for some $V \in \mathrm{Ob}(\mathbf{Vect}_k)$. This may be interpreted as a functor

$$\begin{aligned} \rho\colon \underline{H} &\longrightarrow \mathbf{Vect}_k \\ * &\longmapsto V. \end{aligned}$$

In this sense, $\mathbf{Rep}_k(\underline{H}) \cong \mathbf{Fun}_k(\underline{H}, \mathbf{Vect}_k)$.

We might ask when we can extend a representation of H to a representation of G. This question is the same as an extension question:

$$\begin{array}{ccc} \underline{H} & \xrightarrow{\ \rho\ } & \mathbf{Vect}_k \\ \big\downarrow{\scriptstyle i} & \nearrow & \\ \underline{G} & & \end{array}$$

Then the left Kan extension of ρ along i is the **induced representation**, and the right Kan extension is the **coinduced representation**.

$$\begin{aligned} \mathrm{Lan}_i(\rho) &= \mathrm{Ind}_H^G(V) \cong k[G] \otimes_{k[H]} V \\ \mathrm{Ran}_i(\rho) &= \mathrm{Coind}_H^G(V) \cong \mathrm{Hom}_{k[H]}(k[G], V) \end{aligned}$$

These are the left and right adjoints of the restriction functor

$$\mathrm{Res}_H^G\colon \mathbf{Rep}_k(G) \to \mathbf{Rep}_k(H).$$

The next two examples give evidence to the slogan that "all concepts are Kan extensions."

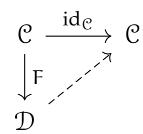
Example 1.6.19. Let $F\colon \mathcal{C} \to \mathcal{D}$ and let $G\colon \mathcal{C} \to \underline{1}$ be the unique functor whose codomain is a one-object category. Any functor $H\colon \underline{1} \to \mathcal{D}$ is a choice of an object in $\mathcal{D}$, and any natural transformation $\eta\colon F \implies H \circ G$ is equivalent to finding the universal natural transformation from F to a constant. Therefore, in this situation,

$$\mathrm{Lan}_G(F) \cong \mathrm{colim}_{\mathcal{C}}(F)$$

Example 1.6.20. Given a pair of adjoint functors $F\colon \mathcal{C} \leftrightarrows \mathcal{D}\colon G$, there are **unit** and **counit** morphisms

$$\begin{aligned} \eta\colon \mathrm{id}_{\mathcal{C}} &\implies GF \\ \varepsilon\colon FG &\implies \mathrm{id}_{\mathcal{D}} \end{aligned}$$

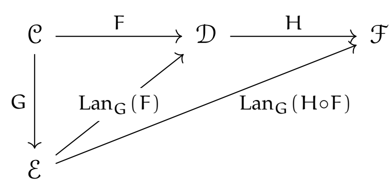
What is the left Kan extension of $\mathrm{id}_{\mathcal{C}}$ along $F\colon \mathcal{C} \to \mathcal{D}$?



The answer is $\mathrm{Lan}_F(\mathrm{id}_{\mathcal{C}}) \cong G$ with universal natural transformation η.

Similarly, the right Kan extension of $\mathrm{id}_{\mathcal{D}}$ along $G\colon \mathcal{D} \to \mathcal{C}$ is $\mathrm{Ran}_G(\mathrm{id}_{\mathcal{D}}) \cong F$ with universal natural transformation ε.

Consider the following situation:



(1.6.1)

By the universal property of Kan extensions, we have a natural transformation

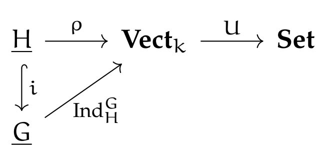
$$\xi\colon \mathrm{Lan}_G(H \circ F) \implies H \circ \mathrm{Lan}_G(F). \tag{1.6.2}$$

Definition 1.6.21. Consider the situation of the diagram (1.6.1).

(a) We say that H preserves left Kan extensions of F along G if both $\mathrm{Lan}_G(F)$ and $\mathrm{Lan}_G(H \circ F)$ exist and the natural transformation ξ from (1.6.2) is an isomorphisms.

(b) The left Kan extension $\mathrm{Lan}_G(F)$ is called **absolute** if it is preserved by all functors H.

Example 1.6.22. In the situation of Example 1.6.18, take the forgetful functor $U\colon \mathbf{Vect}_k \to \mathbf{Set}$. Then



U does not preserve left Kan extensions here. There are two very different constructions:

$$U\,\mathrm{Lan}_i(\rho) = U(k[G] \otimes_{k[H]} V)$$

$$\mathrm{Lan}_i(U \circ \rho) = G \times_H U(V)$$

Notice that U has no right adjoint, so it is not a left adjoint.

Exercise 1.6.23. Show that left adjoints always preserve left Kan extensions.

Example 1.6.24. Let $\mathcal{C}, \mathcal{D}$ be model categories, and $F\colon \mathcal{C} \to \mathcal{D}$ a functor.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow{\scriptstyle\gamma} & \nearrow & \\ \mathbf{Ho}(\mathcal{C}) & & \end{array}$$

A left Kan extension in this situation is a **total right-derived functor** and denoted $\mathbb{L}F$ and a right Kan extension is a **total left-derived functor** and denoted $\mathbb{R}F$. If $F\colon \mathcal{C} \leftrightarrows \mathcal{D}\colon G$ is a Quillen pair, then the left and right derived functors are absolute Kan extensions by a theorem of Maltsionitis in 2007. REF?

Definition 1.6.25. A **pointwise Kan extension** is one preserved by all representable functors.

Recall given a pair of adjoint functors $F\colon \mathcal{C} \leftrightarrows \mathcal{D}\colon G$, there are natural transformations

$$\begin{aligned} \eta\colon \mathrm{id}_{\mathcal{C}} &\implies G \circ F \\ \varepsilon\colon F \circ G &\implies \mathrm{id}_{\mathcal{D}} \end{aligned} \tag{1.6.3}$$

called the **unit** and **counit**, respectively. These satisfy the following identities:

$$\begin{array}{ccc} F & \xRightarrow{F\eta} & FGF \\ & {\scriptstyle\mathrm{id}_F}\searrow & \Downarrow{\scriptstyle\varepsilon F} \\ & & F \end{array} \qquad \begin{array}{ccc} G & \xRightarrow{\eta G} & GFGs \\ & {\scriptstyle\mathrm{id}_G}\searrow & \Downarrow{\scriptstyle G\varepsilon} \\ & & G. \end{array} \tag{1.6.4}$$

Lemma 1.6.26. *If $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ is a pair of functors given with morphisms (1.6.3) satisfying relations (1.6.4), then $F\colon \mathcal{C} \leftrightarrows \mathcal{D}\colon G$ are adjoint functors, with natural isomorphism $\psi_{A,B}\colon \mathrm{Hom}_{\mathcal{D}}(FA, B) \to \mathrm{Hom}_{\mathcal{C}}(A, GB)$ given by*

$$\psi_{A,B}(f\colon F(A) \to B) = (A \xrightarrow{\eta_A} GF(A) \xrightarrow{G(f)} G(B)).$$

Proof. One checks that $\psi_{A,B}$ is a natural bijection with inverse $\psi^{-1}_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, G(B)) \to \mathrm{Hom}_{\mathcal{D}}(F(A), B)$ given by

$$\psi^{-1}_{A,B}(g\colon A \to G(B)) = (F(A) \xrightarrow{F(g)} FG(B) \xrightarrow{\varepsilon_B} B).$$

□

Proposition 1.6.27. *Let $f\colon \mathcal{C} \leftrightarrows \mathcal{D}$ be adjoint functors. Take any category $\mathcal{A}$ and consider two new pairs of functors:*

$$\begin{array}{rcl} G^*\colon \mathbf{Fun}(\mathcal{C}, \mathcal{A}) & \rightleftarrows & \mathbf{Fun}(\mathcal{D}, \mathcal{A})\colon F^* \\ (\mathcal{C} \to \mathcal{A}) & \longmapsto & (\mathcal{D} \xrightarrow{G} \mathcal{C} \to \mathcal{A}) \\ (\mathcal{C} \xrightarrow{F} \mathcal{D} \to \mathcal{A}) & \longleftarrow\!\shortmid & (\mathcal{D} \to \mathcal{A}) \end{array} \tag{1.6.5}$$

$$\begin{array}{ccc} F_*\colon \mathbf{Fun}(\mathcal{A},\mathcal{C}) & \rightleftarrows & \mathbf{Fun}(\mathcal{A},\mathcal{D})\colon G_* \\ (\mathcal{A}\to\mathcal{C}) & \longmapsto & (\mathcal{A}\to\mathcal{C}\xrightarrow{F}\mathcal{D}) \\ (\mathcal{A}\to\mathcal{D}\xrightarrow{G}\mathcal{C}) & \longleftarrow & (\mathcal{A}\to\mathcal{D}) \end{array} \tag{1.6.6}$$

If $F \dashv G$ *is an adjunction, then so is* $G^* \dashv F^*$ *and* $F_* \dashv G_*$.

Proof. Given a unit η and counit ε for the adjunction $F \dashv G$, define a natural transformation for any $R\colon \mathcal{C} \to \mathcal{A}$

$$R\eta\colon R \to RGF$$

by

$$R(\eta_A)\colon R(A) \to R(GF(A))$$

for any $A \in \mathrm{Ob}(\mathcal{C})$. Letting R vary, we get

$$\eta^*\colon \mathrm{id}_{\mathbf{Fun}(\mathcal{C},\mathcal{A})} \implies F^*G^* = (GF)^*$$

Dually, we obtain

$$\varepsilon^*\colon G^*F^* \implies \mathrm{id}_{\mathbf{Fun}(\mathcal{D},\mathcal{A})}.$$

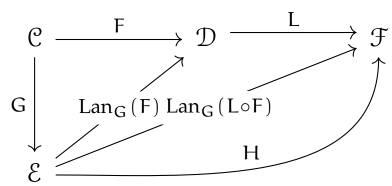
Then, because ε, η satisfy (1.6.4), so do ε^*, η^*. Therefore, $G^* \dashv F^*$.

A similar argument shows that $F_* \dashv G_*$. □

Let's return to Kan extensions.

Proposition 1.6.28. *Left adjoints preserve Kan extensions.*

Proof. Consider a diagram



(1.6.7)

Assume that L has a right adjoint $R\colon \mathcal{F} \to \mathcal{D}$. We will use the Yoneda lemma. By Proposition 1.6.27, we have a chain of bijections

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Fun}(\mathcal{E},\mathcal{F})}(\mathrm{Lan}_G(L\circ F), H) &\cong \mathrm{Hom}_{\mathbf{Fun}(\mathcal{C},\mathcal{F})}(L\circ F, G^*H) \\ &\cong \mathrm{Hom}_{\mathbf{Fun}(\mathcal{C},\mathcal{D})}(F, R\circ H\circ G) \\ &\cong \mathrm{Hom}_{\mathbf{Fun}(\mathcal{E},\mathcal{F})}(L\circ \mathrm{Lan}_G(F), H) \\ &\cong \mathrm{Hom}(\mathrm{Lan}_G(F), R\circ H) \\ &\cong \mathrm{Hom}(F, R\circ H\circ G) \end{aligned}$$

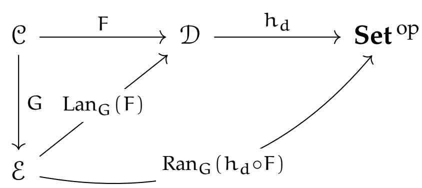
Then by the Yoneda lemma, $\mathrm{Lan}_G(L\circ F) \cong L\circ \mathrm{Lan}_G(H)$. □

Definition 1.6.29.

(a) A right Kan extension is called **pointwise** if it is preserved by all (covariant) representable functors $h^d = \mathrm{Hom}_{\mathcal{D}}(d, -)\colon \mathcal{D} \to \mathbf{Set}$.

(b) A left Kan extension is called **pointwise** if every representable functor $h_d\colon \mathrm{Hom}_{\mathcal{D}}(-, d)\colon \mathcal{D} \to \mathbf{Set}^{\mathrm{op}}$ maps $\mathrm{Lan}_G(F)$ to a right Kan extension: there is a natural isomorphism

$$\mathrm{Ran}_G(h_d \circ F) \xleftarrow{\cong} h_d \circ \mathrm{Lan}_G(F).$$



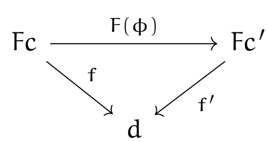
We will give a characterization for pointwise Kan extensions.

1.6.4 Comma Categories

Definition 1.6.30. Given $F\colon \mathcal{C} \to \mathcal{D}$ and $d \in \mathrm{Ob}(\mathcal{D})$, define the **comma category** F/d (or $F \downarrow d$) as follows: the objects are

$$\mathrm{Ob}(F/d) = \{(c, f)\colon c \in \mathrm{Ob}(\mathcal{C}), f \in \mathrm{Hom}_{\mathcal{D}}(Fc, d)\}.$$

The morphisms in F/d between (c, f) and (c', f') are those $\phi\colon c \to c'$ in $\mathrm{Mor}(\mathcal{C})$ such that the following commutes in $\mathcal{D}$:



Similarly, we define $d\backslash F = d \downarrow F$.

Note that there is a forgetful functor $U\colon F/d \to \mathcal{C}$ given by $(c, f) \mapsto c$. This can be thought of as a "fibration" of categories:

$$F/d \xrightarrow{U} \mathcal{C} \xrightarrow{F} \mathcal{D}.$$

Example 1.6.31. Let $F = \mathrm{id}_{\mathcal{D}}\colon \mathcal{D} \to \mathcal{D}$. The category F/d is often written $\mathcal{D} \downarrow d$ and is called the **overcategory** over d. Likewise, $d\backslash F$ is written $d \downarrow \mathcal{D}$ and is called the **undercategory** under d.

Example 1.6.32. Take F to be the standard cosimplicial simplicial set $h_*\colon \mathbb{\Delta} \to$ **sSet**, $[n] \mapsto \Delta[n]_*$. For any simplicial set X, we call the category h_*/X the **category of simplicies of** X and write

$$\mathbb{\Delta}X := h_*/X.$$

The objects of $\mathbb{\Delta}X$ are the union of all simplicies in X.

$$\mathrm{Ob}(\mathbb{\Delta}X) = \left\{ ([n], x) \;\middle|\; [n] \in \mathrm{Ob}(\mathbb{\Delta}), x \in \mathrm{Hom}_{\mathbf{sSet}}(\Delta[n]_*, X) \cong X_n \right\} = \coprod_{n \geq 0} X_n$$

The morphisms between $x \in X_n$ and $y \in Y_m$ are those $f \in \mathrm{Hom}_{\mathbb{\Delta}}([n],[m])$ such that $X(f)(y) = x$.

Another way to define $\mathbb{\Delta}X$ is to consider $X\colon \mathbb{\Delta}^{\mathrm{op}} \to$ **Set** as a functor and take the Grothendieck construction (or the Bousfield–Kan constructions if we consider **Set** as a discrete category): this is the category

$$\mathbb{\Delta}^{\mathrm{op}} \int X$$

with objects the same as above, but morphisms go the other way. Hence,

$$(\mathbb{\Delta}X) \cong \left(\mathbb{\Delta}^{\mathrm{op}} \int X \right)^{\mathrm{op}}.$$

Example 1.6.33. Let $\mathcal{C}$ be a small category. Take the functor $\mathbf{\Delta}\colon \mathbb{\Delta} \to$ **Cat** that sends $[n] \in \mathrm{Ob}(\mathbb{\Delta})$ to the poset $\vec{n}$ associated to $[n]$:

$$\vec{n} = \{0 \to 1 \to \cdots \to n\}.$$

This is a fully faithful functor. The **simplex category** of $\mathcal{C}$ is

$$\mathbf{\Delta}/\mathcal{C}.$$

Let's again decode this definition. The objects are

$$\mathrm{Ob}(\mathbf{\Delta}/\mathcal{C}) = \left\{ ([n], f)\colon [n] \in \mathbb{\Delta}, f\colon \vec{n} \to \mathcal{C} \right\} \cong \coprod_{n \geq 0} \mathcal{N}_n \mathcal{C}$$

A little more thought shows that this is just the simplex category of the nerve of $\mathcal{C}$.

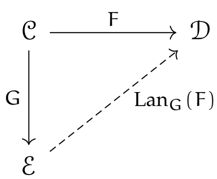
$$\mathbf{\Delta}/\mathcal{C} \cong \mathbb{\Delta}(\mathcal{N}\mathcal{C})$$

Remark 1.6.34. $\mathbb{\Delta}X$ and $(\mathbb{\Delta}X)^{\mathrm{op}}$ are examples of Reedy categories (with fibrant or cofibrant , respectively).

1.6.5 Computing Kan extensions

Theorem 1.6.35. *A left Kan extension is pointwise if and only if it can be computed by the following formula:*

$$\mathrm{Lan}_G(F)(e) = \mathrm{colim}_{G/e}\left(G/e \xrightarrow{U} \mathcal{C} \xrightarrow{F} \mathcal{D}\right)$$



Dually, a right Kan extension is pointwise if and only if it can be computed by the following formula:

$$\mathrm{Ran}_G(F)(e) = \mathrm{colim}_{e\backslash G}(e\backslash G \xrightarrow{U} \mathcal{C} \xrightarrow{F} \mathcal{D}).$$

Proof. Proof of Theorem 1.6.35 It suffices (and is more convenient) to prove the theorem for right Kan extensions. Indeed, $\mathrm{Lan}_G(F)$ is characterized by

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Fun}(\mathcal{E},\mathcal{D})}(\mathrm{Lan}_G(F), H) &\cong \mathrm{Hom}_{\mathbf{Fun}(\mathcal{C},\mathcal{D})}(F, H \circ G)\\ &\cong \mathrm{Hom}_{\mathbf{Fun}(\mathcal{C},\mathcal{D})^{\mathrm{op}}}(H \circ G, F)\\ &\cong \mathrm{Hom}_{\mathbf{Fun}(\mathcal{E},\mathcal{D})^{\mathrm{op}}}(H, \mathrm{Lan}_G(F))\end{aligned}$$

But $\mathbf{Fun}(\mathcal{E},\mathcal{D})^{\mathrm{op}} \cong \mathbf{Fun}(\mathcal{E}^{\mathrm{op}},\mathcal{D}^{\mathrm{op}})$, so $\mathrm{Ran}_{G^\circ}(F^\circ) \cong \mathrm{Lan}_G(F)^\circ$, where we denote by c° an object c $\mathcal{C}$ considered as an object of $\mathcal{C}^{\mathrm{op}}$.

Now we begin the proof proper. Limits commute with representable functors: given $F\colon \mathcal{J} \to \mathcal{D}$, we have

$$\mathrm{Hom}_{\mathcal{D}}(d, \lim_J F) \cong \lim_J \mathrm{Hom}_{\mathcal{D}}(d, F(-)).$$

So if $\mathrm{Ran}_G(F)$ is given by the limit formula, then it automatically commutes with representable functors $h^d = \mathrm{Hom}_{\mathcal{D}}(d,-)$ for all $d \in \mathrm{Ob}(\mathcal{D})$.

Assume that $\mathrm{Ran}_G(F)$ is pointwise. Then for all $d \in \mathrm{Ob}(\mathcal{D})$ and all $e \in \mathrm{Ob}(\mathcal{E})$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}}(d, \mathrm{Ran}_G(F)(e)) &= h^d(\mathrm{Ran}_G(F)(e))\\ &= (h^d \circ \mathrm{Ran}_G(F))(e)\\ &\cong \mathrm{Hom}_{\mathbf{Fun}(\mathcal{E},\mathbf{Set})}(h^e, h^d \circ \mathrm{Ran}_G(F)) && \text{by Yoneda}\\ &\cong \mathrm{Hom}_{\mathbf{Fun}(\mathcal{E},\mathbf{Set})}(h^e, \mathrm{Ran}_G(h^d \circ F)) && \text{by assumption}\\ &\cong \mathrm{Hom}_{\mathbf{Fun}(\mathcal{C},\mathbf{Set})}(h^e \circ G, h^d \circ F)\end{aligned}$$

Expanding the definition of the functors h^e and h^d, we arrive at

$$\mathrm{Hom}_{\mathbf{Fun}(\mathcal{C},\mathbf{Set})}\Big(\mathrm{Hom}_{\mathcal{E}}(e, G(-)), \mathrm{Hom}_{\mathcal{D}}(d, F(-))\Big).$$

Recalling that the objects of $e\backslash G$ are pairs $(c, f\colon e \to G(c))$ for some $c \in \mathrm{Ob}(\mathcal{C})$, then we recognize this last set as the set of all cones under $d \in \mathrm{Ob}(\mathcal{D})$ of the functor FU. It therefore in bijection with the set

$$\cong \mathrm{Hom}_{\mathbf{Fun}(e\backslash G,\mathcal{D})}(\mathrm{const}_d, FU) \cong \mathrm{Hom}(d, \lim_{e\backslash G}(FU)),$$

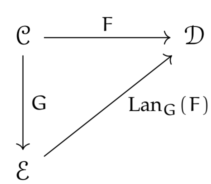
where the last bijection follows from the adjunction const $\dashv$ lim. By the Yoneda lemma,

$$R_G(F)(e) \cong \lim_{e\backslash G}(FU). \qquad \square$$

Corollary 1.6.36.

(a) *If $\mathcal{D}$ is cocomplete, then every Kan extension of $F\colon \mathcal{C} \to \mathcal{D}$ exists and is pointwise.*

(b) *If $\mathcal{D}$ is cocomplete and $\mathcal{G}$ is fully faithful, the universal natural transformation $\eta\colon F \implies \mathrm{Lan}_G(F) \circ G$ is a natural isomorphism, i.e. the diagram*



Proof of Corollary 1.6.36(b). Take any $c \in \mathrm{Ob}(\mathcal{C})$ and consider $G/G(c)$. Since G is fully faithful, $G/G(c)$ has a terminal object given by the pair $(c, \mathrm{id}_G(c))$. Indeed,

$$\mathrm{Hom}_{G/G(c)}((c', f'), (c, \mathrm{id}_{G(c)})) = \left\{ h\colon c' \to c \;\middle|\; \begin{array}{ccc} G(c') & \xrightarrow{G(h)} & G(c) \\ & {\scriptstyle f} \searrow \quad \swarrow {\scriptstyle \mathrm{id}_{G(c)}} & \\ & G(c) & \end{array} \right\}$$

Since one of the legs of this triangle is an identity, this is the same as the set

$$\mathrm{Hom}_{G/G(c)}((c', f'), (c, \mathrm{id}_{G(c)})) = \{h\colon c' \to c \mid G(h) = f'\} = G^{-1}(f').$$

Since G is fully faithful, the preimage of f' has a unique element. Thus, $(c, \mathrm{id}_{G(c)})$ is terminal.

Now, for any $c \in \mathrm{Ob}(\mathcal{C})$, take $e = G(c)$ and apply the colimit formula for the left Kan extension.

$$(\mathrm{Lan}_G(F) \circ G)(c) \cong \mathrm{colim}_{G/G(c)}(G/G(c) \xrightarrow{U} \mathcal{C} \xrightarrow{F} \mathcal{D}) \cong FU(c, \mathrm{id}_{G(c)}) = F(c).$$

$\square$

Example 1.6.37 (Co-Yoneda Lemma)**.** The simplest version of the Co-Yoneda lemma is that any functor may be written as a colimit. Consider the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ {\scriptstyle \mathrm{id}_{\mathcal{C}}}\downarrow & \nearrow_{F} & \\ \mathcal{C} & & \end{array}$$

which demonstrates that $\mathrm{Lan}_{\mathrm{id}_{\mathcal{C}}}(F) \cong F$ by Corollary 1.6.36. Therefore,

$$F(c) \cong \mathrm{colim}_{\mathcal{C}\downarrow c}(\mathcal{C} \downarrow c \xrightarrow{U} \mathcal{C} \xrightarrow{F} \mathcal{D}),$$

where $\mathcal{C} \downarrow c = \mathrm{id}_{\mathcal{C}}/c$.

Example 1.6.38. Let $\widehat{\mathcal{C}} = \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$ and consider the Yoneda embedding $h\colon \mathcal{C} \to \widehat{\mathcal{C}}$ given by $c \mapsto h_c = \mathrm{Hom}_{\mathcal{C}}(-, c)$. Applying the previous example, we see that every presheaf on a small category $\mathcal{C}$ is canonically a colimit of representable presheaves.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{h} & \widehat{\mathcal{C}} \\ \downarrow{\scriptstyle h} & \nearrow_{\mathrm{Lan}_h(h) \cong \mathrm{id}_{\mathcal{C}}} & \\ \widehat{\mathcal{C}} & & \end{array}$$

In particular, for all $X \in \mathrm{Ob}(\widehat{\mathcal{C}})$,

$$X \cong \mathrm{colim}_{h/X}(h/X \xrightarrow{U} \mathcal{C} \xrightarrow{h} \widehat{\mathcal{C}}).$$

Example 1.6.39. Let $\mathcal{C} = \mathbb{\Delta}$, so $\widehat{\mathcal{C}} = \mathbf{sSet}$. Recall that $h/X \cong \mathbb{\Delta}X$ is the category of simplicies of X (Example 1.6.32).

$$\begin{array}{ccc} \mathbb{\Delta} & \xrightarrow{h} & \mathbf{sSet} \\ \downarrow{\scriptstyle h} & \nearrow & \\ \mathbf{sSet} & & \end{array}$$

The previous example shows that every simplicial set X can be written as a colimit of standard simplicies:

$$X \cong \mathrm{colim}_{\Delta[n]_* \to X} \Delta[n].$$

There is another formula for left Kan extensions in terms of coends:

Proposition 1.6.40. *If $\mathcal{C}$ is small, $\mathcal{D}$ is cocomplete, and $\mathcal{E}$ is locally small, then*

$$\mathrm{Lan}_G(F)(e) = \int^{c \in \mathcal{C}} \mathrm{Hom}_{\mathcal{E}}(Gc, e) \cdot F(c),$$

where for any set X *and any* $d \in \mathrm{Ob}(\mathcal{D})$*,* $X \cdot d$ *denotes application of the bifunctor*

$$\begin{aligned} \mathbf{Set} \times \mathcal{D} &\longrightarrow \mathcal{D} \\ (X, d) &\longmapsto \coprod_{x \in X} d, \end{aligned}$$

and $\int$ *is coend of the bifunctor*

$$\begin{aligned} \mathcal{C}^{op} \times \mathcal{C} &\longrightarrow \mathcal{D} \\ (c', c) &\longmapsto \mathrm{Hom}_{\mathcal{E}}(Gc', e) \cdot F(c) \end{aligned}$$

Definition 1.6.41. In the situation of the proposition above, $\mathcal{D}$ is said to be **tensored** over **Set**.

Exercise 1.6.42. Apply Proposition 1.6.40 to group representations: if $H \leq G$ is a subgroup, and $\rho\colon \underline{H} \to \mathbf{Vect}_k$ defines a representation of H, and $i\colon \underline{H} \to \underline{G}$ is the inclusion, show that

$$\mathrm{Lan}_i(\rho) = \mathrm{Ind}_H^G(\rho) \cong k[G] \otimes_{k[H]} V$$

for $V = \rho(*)$.

Exercise 1.6.43. Consider the inclusion of categories $i\colon \mathbb{\Delta} \hookrightarrow \mathbb{\Delta}\mathbf{C}$ right adjoint to the forgetful functor. This induces a forgetful functor from cyclic sets to simplicial sets, $U\colon \mathbf{Set}^{\mathbb{\Delta}\mathbf{C}} \to \mathbf{sSet}$. Use Proposition 1.6.40 to find a left adjoint to U.

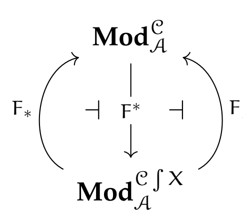
Shapiro's lemma

Let $\mathcal{C}$ be a small category and let $X\colon \mathcal{C} \to \mathbf{Set}$ be any diagram of shape $\mathcal{C}$. Then we define the Grothendieck construction $\mathcal{C} \int X$ as in Definition 1.4.15. There is a functor $F\colon \mathcal{C} \int X \to \mathcal{C}$ given on objects by $(c, x) \mapsto c$.

Take any abelian category $\mathcal{A}$, such as $\mathcal{A} = \mathbf{Vect}_k$ or $\mathcal{A} = R\text{-}\mathbf{Mod}$. Assume that $\mathcal{A}$ is both complete and cocomplete. Write $\mathbf{Mod}_{\mathcal{A}}^{\mathcal{C}} := \mathbf{Fun}(\mathcal{C}, \mathcal{A})$. We have a functor

$$F^*\colon \mathbf{Mod}_{\mathcal{A}}^{\mathcal{C}} \to \mathbf{Mod}_{\mathcal{A}}^{\mathcal{C}\int X}.$$

Since $\mathcal{A}$ is both complete and cocomplete, F has both left and right adjoints given by the left and right Kan extensions, which se call F_* and $F_!$, respectively.



We want to compute F_* explicitly.

Lemma 1.6.44 (Shapiro's Lemma)**.** $F_*\colon \mathbf{Mod}_{\mathcal{A}}^{\mathcal{C}\int X} \to \mathbf{Mod}_{\mathcal{A}}^{\mathcal{C}}$ *is defined as follows. For* $M\colon \mathcal{C}\int X \to \mathcal{A}$*,*

$$F_*(M)\colon \mathcal{C} \to \mathcal{A}$$

is given by

$$F_*(M)(d) = \bigoplus_{x\in X(d)} M(d,x)$$

Example 1.6.45. When $A = \mathbf{Vect}_k$, take $M\colon \mathcal{C}\int X \to \mathbf{Vect}_k$ to be the trivial $(\mathcal{C}\int X)$-module $M(c,x) = k$ and $M(\phi) = \mathrm{id}_k$. Then $F_*(k)\colon \mathcal{C} \to \mathbf{Vect}_k$ is the composition $k[-] \circ X$, where $k[-]\colon \mathbf{Set} \to \mathbf{Vect}_k$ is the free vector space functor. This is often denoted by $k[X]\colon d \mapsto k[X(d)]$.

Proof of Lemma 1.6.44. Consider the diagram

$$\begin{array}{ccc} \mathcal{C}\int X & \xrightarrow{\quad M \quad} & \mathcal{A} \\ \Big\downarrow F & \nearrow_{\mathrm{Lan}_F(M) = F_*(M)} & \\ \mathcal{C} & & \end{array}$$

For $d \in \mathrm{Ob}(\mathcal{C})$,

$$\mathrm{Lan}_F(M)(d) = \mathrm{colim}_{F/d}\left(F/d \xrightarrow{u} \mathcal{C}\int X \xrightarrow{M} \mathcal{A}\right).$$

Let's look at F/d. The objects in this category are

$$\mathrm{Ob}(F/d) = \left\{(c,f) \;\middle|\; c \in \mathrm{Ob}\left(\mathcal{C}\int X\right), f\colon F(c) \to d\right\}.$$

Equivalently,

$$\mathrm{Ob}(F/d) = \Big\{(c,x,f) \mid c \in \mathrm{Ob}(\mathcal{C}), x \in X(c), f \in \mathrm{Hom}_{\mathcal{C}}(c,d)\Big\}.$$

The morphisms in F/d are given by:

$$\mathrm{Hom}_{F/d}((c,x,f),(c',x',f')) = \Big\{\phi \in \mathrm{Hom}_{\mathcal{C}}(c,c') = X(\phi)(x) = x' \text{ and } f' \circ \phi = f\Big\}$$

Consider now the forgetful functor

$$\begin{aligned} F/d &\xrightarrow{\;u\;} \mathcal{C}\int X \\ (c,x,f) &\longmapsto (c,x) \\ \phi &\longmapsto \phi \end{aligned}$$

By definition, $\operatorname{colim}_{F/d}(M \circ U)$ is an object $A \in \mathrm{Ob}(\mathcal{A})$ given with a universal natural transformation $\eta\colon M \circ U \implies \mathrm{const}_A$. We claim that

$$A = \bigoplus_{y \in X(d)} M(d, y)$$

with η given by

$$\eta_{(c,x,f)}\colon M(c, x) \xrightarrow{M(\phi_f)} \bigoplus_{y \in X(d)} M(d, y)$$

where $y = X(f)(x)$ for $x \in X(c)$ and $f\colon c \to d$. The morphism ϕ_f is just f itself, viewed as a morphism

$$\phi_f \in \mathrm{Hom}_{F/d}\big((c, x, f), (d, y, \mathrm{id}_d)\big).$$

This is well-defined because $X(f)(x) = y$ and $f = f \circ \mathrm{id}_d$. Therefore,

$$M(\phi_f) = M(f)\colon M(c, x) \hookrightarrow M(d, y) \hookrightarrow A.$$

We need to check that η is a natural transformation of functors. Given $\phi\colon (c, x, f) \to (c', x', f')$, we have

$$\begin{array}{ccc} MU(c,x,f) & \xrightarrow{\eta_{(c,x,f)}} & A \\ \downarrow{\scriptstyle M(\phi)} & & \| \\ MU(c',x',f') & \xrightarrow{\eta_{(c',x',f')}} & A \end{array}$$

Commutativity of this diagram corresponds to the commutativity of the diagram

$$\begin{array}{ccc} M(c,x) & \xrightarrow{M(\phi_f)} & A \\ \downarrow{\scriptstyle M(\phi)} & & \| \\ M(c',x') & \xrightarrow{M(\phi_{f'})} & A \end{array}$$

We have for $m \in M(c, x)$,

$$\begin{array}{ccccc} m & \longmapsto & M(f)(m) & & \\ \downarrow & & & \diagdown\!\!\diagdown & \\ m' = M(\phi)(m) & \longmapsto & M(f')(m') & = & M(f')M(\phi)(m). \end{array}$$

So this diagram commutes.

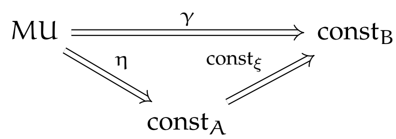
We need also check that η is universal. Given $B \in \mathrm{Ob}(\mathcal{A})$, and any other natural transformation $\gamma\colon MU \to \mathrm{const}_B$ with components $\gamma_{(c,x,f)}\colon M(c, x) \to B$, define

$$\xi\colon A = \bigoplus_{y \in X(d)} M(d, y) \to B$$

in Mor($\mathcal{A}$) by

$$\xi|_{M(d,y)} = \gamma(d, y, \mathrm{id}_d) \colon M(d, y) \to B.$$

Then one can check that the following diagram commutes



□

Remark 1.6.46. If $\mathcal{A} = \mathbf{Mod}_k$, we have natural isomorphisms for all $i \geq 0$, all $M \in \mathbf{Mod}_k^{\mathcal{C} \int X}$, and $N \in \mathbf{Mod}_k^{\mathcal{C}}$:

$$\mathrm{Tor}_i^{\mathcal{C}}(F_*(M), N) \cong \mathrm{Tor}_i^{\mathcal{C} \int X}(M, F^*(N)).$$

Here, $\mathrm{Tor}^{\mathcal{C}}$ are the derived functors of

$$(- \otimes_{C,k} -)$$

This is usually called **Shapiro's Lemma** when $M = k$ and $\mathcal{C}$ is a small category.

1.7 Fundamental constructions

1.7.1 Skeletons and coskeletons

For fixed $n \geq 0$, define $\Delta_{\leq n}$ to be the full subcategory of Δ with objects $\{[0], [1], [2], \ldots, [n]\}$. Then

$$i_n \colon \Delta_{\leq n} \hookrightarrow \Delta$$

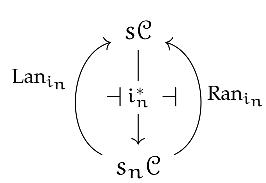
is the obvious inclusion. Given any category $\mathcal{C}$, define

$$s\mathcal{C} := \mathbf{Fun}(\Delta^{op}, \mathcal{C})$$

and

$$s_n\mathcal{C} := \mathbf{Fun}(\Delta\text{-}\leq n^{op}, \mathcal{C}).$$

Then we have restriction and (if $\mathcal{C}$ is complete and cocomplete), it has left and right adjoints.



Definition 1.7.1. The n**-skeleton** of $X \in \mathrm{Ob}(s\mathcal{C})$ is defined by

$$\mathrm{sk}_n(X) := \mathrm{Lan}_{i_n}(i_n^* X) \in \mathrm{Ob}(s\mathcal{C}).$$

The n**-coskeleton** of X is

$$\mathrm{cosk}_n(X) := \mathrm{Ran}_{i_n}(i_n^* X) \in \mathrm{Ob}(s\mathcal{C}).$$

By Theorem 1.6.35, we have the colimit formula for the skeleton.

$$\mathrm{sk}_n(X)_m \cong \mathrm{colim}_{\substack{\phi\colon [m]\to[k] \\ k \le n}} (\phi^*(X_k)).$$

Note that any $\phi\colon [m] \to [k]$ can be factored uniquely in Δ as a surjection followed by an injection, so we may take this colimit over only the surjective maps

$$\mathrm{sk}_n(X)_m \cong \mathrm{colim}_{\substack{\phi\colon [m]\twoheadrightarrow[k] \\ k \le n}} (\phi^*(X_k)).$$

Example 1.7.2. This implies (when $\mathcal{C} = \mathbf{Set}$) that $\mathrm{sk}_n(X)_m = X_m$ if $m \le n$. In particular, $\mathrm{sk}_n(X)$ is the simplicial subset of X generated by nondegenerate simplicies in degrees at most n.

Remark 1.7.3. For all n, there are natural morphisms of simplicial sets

$$\mathrm{sk}_n(X) \hookrightarrow \mathrm{sk}_{n+1}(X) \hookrightarrow \cdots \hookrightarrow X$$

such that

$$\mathrm{colim}_{n\to\infty} \mathrm{sk}_n(X) \cong X.$$

Definition 1.7.4. We may work relatively: if $f\colon X \to Y$ is a map of simplicial sets, we may define the **relative skeleton** $\mathrm{sk}_n^X(Y)$ via the pushout diagram

$$\begin{array}{ccc} \mathrm{sk}_n(X) & \longrightarrow & X \\ \big\downarrow{\scriptstyle \mathrm{sk}_n(f)} & \lrcorner & \big\downarrow \\ \mathrm{sk}_n(Y) & \longrightarrow & \mathrm{sk}_n^X(Y) \end{array}$$

1.7.2 Augmented simplicial sets

Example 1.7.5. Δ has a terminal object $[0]$ but no initial object. To fix this, we define the **augmented simplicial category** Δ_+ to have objects

$$\mathrm{Ob}(\Delta_+) = \mathrm{Ob}(\Delta) \cup \{[-1]\}$$

and morphisms

$$\mathrm{Hom}_{\Delta_+}([n],[m]) = \begin{cases} \mathrm{Hom}_{\Delta}([n],[m]) & (n, m \ge 0) \\ \{[-1] \to [m]\} & (n = -1) \\ \varnothing & (m = -1). \end{cases}$$

Definition 1.7.6. For any category $\mathcal{C}$, we define an **augmented simplicial object in** $\mathcal{C}$ as a functor $X\colon {\Delta_+}^{op}$ to$\mathcal{C}$. We denote the category of such objects by $s_+\mathcal{C} = \mathbf{Fun}({\Delta_+}^{op}, \mathcal{C})$.

Explicitly, $X \in Ob(s_+\mathcal{C})$ is given by $X \in Ob(s\mathcal{C})$ together with $X_{-1} \in Ob(\mathcal{C})$ and $\varepsilon\colon X_0 \to X_{-1}$ in $Mor(\mathcal{C})$ such that ε equalizes d_0 and d_1.

$$X_{-1} \xleftarrow{\ \varepsilon\ } X_0 \underset{d_1}{\overset{d_0}{\leftleftarrows}} X_1$$

We can denote $\varepsilon = d_0$ and describe the simplicial relation $d_i d_j = d_{j-1} d_i$ for $(i < j)$ to include ε: $d_0 d_1 = d_0 d_0$.

Equivalently, we may view X_{-1} as a constant simplicial object in $\mathcal{C}$ and then augmentation $\varepsilon_*\colon X_* \to X_{-1}$ is a morphism of simplicial objects.

Consider the inclusion $i_+\colon \Delta \hookrightarrow \Delta_+$. This gives an inclusion $i_+^*\colon s_+\mathcal{C} \to s\mathcal{C}$. If $\mathcal{C}$ is both complete and cocomplete, then this has both right and left adjoints. Take $\mathcal{C} = \mathbf{Set}$.

$$L \dashv i_+^* \dashv R, \qquad s_+\mathcal{C} \xrightarrow{\ i_+^*\ } s\mathcal{C}, \qquad L, R\colon s\mathcal{C} \to s_+\mathcal{C}$$

What are L and R? As usual, they are given by left and right Kan extensions.

$$\begin{array}{ccc} \Delta^{op} & \xrightarrow{\ X\ } & \mathbf{Set} \\ {\scriptstyle i_+}\big\downarrow & \nearrow_{L(X)} & \\ {\Delta_+}^{op} & & \end{array}$$

By the formula for left Kan extensions, Theorem 1.6.35,

$$L(X)_n = \mathrm{Lan}_{i_+}(X)_n = \mathrm{colim}_{\Delta_+^{op}/[n]} \left(\Delta_+^{op}/_{[n]} \xrightarrow{U} \Delta^{op} \xrightarrow{X} \mathbf{Set} \right)$$

If $n \geq 0$, then the objects of $\Delta_+^{op}/[n]$ are pairs $([m], f)$ with $[m] \in Ob(\Delta^{op})$ and $f\colon [m] \to [n] \in \Delta_+^{op}$. This has a terminal object: $([n], id_{[n]})$, so

$$L(X)_n = \mathrm{Lan}_{i_+}(X)_n = (X \circ U)([n], id_{[n]}) = X([n]) = X_n,$$

as it should be.

For $n = -1$, the objects of $\Delta_+^{op}/[-1]$ are pairs $([m], f)$ with $[m] \in Ob(\Delta^{op})$ and $f\colon [m] \to [-1]$ in ${\Delta_+}^{op}$. Since $[-1]$ is initial in Δ_+, it is terminal in ${\Delta_+}^{op}$, and

f is redundant data. Hence, the objects of $\Delta_+^{\mathrm{op}}/[-1]$ are the same as those of Δ^{op}. Therefore,

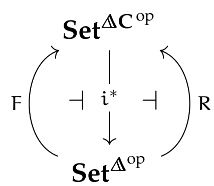
$$L(X)_{-1} = \mathrm{Lan}_{i_+}(X)_{-1} \cong \mathrm{colim}_{\Delta^{\mathrm{op}}}(X) \cong \mathrm{coeq}\left(X_1 \overset{d_0}{\underset{d_1}{\rightrightarrows}} X_0 \right) = \pi_0(X).$$

Similarly, we may compute the right adjoint $R = \mathrm{Ran}_{i_+}$, and we get the trivial augmentation with $X_{-1} = \{*\}$.

1.7.3 Simplicial and cyclic sets

Recall that $\Delta\mathbf{C}$ has the same objects as Δ and morphisms determined by the property that every $f \in \mathrm{Hom}_{\Delta\mathbf{C}}([n],[m])$ can be factored uniquely as $f = \phi \circ \gamma$ with $\phi \in \mathrm{Hom}_{\Delta}([n],[m])$ and $\gamma \in \mathrm{Aut}_{\Delta\mathbf{C}}([n]) = \mathbb{Z}/(n+1)$.

There is a natural inclusion functor $i\colon \Delta \hookrightarrow \Delta\mathbf{C}$. We ask for left and right adjoints to the functor $U = i^*\colon \mathbf{Set}^{\Delta\mathbf{C}^{\mathrm{op}}} \to \mathbf{Set}^{\Delta^{\mathrm{op}}}$.



Recall the following trick: for any $g \in \mathrm{Aut}_{\Delta\mathbf{C}}([m])$ and any $a \in \mathrm{Hom}_{\Delta\mathbf{C}}([n],[m])$, we define $f = g \circ a$ and apply the defining property of morphisms to f: there are unique morphisms

$$\begin{aligned} \phi &= g^*(a) \in \mathrm{Hom}_{\Delta}([n],[m]) \\ \gamma &= a_*(g) \in \mathrm{Aut}_{\Delta\mathbf{C}}([n]) = C_n = \mathbb{Z}/(n+1) \end{aligned}$$

such that

$$g \circ a = g^*(a) \circ a_*(g) \tag{1.7.1}$$

in $\Delta\mathbf{C}$. In particular, for fixed $g \in \mathrm{Aut}_{\Delta\mathbf{C}}([m])$, we have

$$g^*\colon \mathrm{Hom}_{\Delta\mathbf{C}}([n],[m]) \to \mathrm{Hom}_{\Delta}([n],[m])$$

and for fixed $a \in \mathrm{Hom}_{\Delta\mathbf{C}}([n],[m])$, we have

$$a_*\colon \mathrm{Aut}_{\Delta\mathbf{C}}([m]) \to \mathrm{Aut}_{\Delta\mathbf{C}}([n]).$$

Dually, in $\Delta\mathbf{C}^{\mathrm{op}}$, we have for fixed $g \in \mathrm{Aut}_{\Delta\mathbf{C}^{\mathrm{op}}}([m])$,

$$g^*\colon \mathrm{Hom}_{\Delta\mathbf{C}^{\mathrm{op}}}([m],[n]) \to \mathrm{Hom}_{\Delta^{\mathrm{op}}}([m],[n])$$

and for fixed $a \in \mathrm{Hom}_{\Delta\mathbf{C}^{op}}([m],[n])$, we have

$$a_*\colon \mathrm{Aut}_{\Delta\mathbf{C}^{op}}([m]) \to \mathrm{Aut}_{\Delta\mathbf{C}^{op}}([n]).$$

The equation (1.7.1) becomes

$$a \circ g = a_*(g) \circ g^*(a) \tag{1.7.2}$$

Claim that the left adjoint to $U\colon \mathbf{Set}^{\Delta\mathbf{C}^{op}} \to \mathbf{Set}^{\Delta^{op}}$ is given by

$$\begin{aligned} F\colon \mathbf{Set}^{\Delta^{op}} &\longrightarrow \mathbf{Set}^{\Delta\mathbf{C}^{op}} \\ Y_* &\longmapsto F(Y)_* \end{aligned}$$

where for all $n \geq 0$,

$$F(Y)_n = C_n \times Y_n = {}^{\mathbb{Z}}/_{(n+1)} \times Y_n$$

and $F(Y)(a)$ is defined by

$$\begin{aligned} F(Y)_m = C_m \times Y_n &\longrightarrow F(Y)_n = C_n \times Y_n \\ (g, y) &\longmapsto \big(a_*(g), Y(g^*(a))y\big) \end{aligned}$$

Lemma 1.7.7. *The counit of the adjunction $F \dashv U$ is given by*

$$\varepsilon = \mathrm{ev}_*\colon FU \implies \mathrm{id}_{\mathbf{Set}^{\Delta\mathbf{C}^{op}}}$$

i.e. for all $X \in \mathbf{Set}^{\Delta\mathbf{C}^{op}}$, the morphism

$$\mathrm{ev}_*(X)\colon F(X)_* \to X_*$$

in $\mathbf{Set}^{\Delta\mathbf{C}^{op}}$ is given by the evaluation map

$$\begin{aligned} C_n \times X_n &\longrightarrow X_n \\ (g, x) &\longmapsto X(g)(x) \end{aligned}$$

We will use the notation $g_*(x) := X(g)(x)$.

Proof. To verify that this is the counit, we must check first that $\mathrm{ev}_*(X)$ is a morphism of cyclic sets: for all $a \in \mathrm{Hom}_{\Delta\mathbf{C}^{op}}([m],[n])$, the following diagram must commute:

$$\begin{array}{ccc} C_m \times X_m & \xrightarrow{\mathrm{ev}_m} & X_m \\ \downarrow{\scriptstyle F(X)a} & & \downarrow{\scriptstyle a_*} \\ C_n \times X_n & \xrightarrow{\mathrm{ev}_n} & X_n \end{array}$$

$$\begin{aligned}
(\mathrm{ev}_n \circ F(X)(a))(g,x) &= \mathrm{ev}_n([F(x)(a)](g,x)) \\
&= \mathrm{ev}_n(a_*(g), X(g^*(a))(x)) && \text{def of } F \\
&= \Big(X(a_*(g)) \circ X(g^*(a))\Big)(x) && (1.7.2) \\
&= X(a \circ g)(x) && \text{functoriality} \\
&= (X(a) \circ X(g))(x) \\
&= X(\mathrm{ev}_m(g,x))
\end{aligned}$$

□

Proposition 1.7.8. *The functor* $F\colon \mathbf{Set}^{\Delta^{op}} \to \mathbf{Set}^{\Delta \mathbf{C}^{op}}$ *defined above is a left adjoint to* $U\colon \mathbf{Set}^{\Delta \mathbf{C}^{op}} \to \mathbf{Set}^{\Delta^{op}}$.

Proof. To prove this, we show that

$$\Phi\colon \mathrm{Hom}_{\mathbf{Set}^{\Delta^{op}}}(F(Y), X) \leftrightarrows \mathrm{Hom}_{\mathbf{Set}^{\Delta^{op}}}(Y, U(X))\colon \Psi$$

is a bijection.

Define $\eta_*\colon Y_* \to UF(Y)_*$ by

$$Y_n \ni \eta_n(y) = (1, y) \in F(Y)_n = C_n \times Y_n.$$

Then Φ and Ψ are given by

$$\Phi\colon \Big(F(Y) \xrightarrow{\alpha} X\Big) \mapsto \Big(Y \xrightarrow{\eta} UF(Y) \xrightarrow{\alpha} UX\Big).$$

$$\Psi\colon \Big(Y \xrightarrow{\beta} UX\Big) \mapsto \Big(F(Y) \xrightarrow{F(\beta)} FU(Y) \xrightarrow{\mathrm{ev}} Y\Big)$$

□

Exercise 1.7.9. Compute F using the colimit formula.

1.7.4 Adjunctions and cosimplicial objects

Definition 1.7.10. Given two locally small categories $\mathcal{C}$ and $\mathcal{D}$, define a cateogry $\mathbf{Adj}(\mathcal{C}, \mathcal{D})$ whose objects are triples (L, R, ϕ) of adjoint functors $L\colon \mathcal{C} \leftrightarrows \mathcal{D}\colon R$ together with a natural isomorphism

$$\phi\colon \mathrm{Hom}_{\mathcal{D}}(L(-), -) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(-, R(-)).$$

A morphism in $\mathbf{Adj}(\mathcal{C}, \mathcal{D})$ from (L, R, ϕ) to (L', R', ϕ') is a pair of natural transformations (α, β), $\alpha\colon L \implies L'$ and $\beta\colon R \implies R'$, such that the following diagram commutes.

$$\begin{array}{ccc} \mathrm{Hom}(L'(c), d) & \xrightarrow{\alpha_c^*} & \mathrm{Hom}(L(c), d) \\ \big\downarrow \phi'_{c,d} & & \big\downarrow \phi_{c,d} \\ \mathrm{Hom}(c, R(d)) & \xrightarrow{(\beta_d)_*} & \mathrm{Hom}(c, R'(d)) \end{array}$$

Theorem 1.7.11. *Let $\mathcal{C}$ be a small category and $\mathcal{D}$ a locally small, cocomplete category. Then there is a natural equivalence of categories*

$$\mathcal{D}^{\mathcal{C}} \cong \mathbf{Adj}(\widehat{\mathcal{C}}, \mathcal{D}),$$

where $\mathcal{D}^{\mathcal{C}} = \mathbf{Fun}(\mathcal{C}, \mathcal{D})$ and $\widehat{\mathcal{C}} = \mathbf{Fun}(\mathcal{C}^{op}, \mathbf{Set})$.

Proof. We construct two mutually inverse functors

$$\Phi\colon \mathcal{D}^{\mathcal{C}} \rightleftarrows \mathbf{Adj}(\widehat{\mathcal{C}}, \mathcal{D})\colon \Psi.$$

Define Ψ simply by restriction:

$$\Psi(L, R, \phi) = h^*(L) = L \circ h\colon \mathcal{C} \xrightarrow{h} \widehat{\mathcal{C}} \xrightarrow{L} \mathcal{D},$$
$$\Psi(\alpha, \beta) = h^*(\alpha) = \alpha \circ h,$$

where $h\colon \mathcal{C} \to \widehat{\mathcal{C}}$ is the Yoneda embedding.

To construct Φ, define for a functor $F\colon \mathcal{C} \to \mathcal{D}$,

$$\Phi(F) = (L(F), R(F), \phi(F)),$$

where $L(F) = \mathrm{Lan}_h(F)$ and $R(F)$ is defined by

$$\begin{aligned} R(F)\colon \mathcal{D} &\longrightarrow \widehat{\mathcal{C}} \\ d &\longmapsto \mathrm{Hom}_{\mathcal{D}}(F(-), d). \end{aligned}$$

In other words, $R(F)(d) = h_d \circ F$. We must check that $L(F)$ and $R(F)$ are adjoints, and then we may define $\phi(F)$ as the natural isomorphism coming from the adjunction.

Take $c \in \mathrm{Ob}(\mathcal{C})$ and consider the functor $h_c \in \mathrm{Ob}(\widehat{\mathcal{C}})$. To show that the adjunction holds for representable presheaves, we have:

$$\begin{aligned} \mathrm{Hom}_{\widehat{\mathcal{C}}}(h_c, R(F)(d)) &\cong (R(F)d)(c) && \text{by Yoneda} \\ &= \mathrm{Hom}_{\mathcal{D}}(F(c), d) \\ &\cong \mathrm{Hom}_{\mathcal{D}}(L(F)(h_c), d) \end{aligned}$$

the last isomorphism holds because $\mathrm{Lan}_h(F)$ is a Kan extension along a fully faithful functor, so $\mathrm{Lan}_h(F) \circ h \cong F$.

To extend the adjunction to all presheaves $X \colon \mathcal{C}^{op} \to \mathbf{Set}$, we use two facts:

(a) The Co-Yoneda lemma (Example 1.6.37). Every presheaf X is canonically a colimit of representable presheaves;

$$X \cong \mathrm{colim}_{h/X}\left(h/X \xrightarrow{U} \mathcal{C} \xrightarrow{h} \widehat{\mathcal{C}}\right) = \mathrm{colim}_{h/X}(h)$$

(b) The formula to compute a left Kan extension (Theorem 1.6.35).

$$\mathrm{Lan}_h(F) = \mathrm{colim}_{h/X}\left(h/X \xrightarrow{U} \mathcal{C} \xrightarrow{F} \mathcal{D}\right) = \mathrm{colim}_{h/X}(F).$$

Since colimits commute with colimits, and $L(F)$ preserves colimits, we have for all $d \in \mathrm{Ob}(\mathcal{D})$ and all $X \in \mathrm{Ob}(\widehat{\mathcal{C}})$,

$$\begin{aligned}
\mathrm{Hom}_{\widehat{\mathcal{C}}}(X, R(F)(d)) &\cong \mathrm{Hom}_{\widehat{\mathcal{C}}}(\mathrm{colim}_{h/X}(h_c), R(F)(d)) \\
&\cong \lim_{h/X} \mathrm{Hom}_{\widehat{\mathcal{C}}}(h_c, R(F)(d)) \\
&\cong \lim_{h/X} \mathrm{Hom}_{\mathcal{D}}(L(F)(h_c), d) \\
&\cong \mathrm{Hom}_{\mathcal{D}}(\mathrm{colim}_{h/X}(L(F)(h_c)), d) \\
&\cong \mathrm{Hom}_{\mathcal{D}}(L(F)\,\mathrm{colim}_{h/X}(h_c), d) \\
&\cong \mathrm{Hom}_{\mathcal{D}}(L(F)(X), d)
\end{aligned}$$

Thus, $L(F)$ and $R(F)$ are adjoint functors, and so Φ is well-defined. One checks that they are mutually inverse. □

1.8 Geometric realizations and totalization

Corollary 1.8.1 (Corollary to Theorem 1.7.11)**.** *Let $\mathcal{D}$ be a locally small, cocomplete category and let $\mathcal{D} = \mathbb{\Delta}$. Then there is a natural equivalence*

$$\mathcal{D}^{\mathbb{\Delta}} \cong \mathbf{Adj}(\mathbf{sSet}, \mathcal{D}).$$

This corollary says that in particular, every left adjoint functor on **sSet** with values in $\mathcal{D}$ comes from a cosimplicial object in $\mathcal{D}$!

Given $\Delta^\bullet \colon \mathbb{\Delta} \to \mathcal{D}$, we have adjoint functors $L \colon$ **sSet** $\leftrightarrows \mathcal{D} \colon R$, where

$$L(X) = \mathrm{Lan}_h(X) = \mathrm{colim}_{\mathbb{\Delta}X}(\Delta^\bullet).$$

(Recall that in this case, $h/X = \mathbb{\Delta}X$ is the category of simplicies with objects $\mathrm{Ob}(\mathbb{\Delta}X) = \coprod_{n \geq 0} X_n$.) The functor $R \colon \mathcal{D} \to$ **sSet** is defined by

$$d \mapsto Rd = \{(Rd)_n = \mathrm{Hom}_{\mathcal{D}}(\Delta^n, d)\}\}_{n \geq 0}.$$

Example 1.8.2. Take $\mathcal{D} = \mathbf{Top}$, and let $\Delta^\bullet \colon \mathbb{\Delta} \to \mathbf{Top}$ be the functor

$$[n] \mapsto \Delta^n = \left\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i=0}^{n} x_i = 1, x_i \geq 0 \right\}.$$

Theorem 1.7.11 yields two adjoint functors

$$|-| \colon \mathbf{sSet} \leftrightarrows \mathbf{Top} \colon \delta$$

where

$$|X| \cong \operatorname{colim}_{\mathbb{\Delta} X}(\Delta^\bullet)$$

and $S(Y)$ is the simplicial set with n-simplicies

$$(SY)_n = \operatorname{Hom}_{\mathbf{Top}}(\Delta^n, Y)$$

?

Definition 1.8.3. The functor $|-|$ is called the **geometric realization** and S is called the **singular simplicial set** or **singular complex** functor.

Example 1.8.4. Let $\mathcal{D} = \mathbf{Cat}$, and consider the functor $\Delta^\bullet \colon \mathbb{\Delta} \to \mathbf{Cat}$ defined by

$$[n] \mapsto \{0 \to 1 \to \cdots \to n\} = \vec{n}.$$

This gives an adjunction

$$\mathrm{ho} \colon \mathbf{sSet} \leftrightarrows \mathbf{Cat} \colon \mathcal{N},$$

where $\mathcal{N}$ is the nerve functor. The functor ho is called the **categorization functor**, and is due to [Tho79].

If we apply **??** to the cyclic category $\mathbb{\Delta}\mathbf{C}$ instead of the simplicial category $\mathbb{\Delta}$, we may define geometric realization of cyclic sets.

Definition 1.8.5. The **standard cocyclic space**

$$\Delta^\bullet \colon \mathbb{\Delta}\mathbf{C} \to \mathbf{Top}$$

is defined as the standard cosimplicial space $\Delta^\bullet \colon \mathbb{\Delta} \to \mathbf{Top}$, but with the cyclic morphisms permuting the standard basis vectors of $\mathbb{R}^{n+1}$:

$$\begin{aligned} \Delta^n &\xrightarrow{\tau_n} \Delta^n \\ e_0 &\longmapsto e_n \\ e_i &\longmapsto e_{n-1} \qquad (1 \leq i \leq n). \end{aligned}$$

In barycentric coordinates,

$$\tau_n(x_0, \dots, x_n) = \tau_n\left(\sum_{i=0}^{n} x_i e_i\right) = \sum_{i=0}^{n} x_i \tau_n(e_i) = x_0 e_n + \sum_{i=1}^{n} x_i e_{i-1} = (x_1, x_2, \dots, x_n, x_0).$$

Let's check the cyclic identities. Recall that

$$d^i(x_0, \dots, x_n) = (x_0, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n).$$

We have:

$$\begin{aligned} \tau_n d^i(x_0, \dots, x_n) &= \tau_n(x_0, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) \\ &= (x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n, x_0). \end{aligned}$$

On the other hand, we have:

$$\begin{aligned} d^i \tau_{n-1}(x_0, \dots, x_n) &= (x_1, \dots, x_n, x_0) \\ &= (x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n, x_0) \end{aligned}$$

These two are the same. One may similarly check the cyclic relations to see that τ_n is compatible with both cofaces d^i and codegeneracies s^j.

This yields a **cyclic realization** of a cyclic set:

$$\begin{aligned} |-|^{\text{cyc}} \colon \mathbf{Set}^{\mathbb{\Delta}\mathbf{C}^{\text{op}}} &\longrightarrow \mathbf{Top} \\ X &\longmapsto \operatorname{colim}_{h^{\text{cyc}}/X}(\Delta^\bullet) \end{aligned}$$

Remark 1.8.6. There are at least two other constructions of cyclic realization:

(a) Given a cyclic set X, geometric realization gives an S^1-space $|X|$, and the Borel construction gives the cyclic realization $|X|^{\text{cyc}} = ES^1 \times_{S^1} |X|$.

(b) The **fat cyclic realization** is

$$\|X\|^{\text{cyc}} := \operatorname{hocolim}_{\mathbb{\Delta}\mathbf{C}^{\text{op}}}(X) \cong |\mathcal{N}\, \mathbb{\Delta}\mathbf{C}_X^{\text{op}},$$

where $\mathbb{\Delta}\mathbf{C}_X^{\text{op}}$ is the Bousfield–Kan construction for the functor $X\colon \mathbb{\Delta}\mathbf{C}^{\text{op}} \to \mathbf{Set}$.

Example 1.8.7. Consider the functor

$$\begin{aligned} \mathbb{\Delta} &\hookrightarrow \mathbf{Cat} \\ [n] &\longmapsto \vec{n} = \{0 \to 1 \to 2 \to \cdots \to n\} \end{aligned}$$

The basic construction gives an adjunction:

$$\text{ho}\colon \mathbf{sSet} \rightleftarrows \mathbf{Cat} \colon \mathcal{N}$$

The functor $\mathcal{N}$ is defined on objects by

$$(\mathcal{N}\mathcal{C})_n = \{\mathrm{Hom}_{\mathbf{Cat}}(\vec{n}, \mathcal{C})\} = \left\{ c_0 \xrightarrow{f_0} c_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} c_n \right\}$$

In fact, $\mathcal{N}$ is exactly the nerve of $\mathcal{C}$.

For a simplicial set X, ho(X) is defined to be the category freely generated by the graph

$$X_0 \underset{d_1}{\overset{d_0}{\leftleftarrows}} X_1$$

modulo the relations

$$X_0 \xrightarrow{s_0} X_1 \;\underset{d_2}{\overset{d_0}{\leftleftarrows\!\!\leftarrow}}\; X_2 \quad (\text{middle arrow } d_1)$$

(a) $d_2(x) \circ d_0(x) = d_1(x)$ forall $x \in X_2$.

(b) $s_0(x) = \mathrm{id}_x$ for all $x \in X_0$.

Equivalently, if $\mathcal{C} = \mathrm{ho}(X)$, then

$$\mathrm{Ob}(\mathcal{C}) \;\underset{s}{\overset{t}{\leftleftarrows}}\; (\xrightarrow{i})\; \mathrm{Mor}(\mathcal{C}) \;\underset{d_2}{\overset{d_0}{\leftleftarrows\!\!\leftarrow}}\; (d_1)\; \mathrm{Mor}(\mathcal{C}) \times_{\mathrm{Ob}(\mathcal{C})} \mathrm{Mor}(\mathcal{C})$$

where s and t are source and target maps for a morphism, and i is the identity morphism

$$d_0(f, g) = f$$
$$d_1(f, g) = f \circ g$$
$$d_2(f, g) = g$$

Remark 1.8.8. (a) $\mathrm{ho} \circ \mathcal{N} \xrightarrow{\sim} \mathrm{id}_{\mathbf{Cat}}$ is an isomorphism. Therefore, $\mathcal{N}$ is fully faithful, so we may think of small categories as simplicial sets.

(b) ho(X) is uniquely determined only by X_0, X_1, and X_2, and morphisms between them. One might ask if we can extend this construction to include the data of X_3, X_4, etc. This leads to infinity categories and the homotopy coherent nerve.

Example 1.8.9. Take $\mathcal{D} = \mathbf{sSet}$, and fix an object $Y \in \mathrm{Ob}(\mathbf{sSet})$. Consider the functor $F\colon \mathbb{\Delta} \to \mathbf{sSet}$ given by the cartesian product of the standard n-simples $\Delta[n]$ with Y:

$$F\colon [n] \mapsto \Delta[n] \times Y.$$

Then the functors corresponding to this cosimplicial object are

$$L\colon \mathbf{sSet} \rightleftarrows \mathbf{sSet}\colon R$$

where

$$R(Z)_n = \mathrm{Hom}_{\mathbf{sSet}}(\Delta[n] \times Y, Z).$$

The left adjoint is given by

$$L(X) = \mathrm{colim}_{\Delta X}(\Delta[n] \times Y) \cong \mathrm{colim}_{\Delta X}(\Delta[n]) \times Y \cong X \times Y$$

Thus, $R(Y)$ plays the role of the internal hom in **sSet**:

$$\mathrm{Hom}_{\mathbf{sSet}}(X \times Y, Z) \cong \mathrm{Hom}_{\mathbf{sSet}}(X, R(Z)). \tag{1.8.1}$$

Definition 1.8.10. The **function space of simplicial sets** is

$$\underline{\mathrm{Hom}}(Y, Z) := R(Z) = \{\mathrm{Hom}_{\mathbf{sSet}}(\Delta[n] \times Y, Z)\}_{n \geq 0}.$$

This is the internal hom in the category of simplicial sets.

Remark 1.8.11.

(a) In the previous example, the Yoneda lemma immediately tells us the definition of $\underline{\mathrm{Hom}}(Y, Z)$ – take $X = \Delta[n]$ in (1.8.1).

(b) The equation (1.8.1) is the degree zero part of an enriched adjunction. There is an isomorphism of simplicial sets:

$$\underline{\mathrm{Hom}}(X \times Y, Z) \cong \underline{\mathrm{Hom}}(X, \underline{\mathrm{Hom}}(Y, Z))$$

Example 1.8.12. Consider the category **Grd** of small groupoids. There are functors

$$\mathbf{Grd} \xrightarrow{i} \mathbf{Cat} \xrightarrow{\mathcal{N}} \mathbf{sSet},$$

both of which have left adjoints: $\mathrm{ho} \dashv \mathcal{N}$ and $\tau \dashv i$. The left adjoint $\tau \dashv i$ is defined by

$$\tau(\mathcal{C}) = \mathcal{C}[\mathrm{Mor}(\mathcal{C})^{-1}].$$

We call the composition $\Pi = \mathrm{ho} \circ \tau\colon$ **sSet** $\to$ **Grd** the **fundamental groupoid functor**.

1.8.1 Applications to affine algebraic groups

Definition 1.8.13. An **affine algebraic group scheme** over a field k is a representable functor

$$G\colon \mathbf{CommAlg}_k \to \mathbf{Group}\,.$$

The representative of G is denoted $\mathcal{O}(G)$, and

$$\mathrm{Hom}_{\mathbf{CommAlg}}(\mathcal{O}(G), A) \cong G(A).$$

The functor G comes with multiplication, inverse, and unit natural transformations:

$$\begin{aligned} m_A &\colon G(A) \times G(A) \to G(A), \\ i_A &\colon G(A) \to G(A), \\ e_A &\colon * \to G(A). \end{aligned}$$

Since these are natural transformations, they give morphisms

$$\begin{aligned} \Delta &\colon \mathcal{O}(G) \to \mathcal{O}(G \times G) \cong \mathcal{O}(G) \otimes \mathcal{O}(G) \\ S &\colon \mathcal{O}(G) \to \mathcal{O}(G) \\ \varepsilon &\colon \mathcal{O}(G) \to k \end{aligned}$$

making $\mathcal{O}(G)$ into a commutative Hopf algebra. We have an anti-equivalence of categories between the category of affine algebraic group schemes over k and commutative Hopf k-algebras, given by $\mathcal{O}\colon \mathbf{AffGrSch}_k \to \mathbf{CommHopfAlg}_k$ and $\mathrm{Spec}\colon \mathbf{CommHopfAlg}_k \to \mathbf{AffGrSch}_k$.

Example 1.8.14.

(a) The **additive group** $\mathbb{G}_a\colon \mathbf{CommAlg}_k \to \mathbf{Group}$ is the functor taking a commutative algebra A to its additive group, $A \mapsto (A, +, 0)$. $\mathcal{O}(\mathbb{G}_a) \cong k[x]$.

(b) The **multiplicative group** $\mathbb{G}_m\colon \mathbf{CommAlg}_k \to \mathbf{Group}$ is the functor taking a commutative algebra A to its group of units, $A \mapsto (A^\times, \cdot, 1)$. $\mathcal{O}(\mathbb{G}_m) \cong k[x, x^{-1}]$.

(c) For $n \geq 1$, $\mathrm{GL}_n\colon \mathbf{CommAlg}_k \to \mathbf{Group}$ is the functor $A \mapsto \mathrm{GL}_n(A) = M_n(A)^\times$. In this case,

$$\mathcal{O}(\mathrm{GL}_n) \cong k\left[\{x_{ij}\}_{i,j=1}^n\right][\det(x_{ij})^{-1}].$$

(d) For $n \geq 1$, $\mathrm{SL}_n\colon \mathbf{CommAlg}_k \to \mathbf{Group}$ is the functor

$$A \mapsto \{A \in M_n(A) \mid \det(A) = 1\}.$$

$$\mathcal{O}(\mathrm{SL}_n) \cong k\left[\{x_{ij}\}_{i,j=1}^n\right] / \langle \det(x_{ij}) - 1 \rangle.$$

Now fix G and consider the composite

$$\begin{array}{rl} B_*G\colon \mathbf{CommAlg}_k \xrightarrow{G} \mathbf{Group} \hookrightarrow \mathbf{Cat} \xrightarrow{N} \mathbf{sSet} & \\ A \longmapsto B_*[G(A)] = \{G(A)^{\times n}\}_{n\geq 0} & \end{array}$$

We may consider B_*G as a functor from $\mathbf{CommAlg}_k$ to $\mathbf{Fun}(\Delta^{op}, \mathbf{Set})$, or alternatively by adjunction,

$$B_*G\colon \Delta^{op} \to \mathbf{Fun}(\mathbf{CommAlg}_k, \mathbf{Set}).$$

Explicitly,

$$\begin{array}{rcl} B_*G\colon \Delta^{op} & \longrightarrow & \mathbf{Fun}(\mathbf{CommAlg}_k, \mathbf{Set}) \xleftarrow{\text{Yoneda}} {\mathbf{CommAlg}_k}^{op} \\ [n] & \longmapsto & [A \mapsto B_n[G(A)]] \end{array}$$

We may also define a cosimplicial commutative algebra $\mathcal{O}(B_*G)$ by

$$\begin{array}{rcl} \mathcal{O}(B_*(G))\colon \Delta & \longrightarrow & \mathbf{CommAlg}_k \\ [n] & \longmapsto & \mathcal{O}(G^n) = \mathcal{O}(G)^{\otimes n} \end{array}$$

The codegeneracies $d^i\colon \mathcal{O}(G^{n-1}) \to \mathcal{O}(G)$ are given by

$$(d^i f)(g_1, \ldots, g_n) = \begin{cases} f(g_2, \ldots, g_n) & (i = 0) \\ f(g_1, \ldots, g_i g_{i+1}, \ldots, g_n) & (1 \leq i \leq n-1) \\ f(g_1, \ldots, g_{n-1}) & (i = n). \end{cases}$$

for $f \in \mathcal{O}(G^{n-1})$.

The cofaces $s^j\colon \mathcal{O}(G^n) \to \mathcal{O}(G^{n-1})$ are given by

$$(s^j f)(g_1, \ldots, g_{n-1}) = f(g_1, \ldots, g_{j-1}, e, g_j, \ldots, g_{n-1})$$

for $f \in \mathcal{O}(G^n)$.

Remark 1.8.15. The realization of Δ in $\mathbf{CommAlg}_k$ (depending on G) is similar to the usual geometric realization of Δ in **Top**.

By Theorem 1.7.11, we have adjoint functors

$$L\colon \mathbf{sSet} \leftrightarrows \mathbf{CommAlg}_k \colon R$$

where the n-simplicies of $R(A)$ are

$$R(A)_n = \mathrm{Hom}_{\mathbf{CommAlg}_k}(\mathcal{O}(B_n G), A) \cong G(A)^{\times n} = B_n[G(A)].$$

Thus, the right adjoint is $R(A) = B_*[G(A)]$.

For a simplicial set X,

$$L(X) \cong \operatorname{colim}_{\Delta X}(\mathcal{O}(B_* G)) = \operatorname{colim}_{\Delta[n]_* \to X}(\mathcal{O}(G)^{\otimes n}).$$

In particular, $L(\Delta[n]) \cong \mathcal{O}(G)^{\otimes n}$.

Example 1.8.16. If Γ is a discrete group, and $X = B_*\Gamma \in \mathrm{Ob}(\mathbf{sSet})$, what is $L(B_*\Gamma)$? We have

$$\begin{aligned} \mathrm{Hom}_{\mathbf{CommAlg}_k}(L(B_*\Gamma), A) &\cong \mathrm{Hom}_{\mathbf{sSet}}(B_*\Gamma, B_*[G(A)]) \\ &= \mathrm{Hom}_{\mathbf{sSet}}(\mathcal{N}_*(\underline{\Gamma}), \mathcal{N}_*(\underline{G(A)})) \\ &\cong \mathrm{Hom}_{\mathbf{Cat}}(\underline{\Gamma}, \underline{G(A)}) && \mathcal{N} \text{ is fully faithful} \\ &\cong \mathrm{Hom}_{\mathbf{Group}}(\Gamma, G(A)) \end{aligned}$$

Hence, by the Yoneda lemma, $L(B\Gamma) = \mathcal{O}(\mathrm{Rep}_G(\Gamma))$.

More generally, if X is a reduced simplicial set, (and so $|X|$ is a pointed connected space),

$$L(X) \cong \mathcal{O}(\mathrm{Rep}_G(\pi_1(|X|, *))).$$

Exercise 1.8.17. Calculate $L(X)$ (in some explicit form) for any simplicial set X.

1.9 Homotopy coherent nerve

We want to refine the nerve/categorization construction

$$\mathrm{ho}\colon \mathbf{sSet} \leftrightarrows \mathbf{Cat} \colon \mathcal{N}$$

to an adjunction

$$\mathfrak{C}\colon \mathbf{sSet} \leftrightarrows \mathbf{sCat}_0 \colon \mathfrak{N}.$$

The right adjoint $\mathfrak{N}$ is called the **homotopy coherent nerve**. First, we must discuss simplicial categories.

A simplicial object in **Cat** is a functor

$$C_*\colon \Delta^{\mathrm{op}} \to \mathbf{Cat}$$

together with functors

$$\begin{aligned} d_i\colon C_n &\to C_{n-1} && (0 \le i \le n, n \ge 1) \\ s_j\colon C_n &\to C_{n+1} && (0 \le n \le n, n \ge 0). \end{aligned}$$

But this is too general for our purposes.

Definition 1.9.1. A **simplicial category** $\underline{C}$ is a simplicial object C_* in **Cat** such that the morphisms d_i and s_j are identity maps on objects.

Thus, $\mathrm{Ob}(C_i) = \mathrm{Ob}(C_j)$ for all i, j. Define $\mathrm{Ob}(\underline{C}) := \mathrm{Ob}(C_0)$. It is called the **underlying category**. On morphisms, we can form for $c_1, c_2 \in \mathrm{Ob}(\underline{C})$ a simplicial set $\underline{\mathrm{Hom}}_{\underline{C}}(c_1, c_2)$ with n simplicies

$$\underline{\mathrm{Hom}}_{\underline{C}}(c_1, c_1)_n = \mathrm{Hom}_{C_n}(c_1, c_2).$$

We have just proved:

Proposition 1.9.2. *The data of a simplicial category is equivalent to the data of a category enriched over simplicial sets.*

Remark 1.9.3. We can define such enrichment for large categories, such as **sSet**, as well, by setting $\mathrm{Ob}(\underline{\mathbf{sSet}}) = \mathrm{Ob}(\mathbf{sSet})$ and

$$\underline{\mathrm{Hom}}_{\underline{\mathbf{sSet}}}(X, Y)_n = \mathrm{Hom}_{\mathbf{sSet}}(\Delta[n]_* \times X, Y)$$

is the same as the enrichment of simplicial sets over itself, as in Definition 1.8.10.

Definition 1.9.4 (Notation)**.** Write $\underline{C}_*(c_1, c_2) := \underline{\mathrm{Hom}}_{\underline{C}}(c_1, c_2)$ for the simplicial set of morphisms between $c_1, c_2 \in \mathrm{Ob}(\underline{C})$.

Definition 1.9.5. A **simplicial functor** $F\colon \underline{C} \to \underline{D}$ consists of a morphism

$$F\colon \mathrm{Ob}(\underline{C}) \to \mathrm{Ob}(\underline{D})$$

between object sets and for all $c_1, c_2 \in \mathrm{Ob}(\mathcal{C})$, a map of simplicial sets:

$$F\colon \underline{C}(c_1, c_2) \to \underline{D}(Fc_1, Fc_2).$$

Equivalently, F is a collection of functors $F_n\colon C_n \to D_n$.

Definition 1.9.6. Given simplicial functors $F, G\colon \underline{C} \to \underline{D}$, a **simplicial natural transformation** $\xi\colon F \implies G$ consists of the data:

(a) a natural transformation $\xi_c\colon F_0 \implies G_0$ of functors between the underlying categories;

(b) for all $n \geq 1$, a natural transformation $s_0^n(\xi_c)\colon F_n \implies G_n$ between functors between C_n and D_n

Definition 1.9.7. Let $\mathbf{sCat}_0$ be the category of small simplicial categories.

1.9.1 Barr–Beck construction

Definition 1.9.8. A **monad** (T, η, μ) on a category $\mathcal{C}$ is given by an endofunctor $T\colon \mathcal{C} \to \mathcal{C}$ with two natural transformations

$$\eta\colon \mathrm{id}_{\mathcal{C}} \implies T \qquad \text{the } \textbf{unit}$$
$$\mu\colon T \circ T \implies T \qquad \text{the } \textbf{multiplication}$$

satisfying two conditions:

- (Associativity) The following diagram commutes:

$$\begin{array}{ccc} T \circ T \circ T & \overset{T\mu}{\Longrightarrow} & T \circ T \\ \Downarrow{\scriptstyle \mu_T} & & \Downarrow{\scriptstyle \mu} \\ T \circ T & \overset{\mu}{\Longrightarrow} & T \end{array} \tag{1.9.1}$$

- (Unitality) The following diagram commutes:

$$\begin{array}{ccccc} T & \overset{T\eta}{\Longrightarrow} & T \circ T & \overset{\eta_T}{\Longleftarrow} & T \\ & \diagdown\!\!\diagdown & \Downarrow{\scriptstyle \mu} & /\!\!/ & \\ & & T & & \end{array} \tag{1.9.2}$$

Definition 1.9.9. A **comonad** (S, ε, δ) on a category $\mathcal{D}$ is given by an endofunctor $S\colon \mathcal{D} \to \mathcal{D}$, and two natural transformations

$$\varepsilon\colon S \implies \mathrm{id}_{\mathcal{D}} \qquad \text{the } \textbf{counit}$$
$$\delta\colon S \implies S \circ S \qquad \text{the } \textbf{comultiplication}$$

satisfying the coassociative and counital laws dual to (1.9.1) and (1.9.2).

Example 1.9.10. Consider a pair of adjoint functors $F\colon \mathcal{C} \leftrightarrows \mathcal{D}\colon G$ with unit $\eta\colon \mathrm{id}_{\mathcal{C}} \implies GF$ and counit $\varepsilon\colon FG \implies \mathrm{id}_{\mathcal{D}}$. Then we define a monad (T, η, μ) on $\mathcal{C}$ with $T = GF$, η the unit of the adjunction, and $\mu = G\varepsilon_F$. Dually, we have a comonad (S, ε, δ) with $S = FG$, ε the counit of the adjunction, and $\delta = F\eta_G$. The fact that this defines a monad and comonad follows from the triangular identities for the adjunction natural transformations.

Remark 1.9.11. The above example is universal in the following sense: given any monad, there is an associated adjunction inducing that monad.

Example 1.9.12. Consider the adjunction $F\colon \mathbf{Set} \leftrightarrow R\text{-}\mathbf{Mod}\colon U$, where $F(X) = R[X]$ is the free R-module on X, and U is the forgetful functor. Then there is a comonad $FU\colon R\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$, sending an R-module M to the free module $R[U(M)]$ on the underlying set of M.

Proposition 1.9.13. *Every monad on $\mathcal{C}$ gives a functor $\mathcal{C} \to \mathbf{c}\mathcal{C}$, where $\mathbf{c}\mathcal{C}$ is the category cosimplicial objects in $\mathcal{C}$.*

Dually, any comonad on $\mathcal{D}$ gives a functor $\mathcal{D} \to \mathbf{s}\mathcal{D}$.

Proof. We prove the second assertion. Given (S, ε, δ) on $\mathcal{D}$, and $A \in \mathrm{Ob}(\mathcal{D})$, we define

$$S_*\colon \mathcal{D} \to \mathbf{s}\mathcal{D}$$

sending A to the simplicial object S_*A with n-simplicies

$$S_n(A) = S^{n+1}(A) = (\underbrace{S \circ S \circ \cdots \circ S}_{n+1} A)$$

with face maps

$$d_i = S^i \circ \varepsilon \circ S^{n-i}\colon S^{n+1}(A) \to S^n(A)$$

and degeneracies

$$s_j = S^j \circ \delta \circ S^{n-j}\colon S^{n+1}(A) \to S^{n+2}(A)$$

One checks the simplicial identities hold with these faces and degeneracies. □

Remark 1.9.14. The functor $\mathcal{D} \to \mathbf{s}\mathcal{D}$ associated to the comonad $S\colon \mathcal{D} \to \mathcal{D}$ comes with an augmentation given by $S_* d \xrightarrow{\varepsilon} d$. This is called the **standard simplicial resolution** of d.

Example 1.9.15. If R is a ring, and we consider the free-forgetful adjunction $F = R[-]\colon$ **Set** $\leftrightarrows$ **Mod**$(R)\colon U$, then we have an augmented simplicial R-module $(FU)_*M \xrightarrow{\varepsilon} M$. Under the Dold–Kan correspondence, this is a canonical free resolution of M as an R-module.

1.9.2 The homotopy coherent nerve

Recall that we want an adjunction

$$\mathfrak{C}\colon \mathbf{sSet} \leftrightarrows \mathbf{sCat}_0\colon \mathfrak{N}.$$

Equivalently, we may define a functor

$$\begin{aligned} \mathfrak{C}\Delta^*\colon \mathbb{\Delta} &\longrightarrow \mathbf{sCat} \\ [n] &\longmapsto Q\vec{n} := S_*(\vec{n}), \end{aligned}$$

where (S, ε, δ) is the comonad on **Cat** coming from the free-forgetful adjunction

$$F\colon \mathbf{Quiver} \leftrightarrows \mathbf{Cat}\colon U, \tag{1.9.3}$$

where **Quiver** is the category of small **reflexive** directed graphs. Recall that a graph Γ is called **reflexive** if every vertex $v \in V(\Gamma)$ has a distinguished edge $v \xrightarrow{e_v} v$.

The functor $U\colon \mathbf{Cat} \to \mathbf{Quiver}$ forgets the composition laws in a cateogory but remembers objects, codomains, and domains of morphisms, as well as identities.

For a quiver Q, the category $F(Q)$ is the free category generated by the quiver Q with objects the vertices $\mathrm{Ob}(F(Q)) = V(Q)$ and a morphism $f\colon v \to w$ in $\mathrm{Hom}_{F(Q)}(v, w)$ is either a single identity edge or path of non-identity edges between vertices. Composition of morphisms is concatenation of paths.

This adjunction gives a comonad $S = FU\colon \mathbf{Cat} \to \mathbf{Cat}$. By Proposition 1.9.13, there is an associated functor

$$S_*\colon \mathbf{Cat} \to \mathbf{sCat}\,.$$

Define a cosimplicial object in **sCat**

$$\begin{array}{rcl} \mathfrak{C}\Delta^*\colon \mathbb{\Delta} & \hookrightarrow \mathbf{Cat} \xrightarrow{\quad S_* \quad} & \mathbf{sCat} \\ [n] & \longmapsto \vec{n} = \{0 \to 1 \to \cdots \to n\} \longmapsto & (FU)_*\vec{n}. \end{array}$$

For fixed $n \geq 0$, we obtain a simplicial object $(FU)_*\vec{n}$ in **Cat**; this is actually a simplicial category in the sense of Definition 1.9.1. Hence, this functor lands in $\mathbf{sCat}_0$.

By Theorem 1.7.11, we get an adjunction

$$\mathfrak{C}\colon \mathbf{sSet} \leftrightarrows \mathbf{sCat}_0 : \mathfrak{N}.$$

The left adjoint is defined by a Kan extension

$$\mathfrak{C}X := \mathrm{Lan}_h(\mathfrak{C}\Delta^*)(X) \cong \mathrm{colim}_{\mathbb{\Delta}X}(\mathfrak{C}\Delta^*).$$

The n-simplicies of the right adjoint $\mathfrak{N}(\underline{C})$ are given by

$$\mathfrak{N}_n(\underline{C}) = \mathbf{Fun}(\mathfrak{C}\Delta^n, \underline{C}).$$

Definition 1.9.16. The right adjoint $\mathfrak{N}\colon \mathbf{sCat}_0 \to \mathbf{sSet}$ is called the **homotopy coherent nerve**.

Theorem 1.9.17 (Dugger–Spivak 2013). *For each $n \geq 0$, $\mathfrak{C}\Delta^n$ is a simplicial category with objects*

$$\mathrm{Ob}(\mathfrak{C}\Delta^n) = \{0, 1, \ldots, n\} = \mathrm{Ob}(\vec{n})$$

and for each $i, j \in \{0, 1, \ldots, n\}$,

$$\underline{\mathrm{Hom}}_{\mathfrak{C}\Delta^n}(i, j) \cong \mathcal{N}_* P_{ij},$$

where $\mathcal{N}_*$ is the usual nerve and P_{ij} is the poset under inclusion of all subsets of $\{k\colon i \leq k \leq j\}$ containing both i and j.

Remark 1.9.18.

(1) If $i > j$, then $P_{ij} = \varnothing$, and therefore $\underline{\mathrm{Hom}}(i, j) = \varnothing$.

(2) If $j > i$, then P_{ij} is the product of $j - i - 1$ copies of $[1] = \{0 < 1\}$.

(3) In general,

$$\underline{\mathrm{Hom}}_{\mathfrak{C}\Delta^n}(i, j) = \begin{cases} \Delta[1]_*^{\times(j-i-1)} & i < j \\ \Delta[0]_* & i = j \\ \varnothing & i > j \end{cases}$$

Remark 1.9.19. Both **sSet** and **sCat$_0$** have natural model structures. The model structure on **sCat$_0$** is called the **Dwyer–Kan model structure** where a functor $F\colon \underline{C} \to \underline{D}$ between simplicial categories is a weak equivalence if it is both a weak equivalence on all Hom-simplicial sets and $\pi_0(F)\colon \pi_0(\underline{C}) \xrightarrow{\sim} \pi_0(\underline{D})$ is an equivalence of categories.

The corresponding model structure on **sSet** has fibrant objets which are exactly quasi-categories.

In this case, the adjunction $\mathfrak{C} \dashv \mathfrak{N}$ is a Quillen equivalence between the categories of simplicial sets and simplicial categories, which shows that quasi-categories and simplicial categories give two equivalent models of $(\infty, 1)$-categories.

Remark 1.9.20. A category $\mathcal{C}$ has two simplicial thickenings: on one hand, we have $(FU)_*\mathcal{C}$ from the adjunction (1.9.3) and on the other hand, we have Proposition 1.9.13 and $\mathfrak{C}(\mathfrak{N}_*\mathcal{C})$. It is a theorem of Emily Riehl that these are the same: $(FU)_*\mathcal{C} \cong \mathfrak{C}(\mathfrak{N}_*\mathcal{C})$ for any category $\mathcal{C}$.

Chapter 2

The chapter numbers are arbitrary and don't mean anything

2.1 Enriched categories

Definition 2.1.1. A category $\mathcal{S}$ is called **symmetric monoidal** if

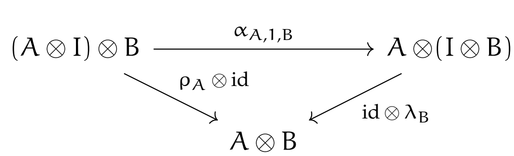
- there exists a bifunctor $\otimes\colon \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ called the **tensor product**,
- there is an object $I \in \mathrm{Ob}(\mathcal{S})$ called the **unit object**,

such that for all $A, B, C \in \mathrm{Ob}(\mathcal{S})$, there are natural isomorphisms

$$\begin{aligned} A \otimes B &\cong B \otimes A, \\ \alpha_{A,B,C}\colon A \otimes (B \otimes C) &\cong (A \otimes B) \otimes C, \\ \lambda_A\colon A &\cong I \otimes A, \\ \rho_A\colon A &\cong A \otimes I. \end{aligned}$$

These must be compatible in the sense that they satisfy two axioms:

- the **triangular identities:**



- the **pentagon axiom:**

$$
\begin{array}{ccccc}
((A \otimes B) \otimes C) \otimes D & \xrightarrow{\alpha_{A,B,C} \otimes \mathrm{id}} & A \otimes ((B \otimes C) \otimes D) & & \\
& & & \searrow^{\alpha_{A,B \otimes C,D}} & \\
\downarrow{\scriptstyle \alpha_{A \otimes B,C,D}} & & & & A \otimes ((B \otimes C) \otimes D) \\
& & & \swarrow_{\mathrm{id} \otimes \alpha_{B,C,D}} & \\
(A \otimes B) \otimes (C \otimes D) & \xrightarrow{\alpha_{A,B,C \otimes D}} & A \otimes (B \otimes (C \otimes D)) & &
\end{array}
$$

Definition 2.1.2. A symmetric monoidal category $\mathcal{S}$ is **closed** if there is a bifunctor

$$\underline{\mathrm{Hom}}_{\mathcal{S}}(-,-)\colon \mathcal{S}^{\mathrm{op}} \times \mathcal{S} \to \mathcal{S}$$

called the **internal hom** such that

$$\underline{\mathrm{Hom}}_{\mathcal{S}}(A \otimes B, C) \cong \underline{\mathrm{Hom}}_{\mathcal{S}}(A, \underline{\mathrm{Hom}}_{\mathcal{S}}(B, C)).$$

For each $\mathcal{B} \in \mathrm{Ob}(\mathcal{S})$, there is an adjunction

$$- \otimes B\colon \mathcal{S} \leftrightarrows \mathcal{S}\colon\ \underline{\mathrm{Hom}}_{\mathcal{S}}(b, -).$$

Remark 2.1.3. There is a natural map

$$\underline{\mathrm{Hom}}(B, C) \otimes \underline{\mathrm{Hom}}(A, B) \to \underline{\mathrm{Hom}}(A, C)$$

which is adjoint to the composite map:

$$
\begin{array}{ccc}
\underline{\mathrm{Hom}}(B, C) \otimes \underline{\mathrm{Hom}}(A, B) \otimes A & \xrightarrow{\mathrm{id} \otimes \varepsilon} & \underline{\mathrm{Hom}}(B, C) \otimes B \\
& \searrow & \downarrow{\scriptstyle \varepsilon} \\
& & C
\end{array}
$$

Definition 2.1.4. A symmetric monoidal category $\mathcal{S}$ is called **Cartesian** if the tensor product is the Cartesian product and the unit is a terminal object of $\mathcal{S}$.

We will often abuse notation and write $(\mathcal{S}, \times, *)$ in general.

Example 2.1.5. Let $\mathcal{S} = \mathbf{sSet}$, with $\otimes = \times$. This has an internal hom given by Definition 1.8.10: the n-simplicies of $\underline{\mathrm{Hom}}(Y, Z)$ are

$$\underline{\mathrm{Hom}}(Y, Z)_n = \mathrm{Hom}_{\mathbf{sSet}}(Y \times \Delta[n], Z).$$

This satisfies the relation for all simplicial sets X, Y and Z:

$$\underline{\mathrm{Hom}}(X \times Y, Z) \cong \underline{\mathrm{Hom}}(X, \underline{\mathrm{Hom}}(Y, Z)).$$

The composition

$$\circ\colon \underline{\mathrm{Hom}}(Y,Z)\times\underline{\mathrm{Hom}}(X,Y)\to\underline{\mathrm{Hom}}(X,Z)$$

of n-simplicies $\left(Y\times\Delta[n]\xrightarrow{f} Z\right)\in\underline{\mathrm{Hom}}(Y,Z)_n$ and $\left(X\times\Delta[n]\xrightarrow{g} Y\right)\in\underline{\mathrm{Hom}}(X,Y)_n$ is:

$$X\times\Delta[n]\xrightarrow{\mathrm{id}\times\mathrm{diag}} X\times\Delta[n]\times\Delta[n]\xrightarrow{g\times\mathrm{id}} Y\times\Delta[n]\xrightarrow{f} Z.$$

Example 2.1.6. Let $(\mathcal{S},\times,*)$ be a Cartesian closed symmetric monoidal category that is both complete and cocomplete. Let $\mathcal{S}_* = *\downarrow\mathcal{S}$ be the slice category under the terminal object $*$, with objects $\mathrm{Ob}(\mathcal{S}_*)=\{(*\to v)\mid v\in\mathrm{Ob}(\mathcal{S})\}$.

There is a canonical way to make $\mathcal{S}_*$ into a symmetric monoidal category. The unit of this monoidal structure will be

$$1_{\mathcal{S}_*} = *\sqcup *.$$

Given $(*\xrightarrow{i_v} v)$ and $(*\xrightarrow{i_w} w)$, define

$$f_v\colon v\xrightarrow{\cong} v\times *\xrightarrow{\mathrm{id}_v\times i_w} v\times w$$
$$f_w\colon w\xrightarrow{\cong} *\times w\xrightarrow{i_v\times\mathrm{id}_w} v\times w$$

Define the tensor product $v\wedge w := \mathrm{cofib}(f_v\sqcup f_w)$ to be the pushout of the diagram

$$\begin{array}{ccc} v\sqcup w & \xrightarrow{f_v\sqcup f_w} & v\times w\\ \downarrow & & \downarrow\\ * & \longrightarrow & v\wedge w. \end{array}$$

Dually, we have an internal hom defined by the pullback:

$$\begin{array}{ccc} \underline{\mathrm{Hom}}_{\mathcal{S}_*}(v,w) & \longrightarrow & \underline{\mathrm{Hom}}_{\mathcal{S}}(v,w)\\ \downarrow & & \downarrow{\scriptstyle i_0^*}\\ *\cong\underline{\mathrm{Hom}}_{\mathcal{S}}(*,*) & \xrightarrow{(i_w)_*} & \underline{\mathrm{Hom}}_{\mathcal{S}}(*,w) \end{array}$$

The closed symmetric monoidal category $(\mathcal{S}_*,\wedge,1_{\mathcal{S}_*},\underline{\mathrm{Hom}}_{\mathcal{S}_*})$ is called the the **category of based objects in** $\mathcal{S}$.

There is a pair of adjoint functors

$$(-)_+\colon\mathcal{S}\leftrightarrows\mathcal{S}_*\colon U$$

where $(-)_+$ is strictly monoidal: $1_{\mathcal{S}_*}\cong(*)_+$ and $v_+\wedge w_+ = (v\times w)_+$.

Example 2.1.7. We may construct symmetric monoidal categories of based objects for both **Top** and **sSet**:

(a) The category of based objects for $(\mathbf{Top}, \times, *)$ is $(\mathbf{Top}_*, \wedge, S^0)$;

(b) The category of based objects for $(\mathbf{sSet}, \times, *)$ is $(\mathbf{sSet}_*, \wedge, \partial\Delta^1)$.

In both cases, $\wedge$ is called the **smash product** of spaces/simplicial sets. Note that neither of the based categories are Cartesian.

Definition 2.1.8. Let $(\mathcal{S}, \otimes, 1)$ be a closed symmetric monoidal category. An $\mathcal{S}$**-category** or **category enriched in** $\mathcal{S}$ is a category $\mathcal{M}$ such that

(S1) For all $X, Y \in \mathrm{Ob}(\mathcal{M})$, there is an object $\underline{\mathrm{Hom}}(X, Y) \in \mathrm{Ob}(\mathcal{S})$.

(S2) For all $X, Y, Z \in \mathrm{Ob}(\mathcal{M})$, there is a morphism in $\mathcal{S}$

$$c_{X,Y,Z} \colon \underline{\mathrm{Hom}}_{\mathcal{M}}(Y, Z) \times \underline{\mathrm{Hom}}_{\mathcal{M}}(X, Y) \to \underline{\mathrm{Hom}}_{\mathcal{M}}(X, Z)$$

natural in X, Y, Z, called the **composition law**.

(S3) For all $X \in \mathrm{Ob}(\mathcal{M})$, there is a morphism of $\mathcal{S}$

$$i_X \colon 1 \to \underline{\mathrm{Hom}}_{\mathcal{M}}(X, X)$$

called the **unit**.

(S4) There are bijections

$$\mathrm{Hom}_{\mathcal{S}}(*, \underline{\mathrm{Hom}}(X, Y)) \cong \mathrm{Hom}_{\mathcal{M}}(X, Y)$$

for all $X, Y \in \mathrm{Ob}(\mathcal{M})$. Under this isomorphism, i_X corresponds to the identity $\mathrm{id}_X \in \mathrm{Mor}(\mathcal{M})$.

These data satisfy compatibility axioms (triangle and pentagon) similar to the ones for $\mathcal{S}$.

Example 2.1.9. A category enriched over $(\mathbf{sSet}, \times, *)$ is a simplicial category. This is equivalent to Definition 1.9.1.

Definition 2.1.10. An $\mathcal{S}$-category is called **tensored** over $\mathcal{S}$ if there is a bifunctor

$$\boxtimes \colon \mathcal{S} \times \mathcal{M} \to \mathcal{M},$$

called the **action of** $\mathcal{S}$ **on** $\mathcal{M}$, such that

$$\underline{\mathrm{Hom}}_{\mathcal{M}}(\nu \boxtimes X, Y) \cong \mathrm{Hom}_{\mathcal{S}}(\nu, \underline{\mathrm{Hom}}(X, Y))$$

for all $\nu \in \mathrm{Ob}(\mathcal{S})$ and $X, Y \in \mathrm{Ob}(\mathcal{M})$.

Definition 2.1.11. An $\mathcal{S}$-category is called **cotensored** over $\mathcal{S}$ if there is a bifunctor

$$\begin{aligned} \mathcal{S}^{op} \times \mathcal{M} &\longrightarrow \mathcal{M} \\ (v, Y) &\longmapsto Y^v, \end{aligned}$$

called the **coaction of** $\mathcal{S}$ **on** $\mathcal{M}$, such that

$$\mathrm{Hom}_{\mathcal{S}}(v, \underline{\mathrm{Hom}}_{\mathcal{M}}(X, Y)) \cong \underline{\mathrm{Hom}}_{\mathcal{M}}(X, Y^v).$$

Example 2.1.12. Let $\mathcal{M}$ be any locally small category which is both complete and cocomplete. Then $\mathcal{M}$ is both tensored and cotensored over $(\mathbf{Set}, \times, \{*\})$. The action of **Set** on $\mathcal{M}$ is given by

$$\begin{aligned} \boxtimes \colon \mathbf{Set} \times \mathcal{M} &\longrightarrow \mathcal{M} \\ (K, X) &\longmapsto \textstyle\coprod_{k \in K} X, \end{aligned}$$

and the coaction is given by

$$\begin{aligned} \mathbf{Set} \times \mathcal{M} &\longrightarrow \mathcal{M} \\ (K, X) &\longmapsto Y^K = \textstyle\prod_{k \in K} Y. \end{aligned}$$

Our main application will be using $\mathcal{S} = (\mathbf{sSet}, \times, *)$ and $\mathcal{M}$ any simplicial category both tensored and cotensored over $\mathcal{S}$, for example $\mathcal{M} = s\mathcal{C}$. In this setting, we may introduce an (internal) geometric realization.

Remark 2.1.13. If we think of a simplicial category $\mathcal{M}$ as an object in **sCat**, then this agrees with Definition 1.9.1. In particular, by the Yoneda lemma

$$\mathrm{Hom}_{\mathcal{S}}\left(\Delta[0], \underline{\mathrm{Hom}}_{\mathcal{M}(X,Y)}\right) \cong \underline{\mathrm{Hom}}_{\mathcal{M}}(X, Y)_0 = \mathrm{Hom}_{\mathcal{M}_0}(X, Y).$$

Thus, $\mathrm{Hom}_{\mathcal{M}}(X, Y) = \mathrm{Hom}_{\mathcal{M}_0}(X, Y)$.

Example 2.1.14. Let $\mathcal{C}$ be any complete and cocomplete category. Let $\mathcal{M} = s\mathcal{C}$. Then $\mathcal{M}$ is canonically tensored and cotensored over $\mathcal{S}et$. The tensor is defined by

$$\begin{aligned} \boxtimes \colon \mathbf{sSet} \times s\mathcal{C} &\longrightarrow s\mathcal{C} \\ (K, X) &\longmapsto \left\{\textstyle\coprod_{K_n} X_n\right\}_{n \geq 0} \end{aligned}$$

and the cotensor is defined by

$$\underline{\mathrm{Hom}}_{s\mathcal{C}}(X, Y) = \{\mathrm{Hom}(\Delta[n] \boxtimes X, Y)\}_{n \geq 0}$$

Note that the tensor satisfies associativity and unit laws:

$$\begin{aligned} (K \times L) \boxtimes X &\cong K \boxtimes (L \boxtimes X) \\ \Delta[0] \boxtimes X &\cong X \end{aligned}$$

Remark 2.1.15. If $s\mathcal{C}$ is tensored over **sSet**, then for any set K (viewed as a discrete simplicial set) and $X \in \mathrm{Ob}(s\mathcal{C})$,

$$K \boxtimes X = \coprod_K X,$$

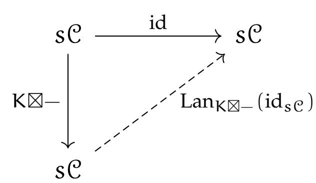
because

$$K \boxtimes X \cong (K \times \Delta[0]) \boxtimes X \cong \left(\coprod_K \Delta[0] \right) \boxtimes X \cong \coprod_K (\Delta[0] \boxtimes X) \cong \coprod_K X.$$

Example 2.1.16. Adopt the setup of Example 2.1.14. Fix a simplicial set K and consider the functor

$$K \boxtimes -\colon s\mathcal{C} \to s\mathcal{C}.$$

Since $\mathcal{C}$ is cocomplete, so is $s\mathcal{C}$, and hence $K \otimes -$ has a right adjoint defined by the left Kan extension



We may then define:

$$Y^K = \mathrm{Lan}_{K\boxtimes -}(\mathrm{id}_{s\mathcal{C}})Y$$

for all $Y \in \mathrm{Ob}(s\mathcal{C})$. Then, by general principles of Kan extensions, we have a bijection

$$\mathrm{Hom}_{s\mathcal{C}}(K \boxtimes X, Y) \cong \mathrm{Hom}_{s\mathcal{C}}(X, Y^k).$$

In fact, this implies that for all $n \geq 0$,

$$\begin{aligned} \underline{\mathrm{Hom}}_{s\mathcal{C}}(K \boxtimes X, Y)_n &= \mathrm{Hom}_{s\mathcal{C}}(\Delta[n] \times K \times X, Y) \\ &\cong \mathrm{Hom}_{s\mathcal{C}}(K \times \Delta[n] \times X, Y) \\ &\cong \mathrm{Hom}_{s\mathcal{C}}(\Delta[n] \times X, Y^K) \\ &= \underline{\mathrm{Hom}}_{s\mathcal{C}}(X, Y^K) \end{aligned}$$

This shows that $s\mathcal{C}$ is also cotensored over **sSet**.

Example 2.1.17. A special case of the previous example is $\mathcal{C} = \mathbf{Set}$ and $\mathcal{M} = \mathbf{sSet}$. Then

$$K \boxtimes X = \left\{ \coprod_{K_n} X_n \right\}_{n \geq 0} = \{K_n \times X_n\}_{n \geq 0} = K \times X.$$

The cotensor is defined by

$$\mathrm{Hom}_{\mathbf{sSet}}(K \boxtimes X, Y) \cong \mathrm{Hom}_{\mathbf{sSet}}(X, Y^k).$$

To determine what Y^K is, put $X = \Delta[n]$. Then

$$\underline{\mathrm{Hom}}_{\mathbf{sSet}}(K, Y)_n = \mathrm{Hom}_{\mathbf{sSet}}(K \boxtimes \Delta[n], Y) \cong \mathrm{Hom}_{\mathbf{sSet}}(\Delta[n], Y^K) \cong (Y^K)_n,$$

the last equality by Yoneda's lemma. Therefore,

$$Y^K \cong \underline{\mathrm{Hom}}_{\mathbf{sSet}}(K, Y).$$

Example 2.1.18. Another special case is $\mathcal{C} = \mathbf{Mod}(R)$, where R is a unital associative ring and $\mathcal{M} = \mathbf{sMod}(R)$. In this case, the coproduct is direct sum, and the tensor product is

$$\begin{aligned} \mathbf{sSet} \times \mathbf{sMod}(R) &\longrightarrow \mathbf{sMod}(R) \\ (K, X) &\longmapsto \left\{ \bigoplus_{K_n} X_n \right\}_{n \geq 0} \end{aligned}$$

The R-module $\bigoplus_{K_n} X_n$ is isomorphic to

$$\bigoplus_{K_n} X_n \cong R[K_n] \otimes_R X_n,$$

where $R[K_n]$ is the free (R, R)-bimodule based on K_n.

For the cotensor,

$$\underline{\mathrm{Hom}}_{\mathbf{sMod}(R)}(X, Y) = \left\{ \mathrm{Hom}_{\mathbf{sMod}(R)}(R[\Delta[n]] \otimes_R X, Y) \right\}_{n \geq 0}$$

and therefore

$$Y^K = \underline{\mathrm{Hom}}_{\mathbf{sSet}}(K, Y)$$

with R-module structure given by R-module structure on the target.

Example 2.1.19. Fix a commutative ring k. Another special case is $\mathcal{C} = \mathbf{CommAlg}_k$ and $\mathcal{M} = \mathbf{sCommAlg}_k$. Then for any simplicial set K,

$$K \boxtimes A = \left\{ \bigotimes_{K_n} A_n \right\}_{n \geq 0}$$

$$\underline{\mathrm{Hom}}_{\mathbf{sCommAlg}_k}(A, B) = \left\{ \mathrm{Hom}_{\mathbf{sCommAlg}_k} \left(\bigotimes_{\Delta[n]} A, B \right) \right\}_{n \geq 0}$$

Example 2.1.20. These examples are quite general. We can take $\mathcal{C}$ to be any algebraic category (e.g. $\mathcal{C} = \mathbf{Group}, \mathbf{Alg}_k$, or even algebras for an operad).

Example 2.1.21. Let $\mathcal{M} = \mathbf{Top}$. Then for any simplicial set K, define

$$K \boxtimes X = |K| \times X$$
$$Y^K = \mathrm{Map}(|K|, Y)$$

Remark 2.1.22. Usually, **Top** is viewed as a topological category, but via the standard adjunction

$$|-|\colon \mathbf{sSet} \leftrightarrows \mathbf{Top}\colon \mathcal{S},$$

we may convert every topological category into a simplicial category, and vice versa. The crucial observation is that geometric realization preserves products:

$$|X \times Y| \cong |X| \times |Y|.$$

2.2 Functor Tensor Products

Let $\mathcal{S}$ be a closed symmetric monoidal category. Let $\mathcal{M}$ be any $\mathcal{S}$-category tensored over $\mathcal{S}$.

Definition 2.2.1. For any small category $\mathcal{C}$ and two functors $F\colon \mathcal{C}^{op} \to \mathcal{S}$ and $G\colon \mathcal{C} \to \mathcal{M}$, define the **functor tensor product**

$$F \boxtimes_{\mathcal{C}} G := \int^{c \in \mathrm{Ob}(\mathcal{C})} F(c) \boxtimes G(c) = \mathrm{coeq}\left(\coprod_{f\colon c \to c'} F(c') \boxtimes G(c) \underset{f^*}{\overset{f_*}{\rightrightarrows}} \coprod_{c \in \mathrm{Ob}(\mathcal{C})} F(c) \boxtimes G(c) \right)$$

where f^* and f_* are defined by

$$f^*\colon F(c') \boxtimes G(c) \xrightarrow{F(f) \boxtimes \mathrm{id}} G(c) \boxtimes F(c) \hookrightarrow \coprod_{c \in \mathrm{Ob}(\mathcal{C})} G(c) \boxtimes F(c)$$

$$f_*\colon F(c') \boxtimes G(c) \xrightarrow{\mathrm{id} \boxtimes G(f)} G(c') \boxtimes F(c') \hookrightarrow \coprod_{c \in \mathrm{Ob}(\mathcal{C})} G(c) \boxtimes F(c)$$

Example 2.2.2. Consider $\mathcal{S} = (\mathbf{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$ and let $\mathcal{M} = \mathcal{S}$. This is a closed symmetric monoidal category enriched and tensored over itself with the internal hom given by the usual one: $\underline{\mathrm{Hom}}_{\mathbf{Ab}} = \mathrm{Hom}_{\mathbf{Ab}} = \mathrm{Hom}_{\mathbb{Z}}$. The tensor product is the usual one: $\boxtimes = \otimes_{\mathbb{Z}}$.

Now take any associative unital ring R. We may consider R as a category $\underline{R}$ with one object enriched over **Ab**. A left R-module M may be realized as an **Ab**-functor $F\colon \underline{R} \to \mathbf{Ab}, * \mapsto M$. Similarly, a right R-module N can be considered as a functor $G\colon \underline{R}^{op} \to \mathbf{Ab}, * \mapsto N$.

Then the functor tensor product reduces to the usual tensor product of right and left R-modules.

$$G \boxtimes_{\underline{R}} F = \int^{* \in \mathrm{Ob}(\underline{R})} N \otimes_{\mathbb{Z}} M \cong {}^{N \otimes_{\mathbb{Z}} M}\big/_{\langle nr \otimes m - n \otimes rm \mid r \in R \rangle} = N \otimes_R M.$$

Example 2.2.3. Let $\mathcal{S} = (\mathbf{sSet}, \times, \Delta^0)$ and take any small category $\mathcal{C}$. Let $\mathcal{M}$ be any simplicial category tensored over **sSet**, and let $F\colon \mathcal{C} \to \mathcal{M}$ be any functor. Let $G\colon \mathcal{C}^{op} \to \mathbf{sSet}$ be the constant functor at the terminal object of **sSet**. Then

$$G \boxtimes_{\mathcal{C}} F \cong \mathrm{colim}_{\mathcal{C}}(F).$$

This follows from two facts:

(a) If $T\colon \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ is constant in the first argument, then the coend of T is isomorphic to $\mathrm{colim}_{\mathcal{C}}(S)$.

(b) The terminal object $* \in \mathcal{S} = \mathbf{sSet}$ acts as the identity on $\mathcal{M}$.

Remark 2.2.4. In general, for any $F\colon \mathcal{C}^{op} \to \mathbf{sSet}$, and $G\colon \mathcal{C} \to \mathcal{M}$, the intuition is that $F \boxtimes_{\mathcal{C}} G$ is a colimit of G weighted by F. Indeed, if $F\colon \mathcal{C}^{op} \to \mathcal{S}$ is the constant diagram at the unit object of $\mathcal{S}$, then

$$F \boxtimes_{\mathcal{C}} G = \mathrm{colim}_{\mathcal{C}}\, G.$$

Example 2.2.5. Let $\mathcal{M} = \mathcal{S} = \mathbf{sSet}$, and let $\mathcal{C}$ be any small category with $c \in \mathrm{Ob}(\mathcal{C})$ a fixed object. Consider any diagram $F\colon \mathcal{C} \to \mathbf{sSet}$ and take

$$h_c = \mathrm{Hom}_{\mathcal{C}}(-, c)\colon \mathcal{C}^{op} \to \mathbf{Set} \hookrightarrow \mathbf{sSet}\,.$$

Then $h_c \boxtimes_{\mathcal{C}} F \cong Fc$ by the Yoneda lemma.

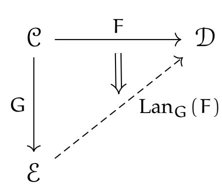
Dually, if $G\colon \mathcal{C}^{op} \to \mathbf{sSet}$ and

$$h^c = \mathrm{Hom}_{\mathcal{C}}(c, -)\colon \mathcal{C} \to \mathbf{Set} \hookrightarrow \mathbf{sSet}\,.$$

Then $G \boxtimes_{\mathcal{C}} h^c \cong G(c)$.

The moral of this example is that representable functors h^c and h_c take the role of free modules. s

Example 2.2.6. Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories with $\mathcal{D}$ cocomplete and hence tensored over **Set**. Consider a left Kan extension



with universal natural transformation $\eta\colon F \implies \mathrm{Lan}_G(F) \circ G$. Fix $e \in \mathrm{Ob}(\mathcal{E})$. Then claim that $\mathrm{Lan}_G(F)(e)$ can be interpreted as a tensor product

$$\mathrm{Lan}_G(F)(e) = (G \circ h_e) \boxtimes_{\mathcal{C}} F = \mathrm{Hom}_{\mathcal{E}}(G(-), e) \boxtimes_{\mathcal{C}} F.$$

We can establish this relation by simply comparing the universal properties.

2.3 Geometric realization

The geometric realization of a simplicial set as a topological space is simply a special case of the previous example.

Consider the standard geometric simplex functor:

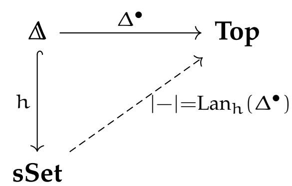
$$\begin{aligned} \mathbb{\Delta} &\xrightarrow{\Delta^\bullet} \mathbf{Top} \\ [n] &\longmapsto \Delta^n = \left\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; \textstyle\sum_i x_i = 1, x_i \geq 0\right\} \end{aligned}$$

Definition 2.3.1. The **geometric realization** of a simplicial set X is

$$|X| = \operatorname{colim}_{\mathbb{\Delta} X}(\Delta^\bullet),$$

where ΔX is the simplex category of X and $\Delta^\bullet$ is the functor above.

This is nothing more than a left Kan extension:



Then

$$\mathrm{Lan}_h(\Delta^\bullet)(X) \cong \operatorname{colim}_{\mathbb{\Delta} X}(\Delta^\bullet)$$

by definition. By the formula

$$\mathrm{Lan}_G(F)e \cong (G \circ h_e) \boxtimes_{\mathcal{C}} F = \mathrm{Hom}_{\mathcal{E}}(G(-), e) \boxtimes_{\mathcal{C}} F$$

from Example 2.2.6, we may rewrite this as:

$$\begin{aligned} \mathrm{Lan}_h(\Delta^\bullet)(X) &\cong \mathrm{Hom}_{\mathbf{sSet}}(\Delta[-]_*, X) \boxtimes_{\mathbb{\Delta}} \Delta^\bullet \\ &\cong X \boxtimes_{\mathbb{\Delta}} \Delta^\bullet \qquad\qquad \text{Yoneda} \qquad (2.3.1) \\ &= \coprod_{n \geq 0} X_n \times \Delta^n / \sim \end{aligned}$$

where $\sim$ is the equivalence relation $(x, f_* u) \sim (f^* x, u)$ for all $f \in \mathrm{Mor}(\mathbb{\Delta})$. This is the classical definition of geometric realization:

Definition 2.3.2 (Classical)**.** The **geometric realization** of a simplicial set X is

$$|X| := \coprod_{n \geq 0} X_n \times \Delta^n / \sim$$

where $\sim$ is the equivalence relation $(x, f_* u) \sim (f^* x, u)$ for all $f \in \mathrm{Mor}(\mathbb{\Delta})$.

For simplicial spaces, we *define* the geometric realization using (2.3.1).

Definition 2.3.3. For a **simplicial space** $X\colon \mathbb{\Delta}^{op} \to \mathbf{sSet}$, define its **geometric realization**

$$|X| := X \boxtimes_{\mathbb{\Delta}} \Delta^{\bullet} \cong \Delta^{\bullet} \boxtimes_{\mathbb{\Delta}^{op}} X = \coprod_{n\geq 0} X_n \times \Delta^n / \sim$$

where $\sim$ is the equivalence relation $(x, f_* u) \sim (f^* x, u)$ for all $f \in \mathrm{Mor}(\mathbb{\Delta})$.

Geometric realization makes sense in much more general contexts.

Definition 2.3.4. Let $\mathcal{M}$ be a category enriched and tensored over **sSet**, define the **internal geometric realization** to be the functor

$$\begin{aligned} |-|\colon s\mathcal{M} &\longrightarrow \mathcal{M} \\ X &\longmapsto \Delta[-]_* \boxtimes_{\mathbb{\Delta}^{op}} X \end{aligned}$$

where $\Delta[-]_*\colon \mathbb{\Delta} \to \mathbf{sSet}$ is the standard simplex functor $[n] \mapsto \Delta[n]_*$.

Example 2.3.5. The category **Top** is tensored over simplicial sets via $(K, Z) \mapsto |K| \times Z$. In this case, the internal geometric realization in **Top** is the usual geometric realization.

Example 2.3.6. If $\mathcal{M} = \mathbf{sSet}$, we may consider an object $X \in \mathrm{Ob}(s\mathcal{M})$ as a bisimplicial set:

$$X_{**} = \{X_{nm}, s_j^h, d_i^h, s_j^v, d_i^v\}_{n,m\geq 0}.$$

There is a natural functor

$$\mathrm{diag}\colon \mathbf{sSet}^{\mathbb{\Delta}^{op}} \to \mathbf{sSet}$$

taking a bisimplicial set X to its diagonal simplicial set

$$[n] \mapsto X_n = \{X_{nn}, d_i = d_i^v d_i^h, s_j = s_j^v s_j^h\}_{n\geq 0}.$$

It is a theorem of Bousfield–Friedlander that $|X| \cong \mathrm{diag}(X)$.

2.4 Homotopy colimits

Why do we need homotopy colimits? They come from topology. Usual colimits are used to build complicated spaces from simpler ones by gluing, but a problem arises if we want to glue homotopy types.

Example 2.4.1. Let $i\colon S^{n-1} \to D^n$ be the inclusion of the $(n-1)$-sphere into the n-disk. There is a natural homotopy equivalence of diagrams

$$\begin{array}{ccccc} D^n & \xleftarrow{i} & S^{n-1} & \xrightarrow{i} & D^n \\ \downarrow{\simeq} & & \downarrow{\simeq} & & \downarrow{\simeq} \\ * & \longleftarrow & S^{n-1} & \longrightarrow & * \end{array}$$

but the colimits of these two diagrams do not have the same homotopy type. In fact,

$$\operatorname{colim}\left(D^n \longleftarrow S^{n-1} \longrightarrow D^n\right) \cong S^n,$$

but on the other hand,

$$\operatorname{colim}\left(* \longleftarrow S^{n-1} \longrightarrow * \right) \cong *.$$

The moral is that the objects (in this case spaces) defined by colimits of diagrams which are only defined up to homotopy are not well defined, even up to homotopy. Homotopy colimits are replacements of usual colimits, when we glue spaces (objects) together with homotopies between gluing maps.

There is another use of homotopy colimits: they provide a natural way to construct deformations (or "quantization") of objects. The idea is that if we want to deform an object (space/algebra/category), decompose it (in a natural way) into a homotopy colimit, and then, instead of deforming the object itself, we deform the underlying diagram from which we obtain the object as a homotopy colimit.

Many kinds of diagrams appear in practice, as seen in the examples below.

2.4.1 Examples

Example 2.4.2 (Mapping Tori)**.** The mapping torus is an example of a homotopy pushout. If $f\colon X \to X$ is a map of spaces, the **mapping torus** of f is defined by

$$T(X,f) := \operatorname{hocolim}\left(X \xleftarrow{(\mathrm{id},f)} X \sqcup X \xrightarrow{(\mathrm{id},\mathrm{id})} X\right) \cong {X \times I}\big/{\langle (x,0) \sim (f(x),1)\rangle}.$$

If f is the identity map, then this is a kind of torus $X \times S^1$ constructed from X. We think of f as the deformation parameter. Note that this comes with a natural projection onto the circle S^1. Indeed,

$$T(*,\mathrm{id}) = \operatorname{hocolim}\left(* \leftarrow * \sqcup * \to *\right) \cong S^1,$$

and the commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{(\mathrm{id},f)} & X \sqcup X & \xrightarrow{(\mathrm{id},\mathrm{id})} & X \\ \downarrow & & \downarrow & & \downarrow \\ * & \longleftarrow & * \sqcup * & \longrightarrow & * \end{array}$$

induces the map of homotopy colimits $T(X, f) \to T(*, f) = S^1$. If X is a closed n-manifold, and $f\colon X \xrightarrow{\sim} X$ a smooth automorphism, then $T(X, f) \to S^1$ is an $(n+1)$-dimensional manifold fibered over S^1.

Example 2.4.3. Let $\Sigma = \Sigma_g$ be a closed orientable surface of genus $g \geq 1$. The **mapping class group** $MCG^+(\Sigma)$ of Σ is the group of orientation-preserving homeomorphisms $\phi\colon \Sigma \to \Sigma$ up to isotopy.

Theorem 2.4.4. *Any orientable Σ-bundle over S^1 has the form*

$$M_\phi(\Sigma) = T(\Sigma, \phi) \to S^1$$

for some $\phi \in MCG^+(\Sigma)$.

The short exact sequence

$$1 \to \pi_1(\Sigma) \to \pi_1(M_\phi(\Sigma)) \to \mathbb{Z} \to 1$$

splits as

$$\pi_1(M_\phi(\Sigma)) \cong \pi_1(\Sigma) \rtimes \mathbb{Z} \hookleftarrow \pi_1(\Sigma).$$

Corresponding to $\pi_1(\Sigma) \subseteq \pi_1(M_\phi(\Sigma))$ is an infinite cyclic covering

$$\Sigma \times \mathbb{R} \to M_\phi(\Sigma).$$

The subgroup $\mathbb{Z}$ is generated by $t\colon \Sigma \times \mathbb{R} \to \Sigma \times \mathbb{R}$, $(x, \lambda) \mapsto (\phi(x), \lambda + 1)$ and

$$M_\phi(\Sigma) \cong {}^{\Sigma \times \mathbb{R}}\big/_{(x,\lambda) \sim (\phi(x), \lambda+1)}.$$

Example 2.4.5 (Suspension)**.** Another homotopy pushout. Given a space X, its **suspension** ΣX is given by

$$\Sigma X = \text{hocolim}\,(* \leftarrow X \to *)$$

As a space, ΣX is obtained from the cylinder $X \times I$ by collapsing both $X \times 0$ and $X \times 1$ to points.

Example 2.4.6 (Join of spaces)**.** Another homotopy pushout. For spaces X and Y, their **join** is the space

$$X * Y \cong \text{hocolim}\left(X \xleftarrow{p_X} X \times Y \xrightarrow{p_Y} Y\right) \cong {}^{X \sqcup (X \times I \times Y) \sqcup Y}\big/_{\sim},$$

where $\sim$ is the relation $(x, 0, y) \sim x$ and $(x, 1, y) \sim y$ for all $x \in X$ and all $y \in Y$.

Example 2.4.7 (Group action)**.** If G is a discrete or topological group acting on a space (simplicial set) X, then we have a natural diagram

$$X\colon \underline{G} \to \mathbf{sSet}$$

given by

$$\mathrm{colim}_{\underline{G}}(X) \cong {}^{X}/_{G}.$$

The **homotopy quotient** or **homotopy orbit space** is

$$X_{hG} := \mathrm{hocolim}_{\underline{G}}(X) \cong EG \times_G X,$$

where EG is a free contractible G-space. The construction $EG \times_G X$ is called the **Borel construction**.

Example 2.4.8. 1. Let G be a discrete group, acting on $\{*\}$ trivially. Then the orbit space of this action is

$$^{*}/_{G} = \mathrm{colim}_{\underline{G}}(*) = *,$$

while the homotopy orbits are

$$*_{hG} = \mathrm{hocolim}_{\underline{G}}(*) \cong BG.$$

2. If G acts on G by left translations, then the homotopy orbits are

$$G_{hG} = \mathrm{hocolim}_{\underline{G}}(G) \cong EG.$$

If G acts on G by the **adjoint representation** (conjugation) $\mathrm{Ad}\colon G \to \mathrm{Aut}(G)$, $\mathrm{Ad}_g(x) = gxg^{-1}$. In this case, the homotopy orbits is the free loop space on BG

$$\mathrm{hocolim}_{\underline{G}}(\mathrm{Ad}) \cong \mathcal{L}BG$$

Example 2.4.9 (Simplicial and cyclic sets)**.** For simplicial sets, the **fat geometric realization** is defined to be

$$\|X\| = \mathrm{hocolim}_{\mathbb{\Delta}^{\mathrm{op}}}(X)$$

By the Bousfield–Kan theorem, $\|X\| \simeq |X|$.

Likewise, for cyclic sets, the **fat cyclic realization** is defined to be

$$\|X\|^{\mathrm{cyc}} = \mathrm{hocolim}_{\mathbb{\Delta}\mathbf{C}^{\mathrm{op}}}(X).$$

By a theorem of Loday–Fiedoriwicz, this is homotopy equivalent to the standard cyclic realization

$$|X|^{\mathrm{cyc}} = ES^1 \times_{S^1} |X|.$$

Example 2.4.10 (Poset diagrams)**.** Let $\mathcal{C}$ be the category associated to a poset. Many geometrically interesting spaces decompose into homotopy colimits of poset diagrams.

Let $\mathcal{B}_n$ be the poset of all nonempty faces in the n-simplex ordered by inclusion.

- For $n = 1$, the poset $\mathcal{B}_1$ is the category $\{0 \leftarrow (01) \rightarrow 1\}$, where (01) is the edge between the zero simplicies of Δ^1.
- For $n = 2$, the poset $\mathcal{B}_2$ is the category with objects $\{0, 1, 2, (01), (02), (12), (012)\}$ and inclusions as inclusions of faces in Δ^2.

In general, such a category $\mathcal{B}_n$ has $2^{n+1} - 1$ objects.

Given a collection $\mathcal{X} = \{X_i\}_{i=1}^n$ of spaces, define

$$\mathcal{D}_{\mathcal{X}} \colon \mathcal{B}_n \to \mathbf{sSet}$$

on objects by

$$A \longmapsto \prod_{i \in A} X_i$$

and on morphisms by

$$(B \supseteq A) \longmapsto P_{AB} \colon \mathcal{D}_{\mathcal{X}}(A) \to \mathcal{D}_{\mathcal{X}}(B)$$

the canonical projection. When $n = 1$, $\mathcal{X} = \{X_0, X_1\}$ and $\mathcal{D}_X$ is the diagram of spaces

$$X_0 \leftarrow X_0 \times X_1 \rightarrow X_1.$$

Theorem 2.4.11 (Ziegler (2001))**.** *The homotopy colimit of the diagram $\mathcal{D}_{\mathcal{X}}$ is the iterated join of spaces:*

$$\operatorname{hocolim}_{B_n}(\mathcal{D}_{\mathcal{X}}) \cong X_0 * X_1 * \cdots * X_n.$$

Corollary 2.4.12. *Assume that $X_i = S^1$ for all i. Then we may modify*

$$\overline{\mathcal{D}}_{\mathcal{X}}(A) = {(S^1)^{|A|}}/{S^1}$$

to be the quotient of the product by the diagonal action. Then

$$\operatorname{hocolim}_{B_n} \overline{\mathcal{D}}_{\mathcal{X}} \cong \mathbb{CP}^n.$$

Remark 2.4.13. More generally, any toric variety can be decomposed in this way.

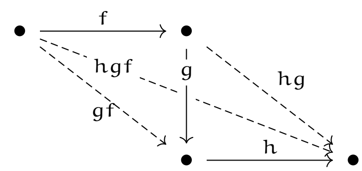
Remark 2.4.14 (References)**.**
- Dwyer–Hirschhorn–Kan–Smith, *Homotopy (co)limits in model categories and homotopy categories* AMS 2004
- M. Shulman *Homotopy limits and colimits in enriched category theory* (2009)
- Farjann *Fundamental groups...* Adv. Math (2006)

2.4.2 Homotopical categories

Definition 2.4.15. A **homotopical category** $\mathcal{M}$ is a category equipped with a class of morphisms $\mathcal{W} \subseteq \mathrm{Mor}(\mathcal{M})$, called the **weak equivalences** satisfying two axioms:

(W1) All identities are in $\mathcal{W}$. For all $X \in \mathrm{Ob}(\mathcal{M})$, $\mathrm{id}_X \in \mathcal{W}$.

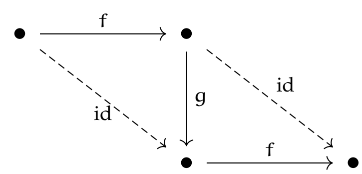
(W2) The **two-out-of-six property**: given a composable triple (f, g, h) of morphisms in $\mathcal{M}$,



such that $gf \in \mathcal{W}$ and $hg \in \mathcal{W}$, the arrows f, g, h, and hgf are also in $\mathcal{W}$.

Remark 2.4.16.

(a) Axioms (W1) and (W2) imply that all isomorphisms of $\mathcal{M}$ are in $\mathcal{W}$: indeed, if fg such that $fg = \mathrm{id}$ and $gf = \mathrm{id}$, then we may draw the diagram



(b) The axiom (W2) implies the usual two-out-of-three property: for a composable pair f, g, if any two of $f, g, gf \in \mathcal{W}$, then so is the third.

These properties allow us to view $\mathcal{W}$ as a subcategory of $\mathcal{M}$, which is **wide** in the sense that it contains all objects in $\mathcal{M}$.

Example 2.4.17. Any category $\mathcal{C}$ can be viewed as a homotopical category if we take $\mathcal{W} = \mathrm{Iso}(\mathcal{C})$. This is called a **minimal homotopical category**.

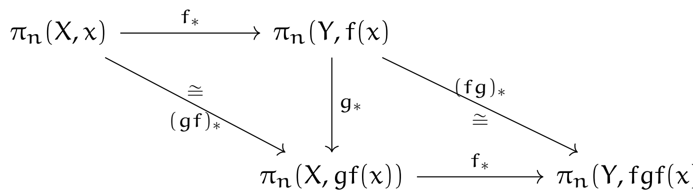
Indeed, $\mathrm{Iso}(\mathcal{C})$ satisfies the two-out-of-six property: given f, g, h such that gf and hg are isomorphisms in $\mathcal{C}$, we need to show that $f, g, h, hgf \in \mathrm{Iso}(\mathcal{C})$. First, $\gamma = f(gf)^{-1}$ is right inverse to g. But g is monic (because $gf_1 = gf_2 \implies hgf_1 = hgf_2 \implies f_1 = f_2$), so γ is also a left inverse to g:

$$g\gamma = \mathrm{id} \implies g(\gamma g) = g \implies g\gamma g = g\mathrm{id} \implies \gamma g = \mathrm{id}.$$

So g is an isomorphisms, which also shows that f is an isomorphism (because gf is an iso) and h is an isomorphism (because hg is an iso).

Remark 2.4.18 (DHKS)**.** The fact that the isomorphisms in any category satisfy the two-out-of-six property is used to prove that homotopy equivalences of spaces are weak homotopy equivalences: given $f\colon X \to Y$ with a homotopy inverse $g\colon Y \to X$ (i.e. $fg \sim \mathrm{id}_Y$ and $gf \sim \mathrm{id}_X$), we must show that $f_*\colon \pi_n(X,x) \to \pi_n(Y, f(x))$ is an isomorphism of groups for all n and all $x \in X$.

Since π_n is homotopy invariant, we have group isomorphisms $(gf)_*$ and $(fg)_*$ as in the diagram below



Then the two-out-of-six property tells us that f_*, g_* are isomorphisms of groups as well.

This is where the two-out-of-six property comes from.

Example 2.4.19. Any model category $\mathcal{M}$ is a homotopical category with the same class of weak equivalences. By the axioms of a model category, the class $\mathcal{W}$ of weak equivalences must satisfy the two-out-of-three property, and in fact, $\mathcal{W}$ also satisfies the two-out-of-six property (although it doesn't follow only from the two-out-of-three property!).

Example 2.4.20. If $\mathcal{M}$ is any homotopical category and $\mathcal{C}$ is any small category, it is immediate that the diagram category $\mathcal{M}^{\mathcal{C}} = \mathbf{Fun}(\mathcal{C}, \mathcal{M})$ is a homotopical category with $\mathcal{W}(\mathcal{M}^{\mathcal{C}})$ defined objectwise: $\alpha\colon F \to G$ is a weak equivalence if and only if $\alpha_c\colon F(c) \to G(c)$ is a weak equivalence for all $c \in \mathrm{Ob}(\mathcal{C})$.

Definition 2.4.21. If $\mathcal{M}$ is a homotopical category with weak equivalences $\mathcal{W}$, then we define its **homotopy category Ho**$(\mathcal{M})$ as a formal localization $\mathbf{Ho}(\mathcal{M}) := \mathcal{M}[\mathcal{W}^{-1}]$.

This is by definition a category $\mathbf{Ho}(\mathcal{M})$ and a functor $\gamma\colon \mathcal{M} \to \mathbf{Ho}(\mathcal{M})$ which is initial among all functors $\rho\colon \mathcal{M} \to \mathcal{N}$ such that $\rho(f) \in \mathrm{Iso}(\mathcal{N})$ for all $f \in \mathcal{W}(\mathcal{M})$.

By convention, we regard $\mathbf{Ho}(\mathcal{M})$ as a minimal homotopical category.

Notice that by definition, for all $f \in \mathcal{W}(\mathcal{M})$, we have $\gamma(f) \in \mathrm{Iso}(\mathbf{Ho}(\mathcal{M}))$. Is the converse always true? The answer is no, but we have a term for those categories where it is.

Definition 2.4.22. A pair $(\mathcal{M}, \mathcal{W})$ of a category $\mathcal{M}$ and a class of morphisms $\mathcal{W}$ is called **saturated** if $\gamma(f) \in \mathrm{Iso}(\mathbf{Ho}(\mathcal{M}))$ implies that $f \in \mathcal{W}(\mathcal{M})$.

Theorem 2.4.23 (Quillen)**.** *Every model category is saturated.*

Lemma 2.4.24. *If* $(\mathcal{M}, \mathcal{W})$ *is a saturated pair such that for all* $X \in \mathrm{Ob}(\mathcal{M})$, $\mathrm{id}_X \in \mathcal{W}$*, then* $(\mathcal{M}, \mathcal{W})$ *is a homotopical category.*

Proof. We must check that the two-out-of-six property holds. Because the pair is saturated,

$$\mathcal{W} = \{f \in \mathrm{Mor}(\mathcal{M}) \mid \gamma(f) \in \mathrm{Iso}(\mathbf{Ho}(\mathcal{M}))\}.$$

Then given $f, g, h \in \mathrm{Mor}(\mathcal{M})$ such that gf and hg are in $\mathcal{W}$, then $\gamma(gf)$ and $\gamma(hg)$ are isomorphisms in $\mathbf{Ho}(\mathcal{M})$. The two-out-of-six property for $\mathbf{Ho}(\mathcal{M})$ shows that $\gamma(f), \gamma(g), \gamma(h), \gamma(hgf)$ are isomorphisms in $\mathbf{Ho}(\mathcal{M})$. Therefore, f, g, h and hgf are weak equivalences. □

Corollary 2.4.25. *Any model category is homotopical.*

Question 2.4.26. Given a saturated homotopical category, does it come from a model structure?

2.5 Poisson Algebras

We work with associative, unital differential graded algebras over a field k of characteristic zero: algebras A with a grading

$$A = \bigoplus_{i \in \mathbb{Z}} A_i$$

such that $A_i A_j \subseteq A_{i+j}$ and a differential $d\colon A \to A$, $d^2 = 0$, $|d| = 1$ satisfying the Leibniz rule:

$$d(ab) = (da)b + (-1)^{|a|} a(db)$$

for all $a, b \in A$.

Definition 2.5.1. Let k be a field. If A is a commutative, differential graded k-algebra, then a **Poisson structure** on A is a bracket

$$\{-,-\}\colon A \times A \to A$$

such that

(a) $\{-,-\}$ is a Lie bracket on A, i.e. $\{-,-\}$ is skew symmetric and satisfies the Jacobi identity.

(b) $\{-,-\}$ satisfies the **Leibniz rule**:

$$\{a, bc\} = b\{a, c\} + \{a, b\}c.$$

Example 2.5.2. Take $A = C^\infty(\mathbb{R}^2)$ or $A = k[x, y]$ for a field $k \supseteq \mathbb{Q}$. $\mathbb{R}^2$ is a symplectic manifold with symplectic form $\omega = dx \wedge dy$, and A is a commutative Poisson algebra with Poisson bracket

$$\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial x}.$$

This is dual to the symplectic form on $\mathbb{R}^2$.

How can we extend this definition to noncommutative algebras? The naïve definition (just forget that A is not commutative in the above) is very restrictive because omits many interesting examples that should otherwise be considered noncommutative Poisson algebras.

Theorem 2.5.3 ((Farkas–Letzter))**.** *If A is a (noncommutative) Noetherian domain, then any Poisson bracket on A is a scalar multiple of the commutator.*

Instead, we will put a differential graded Lie algebra structure on a quotient of A.

Definition 2.5.4. For a differential graded k-algebra A, define

$$A_\natural := {}^{A}/_{[A, A]},$$

where $[A, A]$ is the k-linear span of commutators

$$[A, A] = \mathrm{Span}_k \left\{[a, b] = ab - (-1)^{|a||b|}ba \mid a, b \in A\right\}.$$

Remark 2.5.5. Note that although $A_\natural$ is not an algebra, it is naturally a chain complex with differential induced from A. Indeed, for all $a, b \in A$,

$$\begin{aligned} d([a, b]) &= d(ab - (-1)^{|a||b|}ba) \\ &= (da)b + (-1)^{|a|}a(db) - (-1)^{|a||b|}\big((db)a + (-1)^{|b|}b(da)\big) \\ &= \big((da)b - (-1)^{(|a|+1)|b|}b(da)\big) + (-1)^{|a|}\big(adb) - (-1)^{|a|(|b|+1)}(db)a\big) \\ &= [da, b] + (-1)^{|a|}[a, db] \in [A, A] \end{aligned}$$

Hence, the differential applied to a commutator is again a sum of commutators, so $d\colon A_\natural \to A_\natural$ is well-defined.

Definition 2.5.6. If (V, d_V) is any complex, then the **graded endomorphism ring** is

$$\underline{\mathrm{End}}(V) := \bigoplus_{n\in\mathbb{Z}} \underline{\mathrm{End}}(V)_n$$

where $\underline{\mathrm{End}}(V)_n$ is the set of homomorphisms $f\colon V \to V$ that increase degree by n:

$$\underline{\mathrm{End}}(V)_n := \left\{f\colon V \to V \;\middle|\; f(V_i) \subseteq V_{i+n}\ \forall\, i \in \mathbb{Z}\right\} = \prod_{i\in\mathbb{Z}} \mathrm{Hom}_k(V_i, V_{i+n})$$

Define $d\colon \underline{\mathrm{End}}(V) \to \underline{\mathrm{End}}(V)$ by

$$d(f) = [d_V, f] = d_V \circ f - (-1)^{|f|} f \circ d_V.$$

Since $d_V^2 = 0$, then $d^2 = 0$ as well.

This defines a differential on $\underline{\mathrm{End}}(V)$, making it a differential graded k-algebra with $1 = \mathrm{id}_V$.

We may also think of $\underline{\mathrm{End}}(V)$ as a differential-graded Lie algebra with commutator bracket

$$[f, g] = f \circ g - (-1)^{|f||g|} g \circ f.$$

Definition 2.5.7. Let A be a differential graded k-algebra. A **derivation of degree** r on A is a k-linear homomorphism $\delta\colon A \to A$ such that $\delta \in \underline{\mathrm{End}}(A)_r$ and

$$\delta(ab) = (\delta a)b + (-1)^{|a|\cdot r} a(\delta b)$$

Let $\underline{\mathrm{Der}}(A) \subseteq \underline{\mathrm{End}}(A)$ be the differential graded Lie algebra of all k-linear derivations on A.

If $V = A$ is a differential graded algebra, then $\underline{\mathrm{End}}(A)$ contains $\underline{\mathrm{Der}}(A)$ as a canonical differential graded Lie subalgebra consisting of all graded k-linear derivations of A. Note that $\underline{\mathrm{Der}}(A)$ acts on A naturally so that A becomes a differential graded Lie module over A.

Consider

$$\underline{\mathrm{Der}}(A)^{\natural} := \{\delta \in \underline{\mathrm{Der}}(A) \mid \delta(A) \subseteq [A, A]\}.$$

Then $\underline{\mathrm{Der}}(A)^{\natural}$ is a differential graded Lie ideal in $\underline{\mathrm{Der}}(A)$. Define the quotient differential graded Lie algebra

$$\underline{\mathrm{Der}}(A)_{\natural} := \underline{\mathrm{Der}}(A) / \underline{\mathrm{Der}}(A)^{\natural}.$$

The action of $\underline{\mathrm{Der}}(A)$ on A (by derivation) induces a Lie action of $\underline{\mathrm{Der}}(A)_{\natural}$ on $A_{\natural}$ so that

$$\rho\colon \underline{\mathrm{Der}}(A)_{\natural} \to \underline{\mathrm{End}}(A_{\natural}) \tag{2.5.1}$$

is a differential graded Lie algebra homomorphism given by

$$\delta + \underline{\mathrm{Der}}(A)^{\natural} \longmapsto \begin{pmatrix} \rho(\delta)\colon A_{\natural} \longrightarrow A_{\natural} \\ \overline{a} \longmapsto \overline{\delta(a)} \end{pmatrix}.$$

Definition 2.5.8. A **(noncommutative) Poisson structure** on A is given by a differential graded Lie algebra structure on $A_{\natural}$, i.e.

$$\{-,-\}_{\natural}\colon A_{\natural} \times A_{\natural} \to A_{\natural}$$

such that the corresponding adjoint representation

$$\begin{aligned} \mathrm{ad}\colon A_\natural &\longrightarrow \underline{\mathrm{End}}(A_\natural) \\ \overline{a} &\longmapsto \{\overline{a}, -\}_\natural \end{aligned}$$

factors through ρ (2.5.1).

$$\begin{array}{ccc} A_\natural & \xrightarrow{\quad \mathrm{ad} \quad} & \underline{\mathrm{End}}(A_\natural) \\ & {\scriptstyle \exists\, \alpha} \searrow \quad \nearrow {\scriptstyle \rho} & \\ & \underline{\mathrm{Der}}(A_\natural) & \end{array}$$

Exercise 2.5.9. If A happens to be commtuative, then Definition 2.5.8 and Definition 2.5.1 agree.

Exercise 2.5.10. Repeat this construction for simplicial algebras $\mathbf{sAlg}_k$ so that the Dold–Kan equivalence $\mathcal{N}\colon\ \mathbf{sAlg}_k \to \mathbf{dgAlg}_k$, the two definitions agree.

Example 2.5.11 ((Kontsevich's Bracket))**.** Let $A = k\langle x_1, x_2, \ldots, x_n\rangle$ be a free algebra of rank n, defined by

$$A = \bigoplus_{j \geq 0} A^{(j)},$$

where $A^{(j)}$ is the k-linear span of words in the alphabet $\{x_1, \ldots, x_n\}$ of length j. Notice that for each $j \geq 0$, there is a cyclic operator $\tau_j\colon A^{(j)} \to A^{(j)}$ given by

$$\tau_j(v_1 \cdots v_j) = v_j v_1 \cdots v_{j-1}.$$

For example, $\tau_6(x_1^2 x_3^3 x_2) = x_2 x_1^2 x_3^3$.

Consider the normalization operator

$$N_j = 1 + \tau_j + \tau_j^2 + \ldots + \tau_j^{j-1}\colon A^{(j)} \to A^{(j)}.$$

Call a word w in A of length j **cyclic** if it is a fixed point for N_j: $N_j(w) = w$, and let

$$A^{\mathrm{cyc}} := \bigoplus_{j \geq 0} A^{\mathrm{cyc},(j)} \subseteq A$$

be the subspace spanned by the cyclic words.

Lemma 2.5.12. *The natural map $A^{\mathrm{cyc}} \hookrightarrow A \twoheadrightarrow A_\natural$ is an isomorphism of graded vector spaces.*

Fix $x = x_i$ for some $i \in \{1, \ldots, n\}$ and define the **cyclic derivative**

$$\tfrac{\partial}{\partial x}\colon A \longrightarrow A$$

by the formula

$$w = v_1 \cdots v_j \longmapsto \frac{\partial w}{\partial x} = \sum_{\substack{v_m = x \\ m \leq j}} v_{m+1} \cdots v_j v_1 \cdots v_{m-1}$$

Example 2.5.13. If $A = k\langle x_1, x_2, x_e \rangle$, then

$$\frac{\partial}{\partial x_1}\left(x_1^2 x_2 x_1 x_3\right) = x_1 x_2 x_1 x_3 + x_2 x_1 x_3 x_1 + x_3 x_1^2 x_2$$

By Lemma 2.5.12, this induces a well-defined map

$$\frac{\partial}{\partial x_i} \colon A_\natural \to A_\natural.$$

Definition 2.5.14. Now let $A = k\langle x_1, x_2 \rangle$ and define a Poisson bracket

$$\{-,-\}_\natural \colon A_\natural \times A_\natural \to A_\natural$$

by

$$\left\{\overline{a}, \overline{b}\right\}_\natural = \left(\frac{\partial a}{\partial x_1}\frac{\partial b}{\partial x_1} - \frac{\partial a}{\partial x_2}\frac{\partial b}{\partial x_1}\right) + [A, A].$$

This is called the **Kontsevich bracket**.

Theorem 2.5.15. *This defines a noncommutative Poisson structure on the free algebra $A = k\langle x_1, x_2 \rangle$.*

Remark 2.5.16. Where does this all come from? For a finite dimensional vector space V, recall that we have the representation functor

$$\begin{aligned} (-)_V \colon \mathbf{Alg}_k &\longrightarrow \mathbf{CommAlg}_k \\ A &\longmapsto \mathcal{O}(\mathbf{Rep}_V(A)) \end{aligned}$$

We can think of this functor as a realization (like geometric realization of simplicial sets). Then a noncommutative Poisson structure on A induces a unique classical Poisson structure on the representation variety $(A)_V$ for all V, and in a sense, this is the weakest structure on A that does this.

2.5.1 The category of derived Poisson algebras

Definition 2.5.17. A **morphism of Poisson differential graded algebras** is a homomorphism $f \colon A \to B$ of differential graded algebras such that $f_\natural \colon A_\natural \to B_\natural$ is a morphism of differential graded Lie algebras.

Write $\mathbf{dgAlg}_k^{\mathrm{Poi}}$ for the category of Poisson differential graded algebras.

Recall that both $\mathbf{dgAlg}_k$ and $\mathbf{dgLie}_k$ are both cofibrantly generated model categories, with weak equivalences being quasi-isomorphisms and fibrations being the (degreewise) surjective homomorphisms. Note that there are two forgetful functors $U\colon \mathbf{dgAlg}_k^{\mathrm{Poi}} \to \mathbf{dgAlg}_k$ and $(-)_\natural\colon \mathbf{dgAlg}_k^{\mathrm{Poi}} \to \mathbf{dgLie}_k$.

Definition 2.5.18. A morphism $f\colon A \to B$ of differential graded Poisson k-algebras is a **weak equivalence** if both Uf and $f_\natural$ are weak equivalences in $\mathbf{dgAlg}_k$ and $\mathbf{dgLie}_k$, respectively. Let $\mathcal{W}$ denote the class of all weak equivalences in $\mathbf{dgAlg}_k^{\mathrm{Poi}}$.

Proposition 2.5.19. $\mathbf{dgAlg}_k^{\mathrm{Poi}}$ *is a saturated homotopical category.*

Proof. It suffices to show that the class $\mathcal{W}$ of weak equivalences in $\mathbf{dgAlg}_k^{\mathrm{Poi}}$ is saturated, i.e. $\gamma(f)$ is an isomorphism in $\mathbf{Ho}(\mathbf{dgAlg}_k^{\mathrm{Poi}}) = \mathbf{dgAlg}_k^{\mathrm{Poi}}[\mathcal{W}^{-1}]$ if and only if $f \in \mathcal{W}$. Take $f\colon A \to B$ such that $\gamma(f)$ is an isomorphism in the homotopy category of $\mathbf{dgAlg}_k^{\mathrm{Poi}}$. Since U and $(-)_\natural$ preserve weak equivalences by definition, $\gamma(Uf)$ and $\gamma(f_\natural)$ are isomorphisms in $\mathbf{Ho}(\mathbf{dgAlg}_k)$ and $\mathbf{Ho}(\mathbf{dgLie}_k)$. By Quillen's theorem, Uf and $f_\natural$ are weak equivalences, which by definition means that f is a weak equivalence. □

Conjecture 2.5.20.

(a) *The category $\mathbf{dgAlg}_k^{\mathrm{Poi}}$ has a natural cofibrantly generated model structure with weak equivalences given by $\mathcal{W}$ as in Definition 2.5.18.*

(b) *This model structure should be minimal or weakest in the sense that there is a functor*

$$\mathbf{dgAlg}_k^{\mathrm{Poi}}{}_k \to \mathbf{dgAlg}_k \times^h_{\mathbf{dgCommAlg}_k} \mathbf{dgCommAlg}_k^{\mathrm{Poi}}$$

*where $\times^h$ is the **homotopy fiber product** of model categories (in the sense of Toën 2006) inducing an equivalence on the homotopy categories*

$$\mathbf{Ho}\left(\mathbf{dgAlg}_k^{\mathrm{Poi}}\right) \xrightarrow{\sim} \mathbf{Ho}\left(\mathbf{dgAlg}_k \times^h_{\mathbf{dgCommAlg}_k} \mathbf{dgCommAlg}_k^{\mathrm{Poi}}\right).$$

Definition 2.5.21 (Toën's construction)**.** Given three model categories $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ and tow functors $F_1 : \mathcal{M}_1 \to \mathcal{M}_3$ and $F_2 : \mathcal{M}_2 \to \mathcal{M}_3$.

$$\begin{array}{ccc} & & \mathcal{M}_2 \\ & & \downarrow{\scriptstyle F_2} \\ \mathcal{M}_1 & \xrightarrow{F_1} & \mathcal{M}_3 \end{array}$$

Assume that F_1, F_2 are (e.g. left) Quillen functors. Define $\mathcal{M}_1 \times_{\mathcal{M}_3} \mathcal{M}_2$ to be the category with

- objects are quintuples $(A_1, A_2, A_3, u_1, u_2)$ with $A_i \in \mathrm{Ob}(\mathcal{M}_i)$ and $u_1, u_2 \in \mathrm{Mor}(\mathcal{M}_3)$ as below:

$$F_1(A_1) \xrightarrow{u_1} A_3 \xleftarrow{u_2} F_2(A_2).$$

- morphisms $f\colon (A_1, A_2, A_3, u_1, u_2) \to (B_1, B_2, B_3, \nu_1, \nu_2)$ are triples (f_1, f_2, f_3) with $f_i\colon A_i \to B_i \in \mathrm{Mor}(\mathcal{M}_i)$ such that the diagram below commutes:

$$\begin{array}{ccccc} F_1(A_1) & \xrightarrow{u_1} & A_3 & \xleftarrow{u_2} & F_2(A_2) \\ \downarrow{\scriptstyle F(f_1)} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle F_2(f_2)} \\ F_1(B_1) & \xrightarrow{\nu_1} & B_3 & \xleftarrow{\nu_2} & F_2(B_2) \end{array}$$

Theorem 2.5.22 (Toën). *$\mathcal{M}_1 \times_{\mathcal{M}_3} \mathcal{M}_2$ is a model category with levelwise weak equivalences.*

Remark 2.5.23. Recall, for a finite dimensional vector space V, there is a functor

$$\begin{aligned} (-)_V\colon \mathbf{Alg}_k &\longrightarrow \mathbf{CommAlg}_k \\ A &\longmapsto \mathcal{O}(\mathbf{Rep}_V(A)). \end{aligned}$$

This functor is a left Quillen functor. We form the diagram

$$\begin{array}{ccc} & & \mathbf{dgCommAlg}_k^{\mathrm{Poi}} \\ & & \downarrow{\scriptstyle \text{forget}} \\ \mathbf{dgAlg}_k & \xrightarrow{(-)_V} & \mathbf{dgCommAlg}_k \end{array} \tag{2.5.2}$$

and perform Toën's construction on it to obtain a model category

$$\mathbf{dgAlg}_k \times^h_{\mathbf{dgCommAlg}_k} \mathbf{dgCommAlg}_k^{\mathrm{Poi}}.$$

In reality, we really want this to work for all such $V \cong k^n$, so we should slightly modify the functor across the bottom of (2.5.2) to be countably many functors $(-)_n = (-)_{k^n}$ instead.

We want to compare this category with the category $\mathbf{dgAlg}_k^{\mathrm{Poi}}$ we constructed by hand.

Definition 2.5.24. A **derived Poisson algebra** is an object of $\mathbf{Ho}\left(\mathbf{dgAlg}_k^{\mathrm{Poi}}\right)$.

Proposition 2.5.25. *The reduced cyclic homology $\overline{\mathrm{HC}}_\bullet(A)$ of any derived Poisson algebra carries a natural graded Lie algebra structure.*

Theorem 2.5.26. *If A is any derive Poisson algebra, then for any V, the $\mathrm{GL}_k(V)$-fixed points of representation homology*

$$\mathrm{HR}_*(A, V)^{\mathrm{GL}_k(V)}$$

carries a unique graded Poisson structure such that the derived character map

$$\mathrm{tr}_V(A)_\bullet \colon \overline{\mathrm{HC}}_\bullet(A) \to \mathrm{HR}_*(A, V)^{\mathrm{GL}_k(V)}$$

is a Lie algebra homomorphism.

Example 2.5.27. Let $\mathbf{Top}^1$ be the category of 1-connected topological spaces of **finite rational type** (i.e. $\dim H_i(X; \mathbb{Q}) < \infty$ for all i). Define the **rational homotopy category $\mathbf{Ho}(\mathbf{Top}^1)_\mathbb{Q}$** to be the localization of $\mathbf{Top}^1$ at the class of all morphisms $f \colon X \to Y$ such that $f_* \colon \pi_i(X) \otimes_\mathbb{Z} \mathbb{Q} \to \pi_i(Y) \otimes_\mathbb{Z} \mathbb{Q}$ are isomorphisms of $\mathbb{Q}$-vector spaces for all $i \geq 2$.

Theorem 2.5.28 (Quillen)**.** *There is a zig-zag of (seven) Quillen equivalence between $\mathbf{Top}^1$ and the category $\mathbf{dgLie}_\mathbb{Q}^{red}$ of **reduced** differential graded Lie algebras (i.e. $\mathfrak{a} = \bigoplus_{i>0} \mathfrak{a}_i$, $\mathfrak{a}_0 = 0$). In particular,*

$$\mathbf{Ho}(\mathbf{Top}^1)_\mathbb{Q} \simeq \mathbf{Ho}(\mathbf{dgLie}_\mathbb{Q}^{red})$$

*Consequently, for all $X \in \mathrm{Ob}(\mathbf{Top}^1)$, there is $\mathfrak{a}_X \in \mathbf{dgLie}_\mathbb{Q}^{red}$ which is a complete invariant of the rational homotopy type of X. The algebra $\mathfrak{a}_X$ is called the **Quillen model** for X, and satisfies*

$$H_*(\mathfrak{a}_X) \cong \pi_*(\Omega X) \otimes_\mathbb{Z} \mathbb{Q}.$$

Example 2.5.29. If $X = S^n$ for $n \geq 2$, the corresponding Quillen model $\mathfrak{a}_{S^n}$ is the free Lie algebra $\mathfrak{L}(x)$ on a single generator x of degree $|x| = n - 1$ with differential $d = 0$.

Remark 2.5.30. There is a **Whitehead product** on

$$\pi_*(\Omega X) \otimes_\mathbb{Z} \mathbb{Q} = \pi_{*+1}(\Omega X) \otimes_\mathbb{Z} \mathbb{Q} \cong H_*(\mathfrak{a}_X)$$

2.5.2 Cyclic Homology of Poisson Algebras

Starting with Example 2.5.2, to define a noncommutative Poisson structure, we need to replace the commutative algebra $A = k[x, y]$ with a cofibrant (or free) resolution $R = k\langle x, y, t\rangle$ with generators x, y in degree zero and t in degree 1 and relations

$$dt = [x, y], \qquad dx = dy = 0.$$

We have an acyclic fibration

$$\begin{aligned} R &\xrightarrow{\sim} A \\ x, y &\longmapsto x, y \\ t &\longmapsto 0 \end{aligned}$$

Notice that $R_0 = k\langle x, y\rangle$, so by example Definition 2.5.14, $(R_0)_\natural$ carries a Lie bracket defined in terms of cyclic derivatives (the Kontsevich bracket).

It turns out that $\{-,-\}_\natural$ can be extended to the graded setting

$$\{-,-\}_\natural \colon R_\natural \times R_\natural \to R_\natural$$

so that $\{-,-\}_\natural|_0$ is the Kontsevich bracket. This makes R into a derived Poisson algebra.

What does this induce on $\overline{HC}_*(A)$? Note that

$$\overline{HC}_*(R) \cong \overline{HA}_*(A).$$

By the HKR theorem, since A is a smooth algebra of dimension 2,

$$\overline{HC}_*(A) = \overline{HC}_0(A) \oplus \overline{HC}_1(A),$$

where

$$\begin{aligned} \overline{HC}_0(A) &= \overline{A} = {}^{A}\!/_{k \cdot 1_A} \\ \overline{HC}_1(A) &= {}^{\Omega^1(A)}\!/_{d\Omega^0(A)} = {}^{\Omega^1(A)}\!/_{dA} \end{aligned}$$

where $\Omega^\bullet A$ is the **de Rham algebra** of A:

$$\Omega^0(A) = A\Omega^1(A) \qquad = \{f\,dx + g\,dy \mid f, g, \in A\}$$

Proposition 2.5.25 implies that

$$\overline{HC}_*(A) = \overline{A} \oplus {}^{\Omega^1(A)}\!/_{dA}$$

is a graded Lie algebra with bracket

$$\{-,-\}_\natural \colon \overline{HC}_*(A) \times \overline{HC}_*(A) \to \overline{HC}_*(A).$$

This bracket has only two components. In degree zero, the bracket

$$\{-,-\}_\natural|_0 \colon \overline{HC}_0(A) \times \overline{HC}_0(A) \to \overline{HC}_0(A)$$

is given by the classical Poisson bracket:

$$\begin{aligned} \overline{A} \times \overline{A} &\longrightarrow \overline{A} \\ (\overline{f}, \overline{g}) &\longmapsto \overline{\{f, g\}}. \end{aligned}$$

A direct calculation (exercise) shows that in degree one, the bracket

$$\{-,-\}_\natural|_1\colon \mathrm{HC}_0(A) \times \mathrm{HC}_1(A) \to \mathrm{HC}_1(A)$$

is given by a Lie derivative

$$\begin{aligned} \overline{A} \times {}^{\Omega^1(A)}\!/_{dA} &\longrightarrow {}^{\Omega^1(A)}\!/_{dA} \\ (\overline{f}, \overline{\omega}) &\longmapsto \overline{\mathcal{L}_{\theta_f}(\omega)} \end{aligned}$$

where θ_f is the Hamiltonian vector field corresponding to f: θ_f is the vector field

$$\theta_f = \{f, -\} = \frac{\partial f}{\partial x}\frac{\partial}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial}{\partial x}.$$

2.5.3 Relation to string topology

Recall from Theorem 2.5.28 that for any simply connected space of finite rational type, there is a differential graded Lie algebra $\mathfrak{a}_X$ that completely determines the rational homotopy type of X. The Lie algebra $\mathfrak{a}_X$ is called the **Quillen model** for X.

Theorem 2.5.31 (Jones)**.**

$$\overline{\mathrm{HC}_*}(\mathcal{U}\mathfrak{a}_X) \cong \overline{\mathrm{H}}_*^{S^1}(\mathcal{L}X; \mathbb{Q}).$$

where $\mathcal{L}X = \mathrm{Map}(S^1, X)$ *is the free loop space on* X *and*

$$\mathrm{H}_*^{S^1}(\mathcal{L}X; \mathbb{Q}) = \mathrm{H}_*(ES^1 \times_{S^1} \mathcal{L}X).$$

The reduced equivariant homology $\overline{\mathrm{H}}_*^{S^1}$ *is the kernel of the natural fibration*

$$X \simeq ES^1 \times \mathcal{L}X \to ES^1 \times_{S^1} \mathcal{L}X \to ES^1 \times_{S^1} \{*\} \simeq BS^1$$

Proposition 2.5.32. *Let* X *be a simply connected closed orientable manifold of dimension* $d \geq 2$*. Then* $\mathcal{U}\mathfrak{a}_X$ *has a natural derived Poisson structure.*

Proof Sketch. A theorem of Lambrechts–Stanley (2008) says that there is a finite dimensional (over $\mathbb{Q}$) cochain commutative differential graded algebra $\mathcal{A}$, such that $A \simeq C^*(X, \mathbb{Q})$, where $C^*(X, \mathbb{Q})$ is the differential graded algebra of singular cochains in X with the usual cup product. It comes with a cyclic pairing from Poincaré duality:

$$\langle -, - \rangle\colon \mathcal{A} \otimes \mathcal{A} \to \mathbb{Q}$$

of (cohomological) degree $n = -d$. Cyclic means that

$$\langle a, bc \rangle = \pm \langle ca, b \rangle$$

for all $a, b, c \in A$, with the sign coming from the Koszul sign rule.

Take $C = \mathcal{A}^* = \mathrm{Hom}_{\mathbb{Q}}(\mathcal{A}, \mathbb{Q})$. Then, since $\mathcal{A}$ is finite dimensional, this is a cocommutative coassociative differential graded coalgebra with coproduct dual to the product on $\mathcal{A}$. Then $C \simeq C_*(X, \mathbb{Q})$, where $C_*(X, \mathbb{Q})$ is the singular chains on X. This is equipped with a cyclic pairing $C \otimes_{\mathbb{Q}} C \to \mathbb{Q}$ of homological degree $n = -d$.

Then $\mathcal{U}\mathfrak{a}_X \cong \Omega(C)$ is the algebraic cobar construction on C, i.e. the tensor algebra $(T_{\mathbb{Q}}(C[-1]), d)$ on C shifted by -1. The differential d comes from $\Delta\colon C \to C \otimes C$ and d_C.

The construction with cyclic derivatives can be generalized to the graded free algebra $R = C[-1]$, depending on $\langle -, - \rangle$ on C, and compatible with d_R. This gives a natural Poisson bracket on $R_\natural$, making $\mathcal{U}\mathfrak{a}_X$ a derived Poisson algebra. □

Theorem 2.5.33. *Under the Jones isomorphism*

$$\overline{\mathrm{HC}_*}(\mathcal{U}\mathfrak{a}_X) \cong \overline{\mathrm{H}}_*^{S^1}(\mathcal{L}X; \mathbb{Q}).$$

the graded Lie algebra structure on $\overline{\mathrm{HC}}_*(\mathcal{U}\mathfrak{a}_X)$ *corresponds exactly to the Chas–Sullivan ("string topology") Lie algebra structure on* $\mathrm{H}_*^{S^1}(\mathcal{L}X, \mathbb{Q})$.

Remark 2.5.34. The Chas–Sullivan string topology Lie algebra structure is defined purely geometrically in terms of transverse intersection product of chains on X. The theorem gives an algebraic way to define the Chas–Sullivan structure.

Question 2.5.35. Note that there is a Hodge decomposition on $\overline{\mathrm{HC}}_*(\mathcal{U}\mathfrak{a}_X)$ corresponding to

$$\mathrm{H}_*^{S^1}(\mathcal{L}X, \mathbb{Q}) \cong \bigoplus_{p \geq 1} \mathrm{H}_*^{S^1,(p-1)}(X, \mathbb{Q}).$$

where $\mathrm{H}_*^{S^1,(p-1)}(\mathcal{L}X, \mathbb{Q})$ are the eigenspaces of endomorphisms coming from $S^1 \to S^1, e^{i\theta} \mapsto e^{in\theta}$. Does the Chas–Sullivan bracket preserve this Hodge grading?

See the reference "Hodge decompositions and derived Poisson brackets," Selecta Math (2017) and references therein.

2.6 Derived Functors

Let $(\mathcal{M}, \mathcal{W}_1)$ and $(\mathcal{N}, \mathcal{W}_2)$ be homotopical categories.

Definition 2.6.1. A functor $F\colon \mathcal{M} \to \mathcal{N}$ is **homotopical** if $F(\mathcal{W}_1) \subseteq \mathcal{W}_2$. By the universal mapping property of localization, it induces a diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{F} & \mathcal{N} \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} \\ \mathbf{Ho}(\mathcal{M}) & \xrightarrow{\overline{F}} & \mathbf{Ho}(\mathcal{N}) \end{array}$$

Remark 2.6.2. A functor F is homotopical if and only if δF is homotopical when $\mathbf{Ho}(\mathcal{M})$ is viewed as a homotopical category with $\mathcal{W}_{\mathbf{Ho}(\mathcal{M})} = \mathrm{Iso}(\mathbf{Ho}(\mathcal{M}))$.

Example 2.6.3. If $\mathcal{A}, \mathcal{B}$ are abelian categories, $\mathcal{M} = \mathbf{Ch}^+(\mathcal{A})$, $\mathcal{N} = \mathbf{Ch}^+(\mathcal{B})$. For any additive functor $F\colon \mathcal{A} \to \mathcal{B}$, we can extend to a functor

$$F_\bullet\colon \mathbf{Ch}^+(\mathcal{A}) \to \mathbf{Ch}^+(\mathcal{B}).$$

If we take weak equivalences to be chain homotopy equivalences (i.e. $f\colon X_0 \to Y_0$ such that there exists $g\colon Y_0 \to X_0$ with $fg \sim \mathrm{id}_Y$ and $gf \sim \mathrm{id}_X$), then $F_\bullet$ is homotopical.

If we choose weak equivalences to be quasi-isomorphisms, then $F_\bullet$ is usually *not* homotopical (unless it's exact).

Example 2.6.4. Let $\mathcal{M}$ be a homotopical category and let $\mathcal{C}$ be a small category. Then $\mathcal{M}^{\mathcal{C}}$ is homotopical with objectwise weak equivalences. In general,

$$\mathrm{colim}_{\mathcal{C}}\colon \mathcal{M}^{\mathcal{C}} \to \mathcal{M}$$

is not homotopical.

When $\mathcal{M} = \mathbf{Top}$, and $\mathcal{C} = (\bullet \leftarrow \bullet \rightarrow \bullet)$, then we have two objects $F, F' \in \mathrm{Ob}(\mathcal{M}^{\mathcal{C}})$ as below:

$$\begin{array}{lccccc} F\colon & D^n & \hookleftarrow & S^{n-1} & \hookrightarrow & D^n \\ F'\colon & * & \longleftarrow & S^{n-1} & \longrightarrow & * \end{array}$$

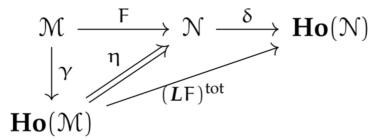
The map $\alpha\colon F \implies F'$ from contracting each D^n to a point is an objectwise weak equivalence,

$$\begin{array}{lccccc} F\colon & D^n & \hookleftarrow & S^{n-1} & \hookrightarrow & D^n \\ \Downarrow{\scriptstyle\alpha} & \downarrow{\scriptstyle\simeq} & & \| & & \downarrow{\scriptstyle\simeq} \\ F'\colon & * & \longleftarrow & S^{n-1} & \longrightarrow & * \end{array}$$

Yet $\mathrm{colim}_{\mathcal{C}}(F) = S^n$ while $\mathrm{colim}_{\mathcal{C}}(F') = *$ are not weak equivalent.

It is a philosophy of homological algebra to replace nonhomotopical functors with a universal homotopical approximation.

Definition 2.6.5 (Quillen)**.** A **total right derived functor** of $F\colon \mathcal{M} \to \mathcal{N}$ is defined by the *right* Kan extension $(\boldsymbol{L}F)^{\mathrm{tot}} = \mathrm{Ran}_{\gamma}(\delta F)$.



This comes with a natural transformation $\eta\colon \mathbb{L}F \circ \gamma \implies \delta F$, as the Right Kan extension.

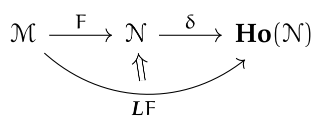
Definition 2.6.6 (DHKS)**.** A **left derived functor** of $F\colon \mathcal{M} \to \mathcal{N}$ is a functor $\boldsymbol{L}F\colon \mathcal{M} \to \mathbf{Ho}(\mathcal{N})$ given together with a comparison morphism

$$\eta\colon \mathbb{L}F \implies \delta F$$

such that

(a) $\boldsymbol{L}F$ is homotopical

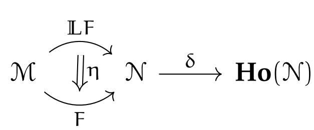
(b) η is terminal among all homotopical functors $G\colon \mathcal{M} \to \mathbf{Ho}(\mathcal{N})$ with a natural transformation $\rho\colon G \implies \delta F$.



Note that by the universal property of localization $\gamma\colon \mathcal{M} \to \mathbf{Ho}(\mathcal{N})$, giving $\boldsymbol{L}F$ is equivalent to giving a total left derived functor $(\boldsymbol{L}F)^{\mathrm{tot}}$.

It's convenient – although not always possible – to "lift" $\boldsymbol{L}F$ to the level of homotopical categories.

Definition 2.6.7. A **pointwise left derived functor** of $F\colon \mathcal{M} \to \mathcal{N}$ is a functor $\mathbb{L}F\colon \mathcal{M} \to \mathcal{N}$ given together with a natural transformation $\mathbb{L}F \implies F$ such that the horizontal composition (pictured below) $\delta \cdot \eta\colon \delta\mathbb{L}F \implies \delta F$ is a derived functor in the sense of Definition 2.6.6.



Remark 2.6.8. $\mathbb{L}F$ may not always exist, and if it does, it's defined only up to homotopy.

2.6.1 Derived functors via deformations

How do we construct derived functors? The idea is that if $F\colon \mathcal{M} \to \mathcal{N}$ is not homotopical, we can often restrict F to a subcategory $\mathcal{M}_Q$ of $\mathcal{M}$ consisting of "good" objects (we say objects "adjusted for F") in the sense that $F\colon \mathcal{M}_Q \hookrightarrow \mathcal{M} \to \mathcal{N}$ becomes homotopical.

Definition 2.6.9. Let $(\mathcal{M}, \mathcal{W})$ be a homotopical category. A **left deformation** of $\mathcal{M}$ is a pair (Q, q) of a functor $Q\colon \mathcal{M} \to \mathcal{M}$ and a natural weak equivalence $q\colon Q \implies \mathrm{id}_{\mathcal{M}}$, i.e. for all $X \in \mathrm{Ob}(\mathcal{M})$,

$$(q_X\colon QX \to X) \in \mathcal{W}.$$

Note that if (Q, q) is a left deformation of $\mathcal{M}$, then Q is automatically a homotopical functor: indeed, for all $f\colon X \to Y$ in $\mathcal{M}$, then there is a commutative diagram

$$\begin{array}{ccc} QX & \xrightarrow{Qf} & QY \\ {\scriptstyle q_X}\downarrow{\scriptstyle \simeq} & & {\scriptstyle q_Y}\downarrow{\scriptstyle \simeq} \\ X & \xrightarrow{f} & Y \end{array}$$

If $f \in \mathcal{W}$, then $f \circ q_X \in \mathcal{W}$, so $q_Y \circ Qf \in \mathcal{W}$ as well. Then the two-out-of-three property implies $Qf \in \mathcal{W}$.

Definition 2.6.10. Given a left deformation (Q, q) of $\mathcal{M}$, call any full subcategory $\mathcal{M}_Q \subseteq \mathcal{M}$ containing the image of Q a **left deformation retract** of $\mathcal{M}$ with respect to (Q, q). It is a homotopical category with weak equivalences $\mathcal{W} \cap \mathcal{M}_Q$.

Lemma 2.6.11. *If $\mathcal{M}_Q$ is any left deformation retract of $\mathcal{M}$, then $i\colon \mathcal{M}_Q \to \mathcal{M}$ induces an equivalence of categories $\mathbf{Ho}(\mathcal{M}_Q) \simeq \mathbf{Ho}(\mathcal{M})$.*

Proof. We have two functors $i\colon \mathcal{M}_Q \leftrightarrows \mathcal{M}\colon Q$, which are both homotopical, and hence induce functors $i\colon \mathbf{Ho}(\mathcal{M}_Q) \leftrightarrows \mathbf{Ho}(\mathcal{M})\colon Q$ which are inverse equivalences. □

Example 2.6.12. If $\mathcal{M}$ is a cofibrantly generated model category, then the Quillen small object argument implies that cofibration/trivial fibration factorization (MC5) may be made functorial, there is a functor $Q\colon \mathcal{M} \to \mathcal{M}$ with $q\colon Q \implies \mathrm{id}_{\mathcal{M}}$ such that for all $X \in \mathrm{Ob}(\mathcal{M})$, there is a factorization of $\varnothing \hookrightarrow X$ as

$$\varnothing \hookrightarrow QX \xrightarrow[\sim]{q_X} X$$

where QX is a functorial cofibrant replacement for X. Then (Q, q) is a left deformation of $\mathcal{M}$, and $\mathcal{M}_Q$ may be taken to be the full subcategory of cofibrant objects.

Definition 2.6.13. Given $F\colon \mathcal{M} \xrightarrow{\sim} \mathcal{N}$, a **left** F**-deformation** is a left deformation (Q, q) such that $F|_{\mathcal{M}_Q}\colon \mathcal{M}_Q \xrightarrow{\sim} \mathcal{M} \xrightarrow{\sim} \mathcal{N}$ is homotopical. If such a left deformation exists for F, then F is called **left deformable.**

Exercise 2.6.14. A left deformation (Q, q) is an F-deformation if and only if

(a) FQ is homotopical, and

(b) $FqQ\colon FQ^2 \implies FQ$ is a natural equivalence of functors. (*Hint: use the two-out-of-three property.*)

Exercise 2.6.15. Any left deformable F has a maximal subcategory $\mathcal{M}_F$ of $\mathcal{M}$ such that $F|_{\mathcal{M}_F}$ is homotopical. (*Hint: use the two-out-of-six property.*)

Theorem 2.6.16 (DHKS)**.** *If* $F\colon \mathcal{M} \xrightarrow{\sim} \mathcal{N}$ *admits a left deformation* (Q, q)*, then*

$$\boldsymbol{L}F := \delta FQ\colon \mathcal{M} \xrightarrow{Q} \mathcal{M} \xrightarrow{F} \mathcal{N} \xrightarrow{\delta} \mathbf{Ho}(\mathcal{N})$$

together with $\eta = \delta Fq\colon \delta FQ \implies \delta F$ *defines a left derived functor of* F*.*

Remark 2.6.17. $\mathbb{L}F := FQ$ is a pointwise left-derived functor.

Proof. We need to check conditions (a) and (b) of Definition 2.6.6.

(a) This one is easy. $\boldsymbol{L}F$ can be factored as

$$\boldsymbol{L}F = \delta \circ F|_{\mathcal{M}_Q} \circ Q,$$

and each is homotopical, so the composition is homotopical as well.

$$\begin{array}{ccccccc} \mathcal{M} & \xrightarrow{Q} & \mathcal{M} & \xrightarrow{F} & \mathcal{N} & \xrightarrow{\delta} & \mathbf{Ho}(\mathcal{N}) \\ & {\scriptstyle Q}\searrow & \uparrow{\scriptstyle i} & \nearrow{\scriptstyle F} & & & \\ & & \mathcal{M}_Q & & & & \end{array}$$

(b) Take any $\rho\colon G \implies \delta F$, with $G\colon \mathcal{M} \to \mathbf{Ho}(\mathcal{N})$ a homotopical functor. Claim that because G is homotopical and q is a weak equivalence, $Gq\colon GQ \implies G$ is an isomorphism of functors.

Indeed, for all $X \in \mathrm{Ob}(\mathcal{M})$, $q_X\colon Q(X) \xrightarrow{\sim} X$ is a weak equivalence. Then because G is a homotopical functor, $G(q_X)\colon GQ(X) \xrightarrow{\sim} G(X)$ is a weak equivalence in $\mathbf{Ho}(\mathcal{N})$, so $G(q_X)\colon GQ(X) \cong G(X)$ is an isomorphism. These are exactly the components of the natural transformation Gq. Therefore, $Gq\colon GQ \implies G$ is a natural isomorphism of functors. Hence, there is an inverse $(Gq)^{-1}\colon G \implies GQ$.

So we can factor ρ as

$$G \xRightarrow{(Gq)^{-1}} QG \xRightarrow{\rho Q} \delta FQ = \boldsymbol{L}F \xRightarrow{\eta = \delta F q} \delta F$$

that is, $\rho = \eta \circ \zeta$, where $\zeta = \rho Q \cdot (Gq)^{-1}$.

It remains to show that ζ is unique. Take any other factorization $\widetilde{\zeta}$

$$\begin{array}{ccc} G & \xRightarrow{\widetilde{\zeta}} & \delta FQ \\ & {\scriptstyle \rho}\searrow & \Downarrow {\scriptstyle \eta = \delta F q} \\ & & \delta F \end{array}$$

Notice that this yields a diagram

$$\begin{array}{ccc} GQ & \xRightarrow{\widetilde{\zeta}Q} & \delta FQ^2 \\ & {\scriptstyle \rho Q}\searrow & \Downarrow {\scriptstyle \eta Q = \delta F q Q} \\ & & \delta FQ \end{array} \tag{2.6.1}$$

Since $F|_{\mathcal{M}_Q}$ is homotopical, then ηQ is a natural weak equivalence and hence an isomorphism of functors (because $\mathbf{Ho}(\mathcal{N})$ is minimal). This means that ηQ is invertible, so $\widetilde{\zeta}Q = (\eta Q)^{-1} \circ \rho Q$ is uniquely determined by ρ.

Now by naturality, we can complete the triangle (2.6.1) to a diagram

$$\begin{array}{ccc} GQ & \xRightarrow{\widetilde{\zeta}Q} & \delta FQ^2 \\ \Downarrow {\scriptstyle Gq} & & \Downarrow {\scriptstyle \eta Q = \delta F q Q} \\ G & \xRightarrow{\widetilde{\zeta}} & \delta FQ \end{array}$$

in which the vertical arrows are isomorphisms, so $\widetilde{\zeta}$ is uniquely determined by $\widetilde{\zeta}Q$, and hence by ρ. □

Remark 2.6.18. The argument above also shows that $\mathbb{L}F = FQ \colon \mathcal{M} \to \mathcal{N}$ is a pointwise derived functor in the sense of Definition 2.6.7.

Remark 2.6.19. Consider the 2-category $\mathbf{HomCat}^L$ whose objects are 4-tuples $(\mathcal{M}, \mathcal{M}_Q, Q, q)$ of a homotopical category $\mathcal{M}$, a left-deformation (Q, q), and a left deformation retract $\mathcal{M}_Q$ of $\mathcal{M}$. The 1-cells of this 2-category are deformable functors $F \colon \mathcal{M} \to \mathcal{M}'$ taking $\mathcal{M}_Q \to \mathcal{M}'_{Q'}$, and the 2-cells are ordinary natural transformations.

There is a pseudofunctor $\mathbb{L} \colon \mathbf{HomCat}^L \to \mathbf{Cat}$ taking objects $(\mathcal{M}, \mathcal{M}_Q, Q, q)$ to $\mathbf{Ho}(\mathcal{M})$, taking 1-cells $F \colon \mathcal{M} \to \mathcal{M}'$ to $\boldsymbol{L}F \colon \mathbf{Ho}(\mathcal{M}) \to \mathbf{Ho}(\mathcal{M}')$, and taking 2-cells α to $\mathbb{L}\alpha \colon \mathbb{L}F \implies \mathbb{L}F'$.

The proof is similar to Hovey's Theorem 1.3.7.9 for left Quillen functors in the case of model categories. The important point is that given $F\colon \mathcal{M} \to \mathcal{M}'$ and $G\colon \mathcal{M}' \to \mathcal{M}''$, then $\mathbb{L}F := FQ$ and $\mathbb{L}G = GQ'$ compose to

$$\mathbb{L}G \circ \mathbb{L}F = GQ' \circ FQ \xrightarrow{Gq'} GFQ = \mathbb{L}(GF).$$

Such a map exists because F sends $\mathcal{M}_Q$ to $\mathcal{M}'_{Q'}$ and G is homotopical on $\mathcal{M}'_{Q'}$. Applying δ, we get an isomorphism of total derived functors:

$$\boldsymbol{L}G \circ \boldsymbol{L}F \cong \boldsymbol{L}(GF).$$

Concisely, $\boldsymbol{L}$ preserves composition.

This is in general not true – we must assume that F sends $\mathcal{M}_Q$ to $\mathcal{M}'_{Q'}$ and G is homotopical on $\mathcal{M}'_{Q'}$.

Proposition 2.6.20 (Maltsionitis 2007)**.** *If F is left deformable, then the total derived functor*

$$\boldsymbol{L}F = \mathrm{Ran}_\gamma(\delta F)\colon \mathbf{Ho}(\mathcal{M}) \to \mathbf{Ho}(\mathcal{N})$$

is a pointwise (in fact, absolute) right Kan extension. That is, $\boldsymbol{L}F$ can be computed as a limit in $\mathbf{Ho}(\mathcal{N})$ even though $\mathbf{Ho}(\mathcal{N})$ is not complete:

$$\boldsymbol{L}F = \mathrm{Ran}_\gamma(\delta F) \cong \lim\left(\gamma_X \backslash \gamma \xrightarrow{u} \mathcal{M} \xrightarrow{\delta F} \mathbf{Ho}(\mathcal{N})\right).$$

We could instead consider **right deformations** (R, r) of a functor $R\colon M \to M$ and natural weak equivalence $r\colon \mathrm{id}_{\mathcal{M}} \implies R$, and arrive at a theory of right derived functors.

Definition 2.6.21. A pair of adjoint functors $F\colon \mathcal{M} \leftrightarrows \mathcal{N}\colon G$ is **deformable** if F is left deformable and G is right deformable.

Theorem 2.6.22 (DHKS)**.** *If $F\colon \mathcal{M} \leftrightarrows \mathcal{N}\colon G$ is a deformable adjoint pair, then $\boldsymbol{L}F$ and $\boldsymbol{R}G$ exist and $\boldsymbol{L}F\colon \mathbf{Ho}(\mathcal{M}) \leftrightarrows \mathbf{Ho}(\mathcal{N})\colon \boldsymbol{R}G$ form an adjunction between the homotopy categories.*

Example 2.6.23. All Quillen pairs between model categories are examples of such a phenomenon.

However, there are some functors that are not Quillen functors, yet are left deformable.

2.6.2 Classical derived functors

Let A be an associative, unital ring or k-algebra. Let $\mathcal{M} = \mathbf{Ch}(A)_{\geq 0}$ be the category of chain complexes of left A-modules. Let $\mathcal{W}$ be the class of quasi-isomorphisms of chain complexes. Then $(\mathcal{M}, \mathcal{W})$ is a homotopical category.

We want to construct a left deformation $Q\colon \mathcal{M} \to \mathcal{M}$ with $q\colon Q \implies \mathrm{id}_{\mathcal{M}}$ which is adjusted for any additive functor on $\mathcal{M}$. This is given by the classical 2-sided bar construction. We follow Quillen's approach.

Let E be an A-bimodule with an A-bimodule homomorphism $\varepsilon\colon E \to A$. We define a differential graded algebra as follows: take the tensor algebra

$$T_A(E) = \bigoplus_{n \geq 0} E \otimes_A E \otimes_A \cdots \otimes_A E$$

with differential $d\colon T_A(E) \to T_A(E)$ extending the homomorphism $\varepsilon\colon E \to A$:

$$d(z_1, \ldots, z_n) = \sum_{i=1}^{n} (-1)^{i-1} (z_1, \ldots, z_{i-1} \cdot \varepsilon(z_i), z_{i+1}, \ldots, z_n),$$

where we denote an element of $E^{\otimes_A n}$ by $(z_1, \ldots, z_n)$. Here, we use the natural isomorphism $E \otimes_A A \otimes_A E \cong E \otimes_A E$ via

$$(x, a, y) \longmapsto (xa, y) = (x, ay).$$

Remark 2.6.24. The tensor algebra $T_A(E)$ has the following universal property: to give a homomorphism of A-algebras $f = (f_0, f_1)\colon T_A(E) \to B$ is equivalent to giving an algebra homomorphism $f_0\colon A \to B$ and an A-bimodule homomorphism $f_1\colon E \to {}_A B_A$, where B is an A-algebra via f_0.

In fact, it is the left adjoint in an adjunction

$$T_A\colon \mathbf{Bimod}(A) \rightleftarrows A \downarrow \mathbf{Alg}\colon U$$

$${}_f B_f \longleftarrow\!\shortmid (A \xrightarrow{f} B)$$

$$E \longmapsto (A \to T_A(E))$$

Lemma 2.6.25 (Quillen)**.** *If $\varepsilon\colon E \to A$ is surjective, then $(T_A(E), d)$ is acyclic (i.e. $H_*(T_A(E), d) = 0$).*

Proof. Fix $z \in E \subset T_A(E)$ of degree 1 such that $dz = \varepsilon(z) = 1$. Then for all $a \in T_A(E)$, we have by the Leibniz rule

$$d(za) = dz \cdot a + (-1)^{|z|} z \cdot da = a - z \cdot da$$

Hence, $a = d(za) + z(da)$. Hence, the homomorphism $a \mapsto z \cdot a$ is a contracting homotopy on $T_A(E)$ as a complex of right A-modules.

Likewise,

$$d(az) = da \cdot z + (-1)^{|a|} a \cdot dz = da \cdot z + (-1)^{|a|} a$$

shows that $a \mapsto (-1)^{|a|} az$ is a contracting homotopy on $T_A(E)$ as a complex of left A-modules. □

Corollary 2.6.26. *Any projective* A*-bimodule* E *with surjective bimodule homomorphism* $\varepsilon \colon E \to A$ *gives a projective resolution* $T_A(E) \to A$ *of* A *as* A*-bimodules:*

$$\cdots \to E \otimes_A E \otimes_A E \xrightarrow{d} E \otimes_A E \xrightarrow{d} E \xrightarrow{\varepsilon} A.$$

Example 2.6.27. Let $E = A \otimes A$, and take $\varepsilon = m \colon A \otimes A \to A, (a, b) \mapsto ab$. In this case, we recover the classical **bar construction**

$$BA_* = (T_A(A \otimes A)_+, b)$$

where the differential is denoted by b

$$b(a_0, \ldots, a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0, \ldots, a_i a_{i+1}, \ldots, a_n).$$

Remark 2.6.28. Quillen's main observation was that to build a cyclic homology theory for algebras, we can start with a projective resolution of A of the form $(T_A(E), d)$ associated to $\varepsilon \colon E \to A$.

Remark 2.6.29. Notice that $(T_A(E), d)$ is the homotopy colimit of the diagram

$$A \xleftarrow{(\mathrm{id},\varepsilon)} T_A(E)_\bullet \xrightarrow{(\mathrm{id},0)} A,$$

where $(\mathrm{id}, \varepsilon)$ is the homomorphism coming from the universal property of the tensor algebra. We think of ε as a deformation parameter.

Example 2.6.30. Let $X = \mathbb{A}^1_{\mathbb{C}}$, and let $A = \mathcal{O}(X) \cong k[q]$. Then $\mathcal{O}(T^*X) \cong k[p, q]$.

$$\mathcal{O}(T^*X) = \operatorname{hocolim}\left(A \xleftarrow{(\mathrm{id},0)} T_A(E) \xrightarrow{(\mathrm{id},0)} A \right)$$

where $E = \mathbb{R}\operatorname{Hom}_{A^e}(A, A^e) = A^!$. Homomorphisms $\varepsilon \colon E \to A$ in $\mathcal{D}^b(A^e)$ are given by

$$\varepsilon \in \operatorname{Hom}_{\mathcal{D}^b(A^e)}(A^!, A) \cong H^0(\mathbb{R}\operatorname{Hom}_{A^e}(\mathbb{R}\operatorname{Hom}_{A^e}(A, A^e), A)) = H_0(A^{!!} \otimes^{\mathbb{L}}_{A^e} A) \cong HH_0(A) \cong A.$$

So the deformation parameters ε are exactly elements of the zeroth Hochschild homology, which in this case is just A itself.

Proposition 2.6.31. *The functor* $Q = BA \otimes_A - \colon \mathbf{Ch}(A)_{\geq 0} \to \mathbf{Ch}(A)_{\geq 0}$ *with* $q_M \colon B_A \otimes_A M \xrightarrow{\varepsilon_*} A \otimes_A M \cong M$ *is a left deformation of* $\mathcal{M} = \mathbf{Ch}(A)_{\geq 0}$ *adjusted for any additive functor.*

Proof. This is essentially all classical results from homological algebra.

(1) q is a natural quasi-isomorphism because $T_A(A \otimes A) \otimes_A M$ is acyclic.

(2) $QM = BA \otimes_A M$ is a complex of free left A-modules, and hence projective. Take $\mathcal{M}_Q$ to be the full subcategory of $\mathcal{M}$ of those chain complexes which have projective terms. Then $\mathcal{M}_Q$ contains the image of Q.

(3) Any additive functor $F\colon \mathbf{Mod}(A) \to \mathbf{Mod}(B)$ and its degreewise extension $F\colon \mathbf{Ch}(A)_{\geq 0} \to \mathbf{Ch}(B)_{\geq 0}$ preserves homotopy equivalences. So since any quasi-isomorphism between projective (nonnegatively graded) complexes is a homotopy equivalence, $F|_{\mathcal{M}_Q}$ is a homotopical functor.

□

2.7 Simplicial groups and spaces

Let $\mathbf{sGr} = \mathbf{Fun}(\Delta^{op}, \mathbf{Gr})$ be the category of simplicial groups. Let

$$G_* = \{G_n\}_{n\geq 0}$$

be a simplicial group and let X_* be a simplicial set.

Definition 2.7.1. A **twisting function** $\tau\colon X_* \to G_{*-1}$ is a family of maps $\tau_n\colon X_n \to G_{n-1}$ for all $n \geq 1$ satisfying the following compatibility conditions:

$$\begin{aligned} d_i(\tau(x)) &= \tau(d_{i+1}(x)) && (i \geq 0) \\ s_j(\tau(x)) &= \tau(s_{j+1}(x)) && (j \geq 0) \\ \tau(s_0(x)) &= 1_{G_n} && (x \in X_n) \end{aligned}$$

Definition 2.7.2. A **twisted Cartesian product** with **fiber** G_* and **base** X_* and twisting function $\tau\colon X_* \to G_{*-1}$ is a simplicial set $E_* := G_* \times_\tau X_*$ with

$$E_n = G_n \times X_n$$

for all $n \geq 0$ and face maps

$$d_i(g,x) := \begin{cases} (\tau(x)\cdot d_0(g), d_0(x)) & (i = 0) \\ (d_i(g), g_i(x)) & (i > 0) \end{cases}$$

and degeneracies

$$s_j(g,x) = (s_j(g), s_j(x))$$

Definition 2.7.3. Let $p\colon E_* \to X_*$ be a principal G_*-fibration. A **local cross-section** $\sigma\colon X_* \to E_*$ of p is a degreewise section of p, $p_n\sigma_n = \mathrm{id}_X$, commuting with all faces except in degree zero and all degeneracies: $d_i\sigma = \sigma d_i$ for all $i > 0$ and $s_j\sigma = \sigma s_j$ for all $j \geq 0$.

?

Proposition 2.7.4. *Any principal G_*-fibration $p\colon E_* \to X_*$ with a right G_*-action on E_* with local cross-section $\sigma\colon X_* \to E_*$ can be identified with a twisted Cartesian product $G \times_\tau X$, where $\tau\colon X_* \to G_{*-1}$ is determined by*

$$d_0\sigma(x) = \sigma(d_0 x)\tau(x).$$

Definition 2.7.5. The **classifying space of a simplicial group** is the reduced simplicial set $\overline{W}(G_*)$ with

$$\begin{aligned} \overline{W}(G)_0 &= \{*\} \\ \overline{W}(G)_1 &= G_0 \\ \overline{W}(G)_n &= G_{n-1} \times G_{n-2} \times \cdots \times G_0 \qquad\qquad (n > 0) \end{aligned}$$

with faces

$$d_0 = d_1 \colon \overline{W}(G)_1 \to \overline{W}(G)_0, g \mapsto *,$$

$$\begin{aligned} d_0(g_{n-1}, \ldots, g_0) &= (g_{n-2}, \ldots, g_0) \\ d_{i+1}(g_{n-1}, \ldots, g_0) &= (d_i g_{n-1}, \ldots, d_1 g_{n-i}, g_{n-i-2} \cdot d_0 g_{n-i-1}, g_{n_i-3}, \ldots, g_0) \end{aligned}$$

and degeneracies

$$s_0 \colon \overline{W}(G)_0 \to \overline{W}(G)_1, * \mapsto 1_{G_0}$$

$$\begin{aligned} s_0(g_{n-1}, \ldots, g_0) &= (1, g_{n-1}, \ldots, g_0) \\ s_{j+1}(g_{n-1}, \ldots, g_0) &= (s_j(g_{n-1}), \ldots, s_0(g_{n-j-1}), 1, g_{n-i-2}, \ldots, g_0) \end{aligned}$$

This is a simplicial set coming with a universal twisting function $\tau(G) \colon \overline{W}(G)_n \to G_{n-1}$ given by

$$(g_{n-1}, \ldots, g_0) \mapsto g_{n-1}.$$

The next lemma explains why $\overline{W}(G)$ is called a classifying space.

Lemma 2.7.6. $\tau(G)$ *is a universal twisting function in the sense that any (principal) twisted product $G \times_\tau X$ can be induced from $G_* \times_{\tau(G)} \overline{W}(G)$ by the unique classifying map $X_* \to \overline{W}(G)_*$ given by*

$$X \ni x_n \longmapsto (\tau(x), \tau(d_0 x), \ldots, \tau(d_0^{n-1} x)) \in \overline{W}(G)_n.$$

Example 2.7.7. If $G = \{G\}_{n \geq 0}$ is a discrete simplicial group $\overline{W}(G) = B_* G$ is the simplicial nerve of G. Then

$$G \times_{\tau(G)} \overline{W}(G) \cong E_* G,$$

the isomorphism given in degree n by

$$\begin{array}{ccc} G^{n+1} & \longleftarrow & E_n(G) = G^{n+1} \\ (g_0, g_0 g_1, \ldots, g_0 g_1 \cdots g_n) & \longleftarrow\!\shortmid & (g_0, \ldots, g_n) \end{array}$$

Definition 2.7.8. Given a simplicial set X_*, define the Kan loop group of X, $\mathbb{G}(X)_* \in \mathrm{Ob}(\mathbf{sGr})$ with

$$\mathbb{G}(X)_n = \mathbb{F}\langle X\rangle \big/ \langle s_0(x) = 1, \forall x \in X_n\rangle \cong \mathbb{F}\langle B_n\rangle,$$

where $B_n = X_{n+1} \setminus s_0(X_n)$, and $\mathbb{F}\langle Y\rangle$ denotes the free group on the set Y. (Note that the B_n do not form a simplicial set!)

Define the faces and degeneracies by defining them on the generating sets

$$d_i^{\mathbb{G}}(x) = \begin{cases} d_1(x) \cdot d_0(x)^{-1} & (i = 0) \\ d_{i+1}(x) & (i > 0) \end{cases}$$

$$s_j^{\mathbb{G}}(x) = s_{j+1}(x) \qquad (j \geq 0)$$

Now define $\tau(X)\colon X_* \to \mathbb{G}(X)_{*-1}$ by the composite

$$\tau_n(X)\colon X_n \hookrightarrow \mathbb{F}\langle X_n\rangle \twoheadrightarrow \mathbb{G}(X)_{n-1}.$$

Definition 2.7.9. Given any simplicial set X_* and any simplicial group G_*, define $\mathrm{Tw}(X_*, G_*)$ the set of all twisting functions $\tau\colon X_* \to G_{*-1}$.

Theorem 2.7.10. *There are natural bijections*

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathbf{sGr}}(\mathbb{G}(X)_*, G_*) & \xrightarrow{\cong} & \mathrm{Tw}(X_*, G_*) & \xleftarrow[\cong]{} & \mathrm{Hom}_{\mathbf{sSet}}(X_*, \overline{W}(G)_*) \\ f & \longmapsto & f \circ \tau(X)_* & & \\ & & \tau(G)_* \circ f & \longleftarrow\!\shortmid & f \end{array}$$

Hence, we have an adjunction

$$\mathbb{G}\colon \mathbf{sSet}_0 \leftrightarrows\colon \mathbf{sGr}\colon \overline{W}.$$

Corollary 2.7.11. *There is a natural bijection between the set of twisting functions* $\mathrm{Tw}(X_*, G_*)$ *and the isomorphism classes of pairs* (E_*, G_*)*, where* E_* *is a principal* G_**-bundle over* X_* *with local section* $\sigma_*\colon X_* \to E_*$*. The bijection is given by*

$$\tau \mapsto (G \times_\tau X, \sigma).$$

where $d_0\sigma_n(x) = \sigma_{n-1}(d_0 x)\tau(x)$.

Proof. Combine Proposition 2.7.4, Lemma 2.7.6, and Theorem 2.7.10. □

Theorem 2.7.12 (Kan)**.**

(a) For a simplicial set X_* and a simplicial group G_*, there are natural weak homotopy equivalences

$$|\mathbb{G}(X)_*| \simeq \Omega|X|,$$
$$|\overline{W}(G)_*| \simeq B|G|.$$

(b) The adjoint functors $(\mathbb{G}, \overline{W})$ give a Quillen equivalence between the model categories of reduced simplicial sets and simplicial groups:

$$\mathbb{G}\colon \mathbf{sSet}_0 \leftrightarrows \mathbf{sGr}\colon \overline{W}.$$

That is, $\mathbf{Ho}(\mathbf{sSet}_0) \simeq \mathbf{Ho}(\mathbf{sGr})$.

2.7.1 Relation to spaces

Recall the singular complex

$$\begin{aligned} S\colon \mathbf{Top} &\longrightarrow \mathbf{sSet} \\ X &\longmapsto S(X)_* = \{\mathrm{Hom}_{\mathbf{Top}}(\Delta^n, X)\}_{n\geq 0} \end{aligned}$$

Definition 2.7.13. Define the **Eilenberg subcomplex:** $ES\colon \mathbf{Top}_* \to \mathbf{sSet}_0$ given by

$$(X,*) \longmapsto ES(X)_n = \{f\colon \Delta^n \to X\colon f(e_i) = *\forall i\}.$$

Lemma 2.7.14 (Eilenberg)**.** *If $(X,*)$ is connected, then*

$$ES(X)_* \hookrightarrow S(X)_*$$

is a weak equivalence of simplicial sets, i.e.

$$|ES(X)_*| \simeq |S(X)_*| \simeq X.$$

Corollary 2.7.15. *Let $\mathbf{Top}_{0,*}$ be the category of pointed connected spaces. Then*

$$|-|\colon \mathbf{sSet}_0 \leftrightarrows \mathbf{Top}_{0,*}\colon ES$$

is a Quillen equivalence of model categories, i.e. $\mathbf{Ho}(\mathbf{sSet}_0) \simeq \mathbf{Ho}(\mathbf{Top}_{0,*})$.

As a consequence of Theorem 2.7.12 and Corollary 2.7.15, we have a zig-zag of Quillen equivalences exhibiting

$$\mathbf{Ho}(\mathbf{sGr}) \simeq \mathbf{Ho}(\mathbf{sSet}_0) \simeq \mathbf{Ho}(\mathbf{Top}_{0,*}). \tag{2.7.1}$$

Given a pointed connected space X, we may construct the simplicial group $\mathbb{G}(ES(X)_*)$, which is a *huge* simplicial group. For reduced CW-complexes, we can construct a much smaller simplicial group which models the same thing.

2.8 Free diagrams

Let I be any small indexing category and let $\mathcal{C}$ be any category with colimits (and hence, coproducts). The diagrams $\mathcal{C}^I = \mathbf{Fun}(I, \mathcal{C})$ of shape I in $\mathcal{C}$ can be thought of as a kind of I-modules.

Question 2.8.1. What are the free I-diagrams (i.e. analogues of free modules)?

Let I^δ be the category I **made discrete**, i.e. with no arrows other than identities: $\mathrm{Ob}(I^\delta) = \mathrm{Ob}(I)$ and

$$\mathrm{Hom}_{I^\delta}(i, j) = \begin{cases} \varnothing & (i \neq j), \\ \mathrm{id} & (i = j). \end{cases}$$

There is a natural inclusion $I^\delta \hookrightarrow I$.

Definition 2.8.2. An I-diagram $X \colon I \to \mathcal{C}$ is **free** if it is the left Kan extension of some I^δ-diagram $Y \colon I^\delta \to \mathcal{C}$ along the inclusion $i \colon I^\delta \to I$.

$$\begin{array}{ccc} I^\delta & \xrightarrow{Y} & \mathcal{C} \\ {\scriptstyle i}\big\downarrow & \nearrow_{\mathrm{Lan}_i(Y) \cong X} & \\ I & & \end{array}$$

By properties of left Kan extensions, for all $i \in I$,

$$X(i) \cong \operatorname*{colim}_{(f \colon j \to i) \in I/i} (Y_j) \cong \coprod_{f \colon j \to i} Y_j,$$

where the coproduct is taken over all $f \colon j \to i$ in I.

Example 2.8.3. If $\mathcal{C} = \mathbf{Set}$, and I is any diagram category, then $X \colon I \to \mathbf{Set}$ is free if and only if there is a sequence S of objects in I such that X is a disjoint union of corepresentable functors:

$$X \cong \coprod_{s \in S} h^s,$$

where $h^s = \mathrm{Hom}(s, -) \colon I \to \mathbf{Set}$. The h^s are called **elementary** I**-free diagrams**.

Example 2.8.4. Let $I = \{1 \leftarrow 0 \to 2\}$. An I-diagram X in **Set** is a collection of three sets and two functions

$$X_1 \xleftarrow{f_1} X_0 \xrightarrow{f_2} X_2.$$

When is this free? First, let's look at the elementary free diagrams:

$$h^1 = \mathrm{Hom}_I(1, -) \colon I \to \mathbf{Set},$$

which describes the diagram

$$* \leftarrow \emptyset \rightarrow \emptyset.$$

Similarly, h^2 describes the diagram

$$\emptyset \leftarrow \emptyset \rightarrow *,$$

and h^0 describes the diagram

$$* \leftarrow * \rightarrow *.$$

Note that in each of these diagrams above, all of the functions are injective. So the diagram X is free if and only if both f_1 and f_2 are injective.

Example 2.8.5. Let $\mathcal{C} = \mathbf{sSet}$ and let I be any diagram category. Then a diagram $X_*\colon I \to \mathbf{sSet}$ gives for each $i \in \mathrm{Ob}(I)$ a simplicial set $X_*(i)$. We may think of this as a collection of diagrams of sets $X_n\colon I \to \mathbf{Set}$. Hence, X_* is free if and only if X_n is free as an I-diagram in **Sets**, for each $n \geq 0$.

Exercise 2.8.6. Let $\mathcal{C} = \mathbf{Cat}$, and let I be any diagram category. For any $X\colon I \to \mathbf{Cat}$, there are two diagrams of sets: $\mathrm{Ob}(X)\colon I \to \mathbf{Set}$, $i \mapsto \mathrm{Ob}(X(i))$ and $\mathrm{Mor}(X)\colon I \to \mathbf{Set}$, $i \mapsto \mathrm{Mor}(X(i))$. Prove that $X\colon I \to \mathbf{Cat}$ is free if and only if both $\mathrm{Ob}(X)$ and $\mathrm{Mor}(X)$ are free as diagrams of sets.

Example 2.8.7. Let $\mathcal{C} = \mathbf{Grd} \subseteq \mathbf{Cat}$ be the category of small groupoids. Then $X\colon I \to \mathbf{Grd}$ is free if and only if $\mathrm{Ob}(X)\colon I \to \mathbf{Set}$ is free. Thus a free diagram of groupoids need not be free as a diagram of categories.

2.8.1 Simplicial Diagrams

Let $\Delta_+ \subseteq \Delta$ be the subcategory of Δ consisting of all of the objects and only the surjective morphisms, i.e. $\mathrm{Mor}(\Delta_+)$ are generated by the codegeneracy maps $s^j\colon [n+1] \to [n]$ for $0 \leq j \leq n$ and $n \geq 0$.

Definition 2.8.8. A simplicial object $X\colon \Delta^{op} \to \mathcal{C}$ is called **semi-free** if the restriction to Δ_+^{op}, $X|_{\Delta_+^{op}}\colon \Delta_+^{op} \to \mathcal{C}$, is a free diagram.

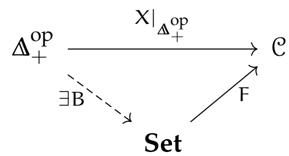
Assume that there is an adjunction between $\mathcal{C}$ and **Set**:

$$F\colon \mathbf{Set} \leftrightarrows \mathcal{C}\colon U.$$

Then, we can make this definition explicit:

$$X\colon \Delta^{op} \to \mathcal{C}$$

is a semi-free simplicial object if and only if $X|_{\Delta_+^{op}} \colon \Delta_+^{op} \to \mathcal{C}$ factors through **Set**:



Let $\Gamma_* \colon \Delta^{op} \to \mathbf{Gr}$ be a simplicial group. Then Γ_* is semi-free if and only if there are subsets $B_n \subseteq \Gamma_n$ for each $n \geq 0$ such that

(1) $\Gamma_n = \mathbb{F}\langle B_n \rangle$

(2) $B := \bigcup_{n \geq 0} B_n$ is closed under the degeneracies $s_j \colon \Gamma_n \to \Gamma_{n-1}$.

Example 2.8.9. The **Kan loop group** of a reduced simplicial set $X \in \mathrm{Ob}(\mathbf{sSet}_0)$ is semi-free:

$$\mathbb{G}(X)_n = {}^{\mathbb{F}\langle X_{n+1} \rangle}\big/_{\langle s_0(x) = 1 \forall x \in X_n \rangle} = \mathbb{F}\langle B_n \rangle$$

where $B_n = X_{n+1} \setminus s_0(X_n)$.

Definition 2.8.10. If Γ_* is a semi-free simplicial group, then we define the set of **nondegenerate generators** $\overline{B}_n$ of Γ in degree n:

$$\overline{B}_n := B_n \setminus \bigcup_{j=0}^{n-1} s_j(B_{n-1})$$

and the set of nondegenerate generators is

$$\overline{B} = \bigcup_{n \geq 0} \overline{B}_n.$$

Note that to give a semi-free simplicial group Γ_*, we need to specify

(1) the set $\overline{B}$,

(2) the values of face maps $d_i \colon \Gamma_n \to \Gamma_{n-1}$ on $\overline{B}_n$

$$\{d_i(x) \subseteq \Gamma_{n-1} \mid x \in \overline{B}_n\}_{n \geq 1}$$

Definition 2.8.11. A set of nondegenerate generators of a semi-free simplicial group Γ_* is called a **CW-basis** if $d_i(x) = 1 \in \Gamma_{n-1}$ for all $x \in \overline{B}_n$ and all $0 \leq i \leq n-1$.

Recall from (2.7.1) that there is a chain of Quillen equivalences

$$\mathbf{Ho}(\mathbf{sGr}) \simeq \mathbf{Ho}(\mathbf{sSet}_0) \simeq \mathbf{Ho}(\mathbf{Top}_{0,*}).$$

A simplicial group Γ_* is called a **simplicial group model** of a space X if its homotopy type corresponds to X under this equivalence.

For all $X \in \mathbf{Ho}(\mathbf{Top}_{0,*})$, we have a "big" functorial model of X

$$\Gamma_*^{\text{big}}(X) = \mathbb{G}(ES(X)_*).$$

One of Kan's main observations is the following:

Theorem 2.8.12 (Kan)**.** *Let X be a reduced CW-complex (i.e. $\mathrm{sk}_0(X) = \{*\}$ and $\mathrm{sk}_n(X) = X$ for all $n \gg 0$). Then there is a semi-free simplicial group model of X, denoted $\Gamma_*^{\text{small}}(X)$, such that*

(1) $|\Gamma_*(X)| \simeq \Omega X$

(2) $\Gamma_*(X)$ *has a CW-basis* $\overline{B} = \bigcup_{n \geq 0} \overline{B}_n$ *such that* $\overline{B}_{n-1}$ *is in bijection with the* n*-cells of* X *for all* $n \geq 1$.

(3) the attaching element $d_n(x)$, $x \in \overline{B}_{n-1}$ *depends only on the homotopy class of the attaching maps of* X*:*

$$[f] \in \pi_{n-1}(\mathrm{sk}_{n-1}(X))$$

Example 2.8.13. When $X = S^1$, $\Gamma_*(X) = \{\mathbb{F}_1\}_{n \geq 0}$.

Remark 2.8.14. Because of the vagueness in the homotopy class of attaching maps, the choice of $d_n(x)$ is not very explicit. In practice, finding d_n is done by guess-and check.

2.9 Homotopy 2-types

2.9.1 The Moore complex of a simplicial group

For $\Gamma_* \in \mathrm{Ob}(\mathbf{sGr})$, the associated **Moore complex** is the chain complex which on level n is the abelian group

$$N_n\Gamma := \bigcap_{i=1}^{n} \ker(d_i \colon \Gamma_n \to \Gamma_{n-1})$$

for $i \neq 0$. If $x \in N_n\Gamma$, then $d_i(x) = 1$ for all $i \geq 1$, so

$$d_k d_0(x) = d_0 d_{k+1}(x) = 1$$

for all $k \geq 0$. Hence, $d_0(x) \in N_{n-1}(\Gamma)$, and $d_0^2(x) = 1$ for all $x \in N_n(\Gamma)$. There is a chain complex

$$N_*\Gamma = \left[N_0\Gamma \xleftarrow{d} N_1\Gamma \xleftarrow{d} N_2\gamma \leftarrow \cdots \right],$$

where $d = d_0|_{N_n\Gamma}$ for all $n \geq 0$.

Definition 2.9.1. The chain complex $N_*\Gamma$ is the **Moore complex** associated to Γ.

Theorem 2.9.2 (Moore)**.** *There are natural isomorphisms*

$$\pi_*(\Gamma) = \pi_*(|\Gamma|) \cong H_*(N_*\Gamma, d).$$

Remark 2.9.3. $|\Gamma|$ is a topological group.

Corollary 2.9.4. *If X is a pointed connected space, and $\Gamma_* = \Gamma_*(X)$ is a simplicial group model of X, then*

$$\pi_i(X) \cong \pi_{i-1}(\Gamma_*(X)) \cong H_{i-1}(N_*\Gamma, d).$$

for all $i \geq 1$.

Proof. Since $|\Gamma_*| \simeq \Omega X$ (by Theorem 2.7.12), then

$$\pi_i(X) \simeq \pi_{i-1}(\Omega X) \simeq \pi_{i-1}(\Gamma_*(X)).$$

for all $i \geq 1$. □

2.9.2 Homotopy 1-types

Definition 2.9.5. A connected space X is called a **homotopy n-type** or **n-coconnected** if $\pi_i(X) = 0$ for $i \geq n+1$. Homotopy 1-types are also called **aspherical spaces**.

Write $\mathbf{Ho}(\mathbf{Top}_{0,*}^{\leq n})$ for the homotopy category of n-types. Since there is a Quillen equivalence, $\mathbf{Ho}(\mathbf{Top}_{0,*}) \simeq \mathbf{Ho}(\mathbf{sGr})$, a natural question is to characterize the image of $\mathbf{Ho}(\mathbf{Top}_{0,*}^{\leq n})$ inside simplicial groups.

For $n = 1$, we have aspherical spaces: in this case,

$$\begin{aligned} \mathbf{Ho}(\mathbf{Top}_{0,*}^{\leq 1}) &\xrightarrow{\cong} \mathbf{Gr} \\ X &\longmapsto \pi_1(X) \\ B\Gamma &\longleftarrow\!\shortmid \Gamma \end{aligned}$$

so $\mathbf{Ho}(\mathbf{Top}_{0,*}^{\leq 1})$ can be identified with discrete simplicial groups $\Gamma_* = \{\Gamma_n\}_{n \geq 0}$. If Γ_* is a discrete simplicial group, then

$$N_n\Gamma = \begin{cases} \Gamma & (n = 0) \\ 1 & (n > 0) \end{cases}$$

so the Moore complex is

$$N_*\Gamma = [1 \leftarrow \Gamma \leftarrow 1 \leftarrow 1 \leftarrow \cdots].$$

We want a similar characterization for 2-types, i.e. $\mathbf{Ho}(\mathbf{Top}_{0,*}^{\leq 2})$. This is given in terms of crossed modules of groups.

2.9.3 Crossed modules

Recall that if A and G are fixed groups, and A is abelian, then it is well-known that the equivalence classes of extensions

$$\xi := \left[0 \to A \xrightarrow{i} E \xrightarrow{\pi} G \to 1\right]$$

are in bijection with cohomology classes in $H^2(G; A)$. To give a class in $H^2(G; A)$, choose a section $s\colon G \to E$ such that $\pi s = \mathrm{id}$ an $s(1) = 1$. Define an action

$$\begin{aligned} G &\longrightarrow \mathrm{Aut}(A) \\ g &\longmapsto \left(\mathrm{Ad}_{s(g)}\colon a \mapsto s(g)as(g)^{-1}\right) \end{aligned}$$

and

$$\begin{aligned} c\colon G \times G &\longrightarrow A \\ (g,h) &\longmapsto s(gh)s(g)^{-1}s(h)^{-1} \end{aligned}$$

Then c is a 2-cocycle, so $[c] \in H^2(G; A)$.

We want a similar interpretation of $H^3(G; A)$.

Definition 2.9.6. A **crossed module** is a group homomorphism $\mu\colon M \to N$ given together with a (left) action of N on M,

$$\begin{aligned} \rho\colon N \times M &\longrightarrow M \\ (n,m) &\longmapsto {}^n m \end{aligned}$$

satisfying

(a) $\mu({}^n m) = n\mu(m)n^{-1}$ in N

(b) ${}^{\mu(m)}m' = mm'm^{-1}$ in M

Remark 2.9.7. The following lifting property is equivalent to axioms (a) and (b): given any group homomorphism $\mu\colon M \to N$, there is a natural commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\mathrm{Ad}} & \mathrm{Aut}(M) \\ \downarrow{\scriptstyle \mu} & \overset{\rho}{\nearrow} & \downarrow{\scriptstyle \mathrm{act}} \\ N & \xrightarrow[\mathrm{Ad}_\mu]{} & \mathrm{Hom}_{\mathbf{Gr}}(M,N) \end{array}$$

Then μ is a crossed module if and only if there is a group homomorphism $\rho\colon N \to \mathrm{Aut}(M)$ (dashed) making the diagram commute. Property (a) is equivalent to the commutativity of the lower-right triangle, and property (b) is equivalent to the commutativity of the upper-left triangle.

Example 2.9.8. If $M \lhd N$, then $M \hookrightarrow N$ is naturally a crossed module, with ${}^{n}m = n \cdot m \cdot n^{-1}$.

Example 2.9.9 (Whitehead)**.** A Serre fibration $F \xrightarrow{i} E \xrightarrow{\pi} B$ naturally yields a crossed module on the level of fundamental groups: $\mu = i_*\colon \pi_1(F) \to \pi_1(E)$.

Example 2.9.10 (Steinberg)**.** If R is a unital ring, recall for $n \geq 3$, the n-th **Steinberg group** $\mathrm{St}_n(R)$ is the group generated by symbols $x_{ij}(r)$ for $r \in R$, $1 \leq i \neq j \leq n$, subject to the **Steinberg relations**:

$$x_{ij}(r) \cdot x_{ij}(s) = x_{ij}(r+s)$$

$$[x_{ij}(r), x_{k\ell}(s)] = \begin{cases} 1 & i \neq k \text{ and } i \neq \ell, \\ x_{i\ell}(rs) & j = k \text{ and } i \neq \ell, \\ x_{kj}(-rs) & j \neq k \text{ and } i = \ell. \end{cases}$$

Note that the elementary matrices

$$e_{ij}(r) = \begin{matrix} & i \\ j & \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & r & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \end{matrix} \in \mathrm{GL}_n(R)$$

satisfy these relations. So there is a well-defined group homomorphism

$$\mu_n\colon \mathrm{St}_n(R) \to E_n(R),$$

such that

$$\begin{array}{ccc} \mathrm{St}_n(R) & \xrightarrow{\mu_n} & \mathrm{GL}_n(R) \\ \downarrow & & \downarrow \\ \mathrm{St}_{n+1}(R) & \xrightarrow{\mu_{n+1}} & \mathrm{GL}_{n+1}(R). \end{array}$$

These homomorphisms assemble to a homomorphism between the colimits of these inclusions:

$$\begin{array}{ccc} \mathrm{St}(R) & \xrightarrow{\quad\mu\quad} & \mathrm{GL}(R) \\ \| & & \| \\ \mathrm{colim}_n \mathrm{St}_n(R) & \xrightarrow{\mathrm{colim}_n \mu_n} & \mathrm{colim}_n \mathrm{GL}_n(R) \end{array}$$

This is actually a crossed module with respect to the natural conjugation action of $\mathrm{GL}(R)$. This follows from the following lemma and theorem.

Lemma 2.9.11 (Whitehead Lemma). *The image of* μ *is the elementary matrices inside* $\mathrm{GL}(R)$*:*

$$\mathrm{im}(\mu) = E(R) = [\mathrm{GL}(R), \mathrm{GL}(R)].$$

Theorem 2.9.12 (Kervaire–Steinberg)**.**

(a) $\mathrm{St}(R)$ *is the universal central extension of* $E(R)$

(b) There is an exact sequence

$$0 \to K_2(R) \to \mathrm{St}(R) \to \mathrm{GL}(R) \to K_1(R) \to 0$$

where $K_i(R)$*,* $i = 1, 2$ *are the algebraic* K*-theory groups of* R*.*

Example 2.9.13. Any group object $\mathcal{G}$ in the category **Cat** (i.e. the functor

$$\mathrm{Hom}_{\mathbf{Cat}}(-, \mathcal{G})\colon \mathbf{Cat} \to \mathbf{Set}$$

naturally factors through the category of groups) naturally determines a crossed module.

Note that the usual nerve functor $\mathcal{N}\colon \mathbf{Cat} \to \mathbf{sSet}$ is right adjoint, and hence sends group objects in **Cat** to group objects in **sSet** (= simplicial groups). So $\mathcal{N}_\bullet\mathcal{G}$ is a simplicial group, and its Moore complex looks like

$$N(\mathcal{N}_\bullet\mathcal{G}) = \left[N_0 \xleftarrow{d} N_1 \leftarrow 1 \leftarrow \cdots\right].$$

The homomorphism $d\colon N_1 \to N_0$ is a crossed module.

Lemma 2.9.14. *Given a crossed module* $(\mu\colon M \to N, \rho)$*, define* $A = \ker(\mu)$ *and* $G = \mathrm{coker}(\mu)$*. Then*

(a) G *and* A *are groups, and* A *is an abelian central subgroup of* M*.*

(b) The action $\rho\colon N \times M \to M$ *induces a well-defined action* $\overline{\rho}\colon G \times A \to A$ *making* A *into a* G*-module.*

Proof. First we prove (a). By the Definition 2.9.6(a), $n \cdot \mu(m) \cdot n^{-1} \in \mathrm{im}(\mu)$ so $\mathrm{im}(\mu)$ is normal in M, so $G = M/\mathrm{im}(\mu)$ is a group. By Definition 2.9.6(b), for all $a \in A$ and $m \in M$,

$$m = {}^{\mu(a)}m = ama^{-1}$$

if and only if $[a, m] = 1$. Hence, A is a central subgroup of M.

The proof of (b) is an exercise. □

Example 2.9.15. If we take the Steinberg crossed module $\mu\colon \mathrm{St}(R) \to \mathrm{GL}(R)$ from Example 2.9.10, then in this case $G = K_1(R)$ and $A = K_2(R)$ by Theorem 2.9.12.

Fix a group G, and an abelian group A with G-module structure. Consider the set CrMod(G, A) of all crossed modules $\mu\colon M \to N$ with $G = \operatorname{coker}(\mu)$ and $A = \ker(\mu)$, and the induced action $\overline{\rho}$ being the given one. We say that two such crossed modules $(\mu\colon M \to N, \rho)$ and $(\mu'\colon M' \to N', \rho')$ are **related** if there are group homomorphisms $\alpha\colon M \to M'$ and $\beta\colon N \to N'$ such that the following commutes:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & A & \longrightarrow & M & \xrightarrow{\mu} & N & \longrightarrow & G & \longrightarrow & 1 \\ & & \| & & \downarrow \alpha & & & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & M' & \xrightarrow{\mu'} & N' & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

Let CrMod$(G, A)/\sim$ be the set of equivalence classes of crossed modules in CrMod(G, A) modulo the equivalence generated by the above relation.

Proposition 2.9.16 (MacLane)**.** *There is a natural isomorphism* CrMod$(G, A)/\sim\, \cong H^3(G; A)$. *The cohomology class* $k(\mu, \rho) \in H^3(G, A)$ *corresponding to* $\mu\colon M \to N$ *is called its MacLane invariant.*

We will give a topological interpretation of this proposition. We need the notion of a **classifying space for crossed modules**.

Construction 2.9.17. Given $(\mu\colon M \to N, \rho)$, define a small category $\mathcal{C}$ as follows: the objects of $\mathcal{C}$ are N, and the morphisms of $\mathcal{C}$ are the crossed product of groups $M \rtimes N$: this is $M \times N$ as a set with multiplication $(m, n) \cdot (m', n') = (m \cdot {}^{n}(m'), nn')$.

The source and target maps for this category are $s, t\colon \operatorname{Mor}(\mathcal{C}) \to \operatorname{Ob}(\mathcal{C})$,

$$s(m, n) = n, \qquad t(m, n) = \mu(m)n.$$

So $(m, n) \in M \rtimes N$ corresponds to the arrow $n \xrightarrow{(m,n)} \mu(m)n$.

Composition is given by

$$\left(n \xrightarrow{(m,n)} \mu(m)n\right) \circ \left(\mu(m)n \xrightarrow{(m',\mu(m)n)} \mu(m')(\mu(m)n)\right) = \left(n \xrightarrow{(\mu(m'm)n,n)} \mu(m'm)n\right).$$

Note that $\mu(m')(\mu(m)n) = \mu(m'm)n$.

The nerve $\mathcal{N}_\bullet\mathcal{C}$ of this category looks like:

$$\begin{aligned} &\mathcal{N}_0\mathcal{C} = N \\ &\mathcal{N}_1\mathcal{C} = M \rtimes N \cong M \times N \\ &\mathcal{N}_2\mathcal{C} = \mathcal{N}_1\mathcal{C} \times_{\mathcal{N}_0\mathcal{C}} \mathcal{N}_1\mathcal{C} = (M \rtimes N) \times_N (M \rtimes N) = M \rtimes (M \rtimes N) \cong M^2 \times N \\ &\qquad \vdots \\ &\mathcal{N}_n\mathcal{C} = M \rtimes (\cdots M \rtimes N \cdots) \cong M^n \times N \end{aligned}$$

One can check that in fact all face and degeneracy maps in $\mathcal{N}_\bullet\mathcal{C}$ are given by group homomorphisms. Hence, $\mathcal{N}_\bullet\mathcal{C}$ is a simplicial group.

Definition 2.9.18. We denote the simplicial group $\mathcal{N}_\bullet \mathcal{C}$ associated to a crossed module $(\mu\colon M \to N, \rho)$ by $N /\!/ M$, where $\mathcal{C}$ is the category as constructed in Construction 2.9.17.

Definition 2.9.19 (MacLane)**.** The **classifying space of the crossed module** $(\mu\colon M \to N, \rho)$ is defined by

$$X(\mu, \rho) := B\,|N /\!/ M|\,.$$

This is the usual classifying space of a topological group.

We want to understand the homotopy types of $X(\mu, \rho)$. To do this, we will use the Kan loop group adjunction

$$\mathbb{G}\colon \mathbf{sSet}_0 \leftrightarrows \mathbf{sGr}\colon \overline{W}.$$

In these terms, we can identify

$$B|N /\!/ M| \cong |\overline{W}(N /\!/ M)|.$$

Then, by Theorem 2.8.12, for all $i \geq 1$,

$$\pi_i(B|N /\!/ M|) = \pi_i|\overline{W}(N /\!/ M)| \cong \pi_{i-1}\Omega|\overline{W}(N /\!/ M)| \cong \pi_{i-1}|\mathbb{G}\overline{W}(N /\!/ M)|.$$

But for any simplicial group Γ_*, the counit $\mathbb{G}\overline{W}\Gamma_* \xrightarrow{\sim} \Gamma_*$ is a weak equivalence. Therefore,

$$\pi_{i-1}|\mathbb{G}\overline{W}(N /\!/ M)| \cong \pi_{i-1}(|N /\!/ M|) \cong \pi_{i-1}(N /\!/ M) \cong H_{i-1}(N_\bullet(N /\!/ M), d),$$

where $(N_\bullet(N /\!/ M), d))$ is the Moore complex of this simplicial group. So we have reduced this problem to finding the Moore complex of this simplicial group.

By definition, $N /\!/ M$ looks like

$$N /\!/ M = \left[\; N \;\underset{d_1}{\overset{d_0}{\leftleftarrows}}\!\!\xrightarrow{s_0}\; M \rtimes N \;\underset{d_2}{\overset{d_0}{\leftarrow}}\; \begin{smallmatrix} \xrightarrow{s_0} \\ \xleftarrow{d_1} \\ \xrightarrow{s_1} \end{smallmatrix} \;\cdots \;\right]$$

where s_0 is the canonical inclusion $N \hookrightarrow M \rtimes N$, so that

$$d_0 s_0 = d_1 s_0 = \mathrm{id}_N,$$

or equivalently $d_0|_N = d_1|_N = \mathrm{id}_N$. A straightforward calculation shows that $N_\bullet(N /\!/ M, d)$ has the form

$$N \xleftarrow{\mu = d} \ker(d_1) \cong M \leftarrow 1 \leftarrow 1 \leftarrow \cdots$$

Claim 2.9.20. This is exactly the crossed module we started with.

The proof of this claim follows from the following two lemmas.

Lemma 2.9.21. *If*

$$\Gamma_* = \left[\Gamma_0 \underset{d_1}{\overset{d_0}{\leftleftarrows}} \Gamma_1 \; \cdots \right]$$

is any simplicial group such that its Moore complex has length 1*, then*

$$[\ker(d_0), \ker(d_1)] = 1 \in \Gamma_1.$$

Proof. For any $x \in \ker(d_0)$ and $y \in \ker(d_1)$, consider

$$[s_0(x), s_0(y)s_1(y)^{-1}] \in \Gamma_2.$$

Therefore, this commutator lives insider $\ker(d_0) \cap \ker(d_1)$. This intersection is 1 by assumption, so

$$[s_0(x), s_0(y)s_1(y)^{-1}] = 1.$$

On the other hand,

$$d_0([s_0(x), s_0(y)s_1(y)^{-1}]) = [x, y],$$

so $[x, y] = 1$. □

Lemma 2.9.22. *In the case of* $N /\!/ M$*, the commutator relation* $[\ker(d_0), \ker(d_1)] = 1$ *is equivalent to Definition 2.9.6(b).*

Corollary 2.9.23.

$$\pi_i(X(\mu, \rho)) = \begin{cases} H_0(N_\bullet(N /\!/ M)) \cong G & (i = 1) \\ H_1(N_\bullet(N /\!/ M)) \cong A & (i = 2) \\ 0 & (i \geq 3) \end{cases}$$

Moreover, the action of π_1 *on* π_2 *agrees with the* G*-module structure on* A*, and* $H^3(B\pi_1; \pi_2) \cong H^3(G; A)$ *and* $k(\mu, \rho)$ *corresponds to Postnikov's* k^3*-invariant.*

For a connected CW complex X, the Postnikov decomposition is given by

$$\mathrm{cosk}_1(X) \twoheadleftarrow \mathrm{cosk}_2(X) \twoheadleftarrow \cdots \twoheadleftarrow \mathrm{cosk}_n(X) \twoheadleftarrow \cdots \twoheadleftarrow X.$$

such that

$$(1)\quad \pi_i(\mathrm{cosk}_n(X)) = \begin{cases} \pi_i(X) & (i \leq n), \\ \pi_i(\mathrm{cosk}_n(X)) & (i > n), \end{cases}$$

(2) for all $n \geq 2$, there is a fibration

$$\mathrm{cosk}_{n-1}(X) \twoheadleftarrow \mathrm{cosk}_n(X) \leftarrow K(\pi_n(X), n)$$

with characteristic classes

$$k^{n+1}(X) \in H^{n+1}(\mathrm{cosk}_{n-1}(X); \pi_n(X))$$

called the $(n+1)$-st **Postnikov invariants**.

In our case, if $X = X(\mu, \rho) = B|N /\!/ M|$, then

$$B\pi_1(X) = \mathrm{cosk}_1(X) \twoheadleftarrow \mathrm{cosk}_2(X) = \cdots = X,$$

because X is 2-coskeletal. Hence,

$$k^3(X) \in H^3(B\pi_1(X); \pi_2(X)) \cong H^3(G, A),$$

and this 3rd Postnikov invariant coincides with the MacLane invariant. As a summary, we have:

Theorem 2.9.24 (Loday)**.** *The following are equivalent:*

(a) a crossed module $(\mu \colon M \to N, \rho)$*,*

(b) group objects in **Cat***,*

(c) simplicial groups Γ_* *with Moore complex of length* 1.

Proof. We prove first (a) $\implies$ (b) $\implies$ (c). First, we may construct a the category $\mathcal{C}(\mu, \rho)$ from Construction 2.9.17, and taking its nerve gives a simplicial group $N /\!/ M$ as in Definition 2.9.18. Finally, to obtain (c) $\implies$ (a), we take the Moore complex of $N /\!/ M$ and argue that this is the crossed module $(\mu \colon M \to N, \rho)$ as in Claim 2.9.20. $\square$

2.9.4 Homotopy Normal Maps

There is another view of this based on the notion of **homotopy normal maps**, due to Farjoun–Seger (2012) and Farjoun–Seger–Hess (2016).

Recall that an injective group homomorphism $\mu \colon M \hookrightarrow N$ is **normal** if it is the kernel of another group homomorphism $N \to \Gamma$. Up to homotopy, any group homomorphism $M \to N$ can be viewed as inclusion, so it's natural to ask how to extend this notion to an arbitrary homomorphism.

Definition 2.9.25. A group homomorphism $\mu \colon M \to N$ is called **homotopy normal** if the corresponding map of spaces $B\mu \colon BM \to BN$ is the homotopy fiber of some fibration

$$BM \xrightarrow{B\mu} BN \xrightarrow{\nu} X$$

where X is some pointed connected space. The homotopy class of $\nu \colon BN \to X$ is called the normal structure on μ.

Remark 2.9.26. If $M \lhd N$ is a normal subgroup in the usual sense, then we have a canonical normal structure on $\mu\colon N \hookrightarrow M$

$$BM \to BN \xrightarrow{B\rho} B\left({}^{N}\!/_{M}\right)$$

coming from the short exact sequence of groups

$$1 \to M \to N \to {}^{N}\!/_{M} \to 1.$$

Theorem 2.9.27 (Farjoun–Seger)**.** *A group homomorphism* $\mu\colon M \to N$ *is homotopy normal if and only if there is an action* $\rho\colon N \to \mathrm{Aut}(M)$ *making* $\mu\colon M \to N$ *a crossed module. The corresponding space for which* $BM \to BN$ *is the homotopy fiber map is* $B|N /\!/ M|$*, the classifying space of* $(\mu\colon M \to N, \rho)$.

An alternative way to state this is through homotopy colimits: given a group homomorphism $\mu\colon M \to N$, we can consider the functor

$$\begin{aligned} \mu\colon \underline{M} &\longrightarrow \mathbf{Set} \hookrightarrow \mathbf{sSet} \\ * &\longmapsto N \\ M \ni m &\longmapsto (\ell_m\colon n \mapsto \mu(m)n)\,. \end{aligned}$$

This functor defines a diagram, and we may take the homotopy quotient (Borel construction)

$$N /\!/ M = \mathrm{hocolim}_{\underline{M}}(\mu) = E_*M \times_M N \in \mathbf{sSet}\,.$$

This simplicial set has simplicies

$$\begin{aligned} (N /\!/ M)_0 &= M \times_M N \cong N \\ (N /\!/ M)_n &= M^{n+1} \times_M N \cong M^n \times N. \end{aligned}$$

Note that $N = (N /\!/ M)_0$ acts by right multiplication on $(N /\!/ M)_n$, so we get an N-simplicial set. Similarly for any simplicial group Γ_*, Γ_0 acts (on the right via the degeneracy s_0) on all Γ_n, so Γ_* is a Γ_0-simplicial set.

Theorem 2.9.28. $M \xrightarrow{\mu} N$ *is homotopy normal if and only if there is a simplicial group* Γ_* *with isomorphism* $\Gamma_0 \cong N$ *which extends to an isomorphism of* Γ_0*-simplicial sets:*

$$\Gamma_* \xrightarrow{\sim} N /\!/ M.$$

Remark 2.9.29. This extends to other monoidal model categories, for example, if $\mu\colon M_* \to N_*$ is a morphism of simplicial groups, we call μ **homotopy normal** if there is a fibration of simplicial sets

$$\overline{W}M_* \xrightarrow{\overline{W}\mu} \overline{W}N_* \xrightarrow{\nu} X_*.$$

Example 2.9.30. If G is a group, and $z \in Z(G)$ is central, then the z-twisted nerve of G is a cyclic set $B_*(G, z)$ such that

$$B_*(G, z)|_{\Delta^{op}} = B_*G$$

and the cyclic action is given by

$$t_n \cdot (g_1, \ldots, g_n) = (z(g_1 g_2 \cdots g_n^{-1}), g_2, \ldots, g_n).$$

The cyclic realization of $B_*(G, z)$ is

$$X(G, z) := |B_*(G, z)|^{cyc} := ES^1 \times_{S^1} |B_*(G, z)|.$$

Consider the crossed module $\gamma\colon \mathbb{Z} \to G$ given by $n \mapsto z^n$, with G acting trivially on $\mathbb{Z}$. Then we may identify

$$X(G, z) \cong B|G /\!/ \mathbb{Z}|,$$

and this tells us that

$$\pi_i X(G, z) = \begin{cases} \operatorname{coker}(\gamma) & (i = 1) \\ \ker(\gamma) & (i = 2) \\ 0 & (i \geq 3). \end{cases}$$

Hence, if z is of infinite order, then $X(G, z) \simeq B(G/\langle z \rangle)$.

2.10 Algebraic K-theory as a derived functor

2.10.1 Quillen's plus construction

Work with pointed connected CW complexes. If X is such a complex, write $\pi = \pi_1(X, *)$.

Theorem 2.10.1. *Let* $N \trianglelefteq \pi_1(X)$ *be a normal perfect (i.e.* $N = [N, N]$*) subgroup. Then there is a pointed connected CW complex* X_N^+ *with a map* $j\colon X \to X_N^+$ *such that*

(a) $\pi_1(j)\colon \pi_1(X) \to \pi_1(X_N^+)$ *is surjective with kernel* N.

(b) $j_*\colon H_*(X; \mathbb{Z}) \xrightarrow{\sim} H_*(X_N^+; \mathbb{Z})$ *is an isomorphism.*

Proof sketch. The idea is to add 2-cells to X to kill $N \trianglelefteq \pi_1(X)$ and then add 3-cells to neutralize the effect of the added cells on (co)homology.

Let $p\colon \widetilde{X} \twoheadrightarrow X$ be a regular covering corresponding to $N \subseteq \pi_1(X, *)$. Take any $x \in N$: since N is perfect, we can write

$$x = \prod_{i=1}^{m} [y_i, z_i]$$

for some $y_i, z_i \in N$. Let $X^1 = \mathrm{sk}^1(X)$, and $\widetilde{X}^1 = \mathrm{sk}^1(\widetilde{X})$ be the 1-skeleta of X and $\widetilde{X}$; thus X^1 is the cell subcomplex of X consisting of all cells e of dimension at most 1. The inclusion $\widetilde{X}^1 \hookrightarrow \widetilde{X}$ induces a surjection

$$p\colon \pi_1(\widetilde{X}^1) \twoheadrightarrow \pi_1(\widetilde{X}) = N,$$

so we may choose $\widetilde{y}_i, \widetilde{x}_i \in \pi_1(\widetilde{X}^1)$ which map onto $y_i, z_i \in N$ under this surjection. Taking the class

$$\alpha = p_* \prod_{i=1}^{m} [\overline{y}_i, \overline{z}_i] \in \pi_1(X) = [S^1, X]$$

we can form a new complex Y by attaching to X a 2-cell with attaching map in the class α.

Then the covering $\widetilde{X} \to X$ extends to a covering $\widetilde{Y} \to Y$, where $\widetilde{Y}$ is obtained from Y by attaching a set of 2-cells, one of which has an attaching map in the class $\bigcap_{i=1}^{m} [\overline{y}_i, \overline{z}_i]$, while the others are translates of the first under the group of covering transformations $G \cong \pi_1(X)/N$.

We proceed as above for a set of elements $x \in N$ generating N as a normal subgroup of $\pi_1(X)$ and write $Y, \widetilde{Y}$ for the result.

Note that $\widetilde{Y}$ is simply connected since we have added cells to X to kill $N \lhd \pi_1(X)$, and thereby added cells to $\widetilde{Y}$ to kill $\pi_1(\widetilde{X}) = N$. So by the Hurewicz theorem, we have

$$\pi_2(\widetilde{Y}) \cong H_2(\widetilde{Y}).$$

By construction, the attaching maps in $\widetilde{Y}$ are null homomorphisms in $\widetilde{Y}^1$, so

$$H_2(\widetilde{Y}) \cong H_2(\widetilde{X}) \otimes F$$

where F is a free module over $\mathbb{Z}[G]$ on generators corresponding to the cells added to X. Let $\{f_\alpha\} \subseteq \pi_2(\widetilde{Y}^2)$ b elements mapping to the $\mathbb{Z}[G]$-basis for F. We construct X_N^+ by attaching 3-cells to Y using the classes $\{p_* f_\alpha\}$. The covering $\widetilde{X}_N^+ \to X_N^+$ can be described in the same way as $\widetilde{Y} \twoheadrightarrow Y$: for each α, we have one 3-cell with attaching maps $[f_\alpha]$ and also all the tranlates of that cell under $G = \pi_1(X)/N$. This implies that the relative cellular chain complex $C_*(\widetilde{X}^+, \widetilde{X})$ has the form

$$\cdots \to 0 \to F \xrightarrow{\cong} F \to 0 \to \cdots$$

and the natural inclusion $C_*(\widetilde{X}) \hookrightarrow C_*(\widetilde{X}^+)$ is a chain homotopy equivalence over $\mathbb{Z}[G]$.

This implies the theorem. □

Definition 2.10.2. We call X_N^+ the **plus construction** with respect to N.

Remark 2.10.3. In Theorem 2.10.1(b), we may equivalently replace homology by cohomology, or we may restate it as

(b′) For any $\pi_1(X_N^+)$-module A (a local system on X_N^+), the induced homomorphism $j^*\colon H^*(X_N^+; A) \xrightarrow{\sim} H^*(X, j^*A)$ is an isomorphism.

Remark 2.10.4. There is an abstract analogue of this construction for various model categories (e.g. DG Lie algebras) where the adjective "perfect" makes sense. The analogy to keep in mind is that π_* for spaces corresponds to H_* for DG objects, and H_* for spaces corresponds to Quillen homology for DG objects.

We record some basic properties of the plus construction in the following:

Proposition 2.10.5.

(P1) (X_N^+, j) *is universal among all pairs* (Y, f) *with* $f\colon X \to Y$ *such that* $\pi_1(f)(N) = 1$ *in* $\pi_1(Y)$*, in the sense that there is a map* $\bar{f}\colon X_N^+ \to Y$*, unique up to homotopy, such that the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle j}\searrow \quad \nearrow{\scriptstyle \exists \bar{f}} & \\ & X_N^+ & \end{array}$$

(P2) If G *is a group such that* $N = [G, G]$ *is perfect, (i.e.* $[[G, G], [G, G]] = [G, G]$*), then* $i\colon N \hookrightarrow G$ *induces* $Bi\colon BN \to BG$ *which gives a universal covering*

$$\alpha = (Bi)^+\colon (BN)_N^+ \to (BG)_N^+$$

for $(BG)_N^+$*:*

$$\begin{array}{ccc} BN & \xrightarrow{Bi} & BG \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} \\ (BN)_N^+ & \xrightarrow{\alpha} & (BG)_N^+ \end{array}$$

Definition 2.10.6. If R is a unital associative ring, define

$$GL(R) := \operatorname{colim} GL_n(R)$$

where the colimit is taken along homomorphisms

$$M \longmapsto \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}$$

and similarly define

$$E(R) = \operatorname{colim} E_n(R)$$

where $E_n(R) \trianglelefteq GL(R)$ is the subgroup of elementary matrices.

Lemma 2.10.7 (Whitehead)**.** $E(R) = [GL(R), GL(R)]$*, and hence* $E(R)$ *is a perfect subgroup of* $GL(R)$*.*

Definition 2.10.8 (Quillen)**.** For $n \geq 1$, the n-th **algebraic** K**-theory** group of R is

$$K_n(R) = \pi_n\left(B\,GL(R)^+\right),$$

where the plus construction is taken relative to $E(R)$.

Applying (P2) above to $E(R) \trianglelefteq GL(R)$, we get a universal covering

$$BE(R)^+ \to B\,GL(R),$$

and therefore $K_n(R) = \pi_n(BE(R)^+)$ for all $n \geq 2$.

We will describe this K-theory as a derived functor.

2.10.2 The Bousfield–Kan completion of a space

This completion is a natural extension of the pro-nilpotent completion of groups to spaces. Recall that if Γ is a discrete group, then its **lower central series** is given by

$$\begin{aligned} \Gamma_1 &= \Gamma \\ \Gamma_2 &= [\Gamma_1, \Gamma] \\ &\vdots \\ \Gamma_n &= [\Gamma_{n-1}, \Gamma] \end{aligned}$$

We have a chain of normal subgroups, called the **lower central series**:

$$\Gamma = \Gamma_1 \trianglerighteq \Gamma_2 \trianglerighteq \Gamma_3 \trianglerighteq \cdots .$$

Note that Γ is abelian if and only if $\Gamma_2 = \{1\}$ and Γ is nilpotent if and only if $\Gamma_n = \{1\}$ for some n.

Definition 2.10.9. The **pronilpotent completion** of Γ is

$$\widehat{\Gamma} = C(\Gamma) := \lim\left(\cdots \twoheadrightarrow \Gamma/\Gamma_3 \xrightarrow{p_2} \Gamma/\Gamma_2 \xrightarrow{p_1} \Gamma/\Gamma_1 = 1 \right).$$

Explicitly, we may describe this limit as

$$C(\Gamma) = \left\{ (\overline{\gamma}_1, \overline{\gamma}_2, \ldots, \overline{\gamma}_n, \ldots) \in \prod_{n\geq 1} \Gamma/\Gamma_n \;\middle|\; p_n(\overline{\gamma}_{n+1}) = \overline{\gamma}_n \ \forall n \geq 1 \right\}$$

This definition has a universal property: the completion map

$$\begin{aligned} p\colon \Gamma &\longrightarrow C(\Gamma) \\ \gamma &\longmapsto (\gamma\Gamma_1, \gamma\Gamma_2, \dots) \end{aligned}$$

is universal (initial) among all maps from Γ to nilpotent groups N.

$$\begin{array}{ccc} \Gamma & \xrightarrow{f} & N \\ & {\scriptstyle p}\searrow \quad \nearrow{\scriptstyle \exists!\bar{f}} & \\ & C(\Gamma) & \end{array}$$

Remark 2.10.10. In categorical terms, this can be described as the right Kan extension of the inclusion of nilpotent groups into groups along itself.

$$\begin{array}{ccc} \mathbf{Gr}^{\mathrm{nil}} & \xhookrightarrow{i} & \mathbf{Gr} \\ \big\downarrow{\scriptstyle i} & \nearrow{\scriptstyle \mathrm{Ran}_i(i)=C} & \\ \mathbf{Gr} & & \end{array}$$

By the description of right Kan extensions as limits, we have

$$C(\Gamma) = \lim_{\Gamma\downarrow\mathbf{Gr}^{\mathrm{nil}}}(i).$$

Next, recall the Kan loop group construction:

$$\mathbb{G}\colon \mathbf{sSet}_0 \leftrightarrows \mathbf{sGr}\colon \overline{W}.$$

Note that

- $\pi_i(\mathbb{G}X) = \pi_i\Omega|X| = \pi_{i+1}(X)$ for all $i \geq 0$,
- $X \xrightarrow{\sim} \overline{W}\mathbb{G}X$ is a weak equivalence for all $X \in \mathrm{Ob}(\mathbf{sSet}_0)$.
- $\mathbb{G}\overline{W}\Gamma \xrightarrow{\sim} \Gamma$ is a weak equivalence for all $\Gamma \in \mathrm{Ob}(\mathbf{sGr})$.

Definition 2.10.11. The **Bousfield–Kan integral completion** of a reduced simplicial set X is defined by

$$\mathbb{Z}_\infty(X) := \overline{W}C\mathbb{G}(X),$$

where $C\colon \mathbf{sGr} \to \mathbf{sGr}$ is the degreewise pronilpotent completion. If $X \in \mathbf{Top}_{0,*}$, then

$$\mathbb{Z}_\infty(X) := |\mathbb{Z}_\infty(ES_*(X))|.$$

We record some basic properties of this object in the following:

Proposition 2.10.12.

(I1) (BK, Lemma I.5.5, pg. 25) For a map $f\colon X \to Y$ *in* $\mathbf{sSet}_0$*, we have*

$$\mathbb{Z}_\infty(f)\colon \mathbb{Z}_\infty(X) \to \mathbb{Z}_\infty(Y)$$

is a homotopy equivalence, if and only if

$$f_*\colon H_*(X;\mathbb{Z}) \xrightarrow{\sim} H_*(Y;\mathbb{Z})$$

is an isomorphism.

(II2) (BK, Proposition V.3.4, pg. 34) For all $X \in \mathrm{Ob}(\mathbf{sSet}_0)$*, the canonical map* $i\colon X \to \mathbb{Z}_\infty(X)$ *is a weak equivalence if and only if* X *is nilpotent (i.e.* $\pi_1(X)$ *is nilpotent and* $\pi_1(X)$ *acts nilpotently on each* $\pi_n(X)$ *for* $n \geq 1$*).*

Proposition 2.10.13 (Farjoun)**.** *Let* R *be a unital associative ring. Then there are natural homotopy equivalences*

(a) $|\mathbb{Z}_\infty \overline{W} E(R)| \cong BE(R)^+|$

(b) $|\mathbb{Z}_\infty \overline{W}\, \mathrm{GL}(R)| \cong B\,\mathrm{GL}(R)^+|$

This proposition shows us that $\mathbb{Z}_\infty$ can be viewed as a simplicial realization of the plus construction.

Corollary 2.10.14. $K_n(R) = \pi_n|\mathbb{Z}_\infty \overline{W}(\mathrm{GL}(R))|$.

We may also realize algebraic K-theory as a derived version of pronilpotent completion.

Theorem 2.10.15. *The functor* $C\colon\ \mathbf{sGr} \to \mathbf{sGr}$ *has a left deformation*

$$Q = \mathbb{G}\overline{W}\colon\ \mathbf{sGr} \to \mathbf{sGr}$$

and therefore it has derived functor $\mathbb{L}C\colon\ \mathbf{Ho}(\mathbf{sGr}) \to \mathbf{Ho}(\mathbf{sGr})$ *such that*

$$\mathbb{L}C(\mathrm{GL}(R)) \cong K_{n+1}(R)$$

for all $n \geq 0$*.*

Remark 2.10.16. This applies to many functors on **Gr** or on **sGr**.

Chapter 3

Derived Algebraic Geometry

In classical algebraic geometry, the basic objects are schemes. Fix a commutative ring k. Consider the category $\mathbf{Sch}_k$ of schemes over k and consider the Yoneda embedding

$$\begin{array}{ccc} \mathbf{Sch}_k & \hookrightarrow & \mathbf{Fun}(\mathbf{Sch}_k{}^{\mathrm{op}}, \mathbf{Set}) \\ X & \longmapsto & h_X \end{array}$$

h_X is called the **functor of points** of X. A scheme X is determined by h_X uniquely up to isomorphism.

Recall that affine schemes over k are schemes of the form $X = \mathrm{Spec}(A)$ where A is a commutative k-algebra: $A \in \mathrm{Ob}(\mathbf{CommAlg}_k)$. The functor $\mathrm{Spec}\colon \mathbf{CommAlg}_k{}^{\mathrm{op}} \hookrightarrow \mathbf{Sch}_k$ is an embedding, and objects in the image are called **affine schemes.** Let $\mathbf{AffSch}_k = \mathbf{CommAlg}_k{}^{\mathrm{op}}$ be the category of affine k-schemes.

Now consider the following diagram

$$\begin{array}{ccc} \mathbf{Sch}_k & \overset{h}{\hookrightarrow} & \mathbf{Fun}(\mathbf{Sch}_k{}^{\mathrm{op}}, \mathbf{Set}) \\ & & \big\downarrow \mathrm{Res} \\ & & \mathbf{Fun}(\mathbf{AffSch}_k{}^{\mathrm{op}}, \mathbf{Set}) \\ & \searrow^{\overline{h}} & \big\downarrow \cong \\ & & \mathbf{Fun}(\mathbf{CommAlg}^{\mathrm{op}}, \mathbf{Set}) \end{array}$$

where $\overline{h} = \mathrm{Res} \circ h$. In fact, the restricted Yoneda embedding $\overline{h}$ is fully faithful.

Lemma 3.0.1 (Enhanced Yoneda Lemma)**.** *The restriction of the functor of points to commutative algebras*

$$\overline{h}\colon \mathbf{Sch}_k \hookrightarrow \mathbf{Fun}(\mathbf{CommAlg}_k, \mathbf{Set})$$

is fully faithful.

Thus, the category of schemes over k can be identified with a certain full subcategory of functors from commutative k-algebras to sets. Corepresentable functors correspond exactly to affine schemes.

Question 3.0.2. How should we characterize this subcategory (the image of $\overline{h}$)?

Definition 3.0.3. Say that a functor $F\colon \mathbf{CommAlg}_k \to \mathbf{Sets}$ is a **scheme-functor** if it comes from to a scheme over k via $\overline{h}$.

Example 3.0.4. Every corepresentable functor $h^A\colon \mathbf{CommAlg}_k \to \mathbf{Set}$, $B \mapsto \mathrm{Hom}_{\mathbf{CommAlg}_k}(A, B)$ is a scheme-functor. In fact, such functors are exactly affine schemes.

For concreteness, if X is the elliptic curve described by $y^2 = x^3 + x^2 + 1$, then $\overline{h}X$ is the functor

$$\begin{aligned} \mathbf{CommAlg}_k &\longrightarrow \mathbf{Set} \\ B &\longmapsto \{(x, y) \in B \times B \mid y^2 = x^3 + x^2 + 1\} \end{aligned}$$

Example 3.0.5. Functors that are not representable may also be scheme-functors. Fix $0 < d < n$, for $n \geq 1$. Define the Grassmannian $\mathbf{Gr}(d, n)$ of d-planes in k^n as the functor that sends a commutative k-algebra B to the set of rank d summands of $B^{\oplus n}$. On morphisms $f\colon B_1 \to B_2$, $\mathbf{Gr}(d, n)$ sends a rank d-summand Q of $B_1^{\oplus n}$ to the rank d-summand $Q \otimes_{B_1} B_2$ of $B_2^{\oplus n}$.

This is the functor of points of the classical Grassmannian.

Exercise 3.0.6. Convince yourself that $\mathbf{Gr}(d, n)$ is not corepresentable.

Theorem 3.0.7 (Grothendieck)**.** *A functor $F\colon \mathbf{CommAlg}_k \to \mathbf{Set}$ has the form $\overline{h}_X$ for some $X \in \mathrm{Ob}(\mathbf{Sch}_k)$ if and only if*

(1) F is a sheaf in the Zariski (Grothendieck) topology on $\mathbf{AffSch}_k = \mathbf{CommAlg}_k{}^{op}$

(2) There exists commutative algebras $A_i \in \mathrm{Ob}(\mathbf{CommAlg}_k)$ and elements

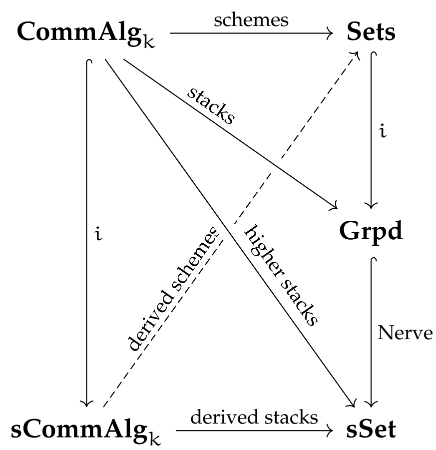
$$\alpha_i \in F(A_i) \overset{\text{Yoneda}}{\cong} \mathrm{Hom}_{\mathbf{Fun}(\mathbf{CommAlg}_k, \mathbf{Set})}(h^{A_i}, F)$$

such that for every field $K \supseteq k$, there is a bijection

$$F(K) \cong \bigcup_i \alpha_i(h^{A_i}(K)),$$

that is, $F(K)$ is covered by $h^{A_i}(K)$ via the α_i.

Derived algebraic geometry (DAG) generalizes classical schemes (viewed as functors $\mathbf{CommAlg}_k \to \mathbf{Set}$) in the following ways:



We need a homotopy theory on simplicial presheaves. Such a thing exists, due to Joyal–Jardine (called local homotopy theory).

Our goal is to describe derived stacks as functors from $\mathbf{sCommAlg}_k \to \mathbf{sSet}$. We will approach this in two steps, first by defining nonderived (higher) stacks as functors $\mathbf{CommAlg}_k \to \mathbf{sSet}$, and then extending to simplicial commutative k-algebras.

3.1 Grothendieck Topology

If X is a topological space, we define (and then replace X by) the category $\mathcal{C} = \mathcal{O}(X)$ of open sets with objects the open sets in X and

$$\operatorname{Hom}_{\mathcal{C}}(U, V) = \begin{cases} \varnothing & U \not\subseteq V \\ \{U \hookrightarrow V\} & U \subseteq V \end{cases}$$

Note that

(1) X is the terminal object in $\mathcal{C}$,

(2) for any finite set I,

$$\bigcap_{i \in I} U_i = \prod_{i \in I} U_i$$

(3) for any set I,

$$\bigcup_{i \in I} U_i = \coprod_{i \in I} U_i$$

(4) A **presheaf** on X is a functor $\mathcal{C}^{op} \to \mathbf{Set}$.

(5) A **sheaf** on X is a presheaf $F \colon \mathcal{C}^{op} \to \mathbf{Set}$ such that for every $U \subseteq X$ open and every open cover $U = \bigcup_{i \in I} U_i$, we have

$$F(U) \cong \operatorname{eq}\left(\prod_{i \in I} F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)\right)$$

Grothendieck's generalization of the notion of topology consists of replacing a space X (or the corresponding category $\mathcal{O}(X)$) by an arbitrary category $\mathcal{C}$ in which we specify systems of coverings for every object $U \in \mathrm{Ob}(\mathcal{C})$.

Definition 3.1.1. A **system of coverings** for $\mathcal{C}$ consists of the data of, for each $U \in \mathrm{Ob}(\mathcal{C})$, a set of morphisms $\mathrm{Cov}(U) = \{U_i \xrightarrow{\phi}_i U\}_{i \in I} \subseteq \mathrm{Mor}(\mathcal{C})$. These data must satisfy:

(C1) For all $U \in \mathrm{Ob}(\mathcal{C})$, $(\mathrm{id} \colon U \to U) \in \mathrm{Cov}(U)$

(C2) For all $f \colon U \to V$ and each $(\phi_i \colon U_i \to U) \in \mathrm{Cov}(U)$, $(U_i \times_U V \to V) \in \mathrm{Cov}(V)$.

(C3) If $(U_i \to U) \in \mathrm{Cov}(U)$ and $(V_{ij} \to U_i) \in \mathrm{Cov}(U_i)$ for each i, then $(V_{ij} \to U_i \to U) \in \mathrm{Cov}(U)$.

Below, we will modify this to take into account the Yoneda embedding $\mathcal{C} \hookrightarrow \widehat{\mathcal{C}} = \mathbf{Fun}(\mathcal{C}^{op}, \mathbf{Set})$. We can define coverings in terms of subfunctors of representable sheaves h_U, called **sieves**.

Let $\mathcal{C}$ be a category. For $X \in \mathrm{Ob}(\mathcal{C})$, let $h_X = \mathrm{Hom}(-, X) \colon \mathcal{C}^{op} \to \mathbf{Set}$. Let $\Pr(\mathcal{C}) = \mathbf{Fun}(\mathcal{C}, \mathbf{Set})$ be the category of presheaves of sets on $\mathcal{C}$. The Yoneda embedding $h \colon \mathcal{C} \hookrightarrow \Pr(\mathcal{C})$ gives a functor $X \mapsto h_X$.

Definition 3.1.2. A **sieve** over $X \in \mathrm{Ob}(\mathcal{C})$ is a presheaf $u \in \Pr(\mathcal{C})$ which comes with a natural $u \hookrightarrow h_X$ (i.e. a subfunctor of h_X).

Note that a sieve u may or may not be representable.

Definition 3.1.3. A **Grothendieck topology** T on $\mathcal{C}$ consists of the data of a family $\mathrm{Cov}(X)$ of sieves over X, for all $X \in \mathrm{Ob}(\mathcal{C})$. These data must satify:

(T1) For all $X \in \mathrm{Ob}(\mathcal{X})$, $h_X \in \mathrm{Cov}(X)$.

(T2) For all $f \colon Y \to X$ and all $u \in \mathrm{Cov}(X)$, $f^*(u) := h_Y \times_{h_X} u \in \mathrm{Cov}(Y)$.

(T3) Let $X \in \mathrm{Ob}(\mathcal{C})$, and let $u \in \mathrm{Cov}(X)$. Let $v \subseteq h_X$ be any sieve over X. If for all $Y \in \mathrm{Ob}(\mathcal{C})$ and all morphisms $f \in u(Y) \subseteq h_X(Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$ such that $f^*(v) \in \mathrm{Cov}(Y)$, then $v \in \mathrm{Cov}(X)$.

Definition 3.1.4. A category $\mathcal{C}$ equipped with a Grothendieck topology T is called a **Grothendieck site**.

Remark 3.1.5. The reason to use presheaves instead of objects in $\mathcal{C}$ to define coverings is that we do not assume $\mathcal{C}$ is cocomplete. In fact, we may not even have coproducts. For example, if X is a topological space, and $\mathcal{C} = \mathcal{O}(X)$ is the category of open subsets of X, objects $U_1, U_2 \in \mathcal{O}(X)$ may not have a coproduct $U_1 \sqcup U_2 \in \mathcal{C}$ if $U_1 \cap U_2 \neq \emptyset$. Moreover, $\{U_1, U_2\}$ is a covering of U if $U_1 \cup U_2 = U$.

To associate a subfunctor to the covering $\{U_1, U_2\}$, we take the coproduct $h_{U_1} \sqcup h_{U_2}$ and consider the two natural maps

$$i_1, i_2 \colon h_{U_1 \cap U_2} \to h_{U_1} \sqcup h_{U_2}$$

coming from the inclusions $i_1 : U_1 \cap U_2 \hookrightarrow U_1$ under the Yoneda embedding. Then the subfunctor associated to this covering is the coequalizer

$$u = \operatorname{coeq}\left(h_{U_1 \cap U_2} \overset{i_1}{\underset{i_2}{\rightrightarrows}} h_{U_1} \sqcup h_{U_2} \right) \hookrightarrow h_U$$

Definition 3.1.6. A **sheaf** on a Grothendieck site $(\mathcal{C}, T)$ is a presheaf $F \colon \mathcal{C}^{op} \to$ **Set** satisfying the **sheaf axiom**: for any $X \in \mathrm{Ob}(\mathcal{C})$ and any $u \in \mathrm{Cov}(X)$, the inclusion $u \hookrightarrow h_X$ induces a bijection of sets

$$F(X) \cong \mathrm{Hom}_{\mathrm{Pr}(\mathcal{C})}(h_X, F) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Pr}(\mathcal{C})}(u, F).$$

Let $\mathrm{Sh}(\mathcal{C})$ be the full subcategory of $\mathrm{Pr}(\mathcal{C})$ satisfying the sheaf axiom.

Lemma 3.1.7. *The inclusion functor* $i \colon \mathrm{Sh}(\mathcal{C}) \hookrightarrow \mathrm{Pr}(\mathcal{C})$ *admits a left adjoint* $a \colon \mathrm{Pr}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{C})$ *which is exact in the sense that it preserves all finite limits.*

For $F \in \mathrm{Pr}(\mathcal{C})$, the sheaf aF is called the **associated sheaf** of F.

3.2 Simplicial Presheaves

Let $(\mathcal{C}, T)$ be a Grothendieck site. Write $\mathrm{sPr}(\mathcal{C})$ for the category of simplicial presheaves in $\mathcal{C}$, that is, simplicial objects in $\mathrm{Pr}(\mathcal{C})$. We may (and will) alternatively think of these as functors from $\mathcal{C}^{op}$ taking values in simplicial sets, because

$$\begin{aligned}
\mathrm{sPr}(\mathcal{C}) &:= \mathbf{Fun}(\Delta^{op}, \mathrm{Pr}(\mathcal{C})) \\
&\cong \mathbf{Fun}(\Delta^{op}, \mathbf{Fun}(\mathcal{C}^{op}, \mathbf{Set})) \\
&\cong \mathbf{Fun}(\Delta^{op} \times \mathcal{C}^{op}, \mathbf{Set}) \\
&\cong \mathbf{Fun}(\mathcal{C}^{op}, \mathbf{Fun}(\Delta^{op}, \mathbf{Set})) \\
&= \mathbf{Fun}(\mathcal{C}^{op}, \mathbf{sSet})
\end{aligned}$$

Definition 3.2.1 (Global Model Structure). Let $f\colon F \to G$ be a morphism in $\mathrm{sPr}(\mathcal{C})$. We say that

(a) f is a **global weak equivalence** if for all $X \in \mathrm{Ob}(\mathcal{C})$, $f(X)\colon F(X) \to G(X)$ is a weak equivalence of simplicial sets.

(b) f is a **global fibration** if for all $X \in \mathrm{Ob}(\mathcal{C})$, $f(X)\colon F(X) \to G(X)$ is a Kan fibration of simplicial sets.

(c) f is a **global cofibration** if it has the left lifting property with respect to all global acyclic fibrations.

Theorem 3.2.2. *This choice of weak equivalences and fibrations makes* $\mathrm{sPr}(\mathcal{C})$ *into a proper cofibrantly generated model category.*

This model structure will only play an auxiliary role. Note that this doesn't depend on the Grothendieck topology T at all. We want a refinement which incorporates data from T. For this, we need to introduce homotopy sheaves of a simplicial presheaf $F \in \mathrm{sPr}(\mathcal{C})$.

Definition 3.2.3. Let $F\colon \mathcal{C}^{op} \to$ **sSet** be a simplicial presheaf on $\mathcal{C}$. We define the **homotopy presheaves** of $F \in \mathrm{sPr}(\mathcal{C})$ as follows. For $i = 0$, define

$$\widetilde{\pi}_0(F)\colon \mathcal{C}^{op} \xrightarrow{(} F)\, \mathbf{sSet} \xrightarrow{\pi_0} \mathbf{Set}$$

For $i > 0$, and for $X \in \mathrm{Ob}(\mathcal{C})$, choose $p \in F(X)_0$ (a zero-simplex in $F(X)$). Then define the i-th homotopy presheaf

$$\begin{aligned} \widetilde{\pi}_i(F,p)\colon (C \downarrow X)^{op} &\longrightarrow \mathbf{Gr} \subseteq \mathbf{Set} \\ \left(Y \xrightarrow{f} X\right) &\longmapsto \pi_i\left(|F(Y)|, f^*(p)\right) \end{aligned}$$

where $f^*\colon F(X) \to F(Y)$ is the pullback $p \mapsto f^*(p)$.

The **homotopy sheaves** of F are the sheafifications of the homotopy presheaves: $\pi_0(F) = a\widetilde{\pi}_0(F)$ and $\pi_i(F,p) = a\widetilde{\pi}_i(F,p)$.

Definition 3.2.4 (Local Model Structure). Given $f\colon F \to G$ in $\mathrm{sPr}(\mathcal{C})$, we say that

(a) f is a **local weak equivalence** if

(i) $\pi_0(f)\colon \pi_0(F) \to \pi_0(G)$ is an isomorphism of sheaves

(ii) for all $X \in \mathrm{Ob}(\mathcal{C})$ and all $p \in F(X)_0$, f induces isomorphisms

$$\pi_i(F,p) \xrightarrow{\sim} \pi_i(G, f^*p)$$

for all $i \geq 1$.

(b) f is a **local cofibration** if it is a global cofibration.

(c) f is a **local fibration** if it has the right lifting property with respect to all local acylic cofibrations.

Theorem 3.2.5 (Jardine,Joyal)**.** *Equipped with the local weak equivalence, local cofibrations, and local fibrations,* $\mathrm{sPr}(\mathcal{C})$ *is a model category.*

When we refer to the model structure on simplicial presheaves, we are referring to the local model structure.

Once we have a model structure, we want to identify the fibrant objects. The claim is that the fibrant objects are exactly the algebraic stacks. We follow a paper of Dugger, Hollandr and Isaksen "Hypercovers and simplicial presheaves" (2004).

The key to finding the fibrant objects of this model structure is to use the notion of a **hypercovering** (Verdier) following Dugger, Hollandr and Isaksen.

Definition 3.2.6. A **hypercovering** of $X \in \mathrm{Ob}(\mathcal{C})$ is a simplicial presheaf $H \in \mathrm{sPr}(\mathcal{C})$ with a morphism $p\colon H \to h_X$, called the **covering map**, such that

(a) for all $n \geq 0$, the presheaf $H_n\colon \mathcal{C}^{op} \xrightarrow{H} \mathbf{sSet} \xrightarrow{(-)_n} \mathbf{Set}$ is a disjoint union of representable presheaves: there are $X_{n,i} \in \mathrm{Ob}(\mathcal{C})$ such that

$$H_n = \coprod_i h_{X_{n,i}}.$$

(b) for all $n \geq 0$, the map of presheaves

$$H_n(-) \cong \mathrm{Hom}(\Delta[n], H) \to \mathrm{Hom}(\partial\Delta[n], H) \times_{\mathrm{Hom}(\partial\Delta[n], h_X)} \mathrm{Hom}(\Delta[n], h_X)$$

induces an epimorphism of the associated presheaves.

Remark 3.2.7. Condition (b) can be equivalently restated as a local lifting property: for any $Y \in \mathrm{Ob}(\mathcal{C})$ and any commutative diagram

$$\begin{array}{ccc} \partial\Delta[n] & \longrightarrow & H(Y) \\ \downarrow & & \downarrow \mathrm{Pr} \\ \Delta[n] & \longrightarrow & h_X(Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X) \end{array}$$

there is a covering sieve $u \in \mathrm{Cov}(Y)$ such that for all $(f\colon Y \to Y) \in u(Y) \subseteq \mathrm{Hom}(U, Y)$, there is a morphism $\Delta[n] \to H(u)$ such that the following diagram commutes:

$$\begin{array}{ccc} \partial\Delta[n] & \longrightarrow & H(u) \\ \downarrow & \nearrow & \downarrow \\ \Delta[n] & \longrightarrow & h_X(u). \end{array}$$

This is a local analogue of the lifting property characterizing acyclic fibrations of **sSet**. In particular, $p\colon H \to h_X$ is a local weak equivalence.

Additionally, if $n \geq 2$, we can restate (b) as follows: $H_n \to \mathrm{Hom}(\partial\Delta[n], H)$ induces an epimorphism of associated sheaves.

Note that this is independent of X and the covering map.

Let $F \in \mathrm{sPr}(\mathcal{C})$. For any $X \in \mathrm{Ob}(\mathcal{C})$ and any hypercovering $p\colon H \to h_X$, define a coaugmented cosimplicial diagram of simplicial sets

$$F(H_*)\colon \Delta \to \mathbf{Set}, \qquad [n] \mapsto F(H_n)$$

as follows:

for $n \geq 0$, a morphism $p\colon H \to h_X \in \mathrm{sPr}(\mathcal{C})$, where we consider $h_X \in \mathrm{Pr}(\mathcal{C}) \hookrightarrow \mathrm{sPr}(\mathcal{C})$ as constant in the simplicial direction, corresponds to a series of maps $\{p_n\colon H_n \to h_X\}_{n\geq 0}$. Applying $\mathrm{Hom}(-,F)$ and using Definition 3.2.6(a), we have

$$\begin{aligned} F_X \cong \mathrm{Hom}_{h_X,F} \xrightarrow{p_n^*(X)} \mathrm{Hom}(H_n, F) &\overset{(a)}{=} \mathrm{Hom}\left(\coprod_i h_{X_{n,i}}, F\right) \\ &\cong \prod_i \mathrm{Hom}(h_{X_{n,i}}, F) \\ &\cong \prod_i F(X_{n,i}) \end{aligned}$$

We call this last object $F(H_n)$. Thus we have a map of cosimplicial diagrams

$$F(X) \to F(X_*)$$

forming the coaugmentation map. Note that $F(X)$ is a constant diagram in the simplicial direction, while $F(H_*)$ is not. This gives

$$\alpha\colon F(X) \to \mathrm{hocolim}_{[n]\in\Delta} F(H_n). \tag{3.2.1}$$

Theorem 3.2.8 (Dugger–Hollandr–Isaksen). *A simplicial presheaf $F \in \mathrm{sPr}(\mathcal{C})$ is fibrant if and only if*

(a) for all $X \in \mathrm{Ob}(\mathcal{C})$, $F(X)$ is fibrant in **sSet** *(i.e. a Kan complex)*

(b) for all $X \in \mathrm{Ob}(\mathcal{C})$ and any hypercovering $p\colon H \to h_X$ over X, the map α is a weak equivalence of simplicial sets.

Definition 3.2.9. A simplicial presheaf is a **stack** if it satisfies Theorem 3.2.8(b).

Therefore, by Theorem 3.2.8, fibrant objects of $\mathrm{sPr}(\mathcal{C})$ are stacks! (This is true because every object is weak equivalent to a fibrant object; we may as well take fibrant replacements.)

Definition 3.2.10. (a) $\mathbf{Ho}(\mathrm{sPr}(\mathcal{C}))$ is the category of stacks.

(b) For all $F, F' \in \mathbf{Ho}(\mathrm{sPr}(\mathcal{C}))$, we write $[F, F'] = \mathrm{Hom}_{\mathbf{Ho}(\mathrm{sPr}(\mathcal{C}))}(F, F')$ for the set of stack morphisms from F to F'.

Proposition 3.2.11. *Take a presheaf* $F\colon \mathcal{C}^{op} \to \mathbf{Set}$*, viewed as a constant simplicial presheaf. Then* F *is a stack if and only if it is a sheaf.*

Proof sketch. This is because the homotopy colimit in (3.2.1) becomes a limit and condition (b) becomes the usual sheaf axiom: for all $X \in \mathrm{Ob}(\mathcal{C})$ and any $u \in \mathrm{Cov}(X)$, $u \hookrightarrow h_X$ induces a bijection $F(X) = \mathrm{Hom}(h_X, F) \xrightarrow{\sim} \mathrm{Hom}(u, F)$. □

The moral is that condition Theorem 3.2.8(b) is a homotopy analogue of the sheaf axiom. Put differently, a presheaf is fibrant in $\mathrm{sPr}(\mathcal{C})$ if and only if it is a sheaf. Therefore, the local homotopy theory of $\mathrm{sPr}(\mathcal{C})$ knows the relation between sheaves and presheaves.

Corollary 3.2.12. *Let* $F \in \mathrm{Pr}(\mathcal{C})$*. Consider* F *as a constant simplicial presheaf, and let* $aF \in \mathrm{Sh}(\mathcal{C})$ *be the associated sheaf. Then for all* $G \in \mathrm{sPr}(\mathcal{C})$*, not necessarily constant, we have*

(a) $[G, F] \cong [G, aF] \cong \mathrm{Hom}_{\mathrm{sPr}(\mathcal{C})}(G, aF)$

(b) Moreover, if $G \in \mathrm{Pr}(\mathcal{C}) \hookrightarrow \mathrm{sPr}(\mathcal{C})$*, then*

$$[G, F] \cong [aG, aF] \cong \mathrm{Hom}_{\mathrm{sPr}(\mathcal{C})}(aG, aF) \cong \mathrm{Hom}_{\mathrm{Pr}(\mathcal{C})}(aG, aF) \cong \mathrm{Hom}_{\mathrm{Sh}(\mathcal{C})}(aG, aF).$$

Corollary 3.2.13. *The natural functor* $\mathrm{Sh}(\mathcal{C}) \hookrightarrow \mathrm{Pr}(\mathcal{C}) \hookrightarrow \mathrm{sPr}(\mathcal{C}) \to \mathbf{Ho}(\mathrm{sPr}(\mathcal{C}))$ *is fully faithful. Moreover, if it has a left adjoint*

$$\begin{array}{ccccc} \pi_0(\mathrm{sPr}(\mathcal{C})) & \longrightarrow & \mathrm{Pr}(\mathcal{C}) & \xrightarrow{a} & \mathrm{Sh}(\mathcal{C}) \\ F & \longmapsto & \widetilde{\pi}_0(F) & \longmapsto & a\widetilde{\pi}_0(F). \end{array}$$

So the category of stacks is a natural extension of the category of sheaves.

Remark 3.2.14 (Warning)**.** In general, sheaves of sets are exactly stacks, but sheaves of simplicial stacks are *far* from being stacks. For example, take a sheaf of groups $G\colon \mathcal{C}^{op} \to \mathbf{Gr}$ and define the simplicial sheaf

$$\begin{array}{rcl} BG\colon \mathcal{C}^{op} & \longrightarrow & \mathbf{sSet} \\ X & \longmapsto & N_*(G(X)) \end{array}$$

where $N_n(G(X)) = G(X)^n$. Then BG is a sheaf of simplicial sets, but it is not, in general a stack. The obstruction lies in $H^1(X, G)$, which may be nontrivial.

3.3 Example: Smooth Manifolds

We will consider simplicial presheaves on smooth manifolds following Hopkins–Freed 2013 and Joyce 2011. Let $\mathcal{C} = \mathbf{Man}$ be the category of smooth, finite dimensional manifolds over $\mathbb{R}$ with smooth maps.

The category **Man** has a natural Grothendieck topology: covering are usual open coverings of manifolds, i.e. a covering of X is a collection $\mathcal{U} = \{U_I\}_{i \in I}$ of manifolds such that $\bigcup_{i \in I} U_i = X$. Associated to $\mathcal{U}$ is a sieve

$$u = \mathrm{eq}\left(\prod_{i,j \in I} h_{U_i \cap U_j} \rightrightarrows \prod_{i \in I} h_{U_i}\right) \hookrightarrow h_X.$$

We identify a smooth manifold $X \in \mathrm{Ob}(\mathbf{Man})$ with the functor it represents, namely $h_X \colon \mathbf{Man}^{\mathrm{op}} \to \mathbf{Set}$, and the associated representable presheaf. Therefore, $\mathrm{Pr}(\mathbf{Man})$ should be considered as a sort of category of generalized manifolds, in the sense that every presheaf $F \colon \mathbf{Man}^{\mathrm{op}} \to \mathbf{Set}$ is determined by its action on test objects ($Y \in \mathbf{Man}$). The point is to extend differential geometric constructions from usual manifolds to these generalized manifolds.

Example 3.3.1 (Differential forms as generalized manifolds)**.** For $p \geq 0$ and $X \in \mathrm{Ob}(\mathbf{Man})$, consider the space $\Omega^p(X)$ of smooth p-forms on X. In local coordinates, we may write $\omega \in \Omega^p(X)$ as

$$\omega = \sum_{i_1, \ldots, i_p} f_{i_1, \ldots, i_p}(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

Since for any $f \colon X \to Y$ in $\mathrm{Mor}(\mathbf{Man})$, there is a pullback map $f^* \colon \Omega^p(Y) \to \Omega^p(X)$, so

$$\Omega^p \colon \mathbf{Man}^{\mathrm{op}} \to \mathbf{Set}$$

is a presheaf on **Man** that is not itself a manifold.

In particular, when $p = 0$, $\Omega^0(X) = C^\infty(X)$ is the set of smooth functions on X, so Ω^0 is a sheaf on **Man**. This is in fact true for all p – Ω^p is a sheaf on manifolds, and therefore a stack on **Man** as well by Proposition 3.2.11.

Example 3.3.2 (G-connections)**.** Let G be a Lie group. Let $\pi \colon P \to X$ be a principal G-bundle on X. Recall that π is a locally trivial fiber bundle such that P is equipped with a right G-action $P \times G \to P$ such that each fiber $\pi^{-1}(X)$, $x \in X$, is preserved under this action and the restriction of this G-action to each fiber $\pi^{-1}(X)$ is free and transitive. That is, for each $p \in \pi^{-1}(X)$, the action map $G \to \pi^{-1}(x)$, $g \mapsto p \cdot g$, is a diffeomorphism.

A **connection** on P is a natural way to relate different fibers. More precisely, a G-connection on P is for each $p \in P$ a direct sum decomposition of the tangent space $T_p P$ which is invariant under the action of G:

$$T_p P \cong H_p \oplus V_p,$$

where $V_p := T_p(\pi^{-1}(\pi(p))) \hookrightarrow T_p P$ is the **subspace of vertical vectors**, contained canonically in $T_p P$. H_p is called the **subspace of horizontal vectors**.

Thus, a connection is determined by the choice of a subspace $H_p \subseteq T_p P$ of horizontal vectors for each $p \in P$. The assignment $p \mapsto H_p$ forms a distribution on P.

Definition 3.3.3 (Ehresmann)**.** A G**-connection** is a distribution on the total space (i.e. a subbundle of $T_* P$) which is G-invariant and transverse to fibers.

Note that for each $p \in P$, the differential of the natural map $i_p \colon G \to \pi^{-1}(\pi(p))$, $g \mapsto p \cdot g$, identifies the Lie algebra of G with the subspace of vertical vectors:

$$\mathfrak{g} = T_e G \cong T_p \pi^{-1}(\pi(p)) = V_p.$$

A G-connection is thus determined by the projection

$$\Theta_p \colon T_p P \to V_p \cong \mathfrak{g}$$

such that $\Theta_p \circ i_p = \mathrm{id}$. That is, by $\mathfrak{g}$-valued 1-form $\Theta \in \Omega^1(P) \otimes \mathfrak{g}$ on P. This is called the **connection form** or simply the **connection**. Write $\Omega^1(P, \mathfrak{g})$ for the $\mathfrak{g}$-valued 1-forms on P.

Formally, Θ is characterized by two properties:

(a) Θ is G-invariant, i.e. $\Theta \in \Omega^1(P, \mathfrak{g})^G$. This means $R_g^* \Theta = \mathrm{Ad}_{g^{-1}} \Theta$ where $R_g \colon P \to P$ is the right action of $g \in G$ on P.

(b) for each $p \in P$, the pullback of Θ along $i_p \colon G \to \pi^{-1}(\pi(p)) \hookrightarrow P$ is equal to the Maurer-Cartan form θ_{MC}.

$$i_p^* \Theta = \theta_{MC}.$$

(If G is a matrix group, then $\theta_{MC} = g^{-1}\, dg$.)

We want to classify principal bundles with connection.

Example 3.3.4. Fix a Lie group G. Define a presheaf $F \colon \mathbf{Man}^{op} \to \mathbf{Set}$ where $F(X)$ is the set of isomorphism classes of G-connections on X.

$$F(X) = \{(\pi \colon P \to X, \theta)\} / \sim$$

where $(P, \Theta) \sim (P', \Theta')$ if and only if there is an isomorphism of principal G-bundles $\phi \colon P \to P'$ over X such that $\Theta = \phi^* \Theta'$.

Note that F is a presheaf but it is *not* a sheaf. Take $X = S^1$, covered by two disjoint open intervals U_1 and U_2, omitting either $0 \in S^1$ or $\pi \in S^1$. If F were a sheaf, then we would have a Cartesian diagram

$$\begin{array}{ccc} F(S^1) & \longrightarrow & F(U_1) \\ \downarrow & & \downarrow \\ F(U_2) & \longrightarrow & F(U_1 \cap U_2). \end{array}$$

However, since (flat) G-connections are determined by equivalence classes of their holonomy representations $\rho \colon \pi_1(X) \to G$ under the adjoint action. The equivalence classes of holonomy representations in this case are

$$\begin{array}{ccc} {}^{G}/_{G} & \longrightarrow & \{*\} \\ \downarrow & & \downarrow \\ \{*\} & \longrightarrow & \{*, *\} \end{array}$$

but this is not a Cartesian diagram, so F is not a sheaf.

So in our quest to classify connection up to isomorphism, we will instead classify connections together with isomorphisms.

Definition 3.3.5. Define a presheaf of groupoids $\underline{B_\nabla G}$ as follows. For a manifold X, the objects of $\underline{B_\nabla G}(X)$ are pairs (π, Θ) with $\pi \colon P \to X$ a principal G-bundle over X and $\Theta \in \Omega^1(P, \mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G.

Morphisms in $\underline{B_\nabla G}(X)$ are commutative diagrams

$$\begin{array}{ccc} P & \xrightarrow[\sim]{\phi} & P' \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi'} & \\ & X & \end{array}$$

such that $\phi^*\Theta = \Theta'$.

Definition 3.3.6. Similarly, define a presheaf of groupoids $\underline{E_\nabla G}$ as follows. For a manifold X, the objects of $\underline{E_\nabla G}(X)$ are triples (π, Θ, s) with $\pi \colon P \to X$ a principal G-bundle, $\Theta \in \Omega^1(P, \mathfrak{g})$, and $s \colon X \to P$ a global section of π. Morphisms in $\underline{E_\nabla G}(X)$ are commutative diagrams of the form

$$\begin{array}{ccc} & X & \\ & {\scriptstyle s}\swarrow \quad \searrow{\scriptstyle s'} & \\ P & \xrightarrow[\sim]{\phi} & P' \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi'} & \\ & X & \end{array}$$

such that $\phi^*\Theta' = \Theta$.

Definition 3.3.7. Define simplicial presheaes $B_\nabla G$ and $E_\nabla G$ by

$$\begin{aligned} B_\nabla G\colon \mathbf{Man} &\longrightarrow \mathbf{sSet} \\ X &\longmapsto \mathcal{N}_*(\underline{B_\nabla G}) \\ E_\nabla G\colon \mathbf{Man} &\longrightarrow \mathbf{sSet} \\ X &\longmapsto \mathcal{N}_*(\underline{E_\nabla G}) \end{aligned}$$

Theorem 3.3.8 (Freed–Hopkins)**.** *Both* $B_\nabla G$ *and* $E_\nabla G$ *are stacks (i.e. they satisfy the homotopy sheaf axiom). Moreover,* $E_\nabla G$ *is weakly equivalent to the sheaf* $\Omega^1 \otimes \mathfrak{g}$*, viewed as a discrete simplicial presheaf.*

Proof sketch. We only define the maps inducing the weak equivalence $E_\nabla G \simeq \Omega^1 \otimes \mathfrak{g}$. At the level of groupoids, we have

$$\alpha\colon \underline{E_\nabla G} \to \Omega^1 \otimes \mathfrak{g} \tag{3.3.1}$$

and its homotopy inverse

$$\beta\colon \Omega^1 \otimes \mathfrak{g} \to \underline{E_\nabla G}.$$

Define α and β for a test manifold X by

$$\begin{aligned} \alpha_X\colon \underline{E_\nabla G} &\longrightarrow \Omega^1(X) \otimes \mathfrak{g} \\ (\pi, \Theta, s) &\longmapsto s^*\Theta \in \Omega^1(X, \mathfrak{g}) \end{aligned}$$

$$\begin{aligned} \beta_X\colon \Omega^1(X) \otimes \mathfrak{g} &\longrightarrow \underline{E_\nabla G}(X) \\ \omega &\longmapsto (\pi_\omega, \Theta_\omega, s_\omega) \end{aligned}$$

where

$$\pi_\omega\colon X \times G \to G$$

is the trival G-bundle,

$$\begin{aligned} s_\omega\colon X &\longrightarrow X \times G \\ x &\longmapsto (x, e_G) \end{aligned}$$

is the identity trivialization, and

$$\Theta_\omega = \pi_G^*\omega + \pi_G^*\theta_{MC}$$

where θ_{MC} is the Maurer-Cartan 1-form on G and $\pi_G\colon X \times G \to G$ is projection onto the second factor.

Evidently, $\alpha_X \circ \beta_X$ is the identity, and conversely, we have the commutative diagram

$$\begin{array}{ccccc}
\underline{E_\nabla G}(X) & \xrightarrow{\alpha_X} & \Omega^1(X, \mathfrak{g}) & \xrightarrow{\beta_X} & \underline{E_\nabla G}(X) \\
\Vert & & & & \downarrow \phi \\
\underline{E_\nabla G} & & \xrightarrow{\mathrm{id}_{\underline{E_\nabla G}}} & & \underline{E_\nabla G}
\end{array}$$

$$\begin{array}{ccccc}
(\pi, \Theta, s) & \xmapsto{\alpha_X} & s^*\Theta = \omega & \xmapsto{\beta_X} & (\pi_\omega, \Theta_\omega, s_\omega) \\
\Vert & & & & \downarrow \phi \\
(\pi, \Theta, s) & & \xmapsto{\mathrm{id}_{\underline{E_\nabla G}}} & & (\pi, \Theta, s)
\end{array}$$

where ϕ is the morphism

$$\begin{array}{ccc}
X \times G & \xrightarrow[\sim]{\phi} & P \\
& {\scriptstyle \pi_\omega} \searrow \quad \swarrow {\scriptstyle \pi} & \\
& X &
\end{array}$$

is given explicitly by $(x, g) \mapsto ?$. □

3.3.1 Differential forms on stacks

If we want to extend a natural geometric construction on manifolds to generalized manifolds $\mathbf{Man}^{op} \to \mathbf{Set}$, we have to express the construction in terms of h_X and then replace h_X by an arbitrary functor $F\colon \mathbf{Man}^{op} \to \mathbf{Set}$.

If we identify a smooth manifold X with the corresponding presheaf $h_X\colon \mathbf{Man}^{op} \to \mathbf{Set}$, $Y \mapsto \mathrm{Hom}(Y, X)$, then by the Yoneda Lemma,

$$\mathrm{Hom}_{\mathrm{Pr}(\mathbf{Man})}(h_X, \Omega^p) \cong \Omega^p(X).$$

Therefore, we may switch both sides of this formula and use it for a definition of differential forms on an arbitrary presheaf $F \in \mathrm{Pr}(\mathbf{Man})$.

Definition 3.3.9. For any $F\colon \mathbf{Man}^{op} \to \mathbf{Set}$, we may define the **differential p-forms on F** by

$$\Omega^p(F) := \mathrm{Hom}_{\mathrm{Pr}(\mathbf{Man})}(F, \Omega^p).$$

For example, take $F = \Omega^q$ for fixed $q \geq 0$. Then look at

$$\Omega^p(\Omega^q) = \mathrm{Hom}_{\mathrm{Pr}(\mathbf{Man})}(\Omega^q, \Omega^p).$$

This is differential p-forms on differential q-forms. Explicitly, an element $\tau \in \Omega^p(\Omega^q)$ is a natural construction of p-forms from q-forms on manifolds in the sense that for all $f\colon X \to Y$, the following diagram commutes

$$\begin{array}{ccc} \Omega^q(Y) & \xrightarrow{\tau_Y} & \Omega^p(Y) \\ \big\downarrow f^* & & \big\downarrow f^* \\ \Omega^q(X) & \xrightarrow{\tau_X} & \Omega^p(X) \end{array}$$

Example 3.3.10. Take $p = q$, and consider the p-form of p-forms given by $\tau = \underline{\omega}^p\colon \Omega^p \to \Omega^p$, $\omega^p_X = \mathrm{id}_{\Omega^p(X)}$. Then $\underline{\omega}^p \in \Omega^p(\Omega^p)$.

Now take $q = 1$ and consider $\Omega^p(\Omega^1)$.

Theorem 3.3.11 (Freed–Hopkins)**.**

(a) $\Omega^p(\Omega^1) = \begin{cases} \mathrm{Span}_{\mathbb{R}}\{d\underline{\omega}^1 \wedge \overset{p/2}{\cdots} d\underline{\omega}^1\} & p \textit{ even} \\ \mathrm{Span}_{\mathbb{R}}\{\underline{\omega}^1 \wedge d\underline{\omega}^1 \wedge \overset{(p-1)/2}{\cdots} d\underline{\omega}^1\} & p \textit{ even} \end{cases}$

(b) The de Rham complex of Ω^1 looks like

$$\Omega^\bullet(\Omega^1) = \left[\mathbb{R} \xrightarrow{0} \mathbb{R} \xrightarrow{1} \mathbb{R} \xrightarrow{0} \cdots\right]$$

so

$$H^p_{dR}(\Omega^1, \mathbb{R}) = \begin{cases} \mathbb{R} & (p = 0) \\ 0 & (p > 0) \end{cases}$$

Definition 3.3.12. For any simplicial presheaf $\mathcal{F} \in \mathbf{Ho}(\mathrm{sPr}(\mathbf{Man}))$, define the **de Rham complex of** $\mathcal{F}$ by

$$\Omega^\bullet(\mathcal{F}) := [\mathcal{F}, \Omega^\bullet].$$

Recall that $[-,-] = \mathrm{Hom}_{\mathbf{Ho}(\mathrm{sPr}(\mathbf{Man}))}(-,-)$. Since each Ω^p is a sheaf and hence a fibrant object in $\mathrm{sPr}(\mathbf{Man})$, by **??** we can compute the de Rham complex

$$\begin{aligned} \Omega^\bullet(\mathcal{F}) &:= [\mathcal{F}, \Omega^\bullet] \\ &\cong \mathrm{Hom}_{\mathrm{sPr}(\mathbf{Man})}(\mathcal{F}, \Omega^\bullet) \\ &\cong \mathrm{eq}\left(\mathrm{Hom}_{\mathrm{Pr}(\mathbf{Man})}(\mathcal{F}_0, \Omega^\bullet) \rightrightarrows \mathrm{Hom}_{\mathrm{Pr}(\mathbf{Man})}(\mathcal{F}_1, \Omega^\bullet)\right) \\ &= \mathrm{eq}\left(\Omega^\bullet(\mathcal{F}_0) \underset{d_0^*}{\overset{d_1^*}{\rightrightarrows}} \Omega^\bullet(\mathcal{F}_1) \right) \\ &\cong \ker\left(\Omega^\bullet(\mathcal{F}_0) \xrightarrow{d_1^* - d_0^*} \Omega^\bullet(\mathcal{F}_1)\right) \end{aligned}$$

Some people take this as the definition of the de Rham complex of a simplicial presheaf $\mathcal{F}$. But this is not evidently homotopy invariant, while our definition is.

3.3.2 Universal G-connection

Definition 3.3.13. Let $F\colon \mathbf{Man}^{op} \to \mathbf{Set}$ be a presheaf on **Man**, and $G = h_G$ a Lie group. A G**-action on** F is defined by

$$a\colon G \times F \to F$$

such that on each test manifold X,

$$a_X\colon \operatorname{Hom}(X, G) \times F(X) \to F(X)$$

is an action of the over group Hom(X, G) on F(X).

Definition 3.3.14. Associated to a G-action on F is a presheaf of action groupoids $\underline{G \ltimes F}$. For a test manifold X, the objects of $\underline{G \ltimes F}(X)$ are F(X) and the morphisms of $\underline{G \ltimes F}(X)$ are $\operatorname{Hom}(X, G) \times F(X)$ where we consider a pair $(g\colon X \to G, \xi)$ as a morphism with source ξ and target $a_X(g, \xi) = g \cdot \xi$.

This has an associated simplicial presheaf $G \ltimes F$ given by applying the nerve construction.

Apply this construction to $F = \Omega^1 \otimes \mathfrak{g}$. This has a natural obvious right G-action, namely

$$a\colon (\Omega^1 \otimes \mathfrak{g}) \times G \to \Omega^1 \otimes \mathfrak{g}$$

with

$$a_X\colon (\Omega^1(X) \otimes \mathfrak{g}) \times \operatorname{Hom}(X, G) \to \Omega^1(X) \otimes \mathfrak{g}$$

given by

$$a_X(\omega, g\colon X \to G) = g^*\theta_{MC} + \operatorname{Ad}_{g^{-1}}(\omega).$$

Define

$$B_\nabla^{\mathrm{triv}} G := (\Omega^1 \otimes \mathfrak{g}) \ltimes G \in \mathrm{sPr}(\mathbf{Man}).$$

There is a natural map of presheaves of groupoids

$$\Psi\colon \underline{B_\nabla^{\mathrm{triv}} G} \to \underline{B_\nabla G}$$

given on objects by

$$\Psi_X\colon \Omega^1(X) \otimes \mathfrak{g} \longrightarrow \{(\pi, \Theta)\}$$

$$\omega \longmapsto (\pi_\omega, \Theta_\omega)$$

where $\pi_\omega\colon X \times G \to X$ is a trivial bundle and $\Theta_\omega = \pi_G^*\omega + \pi_G^*\Theta_{MC}$.

Proposition 3.3.15. Ψ *is a weak equivalence.*

Proposition 3.3.16. *The following diagram commutes*

$$\begin{array}{ccc} \Omega^1 \otimes \mathfrak{g} & \xrightarrow{can} & B^{\mathrm{triv}}_{\nabla} G \\ \mathfrak{P} \downarrow & & \mathfrak{p} \downarrow \\ E_{\nabla} G & \xrightarrow{p} & B_{\nabla} G \end{array}$$

Therefore, we may view $E_{\nabla}G \to B_{\nabla}G$ as a generalized principal G-bundle. Note that β is a G-equivariant map an G acts on $E_{\nabla}G$ freely.

Recall the map $\alpha\colon E_{\nabla}G \to \Omega^1 \otimes \mathfrak{g}$. We have

$$\alpha \in [E_{\nabla}G, \Omega^1 \otimes \mathfrak{g}] \cong [E_{\nabla}G, \Omega^1] \otimes \mathfrak{g} \cong \Omega^1(E_{\nabla}G) \otimes \mathfrak{g}.$$

Hence, α may be considered a G-connection on $E_{\nabla}G \xrightarrow{p} B_{\nabla}G$.

Theorem 3.3.17 (Freed–Hopkins)**.** $\Theta^{un} := \alpha$ *is the universal* G*-connection in the sense that given any pair* (π, Θ) *of a principal* G*-bundle* $\pi\colon P \to X$ *and a connection* $\Theta \in \Omega^1(P, \mathfrak{g})$*, there is a unique classifying map* $f\colon X \to B_{\nabla}G$

$$\begin{array}{ccc} P & \xrightarrow{f} & E_{\nabla}G \\ \downarrow \pi & & \downarrow p \\ X & \xrightarrow{\bar{f}} & B_{\nabla}G \end{array}$$

such that $\Theta = f^*\Theta^{un}$.

Remark 3.3.18. This is to be compared with classical classification of principal bundles G-bundles without connection: given any Lie group G, there are two spaces EG and BG, both infinite–dimensional manifolds, defined up to homotopy, such that for any principal bundle $\pi\colon P \to X$, there is a map $f\colon X \to BG$ such that $P \cong f^*EG$.

$$\begin{array}{ccc} P & \longrightarrow & EG \\ \downarrow \pi & & \downarrow p \\ X & \xrightarrow{f} & BG. \end{array}$$

The map f is defined only up to homotopy, in contrast to the classifying map in Theorem 3.3.17. One point in common is that in both cases, the classifying spaces are not usual manifolds. We leave the category of smooth manifolds for either the homotopy category in the classical case or generalized manifolds in Theorem 3.3.17.

Example 3.3.19. The homomorphism $p^*\colon \Omega^\bullet(B_{\nabla}G) \to \Omega^\bullet(E_{\nabla}G)$ can be realized as a universal Chern–Weil map:

$$\Omega^\bullet(B_{\nabla}G) \cong \mathrm{Sym}(\mathfrak{g}^*)^G \hookrightarrow \Lambda(\mathfrak{g}^*) \otimes \mathrm{Sym}(\mathfrak{g}^*) = W(\mathfrak{g}) \cong \Omega^\bullet(E_{\nabla}G).$$

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