Riehl-Verity 3.5 Summary and Questions

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This section defines a notion of representability for cospans, characterizes representable cospans, and gives a result that specializes to the Yoneda Lemma (albeit not the only one we will see).

Definition 1. The left and right representations of a functor $f : A \to B$ are respectively the comma ∞-categories $Hom_B(f, B)$ and $Hom_B(B, f)$.

Definition 2. A comma ∞-category $Hom_A(f, g) \to C \times B$ is left representable if there is a functor $l : B \to C$ such that $Hom_A(f, g) \simeq Hom_C(l, C)$ over $C \times B$, and right representable if there is a functor $r : C \to B$ such that $Hom_A(f, g) \simeq Hom_B(B, r)$ over $C \times B$.

We now begin a string of theorems relating absolute lifting diagrams to representability of comma ∞-categories.

Theorem 3. *$r : C \to B$ and $\rho : fr \Rightarrow g$ define an absolute right lifting diagram if and only if the 1-cell $y : Hom_B(B, r) \to Hom_A(f, g)$ induced via 1-cell induction by the projection maps from $Hom_B(B, r)$ is a fibered equivalence over $C \times B$.*

Corollary 4. *We can therefore define a functor $f : A \to B$ to be fully faithful if any of the following equivalences hold:*

1. *The identity on A defines an absolute left or right lifting of f along itself.*
2. *For all $X \in \mathcal{K}$ the functor $f_* : hFun(X, A) \to hFun(X, B)$ is fully faithful as a functor of 1-categories.*
3. *The identity 2-cell id_f induces an equivalence $A^{\not\models} \to Hom_B(f, f)$.*

Theorem 5. *For functors $r : C \to B$, $f : B \to A$, $g : C \to A$ there is a bijection between 2-cells $\rho : fr \Rightarrow g$ and fibered isomorphism classes of maps of cospans $y : Hom_B(B, r) \to Hom_A(f, g)$. Moreover, ρ makes r an absolute right lifting if and only if the corresponding y is an equivalence.*

The following lemma was referred to as 'mysterious'.

Lemma 6. *For any functor $f : A \to B$, the codomain projection functor $p_1 : Hom_B(B, f) \to A$ admits a right adjoint right inverse $i : A \to Hom_B(B, f)$ over A induced by the identity 2-cell id_f on f. Moreover, $\eta i = id_i$, $p_1\eta = id_{p_1}$, amd $p_0\eta = \phi$ where ϕ is the canonical 2-cell $p_0 \Rightarrow fp_1$.*

Corollary 7. *For any element $b : 1 \to B$, the identity at b defines a terminal element in $Hom_B(B, b)$.*

We can now prove a 'cheap' version of the Yoneda lemma:

Corollary 8. *For functors $f, g : A \to B$, there are bijections between 2-cells $f \Rightarrow g$ and isomorphism classes of cospan maps of the form $Hom_B(g, B) \to Hom_B(f, B)$ as well as those of the form $Hom_B(B, f) \to Hom_B(B, g)$.*

This is a reasonable thing to call the Yoneda lemma because it states that maps are precisely characterized by their representations, or more explicitly, if $A = 1$ and B is a 1-category, it specializes to a version of the classical Yoneda lemma.

Theorem 9. *The comma ∞-category $Hom_A(f, g)$ associated to a cospan $C \xrightarrow{g} A \xleftarrow{f} B$ is right representable if and only if its codomain projection functor p_1 admits a right adjoint right inverse $i : C \to Hom_A(f, g)$, in which case $p_0 i : C \to B$ defines the representing functor and the 2-cell represented by i defines an absolute lifting of g through f.*

Exercise 10. How might one encode the existence of an adjunction between opposing functors f and u using comma ∞-categories?