Section 3.6: Sliced homotopy 2-categories and fibered equivalences

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This section is a technical necessity, but is not essential to the thread of the story so far.

Previously, in [RV18, Proposition 3.2.10], we saw that the arrow ∞-category $A^{\mathbb{2}}$ satisfies a universal property that characterizes it up to equivalence – but only *over* $A \times A$.

The ambiguity comes in the difference between the slice homotopy 2-category $(\mathfrak{h}\mathcal{K})_{/B}$ and the homotopy 2-category of the slice ∞-cosmos $\mathfrak{h}(\mathcal{K}_{/B})$. In particular, these two ∞-cosmoi have different 2-cells (c.f. [RV18, Exercise 1.4.iv, Proposition 3.6.3]).

Definition 1 (idea)**.** Recall [RV18, Proposition 1.2.19] that the slice ∞-cosmos $\mathcal{K}_{/B}$ is an ∞-cosmos that exists for any $B \in \mathcal{K}$ such that

- objects are isofibrations $p\colon E \twoheadrightarrow B$ with codomain B,
- functor spaces are defined by pullback:

$$\begin{array}{ccc} \mathrm{Fun}_B(E \xrightarrow{p} B, F \xrightarrow{q} B) & \longrightarrow & \mathrm{Fun}(E,F) \\ \big\downarrow & \lrcorner & \big\downarrow q_* \\ \mathbb{1} & \xrightarrow{\quad p \quad} & \mathrm{Fun}(E,B) \end{array}$$

- the terminal object is $\mathrm{id}\colon B \to B$,
- isofibrations, equivalences, pullbacks, and limits of towers are created by the forgetful functor $\mathcal{K}_{/B} \to \mathcal{K}$,
- etc.

However, we can relate $\mathfrak{h}(\mathcal{K}_{/B})$ and $(\mathfrak{h}\mathcal{K})_{/B}$: there is a canonical 2-functor

$$\mathfrak{h}(\mathcal{K}_{/B}) \to (\mathfrak{h}\mathcal{K})_{/B},$$

and this functor has the property that it is **smothering** [RV18, Proposition 3.6.3].

Definition 2 ([RV18, Definition 3.6.1])**.** A 2-functor $F\colon \mathbf{A} \to \mathbf{B}$ is **smothering** if it is surjective on objects, full on 1-cells, full on 2-cells, and **conservative** on 2-cells, i.e. reflects invertible 2-cells.

That the canonical functor

$$\mathfrak{h}(\mathcal{K}_{/B}) \to (\mathfrak{h}\mathcal{K})_{/B}$$

is smothering means that it has many nice properties:

- it reflects equivalences by [RV18, Lemma 3.6.4],
- if $f\colon E \to F$ is a map between isofibrations over B, and f has an inverse g that is *not* over B, then this data is isomorphic to a genuine fibered equivalence between E and F over B by [RV18, Lemma 3.6.5].
- adjunctions in $(\mathfrak{h}\mathcal{K})_{/B}$ may be lifted to adjunctions in $\mathfrak{h}(\mathcal{K}_{/B})$ by [RV18, Lemma 3.6.8].

References

[RV18] Emily Riehl and Dominic Verity. Elements of ∞-category theory. Available online at http://www.math.jhu.edu/~eriehl/elements.pdf, September 2018.