The Cotangent Complex

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1 Introduction

Let k be a commutative ring. To any k-algebra A, there is a simplicial A-module $L_{A/k}$ such that $\pi_* L_{A/k}$ defines a homology theory of algebras, called André–Quillen homology [Qui70]. This gives a very sensitive invariant of the geometry of the algebra. For example:

(1) a morphism $A \to B$ is smooth if and only if $L_{B/A} \to \Omega_{B/A}$ is a weak equivalence and $\Omega_{B/A}$ is projective as an B-module [GS07, Remark 4.33].

(2) a morphism $A \to B$ is étale if and only if $L_{B/A} \simeq 0$ [GS07, Remark 4.33].

(3) a morphism $A \to B$ of Noetherian rings is a locally complete intersection if and only if $\pi_n(M \otimes_B L_{B/A}) = 0$ for all $n \geq 2$ and all B-modules M [Iye07, Theorem 8.4].

(4) a morphism $A \to B$ of Noetherian rings is regular if and only if $\pi_n(M \otimes_B L_{B/A} = 0$ for all $n \geq 1$ and all B-modules M [Iye07, Theorem 9.5].

Moreover, the cotangent complex provides the setting for obstructions of commutative k-algebra structures on k-modules. The Hochschild and cyclic homology of A admit a filtrations by André–Quillen homology [Mor19, Proposition 2.28] and in characteristic zero, this yields a spectral sequence

$$E^2_{i,j} = \pi_i(L^{\wedge j}_{A/k}) \implies HH_{p+q}(A).$$

In this talk, we will define the cotangent complex and André–Quillen homology, state some of its properties, and describe how to perform some calculations.

Remark 1.1 (A note to the reader)**.** Where things have already been done in the literature, I have tried to provide careful citations. When things need more description, however, I have tried to spell it out. Please let me know if there's anything you'd like to see in more detail, or if I have made any errors!

2 Quillen homology of algebras

If we aim to develop a homology theory of algebras, we immediately run into a problem: the category $\mathbf{Alg}_k$ is not an abelian category. Therefore, the usual notion of a resolution in homological algebra doesn't make sense. Instead, another approach is needed. We take the perspective of Quillen homology, following [GS07, Section 4.4].

Definition 2.1. Let $\mathcal{C}$ be a category, and let $A \in \mathrm{Ob}(\mathcal{C})$. We say that A is an **abelian group object** if $C(-, A)\colon \mathcal{C}^{op} \to \mathbf{Set}$ factors through abelian groups.

Equivalently, if $\mathcal{C}$ is nice enough, this states that there are morphisms $m\colon A \times A \to A$ and $u\colon * \to A$ and $i\colon A \to A$ that are associative, unital, and describe inverses.

Let $(\mathcal{C})_{ab}$ be the category of abelian group objects in $\mathcal{C}$. Suppose that the forgetful functor $U\colon (\mathcal{C})_{ab} \to \mathcal{C}$ has a left adjoint $F\colon \mathcal{C} \to (\mathcal{C})_{ab}$ (called **abelianization**) and that both $\mathcal{C}$, $(\mathcal{C})_{ab}$ are model categories making the adjunction $F \dashv U$ into a Quillen equivalence. Then we may define:

Definition 2.2. The **Quillen homology** of $X \in \mathrm{Ob}(\mathcal{C})$ is $\mathbb{L}F(X)$, the total left derived functor of abelianization $F\colon \mathcal{C} \to (\mathcal{C})_{ab}$.

To compute the Quillen homology of an object X, we take a cofibrant replacement QX of X: $\mathbb{L}F(X) \simeq F(QX)$.

Example 2.3. Consider the category **sSet** of simplicial sets. Then $(\mathbf{sSet})_{ab} \simeq \mathbf{sAb}$. If X is a simplicial set, then its abelianization is $\mathbb{Z}[X]$, the free simplicial abelian group whose n-simplicies are the free abelian group on the set X_n. Since all simplicial sets are cofibrant, the Quillen homology of X is $\mathbb{Z}[X]$, with $\pi_*(\mathbb{Z}[X]) = H_*(X; \mathbb{Z})$.

So this is a reasonable framework for constructing homology theories in categories that are not abelian. Let's apply this to commutative algebras.

Let k be a commutative ring. Unfortunately, the only abelian group object in $\mathbf{Alg}_k$ is the zero ring, because any abelian group object A must admit a morphism from the terminal object 0, and therefore $0 = 1$ in A.

The fix is instead to artificially introduce a new terminal object to the category $\mathbf{Alg}_k$. Fix a k-algebra A, and consider the category $\mathbf{Alg}_{k/A}$ of k-algebras over A. Then we may ask what are the abelian objects in this category. It turns out they are not all trivial. The next example gives a nice class of abelian objects.

Example 2.4. If M is an A-module, define a new k-algebra $A \ltimes M$ on the set $A \otimes M$ with multiplication

$$(a_0, m_0) \cdot (a_1, m_1) = (a_0 a_1, a_0 m_1 + a_1 m_0).$$

To see that this is an abelian group object, note that there is a function ϕ

$$\begin{aligned} \mathbf{Alg}_{k/A}(B, A \ltimes M) &\xrightarrow{\ \phi\ } \mathrm{Der}_k(B, M) \\ f &\longmapsto \mathrm{pr}_2 \circ f \end{aligned} \tag{2.5}$$

where $\mathrm{pr}_2 \colon A \ltimes M \cong A \oplus M \to M$ is the projection homomorphism. In fact, this function ϕ is an bijection, and therefore $\mathbf{Alg}_{k/A}(B, A \ltimes M)$ is an abelian group (in fact, a k-module), so $A \ltimes M$ is an abelian group object in $\mathbf{Alg}_{k/A}$.

In fact, every abelian group object in $\mathbf{Alg}_{k/A}$ is of this form.

Proposition 2.6. *There is an equivalence of categories* $A \ltimes (-) \colon \mathbf{Mod}_A \to (\mathbf{Alg}_{k/A})_{\mathrm{ab}}$.

We may use this to determine the abelianization functor on $\mathbf{Alg}_{k/A}$, which is left adjoint to the forgetful functor $U \colon (\mathbf{Alg}_{k/A})_{\mathrm{ab}} \to \mathbf{Alg}_{k/A}$. By the previous proposition, the forgetful functor is equivalent to $A \ltimes (-) \colon \mathbf{Mod}_A \to (\mathbf{Alg}_{k/A})_{\mathrm{ab}}$, so it suffices to find a left adjoint to $A \ltimes (-)$.

Definition 2.7. Let B be a k-algebra and let $I = \ker(B \otimes_k B \to B)$ be the kernel of the multiplication homomorphism. The **module of Kähler differentials of** B **over** k is the B-module $\Omega_{B/k} := I/I^2$.

In fact, $\Omega_{B/k}$ represents the functor $M \mapsto \mathrm{Der}_k(B, M)$.

$$\mathbf{Mod}_B(\Omega_{B/k}, M) \cong \mathrm{Der}_k(B, M). \tag{2.8}$$

This gives us a candidate for the abelianization functor on $\mathbf{Alg}_{A/k}$.

Proposition 2.9. *The abelianization functor on* $\mathbf{Alg}_{A/k}$ *is given by* $B \mapsto A \otimes_B \Omega_{B/k}$.

Proof. Let B be a k-algebra over A, and let M be an A-module. Via $B \to A$, we may also consider M as a B-module. Then combining the isomorphisms (2.5) and (2.8), we have

$$\mathbf{Alg}_{k/A}(B, A \ltimes M) \cong \mathrm{Der}_k(B, M) \cong \mathbf{Mod}_B(\Omega_{B/k}, M) \cong \mathbf{Mod}_A(A \otimes_B \Omega_{B/k}, M). \quad \square$$

To get the Quillen homology, we would take the total left derived functor. However, since $\mathbf{Alg}_{k/A}$ and $\mathbf{Mod}_A$ are not model categories, we can't take derived functors. Instead, we will pass to simplicial k-algebras over A and simplicial A-modules. This is roughly analogous to doing homological algebra not with modules and algebras, but with chain complexes of modules and differential graded algebras.

Before moving on to simplicial k-algebras in the next section, here we record some properties of the functor $\Omega_{(-)/k}$.

Proposition 2.10. *Let* $k \to A \to B$ *be homomorphisms of commutative rings. Then there is an exact sequence*

$$\Omega_{A/k} \otimes_A B \to \Omega_{B/k} \to \Omega_{B/A} \to 0.$$

This sequence is called the **Jacobi–Zariski sequence**.

Proof. Let N be a B-module. By the homomorphism $A \to B$, N may also be considered an A-module, and by the composite $k \to A \to B$, it also becomes a k-module. There is a left-exact sequence

$$0 \to \mathrm{Der}_A(B; N) \to \mathrm{Der}_k(B; N) \to \mathrm{Der}_k(A; N)$$

where the first homomorphism is given by considering an A-linear derivation as a k-linear derivation via $k \to A$, and the second homomorphism is restriction of domain. By the universal property (2.8) of the Kähler differentials, the sequence above is isomorphic to

$$0 \to \mathrm{Hom}_A(\Omega_{A/k}, N) \to \mathrm{Hom}_B(\Omega_{B/k}, N) \to \mathrm{Hom}_B(\Omega_{B/A}, N).$$

Note that the first term is isomorphic to $\mathrm{Hom}_B(\Omega_{A/k} \otimes_A B, N)$ via the restriction/induction adjunction. Naturality of this sequence in N yields the Jacobi–Zariski sequence. □

Remark 2.11. The cotangent complex may be considered the left-derived functor of this right-exact sequence.

Example 2.12. If $\phi\colon k \to A$ is surjective, then $\Omega_{k/A} \cong 0$. To see this, note that a derivation $\delta \in \mathrm{Der}_k(A, M)$ is equivalent to a k-linear homomorphism $A \to M$ that obeys the Leibniz rule and such that $\delta \circ \phi = 0$. In case ϕ is surjective, then every k-linear derivation from A to M is zero. Hence, the $\Omega_{k/A}$ represents the zero functor, and is zero itself.

Example 2.13 ([Iye07, Exercise 2.3]). $\Omega_{k[x_1,\ldots,x_n]/k} \cong \bigoplus_{i=1}^{n} k[x_1,\ldots,x_n]\, dx_i$

3 Simplicial k-algebras

Recall the Dold–Kan correspondence:

Theorem 3.1 (Dold–Kan). *There is an equivalence of categories* $\mathbf{sMod}_k \simeq \mathbf{Ch}_k^+$ *between connective chain complexes of* k*-modules and simplicial* k*-modules.*

There is a standard model structure on $\mathbf{Ch}_k^+$, called the projective model structure, with

- weak equivalences given by quasi-isomorphisms of chain complexes,
- fibrations given by homomorphisms which are surjective in positive degree,
- cofibrations given by injective homomorphisms with projective cokernel.

This translates to a model structure on $\mathbf{sMod}_k$, where $f\colon X \to Y$ is

- a weak equivalence if $f_*\colon \pi_* X \to \pi_* Y$ is an isomorphism,
- a fibration if f is a fibration of the underlying simplicial sets,
- a cofibration if it has the left-lifting property against all acyclic fibrations.

Equivalently, we may say that f is a fibration if $X \to \pi_0 X \times_{\pi_0 Y} \to Y$ is surjective, or if the corresponding homomorphism of chain complexes is a fibration.

To determine a model structure on $\mathbf{sAlg}_k$, we will use the model structure on $\mathbf{sMod}_k$ and transfer it over the free/forgetful adjunction

$$\mathrm{Sym}_k\colon \mathbf{sMod}_k \rightleftarrows \mathbf{sAlg}_k\colon U,$$

where U is the forgetful functor and Sym_k is the symmetric k-algebra functor applied levelwise to simplicial modules.

Theorem 3.2. *There is a simplicial model category structure on* $\mathbf{sAlg}_k$ *where a morphism* $f\colon A \to B$ *is*

(a) a weak equivalence if $\pi_* f\colon \pi_* A \to \pi_* B$ *is an isomorphism,*

(b) a fibration if $A \to \pi_0 A \times_{\pi_0 B} B$ *is surjective,*

(c) a cofibration if it has the left-lifting property with respect to acyclic fibrations.

A characterization of the cofibrations is provided by the following.

Definition 3.3 ([Iye07, Definition 4.1])**.** Say that a k-algebra homomorphism $f\colon A \to B$ is **free** if there is a sequence $X = \{X_n\}_{n\geq 0}$ of sets such that $B_n \cong A_n[X_n]$ and $s_j(X_n) \subseteq X_{n+1}$, and f is isomorphic the inclusion $A_n \hookrightarrow A_n[X_n]$.

Informally, $A \to B$ is free if B is polynomial over A, compatibly with the degeneracies.

Proposition 3.4. *A morphism in* $\mathbf{sAlg}_k$ *is a cofibration if and only if it is a retract of a free morphism. A simplicial* k*-algebra* A *is cofibrant if and only if there are projective* k*-modules* P_j *and isomorphisms*

$$A_n \cong \coprod_{\phi\colon [n]\twoheadrightarrow[j]} \phi^* \operatorname{Sym}_k(P_j).$$

Definition 3.5. If $f\colon A \to B$ is a homomorphism of simplicial k-algebras, then a **simplicial resolution of** B **as an** A**-algebra** is a factorization of f as a cofibration followed by an acyclic fibration $A \hookrightarrow P \xrightarrow{\sim}\!\!\!\!\!\rightarrow B$

Such a simplicial resolution always exists by the axioms of model categories, or alternatively, explicit general constructions can be found in [Lod13, 3.5.1] or [Wei94, paragraph preceding Definition 8.8.2]. Later we will see nicer constructions for specific examples.

Example 3.6 ([Iye07, Construction 4.13][1])**.** Let's compute the a simplicial resolution of k as an $k[y]$-algebra, where y acts by zero on k.

Consider the simplicial bar complex B with n-simplicies

$$B_n = k[y] \otimes_k k[y]^{\otimes n} \otimes_k k$$

face maps

$$d_i(a \otimes a_1 \otimes \cdots \otimes a_n \otimes \lambda) = \begin{cases} aa_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes \lambda & (i = 0) \\ a \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \otimes \lambda & (0 < i < n) \\ a \otimes a_1 \otimes \cdots \otimes a_n \cdot \lambda & (i = n), \end{cases}$$

and degeneracies

$$s_j(a \otimes a_1 \otimes \cdots \otimes a_n \otimes \lambda) = \begin{cases} a \otimes 1 \otimes a_1 \otimes \cdots \otimes a_n \otimes \lambda & (j = 0) \\ a \otimes a_1 \otimes \cdots \otimes a_{i-1} \otimes 1 \otimes a_i \otimes \cdots \otimes a_n \otimes \lambda & (0 < j < n) \\ a \otimes a_1 \otimes \cdots \otimes a_n \otimes 1 \otimes \lambda & (j = n). \end{cases}$$

[1] I'm pretty certain that this is wrong in the cited reference. At the least, there are confusing typos in the definitions of the faces and degeneracies there.

Evidently, B_n is polynomial over $R[y]$; it is isomorphic to $R[y][x_1, \dots, x_n]$ via

$$x_i \mapsto 1 \otimes \cdots \otimes 1 \otimes y \otimes 1 \otimes \cdots \otimes 1$$

with y in the i-th spot. Therefore, the map $R[y] \hookrightarrow B$ is a cofibration. It is standard that $B \to R$ is an acyclic fibration, i.e. surjective quasi-isomorphism.

Proposition 3.7 ([Iye07, Construction 4.16])**.** *There is a weak equivalence of simplicial* $R[y]$-*modules* $B \simeq K$ *where* K *is the Koszul complex*

$$0 \to R[y] \xrightarrow{y} R[y] \to 0$$

given by $\mathrm{id}_{R[y]}$ *in degree zero and* $- \otimes y$ *in degree one.*

4 The cotangent complex

Now that we have model structures on simplicial k-algebras and simplicial k-modules, we can define the cotangent complex. The constructions in Section 2 extend levelwise to functors of simplicial objects

$$\Omega_{(-)/k} \otimes_{(-)} A \colon \mathbf{sAlg}_{k/A} \rightleftarrows \mathbf{sMod}_A \colon A \ltimes (-)$$

It is easy to see that the right adjoint $A \ltimes (-)$ preserves weak equivalences and fibrations, and hence the adjunction is Quillen. Hence, the total left derived functor makes sense and we may take Quillen homology.

Definition 4.1. The **cotangent complex** of any k-algebra A is

$$L_{A/k} := \Omega_{Q/k} \otimes_Q A$$

where Q is any cofibrant replacement for A in $\mathbf{sAlg}_k$.

Definition 4.2. The **André–Quillen homology** of A is the homotopy of the cotangent complex:

$$D_n(A/k) := \pi_n L_{A/k}.$$

There is a natural map $L_{A/k} \to \Omega_{A/k}$ coming from the Jacobi–Zariski sequence for $k \to Q \to A$; see Proposition 2.10.

Example 4.3. If A is a cofibrant k-algebra, then we may take A as its own cofibrant replacement and $L_{A/k} \simeq \Omega_{A/k} \otimes_A A \cong \Omega_{A/k}$. In particular,

$$L_{k[x_1,\dots,x_n]/k} \simeq \Omega_{k[x_1,\dots,x_n]/k} \cong \bigoplus_{i=1}^{n} k[x_1, \dots, x_n] dx_n.$$

Example 4.4 ([Mor19, Example 2.26])**.** For any k-algebra A, there is an isomorphism $\pi_0 L_{A/k} \cong \Omega_{A/k}$.

Example 4.5 (Conormal sequence, [Mor19, Example 2.26] or [Iye07, Exercise 5.5])**.** If $k \to A$ is surjective with cokernel J, then $\pi_0 L_{A/k} = 0$ and $\pi_1 L_{A/k} = J/J^2$.

Example 4.6. Let's compute the cotangent complex of k as an $k[y]$-algebra, where y acts by zero on k. Recall that in Example 3.6, we described a simplicial k-algebra B such that $k[y] \hookrightarrow B \xrightarrow{\sim} k$ is a factorization of $k[y] \to k$ as a cofibration followed by an acyclic fibration. This simplicial k-algebra is isomorphic to a polynomial k-algebra

$$B_n \cong k[y][x_1, \ldots, x_n]$$

with face maps determined by

$$d_i(y) = y, \qquad d_i(x_j) = \begin{cases} y & (i = 0, j = 1) \\ x_{j-1} & (i < j \text{ and } (i,j) \neq (0,1)) \\ x_j & (i > j \text{ or } i = j \neq n) \\ 0 & (i = j = n) \end{cases} \tag{4.7}$$

From Example 2.13, we find that

$$\Omega_{B_n/k[y]} \cong \bigoplus_{i=1}^{n} k[y][x_1, \ldots, x_n] dx_i.$$

To get the n-simplicies of the cotangent complex, we tensor this with k over B_n. Hence,

$$(L_{k/k[y]})_n = \Omega_{B_n/k[y]} \otimes_{B_n} k \cong \left(\bigoplus_{i=1}^{n} B_n \, dx_i \right) \otimes_{B_n} k \cong \bigoplus_{i=1}^{n} k \, dx_i$$

The degeneracy maps from (4.7) determine the face maps on the cotangent complex. In particular, we have k-linear face maps such that

$$d_i(dx_j) = \begin{cases} 0 & (i = j = n \text{ or } i = 0, j = 1) \\ dx_{j-1} & (i < j \text{ and } (i,j) \neq (0,1)) \\ dx_j & (i > j \text{ or } i = j \neq n). \end{cases} \tag{4.8}$$

We may use this to compute the homotopy groups of the cotangent complex by converting it into a chain complex using the Dold–Kan correspondence and then taking the homotopy. This chain complex has the n-simplicies of $L_{k/k[y]}$ in degree n and the differential ∂ of the corresponding chain complex is given by the alternating sum of face maps,

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i.$$

Altogether, the complex looks as follows:

$$0 \xleftarrow{\partial_1} k \, dx_1 \xleftarrow{\partial_2} k \, dx_1 \oplus k \, dx_2 \xleftarrow{\partial_3} k \, dx_1 \oplus k \, dx_2 \oplus k \, dx_3 \longleftarrow \cdots$$

From (4.8), we see that the first two differentials are zero, so we have $\pi_0 L_{k/k[y]} \cong 0$ and $\pi_1 L_{k/k[y]} \cong k$. The third differential $\partial_3 = d_0 - d_1 + d_2 - d_3$ is surjective, sending $dx_1 \mapsto -dx_1$, $dx_2 \mapsto 0$, and $dx_3 \mapsto dx_2$, hence $\pi_2 L_{k/k[y]} \cong 0$. The fourth differential ∂_4 is described by $dx_1 \mapsto 0, dx_2 \mapsto dx_2, dx_3 \mapsto dx_2, dx_4 \mapsto 0$, and its image thus fills the kernel of ∂_3 to given $\pi_3 L_{k/k[y]} \cong 0$. In fact, this pattern continues and $\pi_i L_{k/k[y]} \cong 0$ when $i \neq 1$. One can prove this using Quillen's fundamental spectral sequence [Qui70, Theorem 6.3]. See also [Qui70, Corollary 6.14] or [Iye07, Proposition 5.11].

Remark 4.9. The above calculation of the cotangent complex can be modified to calculate the cotangent complex of the quotient of any ring R by a regular sequence. See [Iye07, Construction 4.16 and Exercise 4.17].

Finally, we summarize some properties of the cotangent complex.

Proposition 4.10.

(a) *The augmentation map* $L_{A/k} \to \Omega_{A/k}$ *induces an isomorphism on* π_0.

(b) *If* A *is a cofibrant* k*-algebra, then* $L_{A/k} \to \Omega_{A/k}$ *is a weak equivalence.*

Proposition 4.11 (Künneth Theorem, [GS07, Lemma 4.30] or [Mor19, Proposition 2.27])**.** *If either* A *or* B *is a flat* k*-algebra, then there is an isomorphism*

$$L_{A \otimes_k B/k} \cong (A \otimes_k L_{B/k}) \oplus (L_{A/k} \otimes_k B).$$

Proposition 4.12 (Flat Base Change, [GS07, Theorem 4.31] or [Mor19, Proposition 2.27])**.** *Let* K *and* A *be* k*-algebras, and suppose that* $k \to A$ *is flat. Then* $K \otimes_k L_{A/k} \to L_{A \otimes_k K/k}$ *is a weak equivalence.*

Theorem 4.13 ([Mor19, Proposition 2.27] or [GS07, Proposition 4.32])**.** *Given a sequence of homomorphisms of commutative rings* $k \to A \to B$*, there is a cofiber sequence of simplicial* k*-algebras*

$$L_{A/k} \otimes_A B \to L_{B/k} \to L_{B/A}.$$

This last theorem yields a long exact sequence in André–Quillen homology.

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