HOMOTOPY THEORY READING GROUP: RECAP ON WEAK 2-LIMITS IN THE HOMOTOPY 2-CATEGORY

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The 2-categorical definition of adjunctions between ∞-categories does not capture the expected universal properties of cones nor do we see how an adjunction $f \dashv u$ induces an isomorphism in "hom-spaces" $\mathrm{Hom}_A(fb, a) \simeq \mathrm{Hom}_B(b, ga)$ for generalized elements. In order to define a notion of ∞-category of cones and these hom-spaces, we need an appropriate notion of comma ∞-category which is in a sense weaker than the standard definition of a comma object in a strict 2-category. This weaker notion of universal property will be expressed via a *smothering functor*.

1. Smothering Functors

Definition 1. A functor $f : A \to B$ between 1-categories is **smothering** if it is surjective on objects, full, and conservative (Namely, it reflects invertibility of morphisms but is neither injective on objects nor faithful).

Example. If $\mathcal{Q}$ is a quasi-category and $\mathcal{J}$ is a 1-category, then one might be interested in understanding the relationship between the homotopy category of $\mathcal{Q}^{\mathcal{J}}$ and the category of $\mathcal{J}$-diagrams in the homotopy category of $\mathcal{Q}$. There is a canonical functor $\mathbf{h}\left(\mathcal{Q}^{\mathcal{J}}\right) \to (\mathbf{h}\mathcal{Q})^{\mathcal{J}}$ defined by applying $\mathbf{h} : \mathcal{Q}\mathbf{Cat} \to \mathbf{Cat}$ to the evaluation functor $\mathcal{Q}^{\mathcal{J}} \times \mathcal{J} \to \mathcal{Q}$ and taking its adjoint mate. One can show that if $\mathcal{J}$ is free on a directed graph, then this canonical functor is smothering (Lemma 3.1.4).

Lemma 2. *Each fibre of a smothering functor is a non-empty connected groupoid.*

Lemma 3. *For any pullback diagram of quasi-categories in which p is an isofibration*

$$\begin{array}{ccc} A \times_B E & \longrightarrow & E \\ \downarrow & & \downarrow p \\ A & \longrightarrow & B \end{array}$$

the canonical functor $\mathbf{h}(A \times_B E) \to \mathbf{h}A \times_{\mathbf{h}B} \mathbf{h}E$ is smothering.

Lemma 4. *For any tower of isofibrations between quasi-categories*

$$\cdots \twoheadrightarrow E_n \twoheadrightarrow E_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow E_2 \twoheadrightarrow E_1 \twoheadrightarrow E_0$$

the canonical functor $\mathbf{h}(lim_n E_n) \to lim_n \mathbf{h}E_n$ is smothering.

Lemma 5. *For any cospan between quasi-categories $C \overset{g}{\to} A \overset{f}{\leftarrow} B$ consider the quasi-category defined by the pullback*

$$\begin{array}{ccc} Hom_A(f, g) & \longrightarrow & A^{\mathbf{2}} \\ {\scriptstyle (p_1, p_0)}\downarrow & & \downarrow{\scriptstyle (cod, dom)} \\ C \times B & \xrightarrow[g \times f]{} & A \times A \end{array}$$

The canonical functor $\mathbf{h}Hom_A(f,g) \to Hom_{\mathbf{h}A}(\mathbf{h}f, \mathbf{h}g)$ *is smothering.*

Now, we see how these smothering functors express a "weak" notion of universal properties in the homotopy 2-category of any ∞-cosmos.

2. ∞-categories of arrows

Definition 6. Let A be an ∞-category. The **∞-category of arrows** in A is the simplicial cotensor $A^{\mathbf{2}}$ together with the canonical endpoint-evaluation isofibration

$$A^{\mathbf{2}} : A^{\Delta[1]} \overset{(p_0,p_1)}{\twoheadrightarrow} A^{\partial\Delta[1]} \cong A \times A$$

induced by $\partial\Delta[1] \to \Delta[1]$.

Lemma 7. *For any ∞-category A, the ∞-category of arrows comes equipped with a canonical 2-cell*

$$A^{\mathbf{2}} \underset{p_1}{\overset{p_0}{\underset{\Downarrow\kappa}{\rightrightarrows}}} A$$

that we refer to as the ***generic arrow*** *with codomain A.*

The idea of this lemma is to use the universal property of the cotensor $A^{\mathbf{2}}$ given by

$$\mathrm{Fun}\left(X, A^{\mathbf{2}}\right) \cong \mathrm{Fun}(X, A)^{\mathbf{2}}$$

and consider $X = A^{\mathbf{2}}$. We get κ by taking the image of the identity under this isomorphism which is a 1-simplex in $\mathrm{Fun}\left(A^{\mathbf{2}}, A\right)$ representing a 2-cell in the homotopy 2-category.

Proposition 8. *The generic arrow with codomain A has a weak universal property in the homotopy 2-category given by three operations (see Proposition 3.2.5): 1-cell induction, 2-cell induction, and 2-cell conservativity.*

Remark. This weak universal property comes from the fact that the natural map

$$\mathbf{h}\mathrm{Fun}\left(X, A^{\mathbf{2}}\right) \to \mathbf{h}\mathrm{Fun}(X, A)^{\mathbf{2}}$$

of homotopy categories is a smothering functor. Surjectivity of objects expresses the notion of 1-cell induction, fullness expresses 2-cell induction, and conservativity expresses 2-cell conservativity.

This universal property is also enjoyed by objects $(e_0, e_1) : E \twoheadrightarrow A \times A$ that are equivalent to the arrow ∞-category $(p_0, p_1) : A^{\mathbf{2}} \twoheadrightarrow A \times A$ in the following sense.

Definition 9. A **fibered equivalence** over an ∞-category B in an ∞-cosmos $\mathcal{K}$ is an equivalence

$$\begin{array}{ccc} E & \xrightarrow{\sim} & B \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

in the sliced ∞-cosmos $\mathcal{K}_{/B}$.

Proposition 10. *For any isofibration $(e_0, e_1) : E \twoheadrightarrow A \times A$ with a fibered equivalence $e : E \overset{\sim}{\to} A^{\mathbf{2}}$, the corresponding 2-cell $\epsilon : e_0 \Rightarrow e_1$ satisfies the defined weak universal properties. Furthermore, arrow ∞-categories are unique up to fibered equivalence.*