Homotopy theory reading group
Recap of sections 3.3 and 3.4

Pullbacks and limits of towers

Proposition 1 (3.1.1)**.** *Pullbacks along an isofibration have a "weak" universal property in $\mathfrak{h}\mathcal{K}$) given by 1-cell induction + 2-cell induction + 2-cell conservativity.*

Then, one can use this result to show that all ∞-cosmoi are right proper (Lemma 3.3.2) and in turn use that to show that "all pullbacks are homotopy pullbacks" (Prop. 3.3.3).

The comma construction

Definition 2. Given a cospan $C \overset{g}{\to} A \overset{f}{\leftarrow} B$ of ∞-categories, the comma ∞-category is defined as the pullback

$$\begin{array}{ccc} \mathrm{Hom}_A(f,g) & \xrightarrow{\phi} & A^2 \\ {\scriptstyle p_1 p_0}\big\downarrow & & \big\downarrow{\scriptstyle p_1 p_0} \\ C \times B & \xrightarrow[g \times f]{} & A \times A \end{array}$$

The 2-cell

$$\begin{array}{ccccc} & & \mathrm{Hom}_A(f,g) & & \\ & {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_0} & \\ C & & \Leftarrow \phi & & B \\ & {\scriptstyle g}\searrow & & \swarrow{\scriptstyle f} & \\ & & A & & \end{array}$$

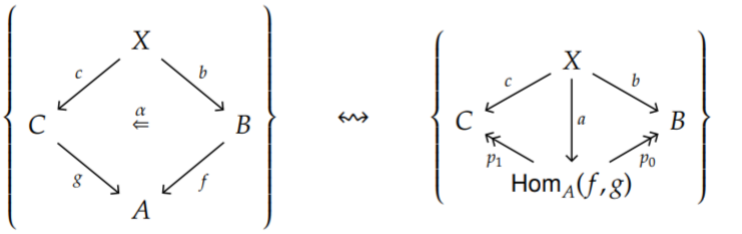
is called the comma cone.

Proposition 3 (3.4.5)**.** *A map f of cospans induces a map between the comma ∞-categories, and if f is a component-wise equivalence (isofibration) [trivial fibration] then the induced map is also an equivalence (isofibration) [trivial fibration].*

The comma cone also satisfies a weak universal property in the homotopy 2-category (Prop. 3.4.6), and is defined up to fibered equivalence.

The weak universal property allows us to deduce a proposition that makes it a bit more familiar.

Proposition 4. *Whiskering with the comma cone induces a bijection between 2-cells as below left*



and fibered iso classes of maps of spans as displayed above right.

Finally, a special instance of the comma ∞-category defines the mapping spaces between elements of an ∞-category.

Definition 5. Given two elements $x, y \colon 1 \rightrightarrows A$ of an ∞-category A, their mapping space is the comma ∞-category given by the pullback

$$\begin{array}{ccc} \mathrm{Hom}_A(x,y) & \xrightarrow{\phi} & A^2 \\ {\scriptstyle p_1 p_0}\downarrow & & \downarrow{\scriptstyle p_1 p_0} \\ 1 \times B & \xrightarrow[y \times x]{} & A \times A \end{array}$$