Topics in Category Theory: Hopf Algebras

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This course really comes in two halves: the first half is monoidal categories and the theory thereof, but it will become more concrete as the weeks go by and we talk about Hopf algebras and their theory. The idea of the course is really as follows:

$$\underline{\text{Algebra}} \xrightarrow{\text{reps}} \underline{\text{Categories}}$$

$$\begin{tikzcd}
\begin{matrix}\text{Hopf Algebra}\\ \text{Quantum Group}\end{matrix} \arrow[r, bend left] & \text{monoidal Category} \arrow[l, bend left]
\end{tikzcd}$$

We can go back and forth between Hopf algebras and the associated monoidal categories to learn more about these structures. These things appear everywhere: universal enveloping algebras, algebraic groups, tensor products, etc.

Example 1. Let R be a commutative ring and let M, N be R-modules. We have a bilinear map $\beta\colon M \times N \to M \otimes_R N$ such that for all $\gamma\colon M \times N \to P$ bilinear, there is a unique $\psi\colon M \otimes_R N \to P$ such that $\psi \circ \beta = \gamma$.

$$\begin{tikzcd}
M \times N \arrow[r, "\beta"] \arrow[dr, "\gamma"'] & M \otimes_R N \arrow[d, dashed, "\psi"] \\
& P
\end{tikzcd}$$

We tend to identify $(M \otimes N) \otimes P$ with $M \otimes (N \otimes P)$ via the isomorphism

$$\begin{aligned}
(M \otimes N) \otimes P &\longrightarrow M \otimes (N \otimes P) \\
(m \otimes n) \otimes p &\longmapsto m \otimes (n \otimes p)
\end{aligned}$$

These two modules classify the trilinear morphisms $M \times N \times P \to Q$.

Example 2. More generally, let $\mathbf{C}$ be a category with finite products. One usually identifies the objects $(X \times Y) \times Z$ and $X \times (Y \times Z)$, but in reality there is just an isomorphism rather than an equality. Both have the universal property of $X \times Y \times Z$. Similarly, $1 \times X \cong X \cong X \times 1$.

Example 3. Let G be a group. A **representation of** G of G**-module** over a commutative ring k is a monoid map $G \xrightarrow{\pi} \mathrm{End}_k(V)$ for some k-module V. We usually write $\pi(g)(v) = g \cdot v$ for $g \in G, v \in V$.

If V, W are G-modules, the tensor product $V \otimes W$ is a G-module via

$$g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w).$$

Example 4. If A is an associative algebra, let A_{Lie} be the Lie algebra with underlying space A and Lie bracket given by $[a, b] = ab - ba$.

If $\mathfrak{g}$ is a Lie algebra, a $\mathfrak{g}$**-module** is a Lie algebra map $\mathfrak{g} \to \mathrm{End}_k(V)_{\mathrm{Lie}}$ for V a k-module. (To explain the notation: note that $A = \mathrm{End}_k(V)$ is an associative algebra, so $\mathrm{End}_k(V)_{\mathrm{Lie}}$ is the associated Lie algebra.)

If V, W are $\mathfrak{g}$-modules, then $V \otimes W$ is a $\mathfrak{g}$-module with

$$x \cdot (v \otimes w) = (x \cdot v) \otimes w + v \otimes (x \cdot w).$$

Example 5. If G is a finite group, k a field, $\phi\colon G \times G \times G \to k^\times$ a 3-cocycle.

Let $V = \bigoplus_{g \in G} V_g$, $W = \bigoplus_{g \in G} W_g$ be G-graded vector spaces. The tensor product of V and W has G-graded structure

$$(V \otimes W)_g = \bigoplus_{g = h\ell} V_h \otimes W_\ell$$

We can define the usual associativity map

$$\begin{array}{ccc} (V \otimes W) \otimes U & \longrightarrow & V \otimes (W \otimes U) \\ (v \otimes w) \otimes u & \longmapsto & v \otimes (w \otimes u) \end{array}$$

but we could also define a map

$$\begin{array}{ccc} (V_{g_1} \otimes W_{g_2}) \otimes U_{g_3} & \longrightarrow & V_{g_1} \otimes (W_{g_2} \otimes U_{g_3}) \\ (v \otimes w) \otimes u & \longmapsto & \phi(g_1, g_2, g_3) v \otimes (w \otimes u). \end{array}$$

This will be an isomorphism if ϕ is a 3-cocycle, and if we replace our associativity map with this one, then we get another monoidal category structure.

Monoidal Categories

So now let's define a monodial category. This will be kind of slow at first, but it will be convenient to have all of this language later on.

Definition 6. A **monoidal category** consists of a category $\mathbf{C}$ with two functors

$$\mathbf{C} \times \mathbf{C} \xrightarrow{\otimes} \mathbf{C}$$

$$\mathbf{1} \longrightarrow \mathbf{C}$$

(with $I \in \mathbf{C}$), and natural isomorphisms

$$\begin{array}{ccc} (X \otimes Y) \otimes Z & \xrightarrow{\alpha_{X,Y,Z}} & X \otimes (Y \otimes Z) \\ I \otimes X & \xrightarrow{\lambda_X} & X \\ X & \xrightarrow{\rho_X} & X \otimes I \end{array}$$

such that the following diagrams commute:

$$\begin{tikzcd} ((X \otimes Y) \otimes Z) \otimes W \arrow[r, "\alpha_{X,Y,Z} \otimes 1"] \arrow[dd, "\alpha_{X \otimes Y, Z, W}"'] & (X \otimes (Y \otimes Z)) \otimes W \arrow[d, "\alpha_{X, Y \otimes Z, W}"] \\ & X \otimes ((Y \otimes Z) \otimes W) \arrow[d, "1 \otimes \alpha_{Y,Z,W}"] \\ (X \otimes Y) \otimes (Z \otimes W) \arrow[r, "\alpha_{X,Y,Z \otimes W}"] & X \otimes (Y \otimes (Z \otimes W)) \end{tikzcd}$$

$$\begin{tikzcd}
X \otimes Y \arrow[r, "\rho_X \otimes 1_Y"] \arrow[d, "1_{X \otimes Y}"'] & (X \otimes I) \otimes Y \arrow[d, "\alpha_{X,I,Y}"] \\
X \otimes Y & X \otimes (I \otimes Y) \arrow[l, "1 \otimes \lambda_Y"]
\end{tikzcd}$$

For notation, we sometimes write $(\mathbf{C}, \otimes, I, \alpha, \lambda, \rho)$ or $(\mathbf{C}, \otimes, I)$, or sometimes even $\mathbf{C}$.

Definition 7. A monoidal category is **strict** if the natural transformations α, λ, ρ as above are identities instead of just isomorphisms, that is, $I \otimes X = X = X \otimes I$ and $X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z$.

Example 8.

(a) If $\mathbf{A}$ is a small category, then the functor category $[\mathbf{A}, \mathbf{A}]$ is strict monoidal with structure $(\mathbf{C}, \otimes, I) = ([\mathbf{A}, \mathbf{A}], \circ, \mathrm{id}_A)$.

(b) A monoid can be regarded as a discrete strict monoidal category with one object.

Lemma 9. In a strict monoidal category, the following diagram commutes:

$$\begin{tikzcd}
W \otimes X \arrow[r, "1_W \otimes \rho_X"] \arrow[dr, "\rho_{W \otimes X}"'] & W \otimes (X \otimes I) \\
& (W \otimes X) \otimes I \arrow[u, "\alpha_{W,X,I}"']
\end{tikzcd}$$

Proof. Since ρ is an isomorphism, it suffices to show that the diagram

$$\begin{tikzcd}
(W \otimes X) \otimes I \arrow[r, "(1_W \otimes \rho_X) \otimes 1_I"] \arrow[dr, "\rho_{W \otimes X} \otimes 1_I"'] & (W \otimes (X \otimes I)) \otimes I \\
& ((W \otimes X) \otimes I) \otimes I \arrow[u, "\alpha_{W,X,I} \otimes 1_I"']
\end{tikzcd}$$

commutes. To that end, consider the following diagram:

$$\begin{tikzcd}
W \otimes (X \otimes I) \arrow[r, "1_W \otimes (\rho_X \otimes 1_I)"] \arrow[rrr, bend left, "1_{W \otimes (X \otimes I)}"] & W \otimes ((X \otimes I) \otimes I) \arrow[r, "1_W \otimes \alpha_{X,I,I}"] & W \otimes (X \otimes (I \otimes I)) \arrow[r, "1_W \otimes (1_X \otimes \lambda_I)"] & W \otimes (X \otimes I) \\
(W \otimes X) \otimes I \arrow[u, "\alpha_{W,X,I}"] \arrow[r, "(1_W \otimes \rho_X) \otimes 1_I"] \arrow[dr, "\rho_{W \otimes X} \otimes 1_I"'] \arrow[drrr, bend right, "1_{(W \otimes X) \otimes I}"'] & (W \otimes (X \otimes I)) \otimes I \arrow[u, "\alpha_{W,X \otimes I,I}"'] & & \\
& ((W \otimes X) \otimes I) \otimes I \arrow[u, "\alpha_{W,X,I} \otimes 1_I"'] \arrow[r, "\alpha_{W \otimes X,I,I}"] & (W \otimes X) \otimes (I \otimes I) \arrow[uu, "\alpha_{W,X,I \otimes I}"'] \arrow[r, "1_{(W \otimes X)} \otimes \lambda_I"] & (W \otimes X) \otimes I \arrow[uu, "\alpha_{W,X,I}"]
\end{tikzcd}$$

With the exception of the triangle, which we want to show commutes, everything else must commute because it's either an axiom or follows from the naturality of α. The commutativity of the other polygons implies the commutativity of the triangle. □

Definition 10. Let $\mathbf{C}, \mathbf{D}$ be monoidal categories. A **monoidal functor $\mathbf{C} \to \mathbf{D}$** is a functor $F\colon \mathbf{C} \to \mathbf{D}$ with a natural transformation $\phi_{X,Y}\colon FX \otimes FY \to F(X \otimes Y)$ and a morphism $\phi_0\colon I_{\mathbf{D}} \to F(I_{\mathbf{C}})$ such that the following diagrams commute.

$$\begin{tikzcd}
(FX \otimes FY) \otimes FZ \arrow[r, "\alpha_{FX,FY,FZ}"] \arrow[d, "\phi_{X,Y} \otimes 1"] & FX \otimes (FY \otimes FZ) \arrow[d, "1 \otimes \phi_{Y,Z}"] \\
F(X \otimes Y) \otimes FZ \arrow[d, "\phi_{X \otimes Y, Z}"] & FX \otimes F(Y \otimes Z) \arrow[d, "\phi_{X, Y \otimes Z}"] \\
F((X \otimes Y) \otimes Z) \arrow[r, "F\alpha_{X,Y,Z}"] & F(X \otimes (Y \otimes Z))
\end{tikzcd}$$

$$\begin{tikzcd}
I \otimes FX \arrow[r, "\phi_0 \otimes 1"] \arrow[d, "\lambda_{FX}"] & FI \otimes FX \arrow[d, "\phi_{I,X}"] \\
FX & F(I \otimes X) \arrow[l, "F(\lambda_X)"]
\end{tikzcd}$$

$$\begin{tikzcd}
FX \arrow[r, "\rho_{FX}"] \arrow[d, "F(\rho_X)"] & FX \otimes I \arrow[d, "1 \otimes \phi_0"] \\
F(X \otimes I) & FX \otimes FI \arrow[l, "\phi_{X,I}"]
\end{tikzcd}$$

These are alternatively called **lax monodial functors** or **(lax) tensor functors**, depending on the author.

Definition 11. A monodial functor is

(a) **strong** when ϕ, ϕ_0 are isomorphisms. Alternatively, this is sometimes included in the definition of monoidal functors.

(b) **normal** if ϕ_0 is an isomorphism.

(c) **strict** if ϕ, ϕ_0 are identities.

Sometimes, the terminology **pseudo-monodial functor** refers to what we call a strong monoidal functor.

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Last time, we defined a monoidal functor $\mathbf{C} \xrightarrow{(F,\phi_0,\phi)} \mathbf{D}$ as a functor F with a morphism $I \xrightarrow{\phi_0} F(I)$, and natural transformation $FX \otimes FY \xrightarrow{\phi_{X,Y}} F(X \otimes Y)$ satisfying some axioms. Every time we have some sort of functor, we want to know that it behaves well under composition, as in the following lemma.

Lemma 12. If $\mathbf{C} \xrightarrow{(F,\phi_0,\phi)} \mathbf{D} \xrightarrow{(G,\chi_0,\chi)} \mathbf{E}$ are monoidal functors, then $GF\colon \mathbf{C} \to \mathbf{D}$ carries a monoidal structure given by

$$G\phi_0 \circ \chi_0 \colon I \to G(I) \to GF(I)$$

$$G\phi_{X,Y} \circ \chi_{FX,FY} \colon GF(X) \otimes GF(Y) \to G(FX \otimes FY) \to GF(X \otimes Y)$$

Exercise 13. Prove Lemma 12. Note also that composition respects these functors if they are strong/strict monoidal.

Definition 14. If (G, χ_0, χ) and (F, ϕ_0, ϕ) are both monoidal functors, then a natural transformation $\tau\colon F \to G$ is a **monodial natural transformation** if the following diagrams commute.

$$\begin{tikzcd} FX \otimes FY \arrow[r, "\phi_{X,Y}"] \arrow[d, "\tau_X \otimes \tau_Y"] & F(X \otimes Y) \arrow[d, "\tau_{X\otimes Y}"] \\ GX \otimes GY \arrow[r, "\chi_{X,Y}"] & G(X \otimes Y) \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "\phi_0"] \arrow[dr, "\chi_0"'] & F(I) \arrow[d, "\tau_I"] \\ & G(I) \end{tikzcd}$$

Remark 15. If $\sigma\colon (G, \chi_0, \chi) \to (H, \psi_0, \psi)$ is another monoidal natural transformation, then $\sigma \circ \tau\colon (F, \phi_0, \phi) \to (H, \psi_0, \psi)$ is monoidal.

So we have a category $\mathbf{Mon}_\ell(\mathbf{C}, \mathbf{D})$ whose objects are monodial functors and whose morphisms are monoidal natural transformations.

Lemma 16. Let $\mathbf{C}, \mathbf{D}, \mathbf{E}$ be monoidal categories and let F, F', G, G' be monoidal functors. Let $\alpha\colon F \to F'$ and $\beta\colon G \to G'$ be monoidal natural transformations.

$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbf{D} \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathbf{E} \end{tikzcd}$$

Then $G\alpha\colon GF \to GF'$ and $\beta_F\colon GF \to G'F$ are monoidal transformations.

Therefore, $\beta_{F'} \circ G\alpha$ is monoidal because both $\beta_{F'}$ and $G\alpha$ are monoidal.

Exercise 17. Prove Lemma 16.

Monoidal Adjunctions

Definition 18. A monoidal functor is a **monoidal equivalence** if it is strong monoidal and an equivalence.

Definition 19. A **monoidal adjunction** is an adjunction $\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{D} \arrow[l, bend left, "G"] \end{tikzcd}$ where $\mathbf{C}, \mathbf{D}$ are monoidal categories, F, G are monoidal functors, and the unit η and the counit ε are monoidal natural transformations.

Exercise 20. Write down the commutative diagrams for Corollary 27. Find the natural monoidal structure on the unit η and counit ε.

Theorem 21 (Doctrinal adjunction)**.** Suppose we have $F\colon A \to B$ is a functor between monoidal categories and $F \dashv G$. Then

(1) If $F \dashv G$ is a monoidal adjunction, then F is strong monoidal.

(2) If F is strong monoidal, then there exists a unique monoidal structure on G that makes $F \dashv G$ a monoidal adjunction.

Proof. Let's prove (1) first. We have a monoidal adjunction $F \dashv G$, so F has a monoidal structure (F, ϕ_0, ϕ) and G has structure (G, χ_0, χ). We want to produce inverses for ϕ_0 and ϕ. To that end, claim that

$$F(X \otimes Y) \xrightarrow{F(\eta_X \otimes \eta_Y)} F(GFX \otimes GFY) \xrightarrow{F(\chi_{FX,FY})} FG(FX \otimes FY) \xrightarrow{\varepsilon_{FX \otimes FY}} FX \otimes FY$$

is an inverse of $\phi_{X,Y}$. To verify this, use the fact that ε and η are monoidal. Similarly,

$$F(I) \xrightarrow{F(\chi_0)} FG(I) \xrightarrow{\varepsilon_I} I$$

is an inverse of ϕ_0.

To prove (2), we are given that (F, ϕ_0, ϕ) is a strong monoidal functor, so $\phi_0, \phi_{X,Y}$ are isomorphisms. We want to show that G has a monoidal structure (G, χ_0, χ). So define $\chi_{X,Y}$ by

$$\begin{tikzcd} GX \otimes GY \arrow[r, dashed, "\chi_{X,Y}"] \arrow[d, "\eta_{GX} \otimes \eta_{GY}"'] & G(X \otimes Y) \\ GF(GX \otimes GY) \arrow[r, "G(\phi_{GX,GY})^{-1}"] & G(FGX \otimes FGY) \arrow[u, "G(\varepsilon_X \otimes \varepsilon_Y)"'] \end{tikzcd}$$

and χ_0 by

$$\chi_0\colon I \xrightarrow{\eta_I} GF(I) \xrightarrow{G\phi_0^{-1}} G(I)$$

One can verify that (G, χ_0, χ) forms a monoidal structure on G. □

Remark 22. We will see later that (op)monoidal functors $\mathbf{1} \to \mathbf{C}$ correspond to (co)monoids in $\mathbf{C}$.

Free (strict) monoidal categories

Definition 23. If $\mathbf{C}$ is a category, define the **free strict monoidal category** $\mathcal{M}(\mathbf{C})$ with objects (possibly empty) lists of objects of $\mathbf{C}$ and with morphisms from $(X_1, \dots, X_n) \to (Y_1, \dots, Y_m)$ only if $n = m$ given by a list $(f_1, \dots, f_n)$ with $f_i\colon X_i \to Y_i$ in $\mathbf{C}$. Compositions are preformed component-wise.

Concatenation of lists gives a strict monoidal structure on $\mathcal{M}(\mathbf{C})$:

$$(X_1, \dots, X_n) \otimes (Y_1, Y_m) = (X_1, \dots, X_m, Y_1, \dots, Y_m),$$

and the unit object is the empty string $I = ()$.

There is a functor $L\colon \mathbf{C} \to \mathcal{M}(\mathbf{C})$ given by $X \mapsto (X)$.

Proposition 24. This construction $\mathcal{M}(\mathbf{C})$ is universal among strict monoidal categories $\mathbf{D}$ with a functor $\mathbf{C} \to \mathbf{D}$.

Definition 25. Denote by $\mathbf{StMon}_S$ the category of strict monoidal categories and strict monoidal functors between them.

Theorem 26. The functor $L\colon \mathbf{C} \to \mathcal{M}(\mathbf{C})$ exhibits $\mathcal{M}(\mathbf{C})$ as the free strict monoidal category on $\mathbf{C}$, in the sense that the functor

$$\mathbf{StMon}_S(\mathcal{M}(\mathbf{C}), \mathbf{A}) \longrightarrow \mathbf{Cat}(\mathbf{C}, \mathbf{A})$$

is an isomorphism, for any strict monoidal category $\mathbf{A}$.

Proof. This proof is the same as the proof that the set of strings on the set is the free monoid on that set. In particular, if $F\colon \mathbf{C} \to \mathbf{A}$ is a functor and $\mathbf{A}$ is a strict monoidal functor, define $\widehat{F}\colon \mathcal{M}(\mathbf{C}) \to \mathbf{A}$ by $\widehat{F}(X_1, \ldots, X_n) = FX_1 \otimes \cdots \otimes FX_n$, etc. □

Corollary 27. The adjunction between the forgetful functor $\mathbf{StMon}_S \to \mathbf{Cat}$ and the free strict monoidal category functor $\mathbf{Cat} \to \mathbf{StMon}_S$ is monadic with monad $\mathbf{C} \mapsto \mathcal{M}(\mathbf{C})$

Exercise 28. Use Beck's Monadicity Theorem to prove Corollary 27, and describe the unit and multiplication.

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Last time we were talking about the free (strict) monoidal category.

Definition 29. We can define a **free monoidal category on one object** $\mathcal{F}$. The objects are defined inductively:

- $(*) \in |\mathcal{F}|$ is the generator
- $(-) \in |\mathcal{F}|$ is the unit
- If $x, y \in |\mathcal{F}|$, then $(x, y) \in |\mathcal{F}|$.

Let $x, y, z \in |\mathcal{F}|$. We also define morphisms as follows.

- $1_x\colon x \to x$ is the identity morphism.
- $((x,y),z) \underset{\bar{\alpha}_{x,y,z}}{\overset{\alpha_{x,y,z}}{\rightleftarrows}} (x,(y,z))$
- $((-),x) \underset{\bar{\lambda}_x}{\overset{\lambda_x}{\rightleftarrows}} x$
- $x \underset{\bar{\rho}_x}{\overset{\rho_x}{\rightleftarrows}} (x,(-))$

- If $x \xrightarrow{\phi} y \xrightarrow{\psi} z$ are morphisms in $\mathcal{F}$, then there is a morphism $\psi \circ \phi \colon x \to z$.
- If $\phi \colon x \to y$, $\psi \colon z \to w$ are morphisms, then there is a morphism $(\phi, \psi) \colon (x, z) \to (y, w)$.

To make sure that this is a monoidal category, we should quotient the set of morphisms by the smallest equivalence relation $\equiv$ generated by the following rules

- 1_x are identities
- Associativity of composition
- $\overline{\alpha}$ is the inverse of α
- $\overline{\lambda}$ is the inverse of λ
- $\overline{\rho}$ is the inverse of ρ
- Naturality of α, λ, ρ
- If $\phi \equiv \psi$, then $(\psi, \chi) \equiv (\phi, \chi)$ and $(\chi, \phi) \equiv (\chi, \phi)$.
- If $\phi \equiv \psi$, then $\xi \circ \phi \equiv \xi \circ \psi$ and $\phi \circ \chi \equiv \psi \circ \chi$.
- $(\psi \circ \phi, 1_x) \equiv (\psi, 1_x) \circ (\phi, 1_x)$
- $(1_x, \psi \circ \phi) \equiv (1_x, \psi) \circ (1_x, \phi)$
- $(1_x, 1_y) = 1_{(x,y)}$.
- Axioms of monoidal categories: the pentagon and two other axioms.

It can be (painfully) checked that $\mathcal{F}$ is a category with monoidal structure induced by the pairing $x, y \mapsto (x, y)$ and unit object $(-)$.

Lemma 30. The map $\mathbf{1} \to \mathcal{F}$ given by $* \mapsto (*) \in \mathcal{F}$ exhibits $\mathcal{F}$ as the free mononidal category on $\mathbf{1}$, in the sense that the map

$$\mathbf{StMon}_s(\mathcal{F}, \mathbf{C}) \longrightarrow [\mathbf{1}, \mathbf{C}] \cong \mathbf{C}$$

is an isomorphism.

Definition 31. If $\mathbf{C}$ is a category, we can define the free monoidal category on $\mathbf{C}$ by constructing the pullback of categories and functors

$$\begin{tikzcd}
\mathcal{F}(\mathbf{C}) \arrow[r] \arrow[d] & \mathcal{F} \arrow[d, "\Gamma_1"] & \ni & (*) \arrow[d, mapsto] \\
\mathcal{M}(\mathbf{C}) \arrow[r, "\mathcal{M}(!)"] & \mathcal{M}(\mathbf{1}) & \ni & 1 \in \mathbb{N}
\end{tikzcd}$$

Note that $\mathcal{M}(\mathbf{1})$ is the monoidal category $(\mathbb{N}, +, 0)$, that is, the objects are natural numbers and tensor is addition, with zero as the unit object.

An object of $\mathcal{F}(\mathbf{C})$ is a pair of a string $(X_1, \ldots, X_n)$ of objects $X_i \in \mathbf{C}$ and an object of $\mathcal{F}$, which we think of as a binary tree describing the associativity of the tensor product of the X_i. For example, think of the object

$$((X_1, X_2, X_3), (((*), (*)), (*)))$$

as

$$(X_1 \otimes X_2) \otimes X_3.$$

The unit object is $((-), (-))$, and for $f\colon A \to B$ in $\mathbf{C}$, we should think of $((f), 1_x)$ as the morphism $f \otimes 1_x\colon A \otimes X \to B \otimes X$.

This category $\mathcal{F}(\mathbf{C})$ is monoidal and the projections are strict monoidal since the category $\mathbf{StMon}_s$ of monoidal categories and strict monoidal functors has pullbacks (this is easy to verify).

Lemma 32. The map $\mathbf{A} \to \mathcal{F}(\mathbf{A})$ exhibits $\mathcal{F}(\mathbf{A})$ as the free monoidal category on $\mathbf{A}$.

Coherence

The associativity morphisms are very difficult to deal with oftentimes, so the coherence theorem gives us a way to reason about these monoidal categories without worrying about the associativity constraints: every diagram with the α, λ, ρ that makes sense should commute.

Proposition 33. Every monoidal category is monoidally equivalent to a strict monoidal category.

Remark 34 (Warning!)**.** This is *not* the coherence theorem.

Say you want the diagram

$$\begin{tikzcd} (I \otimes X) \otimes Y \arrow[r, "\alpha_{I,X,Y}"] \arrow[dr, "\lambda_X \otimes 1_Y"'] & I \otimes (X \otimes Y) \arrow[d, "\lambda_{X \otimes Y}"] \\ & X \otimes Y \end{tikzcd} \tag{1}$$

in a monoidal category $\mathbf{C}$. If $\mathbf{D}$ is a strict monoidal category and $F\colon \mathbf{C} \xrightarrow{\simeq} \mathbf{D}$ is an equivalence, then we have that

$$\begin{tikzcd} (I \otimes FX) \otimes FY \arrow[r, "\alpha_{I,FX,FY}"] \arrow[dr, "\lambda_{FX} \otimes 1"'] & I \otimes (FX \otimes FY) \arrow[d, "\lambda_{FX \otimes FY}"] \\ & FX \otimes FY \end{tikzcd}$$

commutes. But we can't "pull-back" to $\mathbf{C}$ to verify that the original diagram (1) commutes, because we need to know that

$$\begin{tikzcd} I \otimes (FX \otimes FY) \arrow[r, "\phi_0 \otimes \phi_{X,Y}"] \arrow[dr] & FI \otimes F(X \otimes Y) \arrow[d, "\phi_{I,X \otimes Y}"] \\ & F(I \otimes (X \otimes Y)) \end{tikzcd}$$

commutes. So Proposition 33 isn't enough; we would also need some sort of coherence for monoidal functors F.

Proof of Proposition 33. Let $\mathbf{V}$ be a monoidal category. Define a strict monoidal category $E(\mathbf{V})$ whose objects are pairs (S, σ) functors $S \colon \mathbf{V} \to \mathbf{V}$ with $\sigma_{X,Y} \colon X \otimes S(X) \xrightarrow{\sim} S(X \otimes Y)$ a natural isomorphism.

The morphisms in $E(\mathbf{V})$ from $(S, \sigma) \to (T, \tau)$ are natural transformations $\phi \colon S \to T$ such that

$$\begin{tikzcd}
X \otimes S(Y) \arrow[r, "\sigma_{X,Y}"] \arrow[d, "1 \otimes \phi_Y"] & S(X \otimes Y) \arrow[d, "\phi_{X \otimes Y}"] \\
X \otimes T(Y) \arrow[r, "\tau_{X,Y}"] & T(X \otimes Y)
\end{tikzcd}$$

The tensor product $(S, \sigma) \circ (T\tau)$ is $(TS, \sigma \circ \tau)$, where

$$(\sigma \circ \tau)_{X,Y} \colon X \otimes TS(Y) \xrightarrow{\tau_{X,S(Y)}} T(X \otimes SY) \xrightarrow{T\sigma_{X,Y}} TS(X \otimes Y).$$

The unit object is $(1_{\mathbf{V}}, 1)$.

One verifies this is a functor of two variables, and associative and unital, so $E(\mathbf{V})$ is strict monoidal.

Now define $N \colon \mathbf{V} \to E(\mathbf{V})$ on objects X by

$$N(X) = ((- \otimes X), n_X)$$

where

$$(n_x)_{Y,Z} \colon Y \otimes N(X)(Z) = Y \otimes (Z \otimes X) \xrightarrow{\alpha^{-1}_{Y,Z,X}} (Y \otimes Z) \otimes X = N(X)(Y \otimes Z).$$

And on morphisms f by

$$N(f) = (- \otimes f).$$

It's tedious but not too hard to check that N is a functor.

The strong monoidal structure on N is given by

$$\nu_{X,Y} \colon N(X) \circ N(Y) \to N(X \otimes Y)$$

$$(\nu_{X,Y})_W \colon (W \otimes X) \otimes Y \xrightarrow{\alpha} W \otimes (X \otimes Y)$$

$$\nu_0 \colon (1_{\mathbf{V}}, 1) \to N(I)$$

$$(\nu_0)_W \colon W \xrightarrow{\rho_W} W \otimes I$$

The following diagrams commute:

$$\begin{tikzcd}
N(X) \circ N(Y) \circ N(Z) \arrow[rr, "\nu \circ 1"] \arrow[d, "1 \circ \nu"] & & N(X \otimes Y) \circ N(Z) \arrow[d, "\nu"] \\
N(X) \circ N(Y \otimes Z) \arrow[r, "\nu"'] & N(X \otimes (Y \otimes Z)) & N((X \otimes Y) \otimes Z) \arrow[l, "N(\alpha)"]
\end{tikzcd} \tag{2}$$

$$\begin{tikzcd}
(1_{\mathbf{V}},1)\circ N(X) \arrow[r, "\nu_0\circ 1"] \arrow[d, equal] & N(I)\circ N(X) \arrow[d, "\nu"] \\
N(X) & N(I\otimes X) \arrow[l, "N(\lambda_X)"]
\end{tikzcd} \tag{3}$$

$$\begin{tikzcd}
N(X) \arrow[r, "N(\rho_X)"] \arrow[d, equal] & N(X\otimes I) \\
N(X)\circ(1_{\mathbf{V}},1) \arrow[r, "1\circ\nu_0"'] & N(X)\circ N(I) \arrow[u, "\nu"']
\end{tikzcd} \tag{4}$$

If we evaluate (2) on some object W, we get the pentagon axiom. If we evaluate (3) on some object W, we get the unit axiom, and if we evaluate (4), we get the commutative diagram of Lemma 9.

It remains to show that N is full and faithful to show that $\mathbf{V}$ is equivalent to $E(\mathbf{V})$. □

Lecture 4 — 22 January 2016

Last time we were proving that every monoidal category is monoidally equivalent to a strict monoidal category. So let's finish the proof.

Proof of Proposition 33, continued. Last time, we constructed $N\colon \mathbf{V}\to E(\mathbf{V})$ and showed that N was a strong monoidal category.

Recall that $E(\mathbf{V})$ has objects (S,σ) where $S\colon \mathbf{V}\to\mathbf{V}$ and σ is a natural isomorphism $X\otimes S(Y)\to S(X\otimes Y)$.

We still need to show that N is full and faithful, and then $\mathbf{V}$ will be equivalent to its image under N.

To show that N is full, suppose given $N(X)\xrightarrow{\phi} N(Y)$, with components $\phi_W\colon W\otimes X\to W\otimes Y$. Consider the composite

$$f\colon X\xrightarrow{\lambda_X^{-1}} I\otimes X\xrightarrow{\phi_I} I\otimes Y\xrightarrow{\lambda_Y} Y.$$

We can show that $N(f)_W = 1_W\otimes f=\phi$, therefore N is full.

Finally, N is faithful, since $1_I\otimes f = 1_I\otimes g \implies f=g$ because λ is an isomorphism.

Taking the full image of N, we find a strict monoidal category $\mathbf{W}$ such that there is an equivalence $\mathbf{V}\simeq\mathbf{W}$. This concludes the proof of Proposition 33. □

Finally, the theorem we're all waiting for! First, some setup. Let $\mathbf{C}$ be a category. There are functors $\mathbf{C}\to\mathcal{F}(\mathbf{C})$ and $\mathbf{C}\to\mathcal{M}(\mathbf{C})$, and because $\mathcal{F}(\mathbf{C})$ is the free monoidal category on $\mathbf{C}$, then there is a unique strict monoidal functor

$\Gamma_{\mathbf{C}} \colon \mathcal{F}(\mathbf{C}) \to \mathcal{M}(\mathbf{C})$ such that the following commutes:

$$\begin{tikzcd} \mathbf{C} \arrow[r] \arrow[dr] & \mathcal{F}(\mathbf{C}) \arrow[d, dashed, "\Gamma_{\mathbf{C}}"] \\ & \mathcal{M}(\mathbf{C}) \end{tikzcd}$$

Theorem 35 (Coherence Theorem for Monoidal Categories)**.** *$\Gamma_{\mathbf{C}}$ is an equivalence of categories.*

We will prove this theorem at the end of the course, provided there is time.

Remark 36. It is not true in general that there exists a strict monoidal pseudo-inverse to $\Gamma_{\mathbf{C}}$.

So why is this useful?

Corollary 37. If X is a set, regard it as a discrete category. Then all diagrams in $\mathcal{F}(X)$ commute. Meaning that there is at most one morphism between objects.

Proof. $\mathcal{F}(X) \simeq \mathcal{M}(X)$ is the free monoid on X as a discrete category. □

Remark 38. This means that in $\mathcal{F}(X)$, any diagram (which are all formed by tensoring and the monoidal constraints α, λ, ρ and their inverses) commutes.

In practice, we will pretend that all monoidal categories are strict to simplify our lives, because all of the diagrams we want to commute will.

Example 39. If $\mathbf{V}$ is a monoidal category, and we have such a diagram in $\mathbf{V}$ formed from $\alpha, \lambda, \rho, \otimes$. For example, it might be the one we were looking at the other day,

$$\begin{tikzcd} (I \otimes X) \otimes Y \arrow[r, "\alpha"] \arrow[dr, "\lambda \otimes 1"'] & I \otimes (X \otimes Y) \arrow[d, "\lambda"] \\ & X \otimes Y \end{tikzcd}$$

we can show that it commutes as follows. Take the set S formed by the objects different from I in the diagram that are not themselves tensor products of other objects, here for example $S = \{X, Y\}$. Then consider the inclusion $S \to \mathbf{V}$. This induces a map $F \colon \mathcal{F}(S) \to \mathbf{V}$ such that

$$\begin{tikzcd} S \arrow[r] \arrow[dr] & \mathcal{F}(S) \arrow[d, "F"] \\ & \mathbf{V} \end{tikzcd}$$

and all the edges in the diagram are in the image of F, so the diagram commutes in $\mathbf{V}$.

Monoidal Closed Categories

Definition 40. A monoidal **right-closed** category is a monoidal category $\mathbf{V}$ where each $(X \otimes -) \colon \mathbf{V} \to \mathbf{V}$ has a right adjoint, denoted $[X, -]_r \colon \mathbf{V} \to \mathbf{V}$.

A monoidal **left-closed** category is a monoidal category $\mathbf{V}$ where each $(- \otimes X) \colon \mathbf{V} \to \mathbf{V}$ has a right adjoint, denoted $[X, -]_\ell \colon \mathbf{V} \to \mathbf{V}$.

Example 41.

- Any cartesian closed category (that is, monoidal with the categorical product and the unit is the terminal object) is monoidally closed. So **Set**, toposes, ...
- R-**Mod** for a commutative ring R, with $[M, N]_\ell = [M, N]_r = \mathrm{Hom}_R(M, N)$, because of the tensor-hom adjunction.
- $\mathbb{Z}$-graded vector spaces, $\mathbf{grVect}_{\mathbb{Z}}$

$$(V \otimes W)_n = \sum_{i \in \mathbb{Z}} V_i \otimes W_{n-i}, \qquad [V, W]_n = \prod_{i \in \mathbb{Z}} [V_i, W_{i+n}]$$

- If $\mathbf{C}$ is a category, $[\mathbf{C}, \mathbf{C}]$ is left closed if right Kan extensions exist; $[S, T]$ is the right Kan extension of T by S.

Duals

Duals generalize the notion of duals in vector spaces, and come up often in representation theory.

Definition 42. A **dual pair** is a monoidal category $\mathbf{V}$ consists of $X, Y \in \mathbf{V}$, with two maps $e \colon X \otimes Y \to I$ and $n \colon I \to Y \otimes X$ called **evaluation** and **coevaluation**, respectively. These maps satisfy

$$\begin{tikzcd} X \arrow[r, "1_X \otimes n"] \arrow[dr, equal] & X \otimes Y \otimes X \arrow[d, "e \otimes 1"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "n \otimes 1_Y"] \arrow[dr, equal] & Y \otimes X \otimes Y \arrow[d, "1_Y \otimes e"] \\ & Y \end{tikzcd}$$

X is called a **left dual** of Y, and Y is called a **right dual** of X. This is sometimes written $X = {}^\vee Y = {}^*Y$ and $Y = X^\vee = X^*$.

Remark 43. It's no coincidence that these look like the triangular laws of an adjunction!

Exercise 44.

(1) Prove that if both Y and Z are right duals of X, then there is a unique isomorphism $Y \xrightarrow{\sim} Z$ compatible with evaluation and coevaluation (and figure out what compatibility means!).

(2) If X has a right dual X^*, then $[X, -]_r$ exists and $[X, Y]_r \cong X^* \otimes Y$.

(3) Prove that any R-module M has a (left or right) dual if and only if M is projective and finitely presentable.

Example 45. Regarding exercise (2), if V is a finite dimensional vector space, then $V^* \otimes V \cong \mathrm{Hom}(V,V) = [V,V]$.

Lecture 5 — 25 January 2016

Remark 46. Now that we've seen the coherence theorem, we will omit the α, λ, ρ in diagrams understanding that diagrams composed of these will commute.

Monoids and Comonoids

Definition 47. A **monoid** in a monoidal category $\mathbf{V}$ is a triple (A, j, m) where A is an object, $m\colon A \otimes A \to A$, and $j\colon I \to A$ such that the following three diagrams commute:

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "m\otimes 1"] \arrow[d, "1\otimes m"] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "j\otimes 1"] \arrow[dr, "1_A"'] & A \otimes A \arrow[d, "m"] & A \arrow[l, "1\otimes j"'] \arrow[dl, "1_A"] \\ & A & \end{tikzcd}$$

Definition 48. A **morphism of monoids** $(A, j, m) \to (A', j', m')$ is $f\colon A \to A'$ such that the following diagrams commute.

$$\begin{tikzcd} A \otimes A \arrow[r, "f\otimes f"] \arrow[d, "m"] & A' \otimes A' \arrow[d, "m'"] \\ A \arrow[r, "f"] & A' \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "j"] \arrow[dr, "j'"'] & A \arrow[d, "f"] \\ & A' \end{tikzcd}$$

Definition 49. $\mathbf{Mon}(\mathbf{V})$ is the category of monoids in a monoidal category $\mathbf{V}$.

Exercise 50. $\mathbf{Mon}(\mathbf{V})$ is isomorphic to the category $\mathbf{Mon}_\ell(\mathbf{1}, \mathbf{V})$ of monoidal functors $\mathbf{1} \to \mathbf{V}$, and monoidal natural transformations.

Remark 51. It follows from Exercise 50 that if $F\colon \mathbf{V} \to \mathbf{W}$ is a monoidal functor, there is an induced functor $\mathbf{Mon}(F)\colon \mathbf{Mon}(\mathbf{V}) \to \mathbf{Mon}(\mathbf{W})$ such that

$$\begin{tikzcd} \mathbf{Mon}(\mathbf{V}) \arrow[r, "\mathbf{Mon}(F)"] \arrow[d] & \mathbf{Mon}(\mathbf{W}) \arrow[d] \\ V \arrow[r, "F"] & \mathbf{W} \end{tikzcd}$$

commutes, where vertical arrows are forgetful functors. This works because monoidal functors compose, so $\mathbf{1} \xrightarrow{A} \mathbf{V} \xrightarrow{F} \mathbf{W}$ will correspond to the monoid $F(A)$.

Explicitly, FA has multiplication

$$FA \otimes FA \xrightarrow{\phi_{A,A}} F(A \otimes A) \xrightarrow{F(m)} F(A)$$

and unit

$$I \xrightarrow{\phi_0} F(I) \xrightarrow{Fj} F(A).$$

Example 52.

Monoidal Category	Monoids
$(\mathbf{Set}, 1, \times)$	usual monoid
$(R\text{-}\mathbf{Mod}, R, \otimes_R)$	R-algebra
$(\mathbf{Cat}, \mathbf{1}, \times)$	strict monoidal category
$([\mathbf{A}, \mathbf{A}], 1_{\mathbf{A}}, \circ)$	monad on $\mathbf{A}$
$\mathbb{Z}$-graded vector spaces	graded algebras
dgVect = chain complexes	dg-algebras

Definition 53. A **comonoid** is (C, ε, δ), where $C \xrightarrow{\delta} C \otimes C$ and $\varepsilon \colon I \to C$ such that the following diagrams commute:

$$\begin{tikzcd} C \arrow[r, "\delta"] \arrow[d, "\delta"] & C \otimes C \arrow[d, "1\otimes\delta"] \\ C \otimes C \arrow[r, "\delta\otimes 1"] & C \otimes C \otimes C \end{tikzcd} \qquad \begin{tikzcd} & C \arrow[dl, "1_C"'] \arrow[d, "\delta"] \arrow[dr, "1_C"] & \\ C & C \otimes C \arrow[l, "1\otimes\varepsilon"] \arrow[r, "\varepsilon\otimes 1"'] & C \end{tikzcd}$$

Definition 54. A **morphism of comonoids** $(C, \varepsilon, \delta) \to (C', \varepsilon', \delta')$ is a map $f \colon C \to C'$ such that the following diagrams commute.

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\delta"] & C' \arrow[d, "\delta'"] \\ C \otimes C \arrow[r, "f\otimes\delta"] & C' \otimes C' \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\varepsilon"] \arrow[d, "f"] & I \\ C' \arrow[ur, "\varepsilon'"'] & \end{tikzcd}$$

Definition 55. In short, the category of comonoids on $\mathbf{V}$ is $\mathbf{Comon}(\mathbf{V}) = \mathbf{Mon}(\mathbf{V}^{\mathrm{op}})$.

Modules and Comodules

If $\mathbf{V}$ is monoidal, we saw that there is a strong monoidal functor

$$\begin{tikzcd} \mathbf{V} \arrow[rr] \arrow[dr, "N"'] & & {[\mathbf{V}, \mathbf{V}]} \\ & E(\mathbf{V}) \arrow[ur, "\text{strict}"'] & \end{tikzcd}$$

given by $X \mapsto (- \otimes X)$.

In particular, this functor sends monoids to monoids, that is, monoids in $\mathbf{V}$ to monads on $\mathbf{V}$. Explicitly, if (A, j, m) is a monoid in $\mathbf{V}$, then $- \otimes A \colon \mathbf{V} \to \mathbf{V}$ is a monad with multiplication

$$- \otimes A \otimes A \xrightarrow{-\otimes m} - \otimes A$$

and unit

$$- \otimes I \xrightarrow{-\otimes j} - \otimes A.$$

Definition 56. The **category Mod-A of right A-modules** is defined as the Eilenberg-Moore category $\mathbf{V}^{(-\otimes A)}$ of algebras for the monad $(-\otimes A)$. This means that a right A-module is $M \in \mathbf{V}$, with $M \otimes A \xrightarrow{a} M$ such that the following commute.

$$\begin{tikzcd} M\otimes A\otimes A \arrow[r, "a\otimes 1"] \arrow[d, "1\otimes m"] & M\otimes A \arrow[d, "a"] \\ M\otimes A \arrow[r, "a"] & M \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "1\otimes j"] \arrow[dr, "1_M"'] & M\otimes A \arrow[d, "a"] \\ & M \end{tikzcd}$$

Similarly, we have left modules arising from the functor $(A\otimes -)$.

Definition 57. If C is a comonoid, then $(-\otimes C)$ is a comonad, the **category Comod($\mathbf{C}$) of right C-comodules** is the Eilenberg-Moore category $\mathbf{V}^{(-\otimes C)}$ of coalgebras for $-\otimes C$. This means that there is a coaction $\chi\colon M \to M\otimes C$ satisfying the following diagrams.

$$\begin{tikzcd} M \arrow[r, "\chi"] \arrow[d, "\chi"] & M\otimes C \arrow[d, "\chi\otimes 1"] \\ M\otimes C \arrow[r, "1\otimes\delta"] & M\otimes C\otimes C \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\chi"] \arrow[dr, "1_M"'] & M\otimes C \arrow[d, "1\otimes\varepsilon"] \\ & M \end{tikzcd}$$

Coalgebras and comodules in the category of vector spaces

Remark 58. You might think that the theory of monoids and comonoids are completely dual, but that is not necessarily the case once we've fixed a category. The thing is, the theory of monoids is the same as the theory of comonoids in the opposite category, but not every category is the same as its opposite. We'll investigate this in the category k-**Vect** of k-vector spaces, which is a monoidal category with $\otimes$ the multiplication and k the unit.

Remark 59. Confusingly, in **Vect** or R-**Mod**, a comonoid is a called a coalgebra (this is meant in the algebraic sense, *not* as above).

Definition 60. If C is a coalgebra (in the category of vector spaces), a **right coideal** is a subspace $I \subseteq C$ such that $\delta(I) \subseteq I\otimes C \subseteq C\otimes C$, where $\delta\colon C \to C\otimes C$ is the comultiplication.

A **coideal** is a subspace $I \subseteq C$ such that $\delta(I) \subseteq I\otimes C + C\otimes I$.

Example 61. A subspace $V \subseteq C$ of a coalgebra C is a subcoalgebra if and only if V is a left and right coideal.

Theorem 62 (Fundamental Theorem of Coalgebras)**.** Each coalgebra is the union of its finite dimensional subcoalgebras.

This is very different from the case of algebras!

Proof. We will show that if $x \neq 0$ is an element of the coalgebra $\mathbf{C}$, then x belongs to a finite dimensional subcoalgebra. Suppose $\Delta\colon C \to C\otimes C$ is the comultiplication and $\varepsilon\colon C \to k$ is the unit, where k is the field. We write

$$\Delta_2(x) := (\Delta\otimes 1_C)(\Delta(x)) = (1_C\otimes\Delta)(\Delta(x)),$$

and these expressions are equal by coassociativity. Now write

$$\Delta_2(x) = \sum_{i,j=1}^{n} c_i \otimes x_{ij} \otimes d_j$$

where each set $\{c_i\}$, $\{d_j\}$ is linearly independent in C. This is possible because

$$\Delta(x) = \sum_i c_i \otimes c_i'$$

for some linearly independent c_i, and

$$\Delta(c_i') = \sum_j x_{ij} \otimes d_j$$

with the d_j linearly independent.

Let D be the span of $\{x_{ij} \mid 1 \leqslant i, j \leqslant n\}$. Note that D is finite-dimensional. We want to show that D is a subcoalgebra of C, and contains x. First, write

$$x = (\varepsilon \otimes 1_C \otimes \varepsilon)\Delta_2(x) = \sum_{i,j} \varepsilon(c_i)\varepsilon(d_j)x_{ij}$$

and notice that $\varepsilon(c_i)\varepsilon(d_j)$ are scalars, so x is in the span of the x_{ij} and so $x \in D$.

Now by coassociativity, we have that

$$(\Delta \otimes 1_C \otimes 1_C)\Delta_2(x) = (1_C \otimes \Delta \otimes 1_C)\Delta_2(x),$$

so compute

$$\sum_{i,j} \Delta(c_j) \otimes x_{ij} \otimes d_j = \sum_{i,j} c_i \otimes \Delta(x_{ij}) \otimes d_j$$

Since the d_i are linearly independent, then

$$\sum_i \Delta(c_i) \otimes x_{ij} = \sum_i c_i \otimes \Delta(x_{ij}) \in C \otimes C \otimes D$$

Since the c_i are linearly independent, then we conclude that $\Delta(x_{ij}) \in C \otimes D$, by Exercise 14 on the first examples sheet.

A symmetric argument shows that $\Delta(x_i) \in D \otimes C$. Hence, $\Delta(x_{ij}) \in C \otimes D \cap D \otimes C = D \otimes D$. Then because D is the span of the x_{ij}, it follows that D is a subcoalgebra. □

Lecture 6 — 27 January 2016

Let's have some examples of coalgebras.

Example 63.

(1) If A is a finite-dimensional algebra, then A^* is a finite-dimensional coalgebra using the dual of the multiplication $A^* \to (A \otimes A)^* \cong A^* \otimes A^*$. Note

that the isomorphism $A^* \otimes A^* \cong (A \otimes A)^*$ only holds when A is finite dimensional.

For example, if M is a finite monoid (in **Set**), then there is a coalgebra $C = k^M = \{f : M \to k\} \cong k[M]^*$. This is finite-dimensional algebra. If $\gamma : M \to k$, then $\Delta(\gamma) \in k^M \otimes k^M \cong k^{M \times M}$ is defined by $\Delta(\gamma)(m, n) = \gamma(mn)$ and $\varepsilon(\gamma) = \gamma(1)$.

(2) IF V is a vector space, let $T(V)$ be the **tensor algebra** over V,

$$T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}.$$

Then $\Delta : T(V) \to T(V) \otimes T(V)$ is the unique algebra map such that $\Delta(v) = 1 \otimes v + v \otimes 1$ for $v \in V$, and $\varepsilon : T(V) \to k$ is uniquely defined by $\varepsilon(v) = 0$ for $v \in V$. One verifies that this is an algebra map.

(3) Universal enveloping algebras $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. This is in fact very similar to the previous example, as $U(\mathfrak{g})$ is a quotient of the tensor algebra $T(\mathfrak{g})$. Both are bialgebras.

Definition 64 (Notation)**. Sweedler's sigma notation** is a way of simplifying calculations using coalgebras. If C is a coalgebra, and $x \in C$, we can write $\Delta(x)$ as a sum of elementary tensors, say

$$\Delta(x) = \sum_i x_i \otimes x_i'.$$

In Sweedler's notation, we write instead

$$\Delta(x) = \sum x_1 \otimes x_2.$$

Similarly, we write

$$(\Delta \otimes 1)(\Delta(x)) = \sum (x_1)_1 \otimes (x_1)_2 \otimes x_2 \tag{5}$$

But we know that $(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta$, so this is equal to

$$(1 \otimes \Delta)(\Delta(x)) = \sum x_1 \otimes (x_2)_1 \otimes (x_2)_2 \tag{6}$$

We usually rewrite both (5) and (6) as

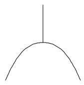
$$\sum (x_1)_1 \otimes (x_1)_2 \otimes x_2 = \sum x_1 \otimes (x_2)_1 \otimes (x_2)_2 = \sum x_1 \otimes x_2 \otimes x_3.$$

Similarly, the counit axioms look like this:

$$(\varepsilon \otimes 1)\Delta(x) = \sum \varepsilon(x_1) x_2 = x$$

$$(1 \otimes \varepsilon)\Delta(x) = \sum \varepsilon(x_2) x_1 = x$$

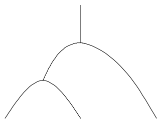
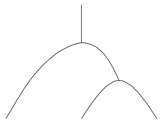
Definition 65 (Notation)**.** In terms of **string diagrams**, we represent comultiplication by

$\Delta =$ 

The coassociativity equation

$$(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta$$

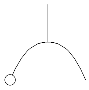
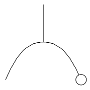
is represented by

 $=$ 

The counit is written as

$\varepsilon =$ 

and the counit law is represented by

$(\varepsilon \otimes 1) \circ \Delta =$  $=$  $= (1 \otimes \varepsilon) \circ \Delta$

Both of these are equal to a single vertical line, which is the identity.

Generally, vertically conjoining diagrams is composition, and putting diagrams next to each other corresponds to tensoring. Flipping things upside down is the dual, so multiplication is represented by a vertical fork, which is a vertical reflection of comultiplication.

Example 66. If $M \xrightarrow{\chi} M \otimes C$ is a comodule, Sweedler notation works like this:

$$\chi(m) = \sum m_0 \otimes m_1$$

with $m_0 \in M$ and $m_1 \in C$. The comodule axioms are given by $(\chi \otimes 1)\chi(m) = (1 \otimes \Delta)(\chi(m))$.

$$(\chi \otimes 1)\chi(m) = \sum \chi(m_0) \otimes m_1 = \sum (m_0)_0 \otimes (m_0)_1 \otimes m_1$$

$$(1 \otimes \Delta)(\chi(m)) = \sum m_0 \otimes \Delta(m_1) = \sum m_0 \otimes (m_1)_1 \otimes (m_1)_2$$

Their common value is written

$$\sum m_0 \otimes m_1 \otimes m_2.$$

Similarly, we have

$$m = (1 \otimes \varepsilon)\chi(m) = \sum \varepsilon(m_1) m_0$$

Braid groups

Definition 67. The n-th **Artin braid group** $\mathbb{B}_n$ is the group with generators $\{\sigma_1, \ldots, \sigma_{n-1}\}$ subject to relations

$$\begin{aligned} \sigma_i\sigma_j &= \sigma_j\sigma_i && \text{if } |i-j| \geqslant 2 \\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} && \text{if } 1 \leqslant i \leqslant n-2 \end{aligned}$$

There is a morphism $\mathbb{B}_n \to S_n$ given by $\sigma_i \mapsto (i, i+1)$. This group is very important and comes up in many different places. Several other definitions are as follows.

Definition 68. The **braid group** on n points $\mathcal{B}_n$ is the fundamental group of the space

$$\{(u_1, \ldots, u_n) \in (\mathbb{R}^2)^n : u_i \neq u_j \ \forall i, j\} \subseteq (\mathbb{R}^2)^n$$

Definition 69. A **geometric braid** b of n strands is a subspace of $\mathbb{R}^2 \times I$ (where $I = [0,1]$ that is the disjoint union of n topological intervals (spaces homeomorphic to I) such that $b \cap \mathbb{R}^2 \times \{0\} = \{(1,0,0), (2,0,0), \ldots, (n,0,0)\}$ and $b \cap \mathbb{R}^2 \times \{1\} = \{(1,0,1), (2,0,1), \ldots, (n,0,1)\}$. Moreover, each strand starts at some point $(i,0,0)$ and ends at $(j,0,1)$.

We say that b and b' are **isotypic** if there is a homotopy $H \colon b \times I \to \mathbb{R}^2 \times I$ where $H(-,0)$ is the inclusion of b into $\mathbb{R}^2 \times I$ and $H(-,1)$ has image b' and each $H(-,t)$ is a topological embedding (homeomorphism onto its image). We also ask that the homotopy doesn't move the endpoints of the strands, by requiring that the maps $t \mapsto H((x,0),t)$ and $t \mapsto H((x,1),t)$ are constant.

The idea is that the isotopy classes of these geometric braids will form a group isomorphic to the Artin braid group, and this will let us draw pictures and be rigorous about it.

Lecture 7 — 29 January 2016

Last time we talked about geometric braids. Let $\mathcal{B}'_n$ be the set of isotopy classes of geometric braids, and let $\mathcal{B}_n = \pi_n(X)$, where $X = \{(u_1, \ldots, u_n) \in (\mathbb{R}^2)^n \mid u_i \neq u_j \forall\, i \neq j\}$.

There is a function $\mathcal{B}'_n \to \mathcal{B}_n$. If b is a geometric braid with strands $b^1, \ldots, b_n$, then write $b_i^t = b^i \cap \mathbb{R}^2 \times \{t\}$ for $0 \leqslant t \leqslant 1$. Set $b_t = (b_t^1, \ldots, b_t^n) \in (\mathbb{R}^2)^n$. Then by the map $t \mapsto b_t$, we get a map $I \to X$. Now fix a path γ from $\{(1,0,1), \ldots, (1,0,n)\}$ to $\{(1,0,0), \ldots, (1,0,n)\}$, define $\gamma \cdot \alpha$ which is a path in X.

This defines a map

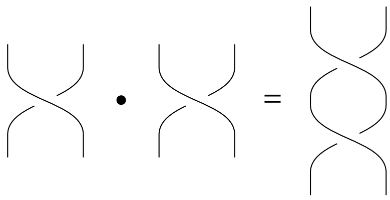
$$\begin{aligned} \Phi \colon \quad \mathcal{B}'_n &\longrightarrow \mathcal{B}_n \\ [\alpha] &\longmapsto [\gamma \cdot \alpha] \end{aligned}$$

where $[\alpha]$ is the isotopy class of α, and $[\gamma \cdot \alpha]$ is the homotopy class.

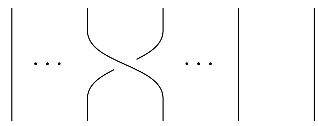
Theorem 70. This map Φ is a bijection.

We already said that $\mathcal{B}_n$ is the braid group, so this gives us a neat way to pictorially represent elements of the braid groups.

The product on B'_n that corresponds to the product of fundamental groups B_n can be described by concatenating braids.

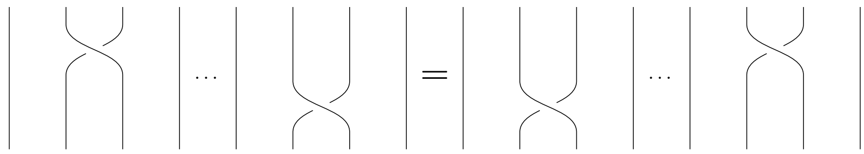


Finally, there is a group morphism from the Artin braid group $\Psi \colon \mathbb{B}_n \to \mathcal{B}_n$ given by the following maps on the generators:

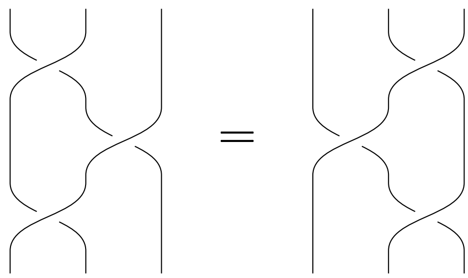
$\sigma_i \longmapsto$ 

And the relations are depicted as

$\sigma_i \sigma_j = \sigma_j \sigma_i$



$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$



Theorem 71. The map $\Psi \colon \mathbb{B}_n \to \mathcal{B}_n$ given above is an isomorphism.

Braided monoidal categories

Definition 72. A **braiding** on a monoidal category **V** is a natural isomorphism $c_{X,Y} \colon X \otimes Y \to Y \otimes X$, such that the following diagrams commute:

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "c_{X,Y} \otimes 1"] \arrow[d, "\alpha"] & (Y \otimes X) \otimes Z \arrow[d, "\alpha"] \\
X \otimes (Y \otimes Z) \arrow[d, "c_{X,Y \otimes Z}"] & Y \otimes (X \otimes Z) \arrow[d, "1 \otimes c_{X,Z}"] \\
(Y \otimes Z) \otimes X \arrow[r, "\alpha"] & Y \otimes (Z \otimes X)
\end{tikzcd}
\qquad
\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "\alpha"] \arrow[d, "c_{X \otimes Y, Z}"] & X \otimes (Y \otimes Z) \arrow[d, "1 \otimes c_{Y,Z}"] \\
Z \otimes (X \otimes Y) \arrow[d, "\alpha^{-1}"] & X \otimes (Z \otimes Y) \arrow[d, "\alpha^{-1}"] \\
(Z \otimes X) \otimes Y & (X \otimes Z) \otimes Y \arrow[l, "c_{X,Z} \otimes 1"']
\end{tikzcd}
\tag{7}$$

Definition 73. A **braided monoidal category** is a monoidal category equipped with a braiding.

Definition 74. A braiding c is a **symmetry** when $c_{X,Y}^{-1} = c_{Y,X}$.

Definition 75. A **symmetric monoidal category** is a braided monoidal category with a braiding that is a symmetry.

Symmetric monoidal categories are much much older in the literature than braided categories. On one hand, they can be viewed as a degenerate form of a higher category, but on the other hand they come from quantum groups, which are a machine for generating braided monoidal categories.

Remark 76. We will see that if $\mathbf{V}$ is braided strict monoidal, then every object $X \in \mathbf{V}$ comes with a "representation" $\mathbb{B}_n \to \mathbf{V}(X^{\otimes n}, X^{\otimes n})$, and the idea is that

$$\sigma_i \mapsto \underbrace{1 \otimes 1 \otimes \cdots \otimes 1}_{i-1} \otimes c_{X,X} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-i-1},$$

and these obey the same relations as in the Artin braid group.

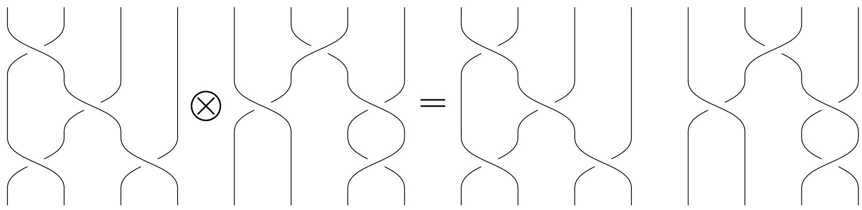
Definition 77. A monoidal functor $F\colon \mathbf{V} \to \mathbf{W}$ between braided monoidal categories is **braided** if

$$\begin{array}{ccc} F(X) \otimes F(Y) & \xrightarrow{\phi_{X,Y}} & F(X \otimes Y) \\ \big\downarrow c_{FX,FY} & & \big\downarrow F(c_{X,Y}) \\ F(Y) \otimes F(X) & \xrightarrow{\phi_{Y,X}} & F(Y \otimes X) \end{array}$$

Example 78. The **braid category** is $\mathbf{B}$ with objects $\mathbb{N}$ and morphisms

$$\mathbf{B}(n,m) = \begin{cases} B_n & \text{if } m = n \\ \varnothing & \text{otherwise} \end{cases}$$

The composition is product in B_n. The monoidal structure is given by $n \otimes m = n + m$, visualized by putting braids next to each other.



This is the free, strict, braided monoidal category on one generator.

Lecture 8 1 February 2016

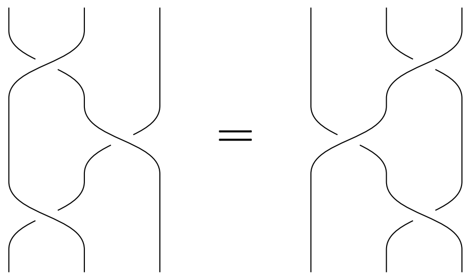
Last time, we saw the definitions of braided monoidal categories. Hopefully today, we'll see the coherence theorem for braided monoidal categories.

Proposition 79. In a braided monoidal category, the following three diagrams commute:

$$\begin{tikzcd}
X\otimes I \arrow[r, "c_{X,I}"] & I\otimes X \arrow[d, "\lambda_X"] \\
X \arrow[u, "\rho_X"] \arrow[r, equal] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
I\otimes X \arrow[r, "c_{I,X}"] \arrow[d, "\lambda_X"'] & X\otimes I \\
X \arrow[ur, "\rho_X"'] &
\end{tikzcd} \tag{8}$$

$$\begin{tikzcd}
(X\otimes Y)\otimes Z \arrow[r, "\alpha"] \arrow[d, "c_{X,Y}\otimes 1"] & X\otimes(Y\otimes Z) \arrow[r, "1\otimes c_{Y,Z}"] & X\otimes(Z\otimes Y) \arrow[r, "\alpha^{-1}"] & (X\otimes Z)\otimes Y \arrow[d, "c_{X,Z}\otimes 1"] \\
(Y\otimes X)\otimes Z \arrow[d, "\alpha"] & & & (Z\otimes X)\otimes Y \arrow[d, "\alpha"] \\
Y\otimes(X\otimes Z) \arrow[d, "1\otimes c_{X,Z}"] & & & Z\otimes(X\otimes Y) \arrow[d, "1\otimes c_{X,Y}"] \\
Y\otimes(Z\otimes X) \arrow[r, "\alpha^{-1}"] & (Y\otimes Z)\otimes X \arrow[r, "c_{Y,Z}\otimes 1"] & (Z\otimes Y)\otimes X \arrow[r, "\alpha"] & Z\otimes(Y\otimes X)
\end{tikzcd} \tag{9}$$

Equation (8) expresses the following in terms of braids



Proof. The diagram in (8) commutes by the axioms (7) of monoidal categories and naturality of c: drawing in the two arrows $c_{X,Y\otimes Z}$ and $c_{X,Z\otimes Y}$ as below, the square in the middle commutes by naturality of c and the left and right rectangles are the two axioms (7) for monoidal categories.

$$\begin{tikzcd}
(X\otimes Y)\otimes Z \arrow[r, "\alpha"] \arrow[d, "c_{X,Y}\otimes 1"] & X\otimes(Y\otimes Z) \arrow[r, "1\otimes c_{Y,Z}"] \arrow[ddd, "c_{X,Y\otimes Z}"] & X\otimes(Z\otimes Y) \arrow[r, "\alpha^{-1}"] \arrow[ddd, "c_{X,Z\otimes Y}"] & (X\otimes Z)\otimes Y \arrow[d, "c_{X,Z}\otimes 1"] \\
(Y\otimes X)\otimes Z \arrow[d, "\alpha"] & & & (Z\otimes X)\otimes Y \arrow[d, "\alpha"] \\
Y\otimes(X\otimes Z) \arrow[d, "1\otimes c_{X,Z}"] & & & Z\otimes(X\otimes Y) \arrow[d, "1\otimes c_{X,Y}"] \\
Y\otimes(Z\otimes X) \arrow[r, "\alpha^{-1}"] & (Y\otimes Z)\otimes X \arrow[r, "c_{Y,Z}\otimes 1"] & (Z\otimes Y)\otimes X \arrow[r, "\alpha"] & Z\otimes(Y\otimes X)
\end{tikzcd}$$

To show that the diagram on the left in (8) commutes, take the left axiom (7) for a monoidal category with $Y = Z = I$. Then attach a bunch of other diagrams to it, each of which is either an axiom of monoidal categories or naturality of c, and observe that the outer diagram commutes.

$$
\begin{tikzcd}
 & (X\otimes I)\otimes I \arrow[r, "c_{X,I}\otimes 1_I"] \arrow[d, "\alpha"] \arrow[ddl, bend right, "\rho_X^{-1}\otimes 1_I"'] & (I\otimes X)\otimes I \arrow[d, "\alpha"] \arrow[dr, bend left, "\lambda_{I,X}\otimes 1_I"] & \\
 & X\otimes(I\otimes I) \arrow[d, "c_{X,I\otimes I}"] \arrow[dl, "1_X\otimes\lambda_I"'] & I\otimes(X\otimes I) \arrow[r, "\lambda_{X\otimes I}"] \arrow[d, "1_I\otimes c_{X,I}"] & X\otimes I \arrow[ddl, bend left, "c_{X,I}"] \\
X\otimes I \arrow[drr, bend right, "c_{X,I}"'] & (I\otimes I)\otimes X \arrow[r, "\alpha"] \arrow[dr, "\lambda_I\otimes 1_X"'] & I\otimes(I\otimes X) \arrow[d, "\lambda_{I\otimes X}"] & \\
 & & I\otimes X &
\end{tikzcd}
$$

The fact that the outer diagram commutes and $c_{X,I}$ is an isomorphism implies that

$$\rho_X^{-1}\otimes 1_I = (\lambda_{I,X}\otimes 1_I)(c_{X,I}\otimes 1_I) = (\lambda_{I,X}\circ c_{X,I})\otimes 1_I.$$

But $(-\otimes I)\colon \mathbf{V}\to\mathbf{V}$ is isomorphic to $1_{\mathbf{V}}\colon \mathbf{V}\to\mathbf{V}$ via ρ. Therefore $(-\otimes I)$ is faithful, so $\rho_X^{-1} = \lambda_{I,X}\circ c_{X,I}$.

The fact that the diagram on the right in Equation 8 commutes is proven similarly, by starting with $X = I = Y$ in the right axiom in (7) of a braiding. $\square$

Coherence for monoidal categories

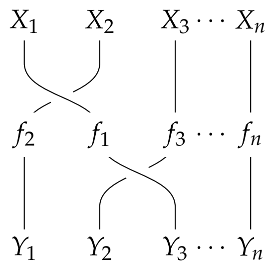
Definition 80. If $\mathbf{A}$ is a category, the **strict braided monoidal category** $\mathbb{B}(\mathbf{A})$. The objects of this category are strings $(X_1,\dots,X_n)$ strings of objects of $\mathbf{A}$. The morphisms $(X_1,\dots,X_n)\to(Y_1,\dots,Y_m)$ only exist if $n=m$ are given by $(\gamma, f_1,\dots,f_n)$ where $f_i\colon X_i\to Y_{\gamma(i)}$ and $\gamma\in\mathbb{B}_n$ is an element of the braid group. By $\gamma(i)$ we mean the result of applying the underlying permutation of γ to i.

The composition of two morphisms is

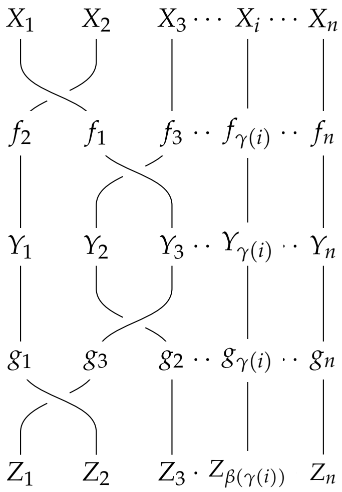
$$(\beta, g_1,\dots,g_n)\circ(\gamma, f_1,\dots,f_n) = (\gamma\beta, g_{\gamma(1)}\circ f_1, g_{\gamma(2)}\circ f_2,\dots,g_{\gamma(n)}\circ f_n).$$

Identities are given by $(1, 1_{X_1},\dots,1_{X_n})$. This is a strict monoidal category with concatenation as the tensor product.

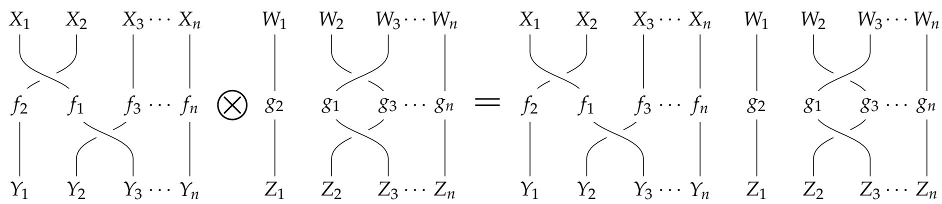
We think of a morphism $(\gamma, f_1,\dots,f_n)$ as, for example,



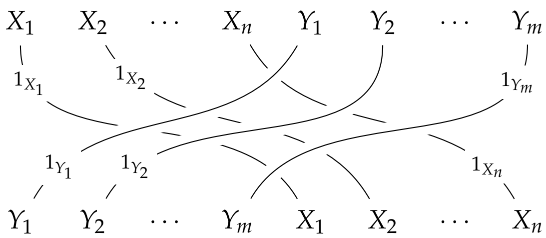
As such a diagram, composition is represented by, for example,



The tensor product is represented by concatenating diagrams horizontally



This category $\mathbb{B}(\mathbf{A})$ has a braiding $(X_1, \dots, X_n) \otimes (Y_1, \dots, Y_m) \to (Y_1, \dots, Y_m) \otimes (X_1, \dots, X_n)$ represented by the string diagram below. Each of the arrows is an identity.



Definition 81. There is a functor $\mathbf{A} \to \mathbb{B}(\mathbf{A})$ given by sending objects X of $\mathbf{A}$ to strings (X) and functions $f \colon X \to Y$ to arrows $(1, f) \colon (X) \to (Y)$.

Theorem 82. *The functor* $\mathbf{A} \to \mathbb{B}(\mathbf{A})$ *exhibits* $\mathbb{B}(\mathbf{A})$ *as the free braided* strict *monoidal category on* $\mathbf{A}$. *There is a bijection between braided strict monoidal functors* $\mathbb{B}(\mathbf{A}) \to \mathbf{C}$ *and functors* $\mathbf{A} \to \mathbf{C}$, *for any braided strict monoidal category* $\mathbf{C}$.

Proof. We'll prove it for $\mathbf{A} = \mathbf{1}$. Then $\mathbb{B}(\mathbf{1}) = \mathbf{B}$ is the braid category as in Equation 7. Given $\mathbf{1} \xrightarrow{x} \mathbf{V}$, that is, an object $X \in \mathbf{V}$, where $\mathbf{V}$ is braided strict monoidal, we want a map $\mathbb{B}(\mathbf{1}) \to \mathbf{V}$. Note that in $\mathbf{V}$, we have a diagram

$$\begin{tikzcd} X \otimes X \otimes X \arrow[r, "c\otimes 1"] & X \otimes X \otimes X \arrow[r, "1\otimes c"] & X \otimes X \otimes X \\ X \otimes X \otimes X \arrow[u, "1\otimes c"] \arrow[r, "c\otimes 1"] & X \otimes X \otimes X \arrow[r, "1\otimes c"] & X \otimes X \otimes X \arrow[u, "c\otimes 1"'] \end{tikzcd} \tag{10}$$

Define $F \colon \mathbf{B} \to \mathbf{V}$ by $F(n) = X^{\otimes n}$, and

$$\begin{array}{rcl} \mathbf{B}(n,n) = \mathbb{B}_n & \longrightarrow & \mathbf{V}(X^{\otimes n}, X^{\otimes n}) \\ \sigma_i & \longmapsto & d_i = \underbrace{1_X \otimes \cdots \otimes 1_X}_{i-1} \otimes c_{X,X} \otimes 1_X \otimes \cdots \otimes 1_X \end{array}$$

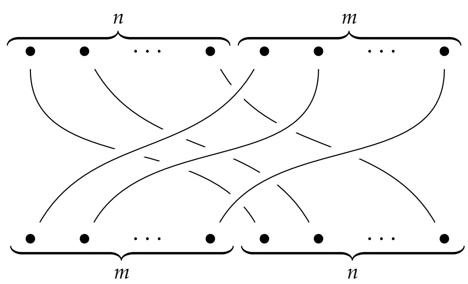
In order for this to define a group morphism on $\mathrm{Aut}_{\mathbf{V}}(X^{\otimes} n)$, we need that to know that these obey the same relations as in the group B_n.

The relation $d_i d_j = d_j d_i$ for $|i - j| \geqslant 2$ follows from functoriality of $\otimes$. And the relation $d_i d_{i+1} d_i = d_{i+1} d_i d_{i+1}$ holds because it's just the diagram (8) tensored on the left and right by some number of identities.

Therefore, we have a functor $F \colon \mathbf{B} \to \mathbf{B}$ which is strict monoidal because the following diagram commutes.

$$\begin{tikzcd} \mathbb{B}_n \times \mathbb{B}_n \arrow[r] \arrow[d] & \mathbf{V}(X^{\otimes n}, X^{\otimes n}) \times \mathbf{V}(X^{\otimes m}, X^{\otimes m}) \arrow[d, "\otimes"] \\ \mathbb{B}_{n+m} \arrow[r] & \mathbf{V}(X^{\otimes (n+m)}, X^{\otimes (n+m)}) \end{tikzcd}$$

The braiding $c_{n,m} \colon n + m \to m + n$ in $\mathbf{B}$ can be seen to be equal to



which in terms of the generators of the braid group is

$$c_{n,m} = (\sigma_{m+n} \sigma_{m+n-1} \cdots \sigma_{m+1}) \cdots (\sigma_{n_1} \sigma_n \sigma_{n-1} \cdots \sigma_2)(\sigma_n \sigma_{n-1} \cdots \sigma_1)$$

The braiding axioms in $\mathbf{V}$ are

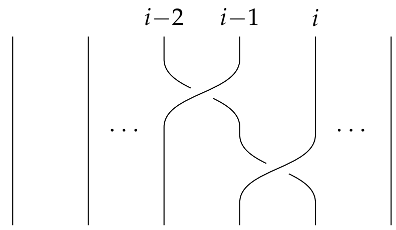
$$(c_{X,Z} \otimes 1_Y)(1_X \otimes c_{Y,Z}) = c_{X,Y \otimes Z}$$

$$(1_Y \otimes c_{X,Z})(c_{X,Y} \otimes 1_Z) = c_{X \otimes Y,Z}$$

Using these, we can see that

$$d_i d_{i-1} = \underbrace{1 \otimes \cdots \otimes 1}_{i-3} \otimes c_{X,X \otimes X} \otimes 1 \otimes \cdots \otimes 1,$$

or in terms of braid diagrams,



Applying these two axioms, we see that $F(c_{n,m}) = c_{X^{\otimes n}, X^{\otimes m}}$. $\square$

Lecture 9 3 February 2016

Today we'll complete the proof of coherence for braided monoidal categories. Last time we saw the free braided strict monoidal category.

Definition 83. We are going to define a category **Fbr**, which is the **free braided monoidal category on one generator**. The objects $|\mathbf{Fbr}|$ of **Fbr** are inductively defined by

- $(*) \in |\mathbf{Fbr}|$, which is the generator.
- $(-) \in |\mathbf{Fbr}|$, which is the unit.
- If $x, y \in |\mathbf{Fbr}|$, then $(x, y) \in |\mathbf{Fbr}|$.

Define a directed graph $X \rightrightarrows |\mathbf{Fbr}|$. We have arrows

- If $x, y, Z \in |\mathbf{Fbr}|$, then $\alpha_{x,y,z} \colon ((x,y),z) \rightleftarrows (x,(y,z)) \colon \overline{\alpha}_{x,y,z}$ is in X
- If $x \in |\mathbf{Fbr}|$, then $\lambda_x \colon ((-),x) \rightleftarrows x \colon \overline{\lambda}_x$ and $\rho_x \colon x \rightleftarrows (x,(-)) \colon \overline{\rho}_x$ are in X
- If $\phi \colon x \to y$ and $\psi \colon y \to z$ are in X, then $\psi \diamond \phi \colon x \to z$.
- If $\phi \colon x \to y$ and $\psi \colon z \to w$ are in X, then $(\phi, \psi) \colon (x,z) \to (y,w) \in X$
- $x \in |\mathbf{Fbr}|$, then $1_x \colon x \to x$ is in X
- If $x, y \in |\mathbf{Fbr}|$, then $c_{x,y} \colon (x,y) \rightleftarrows (y,x) \colon \overline{c}_{x,y}$.

Define the morphisms of **Fbr** to be X quotiented by the smallest equivalence relation $\equiv$ that includes

- $(\chi \diamond \phi) \diamond \psi \equiv \chi \diamond (\phi \diamond \psi)$
- $(1_y \diamond \phi) \equiv \phi$ and $(\phi \diamond 1_x) \equiv \phi$
- $(\psi \diamond \phi, \chi \diamond \tau) \equiv (\psi \diamond \chi, \phi \diamond \tau)$
- $(1_x, 1_y) \equiv 1_{(x,y)}$

- $\overline{\alpha}, \overline{\lambda}, \overline{\rho}$ are inverses of α, λ, ρ, respectively
- The two legs of the naturality diagrams for α, λ, ρ, c must be relationd.
- The axioms of a monoidal category
- The axioms (7) for a braided monoidal category.

Then **Fbr** is a category with morphisms as above, with composition induced by $\diamond$. It's monoidal with $x \otimes y = (x, y)$ and $I = (-)$, and it's braided with c.

Proposition 84. The functor $\mathbf{1} \to \mathbf{Fbr}$ given by $* \mapsto (*)$ exhibits **Fbr** as the free braided monoidal category on **1**. That is, if **V** is a braided monoidal category, then there is a unique strict braided monoidal functor $F\colon \mathbf{Fbr} \to \mathbf{V}$ such that the following commutes:

$$\begin{tikzcd} \mathbf{1} \arrow[rr] \arrow[dr] & & \mathbf{Fbr} \arrow[dl] \\ & \mathbf{V} & \end{tikzcd}$$

Definition 85. Now given a category **A**, the **free braided monoidal category** $\mathbf{Fbr}(\mathbf{A})$ on **A** is the pullback of the square

$$\begin{tikzcd} \mathbf{Fbr}(\mathbf{A}) \arrow[r, "\Gamma'_A"] \arrow[d] & \mathbb{B}(\mathbf{A}) \arrow[d] \\ \mathbf{Fbr} \arrow[r, "\Gamma'_1"] & \mathbf{B} \end{tikzcd}$$

where $\Gamma'_{\mathbf{1}}$ is the unique braided strict monoidal functor $\mathbf{Fbr} \to \mathbf{B}$.

Example 86. More explicitly, $\Gamma'_{\mathbf{1}}$ counts the number of $*$ in an object of **Fbr**.

$$\Gamma'_{\mathbf{1}}(((*), (-)), (*, *)) = (\Gamma'_{\mathbf{1}}(*) \otimes \Gamma'_{\mathbf{1}}(-)) \otimes \Gamma'_{\mathbf{1}}(*, *) = 1 + 0 + 2 = 3 \in \mathbb{N} = \operatorname{ob} \mathbf{B}$$

Remark 87. An object of $\mathbf{Fbr}(\mathbf{A})$ is of the form, for example,

$$((A_1, A_2), A_3, (A_4, (A, 5, A_6))),$$

where each A_i is an object of **A**. It's just an associated list of objects of **A**. $\mathbf{Fbr}(\mathbf{A})$ is the free braided monoidal category on **A**, in the sense that if **V** is a braided monoidal category with a functor $\mathbf{A} \to \mathbf{V}$, then there is a unique braided strict monoidal functor $\mathbf{Fbr}(\mathbf{A}) \to \mathbf{V}$ that makes the following diagram commute.

$$\begin{tikzcd} A \arrow[rr, "\Gamma'_A"] \arrow[dr] & & \mathbf{Fbr}(\mathbf{A}) \arrow[dl] \\ & \mathbf{V} & \end{tikzcd}$$

Theorem 88 (Coherence for braided categories)**.** The braided strict monoidal functor $\Gamma'_{\mathbf{A}}\colon \mathbf{Fbr}(\mathbf{A}) \to \mathbb{B}(\mathbf{A})$ is an equivalence.

Proof. Notice that $\Gamma'_{\mathbf{A}}$ is a pullback of Γ'_1. On the other hand, $\Gamma'_{\mathbf{1}}$ is surjective on objects, since for any $n \in \mathbb{N} = \mathrm{ob}\,\mathbf{B}$ we can always form a bracketing of n-many $*$'s in **Fbr**, which is mapped to n under Γ'_1. This implies that $\Gamma'_{\mathbf{A}}$ is surjective on objects. It will be enough to show that $\Gamma'_{\mathbf{1}}$ is fully faithful, since fully faithful functors are stable under pullback.

To that end, claim that there is a pushout square in **Cat**

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathbb{N} \arrow[d] \\ \mathbf{Fbr} \arrow[r, "\Gamma'_1"] & \mathbf{B} \end{tikzcd} \tag{11}$$

where $\mathcal{F}$ is the free monoidal category on $\mathbf{1}$, and the map $\mathcal{F} \to \mathbf{Fbr}$ is the unique map from the universal property of free monoidal categories:

$$\begin{tikzcd} \mathbf{1} \arrow[r] \arrow[dr] & \mathcal{F} \arrow[d, "\exists!"] \\ & \mathbf{Fbr} \end{tikzcd}$$

Similarly, **N** is the free strict monoidal category, and therefore there is a functor

$$\begin{tikzcd} \mathbf{1} \arrow[r] \arrow[dr] & \mathbb{N} \arrow[d, "\exists!"] \\ & \mathbf{B} \end{tikzcd}$$

$\Gamma_{\mathbf{1}} \colon \mathcal{F} \to \mathbf{N}$ is the unique strict monoidal functor

$$\begin{tikzcd} \mathbf{1} \arrow[r] \arrow[dr] & \mathcal{F} \arrow[d, "\Gamma_{\mathbf{1}}"] \\ & \mathbb{N} \end{tikzcd}$$

We know that $\Gamma_{\mathbf{1}}$ is an equivalence. Now suppose given the pushout **P** of this diagram:

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathbb{N} \arrow[d, "T"] \\ \mathbf{Fbr} \arrow[r, "S"] & \mathbf{P} \end{tikzcd}$$

Claim that **P** can be constructed by adding to the definition of **Fbr** the clauses that α, λ, and ρ are identities. To see this, suppose **P** is constructed in that way, and consider the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathbb{N} \arrow[d, "T"] \arrow[ddr, bend left, "T'"] & \\ \mathbf{Fbr} \arrow[r, "S"] \arrow[drr, bend right, "S'"'] & \mathbf{P} \arrow[dr, dashed] & \\ & & \mathbf{C} \end{tikzcd}$$

The outer square commuting implies that $S'(\alpha)$, $S'(\lambda)$, $S'(\rho)$ are identities. But $\mathbf{P}$ is universal among categories $\mathbf{D}$ with a functor $\mathbf{Fbr} \to \mathbf{D}$ that sends α, ρ, λ to identities. Therefore, there is a unique morphism $\mathbf{P} \to \mathbf{C}$ that makes the diagram above commute. Hence, $\mathbf{P}$ is the pushout of this square.

Now we need to claim that $\mathbf{P}$ is isomorphic to $\mathbf{B}$. Since $\mathbf{B}$ is a category in which α, λ, ρ are sent to identities under the functor $\mathbf{Fbr} \to \mathbf{B}$, then we have

$$\begin{tikzcd} \mathbf{Fbr} \arrow[r, "\Gamma_1'"] \arrow[dr, "S"'] & \mathbf{B} \\ & \mathbf{P} \arrow[u] \end{tikzcd}$$

By construction of $\mathbf{P}$, (the same construction for $\mathbf{Fbr}$ followed by setting $\alpha = 1$, $\lambda = 1$, $\rho = 1$), then $\mathbf{P}$ is also the free strict braided monoidal category; for any braided strict monoidal $\mathbf{V}$, there is a unique map $\mathbf{P} \to \mathbf{V}$ that makes the following commute.

$$\begin{tikzcd} \mathbf{1} \arrow[r] \arrow[dr] & \mathbf{Fbr} \arrow[r] \arrow[d, dashed, "\exists!"] & \mathbf{P} \arrow[dl, dashed, "\exists!"] \\ & \mathbf{V} & \end{tikzcd}$$

So $\mathbf{P}$ and $\mathbf{B}$ are isomorphic, which establishes that $\mathbf{B}$ is the pushout of (11).

Then we can use Lemma 89 below to conclude the proof of coherence. $\square$

Lemma 89. Given a pushout in $\mathbf{Cat}$,

$$\begin{tikzcd} \mathbf{A} \arrow[r, "G"] \arrow[d, "F"] & \mathbf{C} \arrow[d, "T"] \\ \mathbf{B} \arrow[r, "S"] & \mathbf{D} \end{tikzcd}$$

if F is bijective on objects and G is an equivalence with a pseudoinverse G^* with $GG^* = 1$, then S is an equivalence.

Exercise 90. Prove Lemma 89. Hint: $\mathbf{D}$ can be constructed as $\operatorname{ob}\mathbf{D} = \operatorname{ob}\mathbf{C}$, and $\mathbf{D}(N, N') = \mathbf{B}(FG^*N, FG^*N)$ and use composition of $\mathbf{B}$ to make $\mathbf{D}$ into a category.

Corollary 91. In the free braided monoidal category on a set, two morphisms are equal if and only if they have the same underlying braiding.

Proof. If X is a set then we have a functor

$$\mathbf{Fbr}(X) \xrightarrow{\simeq} \mathbb{B}(X) \xrightarrow{\mathbb{B}(!)} \mathbb{B}(\mathbf{1}) = \mathbf{B}$$

that takes a morphism to its underlying braid. Suffices to show that this functor is faithful. Since $\mathbf{Fbr}(X) \xrightarrow{\simeq} \mathbb{B}(X)$ is an equivalence, this reduces to showing that $\mathbb{B}(!)\colon \mathbb{B}(X) \to \mathbf{B}$ is faithful. But $!\colon X \to \mathbf{1}$ is the unique braided strict monoidal functor

$$\begin{tikzcd} X \arrow[r] \arrow[d, "!"] & \mathbb{B}(X) \arrow[d] \\ \mathbf{1} \arrow[r] & \mathbb{B} \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & (*) \arrow[d, mapsto] \\ * \arrow[r, mapsto] & 1 \end{tikzcd}$$

So given a morphism $(X_1, \dots, X_n) \xrightarrow{(\gamma, f_1, \dots, f_n)} (Y_1, \dots, Y_n)$ in $\mathbb{B}(X)$, with $f_i\colon X_i \to Y_{\gamma(i)}$ in X. This implies $f_i = 1_{X_i}$. Therefore, to give a morphism is to give the domain and $\gamma \in \mathbb{B}$.

It follows that two morphisms with the same domain and γ are equal, therefore $\mathbb{B}(!)$ is faithful. $\square$

Lecture 10 — 5 February 2016

Here's a proposition that illustrates the use of the Coherence Theorem.

Proposition 92. Suppose $\mathbf{V}$ is braided, with braiding c. Then the functors $\otimes\colon \mathbf{V} \times \mathbf{V} \to \mathbf{V}$ and $I\colon \mathbf{1} \to \mathbf{V}$ carry a canonical strong monoidal structure given by

$$\begin{tikzcd} (X \otimes Y) \otimes (Z, W) \arrow[r, "\phi_{(X,Y),(Z,W)}"] \arrow[d, "\alpha^{-1}"] & (X \otimes Z) \otimes (Y \otimes W) \\ ((X \otimes Y) \otimes Z) \otimes W \arrow[d, "\alpha \otimes 1_W"] & ((X \otimes Z) \otimes Y) \otimes W \arrow[u, "\alpha_{X \otimes Z, Y, W}"'] \\ (X \otimes (Y \otimes Z)) \otimes W \arrow[r, "(1 \otimes c_{Y,Z}) \otimes 1"] & (X \otimes (Z \otimes Y)) \otimes W \arrow[u, "\alpha^{-1} \otimes 1"'] \end{tikzcd}$$

$$\phi_0\colon I \xrightarrow{\rho_I} I \otimes I$$

(Note that $\mathbf{V} \times \mathbf{V}$ has tensor product $(A, B) \otimes (X, Y) = (A \otimes X, B \otimes Y)$.)

Proof. We will show that this diagram below commutes.

$$\begin{tikzcd} (FX \otimes FY) \otimes FZ \arrow[r, "\alpha_{FX,FY,FZ}"] \arrow[d, "\phi_{X,Y} \otimes 1"] & FX \otimes (FY \otimes FZ) \arrow[d, "1 \otimes \phi_{Y,Z}"] \\ F(X \otimes Y) \otimes FZ \arrow[d, "\phi_{X \otimes Y, Z}"] & FX \otimes F(Y \otimes Z) \arrow[d, "\phi_{X, Y \otimes Z}"] \\ F((X \otimes Y) \otimes Z) \arrow[r, "F\alpha_{X,Y,Z}"] & F(X \otimes (Y \otimes Z)) \end{tikzcd}$$

Take $A = (X, Y)$, $B = (Z, W)$, $C = (U, V)$ and $F = \otimes$. Unlabelled isos in the diagram below are constructed with α, α^{-1}, and tensor products with identities.

$$\begin{tikzcd}
((X\otimes Y)\otimes(Z\otimes W))\otimes(U\otimes V) \arrow[r, "\cong"] \arrow[d, "\alpha"] & (X\otimes(Y\otimes Z)\otimes W)\otimes(U\otimes V) \arrow[r, "((1_X\otimes c_{Y,Z})\otimes 1_W)\otimes(1_U\otimes 1_V)"] & ((X\otimes(Z\otimes Y))\otimes W)\otimes(U\otimes V) \arrow[r, "\cong"] & ((X\otimes Z)\otimes(Y\otimes W))\otimes(U\otimes V) \arrow[d, "\cong"] \\
(X\otimes Y)\otimes((Z\otimes W)\otimes(U\otimes V)) \arrow[d, "\cong"] & & & ((X\otimes Z)\otimes((Y\otimes W)\otimes U))\otimes V \arrow[d, "(1_X\otimes 1_Z)\otimes c_{Y\otimes W,U}\otimes 1_V"] \\
(X\otimes Y)\otimes((Z\otimes(W\otimes U))\otimes V) \arrow[d, "(1_X\otimes 1_Y)\otimes(1_Z\otimes c_{W,U})\otimes 1_V"] & & & (X\otimes Z)\otimes(U\otimes(Y\otimes W))\otimes V \arrow[d, "\cong"] \\
(X\otimes Y)\otimes((Z\otimes(U\otimes W))\otimes V) \arrow[d, "\cong"] & & & ((X\otimes Z)\otimes U)\otimes((Y\otimes W)\otimes V) \arrow[d, "\alpha\otimes\alpha"] \\
(X\otimes Y)\otimes((Z\otimes U)\otimes(W\otimes V)) \arrow[r, "\cong"] & (X\otimes(Y\otimes(Z\otimes U)))\otimes(W\otimes V) \arrow[r, "(1_X\otimes c_{Y,Z\otimes U})\otimes(1_W\otimes 1_V)"] & (X\otimes((Z\otimes U)\otimes Y))\otimes(W\otimes V) \arrow[r, "\cong"] & (X\otimes(Z\otimes U))\otimes(Y\otimes(W\otimes V))
\end{tikzcd}$$

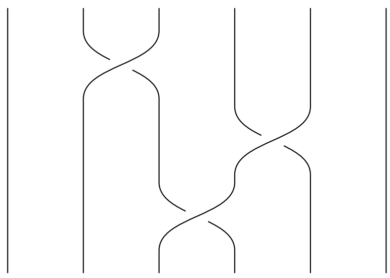
To show that this diagram commutes, use the coherence theorem. For the set $X = \{x, y, z, w, u, v\}$ consider

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \mathbf{V} \\
\mathbf{Fbr}(X) \arrow[ur, "\exists! H"'] &
\end{tikzcd}$$

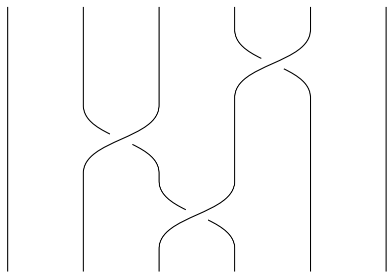
where $H(x) = X$, $H(y) = Y, \ldots, H(v) = V$. Notice that the crazy diagram above is in the image of H. It is enough to show that the diagram commutes in $\mathbf{Fbr}(X)$. To that end, use Corollary 91: from $\mathbf{Fbr}(X) \to \mathbb{B}(X)$, we remove parentheses; from $\mathbb{B}(X) \to \mathbf{B}$, we count the number of objects. So we get

$$\begin{array}{ccccc}
\mathbf{Fbr}(X) & \longrightarrow & \mathbb{B}(X) & \xrightarrow{\mathbb{B}(!)} & \mathbf{B} \\
((X\otimes Y)\otimes(Z\otimes W))\otimes(U\otimes V) & \longmapsto & (X, Y, Z, W, U, V) & \longmapsto & 6 \in \mathbb{B}
\end{array}$$

So we only need to show that the braids are the same. Tracing the diagram around clockwise, we get the braid



and tracing the diagram around counterclockwise, we get the braid



and these braids are the same.

(There are two other axioms to prove, but it's easy to draw the braids and we won't waste that time in lecture today.) □

Remark 93. $\alpha\colon \otimes(\otimes \times 1) \to \otimes(1 \times \otimes)\colon \mathbf{V} \times \mathbf{V} \times \mathbf{V} \to \mathbf{V}$ is a monoidal natural transformation. And similarly for λ and ρ.

Proposition 94. Recall that monoidal functors preserve monoids. That is, if $I \xrightarrow{j} A \xleftarrow{m} A \otimes A$ and F is a monoidal functor, then

$$I \xrightarrow{\phi_0} F(I) \xrightarrow{F(j)} F(A) \xleftarrow{F(m)} F(A \otimes A) \xleftarrow{\phi_{A,A}} FA \otimes FA$$

is a monoid.

Corollary 95.

(1) If (A, j, m) and (A', j', m') are monoids in $\mathbf{V}$, where $\mathbf{V}$ is a braided monoidal category, then

$$I \xrightarrow{\rho_I} I \otimes I \xrightarrow{j \otimes j'} A \otimes A' \xleftarrow{m \otimes m'} A \otimes A \otimes A' \otimes A' \xleftarrow{1 \otimes c_{A,A'} \otimes 1} A \otimes A' \otimes A \otimes A'$$

is a monoid. (Here the monoidal constraints are omitted.)

(2) If (C, ε, δ) and $(C', \varepsilon', \delta')$ are comonoids in $\mathbf{V}$, then

$$I \xleftarrow{\lambda_I} I \otimes I \xleftarrow{\varepsilon \otimes \varepsilon'} C \otimes C' \xrightarrow{\delta \otimes \delta'} C \otimes C \otimes C' \otimes C' \xrightarrow{1 \otimes c_{C,C'} \otimes 1} C \otimes C' \otimes C \otimes C'$$

is a comonoid.

(3) The same is true with c^{-1} instead of c.

Example 96. In $\mathbf{V} = \mathbf{Vect}$, a monoid is an algebra. If A, A' are algebras, then $A \otimes A'$ is an algebra with $(a \otimes a')(b \otimes b') = (ab) \otimes (a'b')$.

Remark 97. If $\mathbf{Mon}(\mathbf{V})$ is the category of monoids of a monoidal category and if $F\colon \mathbf{V} \to \mathbf{W}$ is monoidal, then we get a functor $\mathbf{Mon}(F)\colon \mathbf{Mon}(\mathbf{V}) \to \mathbf{Mon}(\mathbf{W})$. If $\mathbf{V}$ is braided, then $\mathbf{Mon}(\mathbf{V})$ is monoidal, with tensor product (see Corollary 95)

$$\mathbf{Mon}(\mathbf{V}) \times \mathbf{Mon}(\mathbf{V}) = \mathbf{Mon}(\mathbf{V} \times \mathbf{V}) \xrightarrow{M(\otimes)} \mathbf{Mon}(\mathbf{V})$$

and a unit $(I, 1_I, 1_I) \in \mathbf{Mon}(\mathbf{V})$ and α, λ, ρ as in $\mathbf{V}$. Moreover, the forgetful functor $\mathbf{Mon}(\mathbf{V}) \to \mathbf{V}$ is strict monoidal.

Definition 98. A **bimonoid** in a braided monoidal category $\mathbf{V}$ is a comonoid in the monoidal category $\mathbf{Mon}(\mathbf{V})$. Explicitly, a bimonoid (B, j, m) is a monoid in $\mathbf{V}$ with monoid maps

$$I \xleftarrow{\ \varepsilon\ } (B, j, m) \xrightarrow{\ \delta\ } (B, j, m) \otimes (B, j, m)$$

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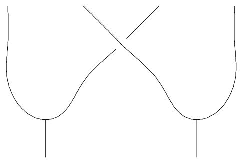
Last time, we saw that if $\mathbf{V}$ is braided, then $\mathbf{Mon}(\mathbf{V})$ is monoidal. If (A, j, m) and (A', j', m') are monoids, then $(A \otimes A', j \otimes j', m \otimes m' \circ 1 \otimes c \otimes 1)$ is their tensor product. Written in string diagrams, the multiplication on $A \otimes A$ is



Remark 99. We also defined bimonoids, which is a comonoid in $\mathbf{Mon}(\mathbf{V})$. We can unpack this definition more explicitly. It consists of a monoid $\widehat{B} = (B, j, m)$ with comultiplication $\widehat{B} \xrightarrow{\delta} \widehat{B} \otimes \widehat{B}$ and a counit $\varepsilon \colon \widehat{B} \to I$, which are morphisms in $\mathbf{Mon}(\mathbf{V})$.

Because the forgetful functor $\mathbf{Mon}(\mathbf{V}) \to \mathbf{V}$ is strict monoidal and faithful, the comonoid axioms for (ε, δ) in $\mathbf{Mon}(\mathbf{V})$ are just the comonoid axioms in $\mathbf{V}$. Then (B, ε, δ) is a comonoid in $\mathbf{V}$ as well. Therefore, δ, ε are monoid morphisms in $\mathbf{V}$. This means that the following diagrams commute.

$$
\begin{tikzcd}
B \otimes B \arrow[r, "\delta\otimes\delta"] \arrow[dd, "m"] & B \otimes B \otimes B \otimes B \arrow[d, "1\otimes c\otimes 1"] \\
 & B \otimes B \otimes B \otimes B \arrow[d, "m\otimes m"] \\
B \arrow[r, "\delta"] & B \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, equal] \arrow[d, "j"] & I \arrow[d, "j\otimes j"] \\
B \arrow[r, "\delta"] & B \otimes B
\end{tikzcd}
\tag{12}
$$

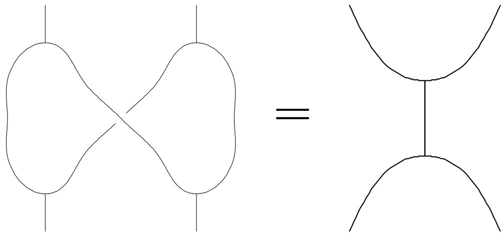
$$
\begin{tikzcd}
B \otimes B \arrow[r, "\varepsilon\otimes\varepsilon"] \arrow[d, "m"] & I \otimes I \arrow[d, "\cong"] \\
B \arrow[r, "\varepsilon"] & I
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, equal] \arrow[d, "j"] & I \arrow[d, equal] \\
B \arrow[r, "\varepsilon"] & I
\end{tikzcd}
\tag{13}
$$

(12) expresses that δ is a monoid morphism, and (13) expresses that ε is a monoid morphism.

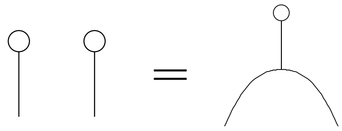
Definition 100. Therefore, a **bimonoid** may be described as

- An object $B \in \mathbf{V}$,
- a monoid structure (B, j, m),
- a comonoid structure (B, ε, δ),
- such the axioms (12) and (13) are satisfied.

In terms of string diagrams, the axioms look like



(14)



(15)

Let's have a look at the axioms for a bimonoid in $\mathbf{Vect}_k$, where c is the usual switch $x \otimes y \mapsto y \otimes x$. A bialgebra is in Sweedler's notation

$$\begin{aligned} \delta(x) &= \sum x_1 \otimes x_2 \\ m(x \otimes y) &= xy \\ j(1_k) &= 1_B \end{aligned}$$

The left hand side of (14) is in Sweedler notation

$$x \otimes y \mapsto \sum x_1 \otimes x_2 \otimes y_1 \otimes y_2 \mapsto \sum x_1 \otimes y_1 \otimes x_2 \otimes y_2 \mapsto \sum x_1 y_1 \otimes x_2 y_2$$

The right hand side of (14) is in Sweedler notation

$$x \otimes y \mapsto xy \mapsto \sum (xy)_1 \otimes (xy)_2.$$

Equating both equations above, (14) is in Sweedler notation

$$\sum (xy)_1 \otimes (xy)_2 = \sum x_1 y_1 \otimes x_2 y_2.$$

Similarly, (**??**) is written in Sweedler notation as

$$\sum (1_B)_1 \otimes (1_B)_2 = 1_B \otimes 1_B.$$

We'll soon see examples of bialgebras when we define Hopf algebras.

We want now to see that (co)modules over a bimonoid form a monoidal category. For this we'll use something more general.

Opmonoidal monads

Recall that a monad $\mathbb{T} = (T, \eta, \mu)$ is a monoid in the monoidal category $([\mathbf{C}, \mathbf{C}], 1_{\mathbf{C}}, \circ)$. Here, we have $T \colon \mathbf{C} \to \mathbf{C}$, $\eta \colon 1_{\mathbf{C}} \implies T$ and $\mu \colon T^2 \implies T$, satisfying

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu_T"] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "T\eta"] \arrow[dr, equal] & T^2 \arrow[d, "\mu"] & T \arrow[l, "\eta_T"'] \arrow[dl, equal] \\ & T & \end{tikzcd}$$

Example 101. If $\mathbf{V}$ is monoidal, and $\widehat{A} = (A, j, m)$ is a monoid in $\mathbf{V}$, then let $T = (A \otimes -) \colon \mathbf{V} \to \mathbf{V}$. Then (T, η, μ) is a monad with

$$\eta = (j \otimes -) \colon 1_{\mathbf{V}} \cong (I \otimes -) \implies (A \otimes -)$$
$$\mu = (m \otimes -) \colon T^2 = A \otimes (A \otimes -) \cong ((A \otimes A) \otimes -) \implies (A \otimes -) = T$$

Definition 102. Given $\mathbb{T} = (T, \eta, \mu)$, it's category of **Eilenberg-Moore** algebras $\mathbf{C}^{\mathbb{T}}$ is the category whose objects are pairs (X, x) with $x \colon Tx \to x$ satisfying

$$\begin{tikzcd} T^2X \arrow[r, "Tx"] \arrow[d, "\mu_X"] & TX \arrow[d, "x"] \\ TX \arrow[r, "x"] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, equal] & TX \arrow[d, "x"] \\ & X \end{tikzcd}$$

and arrows $(X, x) \to (X, y)$ are morphisms $f \colon X \to Y$ in $\mathbf{C}$ such that

$$\begin{tikzcd} TX \arrow[r, "Tf"] \arrow[d, "x"] & TY \arrow[d, "y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commutes.

There is a forgetful functor $\mathbf{C}^{\mathbb{T}} \xrightarrow{U^{\mathbb{T}}} \mathbf{C}$ that sends (X, x) to X, which has a left adjoint $F^{\mathbb{T}} \dashv U^{\mathbb{T}}$ given by $F^{\mathbb{T}}(X) = (TX, \mu_X)$.

Definition 103. An **Opmonoidal monad** on a monoidal category $\mathbf{V}$ is a monad $\mathbb{T} = (T, \eta, \mu)$ where T, η, μ are opmonoidal.

This means that $T \colon \mathbf{V} \to \mathbf{V}$ has an opmonoidal structure.

$$\tau_{X,Y} \colon T(X \otimes Y) \longrightarrow TX \otimes TY$$
$$\tau_0 \colon T(I) \longrightarrow I$$

These arrows satisfy the following diagrams.

$$\begin{tikzcd} T(X \otimes Y \otimes Z) \arrow[r, "\tau_{X,Y\otimes Z}"] \arrow[d, "\tau_{X\otimes Y,Z}"] & TX \otimes T(Y \otimes Z) \arrow[d, "1\otimes\tau_{Y,Z}"] \\ T(X \otimes Y) \otimes TZ \arrow[r, "\tau_{X,Y}\otimes 1"] & TX \otimes TY \otimes TZ \end{tikzcd} \qquad \begin{tikzcd} T(X \otimes I) \arrow[r, "\tau_{X,I}"] & TX \otimes TI \arrow[d, "1\otimes t_0"] \\ TX \arrow[u, "T(\rho_X)"] \arrow[r, "\rho_{TX}"] & TX \otimes I \end{tikzcd}$$

Moreover, $\eta \colon 1_{\mathbf{V}} \to T$ is opmonoidal.

$$\begin{tikzcd} X \otimes Y = 1_{\mathbf{V}}(X \otimes Y) \arrow[r, equal] \arrow[d, "\eta_{X\otimes Y}"] & 1_{\mathbf{V}}(X) \otimes 1_{\mathbf{V}}(Y) = X \otimes Y \arrow[d, "\eta_X\otimes\eta_Y"] \\ T(X \otimes Y) \arrow[r, "\tau_{X,Y}"] & T(X) \otimes T(Y) \end{tikzcd} \qquad \begin{tikzcd} 1_{\mathbf{V}}(I) \arrow[r, equal] \arrow[d, "\eta_I"] & I \arrow[d, equal] \\ T(I) \arrow[r, "\tau_0"] & I \end{tikzcd}$$

And also $\mu\colon T^2 \implies T$ is opmonoidal.

$$\begin{tikzcd}
T^2(X\otimes Y) \arrow[r, "T\tau_{X,Y}"] \arrow[d, "\mu_{X\otimes Y}"] & T(TX\otimes TY) \arrow[r, "\tau_{TX,TY}"] & T^2X\otimes T^2Y \arrow[d, "\mu_X\otimes\mu_Y"] \\
T(X\otimes Y) \arrow[rr, "\tau_{X,Y}"] & & TX\otimes TY
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2(I) \arrow[r, "T\tau_0"] \arrow[d, "\mu_I"] & TI \arrow[r, "\tau_0"] & I \arrow[d, equal] \\
T(I) \arrow[rr, "\tau_0"] & & I
\end{tikzcd}$$

Also T is a monad.

$$\begin{tikzcd}
T^3 \arrow[r, "T\mu"] \arrow[d, "\mu_T"] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, "T\eta"] \arrow[dr, equal] & T^2 \arrow[d, "\mu"] & T \arrow[l, "\eta_T"'] \arrow[dl, equal] \\
& T &
\end{tikzcd}$$

Example 104. The canonical example is a bimonoid $(B, j, m, \varepsilon, \delta)$ in a braided category $\mathbf{V}$. Then $(B\otimes -)\colon \mathbf{V}\to\mathbf{V}$ has a canonical structure of an opmonoidal monad. We know that the multiplication is $\mu_X\colon B\otimes B\otimes X \xrightarrow{m\otimes 1_X} B\otimes X$ and the unit is $\eta_X\colon X \xrightarrow{j\otimes 1} B\otimes X$. Define $\tau_{X,Y}$ as the composite

$$\begin{tikzcd}
B\otimes X\otimes Y \arrow[r, "\tau_{X,Y}"] \arrow[d, "\delta\otimes 1\otimes 1"'] & B\otimes X\otimes B\otimes Y \\
B\otimes B\otimes X\otimes Y \arrow[ur, "1\otimes c\otimes 1"'] &
\end{tikzcd}$$

and $\tau_0 = \varepsilon\otimes 1\colon B\otimes I \to I\otimes I \cong I$.

In the category of vector spaces, this is just tensoring with a Hopf algebra.

Definition 105. Suppose that $\mathbf{V}, \mathbf{W}$ are two monoidal categories, and we have

$$\begin{tikzcd}
\mathbf{V} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{W} \arrow[l, bend left, "G"]
\end{tikzcd}.$$

Then $F \dashv G$ is an **opmonoidal adjunction** if F and G are equipped with opmonoidal structures such that $\varepsilon\colon FG \implies 1$ and $\eta\colon GF \implies 1$ become opmonoidal natural transformations.

Remark 106. By the dual of the Doctrinal Adjunction Theorem (Theorem 21), G has to be *strong*. That is, $G(X\otimes Y)\to GX\otimes GY$ and $G(I)\to I$ are isos.

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Remark 107. $\begin{tikzcd}\mathbf{V} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{W} \arrow[l, bend left, "G"]\end{tikzcd}$ is opmonoidal $\iff$ $\begin{tikzcd}\mathbf{V}^{\mathrm{op}} \arrow[r, bend left, "F^{\mathrm{op}}"] \arrow[r, phantom, "\perp"] & \mathbf{W}^{\mathrm{op}} \arrow[l, bend left, "G^{\mathrm{op}}"]\end{tikzcd}$ is monoidal.

Proposition 108. If $F \dashv G$ is an opmonoidal adjunction, then the induced monad $(T = GF, \eta, \mu = G\varepsilon_F)$ is opmonoidal, where η is the unit and ε is the counit.

Proof. We have seen that (op)monoidal functors compose and that if $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are monoidal categories, E, J, H, S (op)monoidal functors, and $\alpha \colon H \implies J$ is (op)monoidal,

$$\begin{tikzcd} \mathbf{A} \arrow[r, "E"] & \mathbf{B} \arrow[r, bend left, "H"] \arrow[r, bend right, "J"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbf{C} \arrow[r, "S"] & \mathbf{D}, \end{tikzcd}$$

then $S\alpha_E \colon SHE \implies SJF$ is an (op)monoidal transformation.

In the case of $F \dashv G$, GF is opmonoidal, η and ε are opmonoidal by hypothesis, so it remains to show that μ is opmonoidal. But $\mu = G\varepsilon_F$ is opmonoidal by the above with $\mathbf{A} = \mathbf{W}$, $\mathbf{B} = \mathbf{V}$, $\alpha = \varepsilon$, $\mathbf{C} = \mathbf{V}$, $\mathbf{D} = \mathbf{W}$, $E = F$, $H = FG$, $J = 1_{\mathbf{V}}$, $S = G$. ☐

Theorem 109. Suppose that $\mathbb{T} = (T, \eta, \mu)$ is an opmonoidal monad on the monoidal category $\mathbf{V}$. Then $\mathbf{V}^{\mathbb{T}}$ carries a monoidal structure that makes the forgetful functor $U^{\mathbb{T}} \colon \mathbf{V}^{\mathbb{T}} \to \mathbf{V}$ into a *strict* monoidal functor.

Proof. We want to define a tensor product on $\mathbf{V}^{\mathbb{T}}$. Suppose that (A, a) and (B, b) are $\mathbb{T}$-algebras, with $TA \xrightarrow{a} A$, $TB \xrightarrow{b} B$. To define $(A, a) \otimes (B, b)$, since we want U strict monoidal, then we want

$$U((A, a) \otimes (B, b)) = U(A, a) \otimes U(B, b) = A \otimes B.$$

Therefore, we will define $(A, a) \otimes (B, b) = (A \otimes B, a \bullet b)$, where $a \bullet b$ is a map $T(A \otimes B) \to A \otimes B$. There aren't too many things we could do, so define

$$a \bullet b \colon T(A \otimes B) \xrightarrow{\tau_{A,B}} TA \otimes TB \xrightarrow{a \otimes b} A \otimes B.$$

Also, the unit object has to have underlying object $I \in \mathbf{V}$, so define $J = (I, \tau_0)$ to be the unit of $\mathbf{V}^{\mathbb{T}}$; $\tau_0 \colon T(I) \to I$.

This defines a tensor product on objects of $\mathbf{V}^{\mathbb{T}}$, so let's now define it on arrows. Note that we want

$$\begin{tikzcd} \mathbf{V}^{\mathbb{T}} \times \mathbf{V}^{\mathbb{T}} \arrow[r, "\otimes"] \arrow[d, "U \times U"'] & \mathbf{V}^{\mathbb{T}} \arrow[d, "U"] \\ \mathbf{V} \times \mathbf{V} \arrow[r, "\otimes"] & \mathbf{V} \end{tikzcd} \tag{16}$$

Therefore, if $f \colon (A, a) \to (A', a')$ and $g \colon (B, b) \to (B', b')$ are morphisms in $\mathbf{V}^{\mathbb{T}}$, define $f \otimes g$ as their tensor product in $\mathbf{V}$. This is clearly functorial.

Hence, we get a tensor product $\otimes \colon \mathbf{V}^{\mathbb{T}} \times \mathbf{V}^{\mathbb{T}}$ that makes (16) commute.

In order to show that $(\mathbf{V}^{\mathbb{T}}, J, \otimes)$ extends to a monoidal category, it is enough to show that

(1) for $(A, a), (B, b), (C, c)$ in $\mathbf{V}^{\mathbb{T}}$, then $\alpha_{A,B,C} \colon (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ is a morphism of algebras

$$\big((A, a) \otimes (B, b)\big) \otimes (C, c) \to (A, a) \otimes \big((B, b) \otimes (C, c)\big).$$

(2) for $(A, a) \in \mathbf{V}^{\mathbb{T}}$, then $\lambda_A \colon I \otimes A \to A$ is $J \otimes (A, a) \to (A, a)$ and also $\rho_A \colon A \to A \otimes I$ is $A \to (A, a) \otimes J$.

Proof of (1). Want to show that the $\mathbb{T}$-algebra structure commutes with α, that is, α is a morphism of $\mathbb{T}$-algebras.

$$\begin{tikzcd}
T((A \otimes B) \otimes C) \arrow[r, "T\alpha"] \arrow[d, "\tau_{A \otimes B, C}"] & T(A \otimes (B \otimes C)) \arrow[d, "\tau_{A, B \otimes C}"] \\
T(A \otimes B) \otimes T(C) \arrow[d, "\tau_{A,B} \otimes 1"] & T(A) \otimes T(B \otimes C) \arrow[d, "1 \otimes \tau_{B,C}"] \\
(TA \otimes TB) \otimes TC \arrow[r, "\alpha"] \arrow[d, "(a \otimes b) \otimes c"] & TA \otimes (TB \otimes TC) \arrow[d, "a \otimes (b \otimes c)"] \\
(A \otimes B) \otimes C \arrow[r, "\alpha"] & A \otimes (B \otimes C)
\end{tikzcd}$$

The bottom rectangle is naturality of α, and the top rectangle is an axiom of an opmonoidal functor. $\square$

The proof of (2) is an exercise. $\square$

Corollary 110. If $\mathbb{T} = (T, \eta, \mu)$ is opmonoidal, then $F^{\mathbb{T}} \dashv U^{\mathbb{T}}$ is an opmonoidal adjunction.

Proof. We know, by doctrinal adjunction, that if $\mathbf{C}, \mathbf{D}$ are monoidal and $\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "R"] \arrow[r, phantom, "\perp"] & \mathbf{D} \arrow[l, bend left, "S"] \end{tikzcd}$ and S is strong monoidal, then $S \dashv R$ is a monoidal adjunction. Taking opposite categories, $\begin{tikzcd} \mathbf{D} \arrow[r, bend left, "S"] \arrow[r, phantom, "\perp"] & \mathbf{C} \arrow[l, bend left, "R"] \end{tikzcd}$, and S is strong monoidal implies that $R \dashv S$ is an opmonoidal adjunction. Now, we know by Theorem 109 that $U^{\mathbb{T}}$ is strict monoidal. $\square$

Corollary 111. If $(B, j, m, \varepsilon, \delta)$ is a bimonoid in a braided category $\mathbf{V}$, then the category of B-modules (modules for the monoid (B, j, m)) is a monoidal category and the forgetful functor $U \colon B\text{-}\mathbf{Mod} \to \mathbf{V}$ is strict monoidal.

Proof. $B\text{-}\mathbf{Mod} = \mathbf{V}^{(B \otimes -)}$ is the category of algebras for the monad $(B \otimes -, \tau_{X,Y}, \tau_0)$, so apply Corollary 110. Here $\tau_{X,Y} = 1 \otimes c_{X,Y} \otimes 1 \circ \delta \otimes 1 \otimes 1$ and $\tau_0 = \varepsilon \circ \lambda_B$.

More explicitly, if $B \otimes X \xrightarrow{x} X$ and $B \otimes Y \xrightarrow{y} Y$ are B-modules, then $X \otimes Y$ is a B-module with the structure

$$B \otimes X \otimes Y \xrightarrow{\delta \otimes 1 \otimes 1} B \otimes B \otimes X \otimes Y \xrightarrow{1 \otimes c \otimes 1} B \otimes X \otimes B \otimes Y \xrightarrow{x \otimes y} X \otimes Y.$$

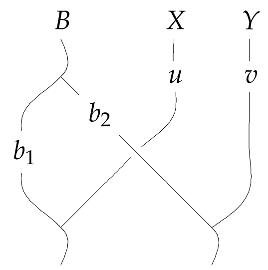
Also, I is a B-module with $B \otimes I \cong B \xrightarrow{\varepsilon} I$. $\square$

Example 112. In Sweedler's notation, if $b \in B$ and $u \otimes v \in X \otimes Y$, then

$$b \cdot (u \otimes v) = \sum b_1 \cdot u \otimes b_2 \cdot v.$$

If $I = k$ is the ground field, then this is a B-module: if $b \in B$, $\alpha \in k$, then $b \cdot \alpha = \varepsilon(b)\alpha \in k$.

In terms of string diagrams, the comultiplication on $X \otimes Y$ is given by



We also have dual statements for comonads. We'll state but not prove them.

Proposition 113 (Dual of Proposition 108)**.** If $F \dashv G$ is monoidal, then the induced comonad $\mathbb{H}$ with $H = FG$ is a monoidal comonad.

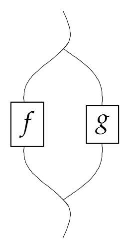
Theorem 114 (Dual of Theorem 109)**.** If $\mathbb{G} = (G, \varepsilon, \delta)$ is a monoidal comonad, then $\mathbf{V}^{\mathbb{G}}$ has monoidal structure and $U^{\mathbb{G}} \colon \mathbf{V}^{\mathbb{G}} \to \mathbf{V}$ is strict monoidal.

Hopf monoids

Recall that if $\mathbf{V}$ is a monoidal category, then $\mathbf{V}(-,-) \colon \mathbf{V}^{\mathrm{op}} \times \mathbf{V} \to \mathbf{Set}$ is a monoidal functor.

Thus, if C is a comonoid and A is a monoid in $\mathbf{V}$, then $\mathbf{V}(C, A)$ is a monoid in **Set**, that is, a regular, everyday, monoid in the sense of algebra. This is because (C, A) is a monoid in $\mathbf{V}^{\mathrm{op}} \times \mathbf{V}$.

This monoidal structure is usually called the **convolution** structure. Given $f, g \colon C \to A$, we write the convolution of f and g as $f * g = m \circ f \otimes g \circ \delta$, and the unit is $C \xrightarrow{\varepsilon} I \xrightarrow{j} A$. (See examples sheet 1).



Definition 115. A bimonoid $\mathbb{H} = (H, j, m, \varepsilon, \delta)$ in a braided category $\mathbf{V}$ is a **Hopf monoid** if it admits an **antipode** $S \colon \mathbb{H} \to \mathbb{H}$ that is the inverse of 1_H in the convolution monoid $\mathbf{V}(H, H)$. In particular,

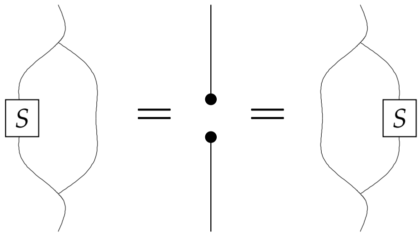
$$1_H * S = j \circ \varepsilon = S * 1_H$$

More explicitly, the following diagram commutes.

$$\begin{tikzcd}
H \arrow[r, "\delta"] \arrow[dd, "\delta"] \arrow[dr, "\varepsilon"] & H \otimes H \arrow[r, "S\otimes 1"] & H \otimes H \arrow[dd, "m"] \\
& I \arrow[dr, "j"] & \\
H \otimes H \arrow[r, "1\otimes S"] & H \otimes H \arrow[r, "m"] & H
\end{tikzcd}$$

Lecture 13

12 February 2016



Definition 116. A Hopf monoid in k**Vect** is called a **Hopf Algebra**.

Let $x \in H$. Then the antipode axioms are, in Sweedler Notation,

$$\begin{tikzcd}
x \arrow[r, mapsto, "\Delta"] \arrow[dd, mapsto, "\Delta"] \arrow[dr, mapsto, "e"] & \sum x_1 \otimes x_2 \arrow[rr, mapsto, "S\otimes 1"] & & \sum S(x_1) \otimes x_2 \arrow[dd, mapsto, "m"] \\
& \varepsilon(x) \arrow[drr, mapsto, "j"] & & \\
\sum x_1 \otimes x_2 \arrow[r, mapsto, "1\otimes S"] & \sum x_1 \otimes S(x_2) \arrow[rr, mapsto, "m"] & & \sum x_1 S(x_2) = \varepsilon(x) 1_H = \sum S(x_1) x_2
\end{tikzcd}$$

$$\sum x_1 S(x_2) = \varepsilon(x) 1_H = \sum S(x_1) x_2$$

Remark 117. Antipodes may not exist, but if they do, they are unique. This is because inverses are unique in the convolution monoid $(\mathbf{V}(H,H), j\varepsilon, *)$ and $S = (1_H)^{-1}$.

Definition 118. A monoid (A, j, m) in a braided category is **commutative** if

$$\begin{tikzcd}
A \otimes A \arrow[rr, "c_{A,A}"] \arrow[dr, "m"'] & & A \otimes A \arrow[dl, "m"] \\
& A &
\end{tikzcd}$$

commutes. Dually, a comonoid (C, ε, δ) is **cocommutative** if the following commutes.

$$\begin{tikzcd}
& C \arrow[dl, "\delta"'] \arrow[dr, "\delta"] & \\
C \otimes C \arrow[rr, "c_{C,C}"] & & C \otimes C
\end{tikzcd}$$

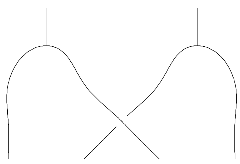
A bimonoid is **(co)commutative** if the underlying (co)monoid is so.

Definition 119. If (A, j, m) and (A', j', m') are monoids, we say a map $f \colon A \to A$ is an **antimorphism** of monoids if it is a morphism of monoids from (A, j, m) to $(A', j', m' \circ c_{A',A'})$.

Example 120. An antihomomorphism of algebras is map $f \colon A \to B$ such that $f(xy) = f(y)f(x)$.

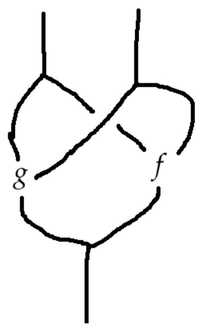
Lemma 121. The antipode of any Hopf monoid H is an antimorphism of monoids and comonoids.

Proof. We are working in some braided monoidal category $\mathbf{V}$. Therefore, $H \otimes H$ is a comonoid with comultiplication given by $(1 \otimes c_{H,H} \otimes 1) \circ (\delta \otimes \delta)$ and counit $\varepsilon \otimes \varepsilon$.

$(1 \otimes c_{H,H} \otimes 1) \circ (\delta \otimes \delta) =$ 

Therefore, $\mathbf{V}(H \otimes H, H)$ is a convolution monoid. What is the convolution here? If $f, g \colon H \otimes H \to H$ then $f * g$ is given by

$$f * g = m \circ f \otimes g \circ (1 \otimes c_{H,H} \otimes 1) \circ (\delta \otimes \delta)$$



and the unit is given by $j \circ (\varepsilon \otimes \varepsilon)$.

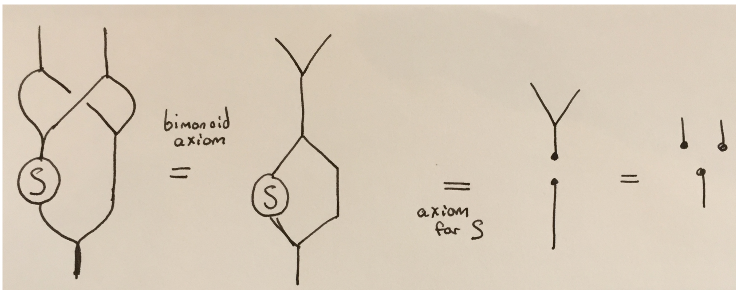
We will show that both the morphisms

$$H \otimes H \xrightarrow{m} H \xrightarrow{S} H \tag{17}$$

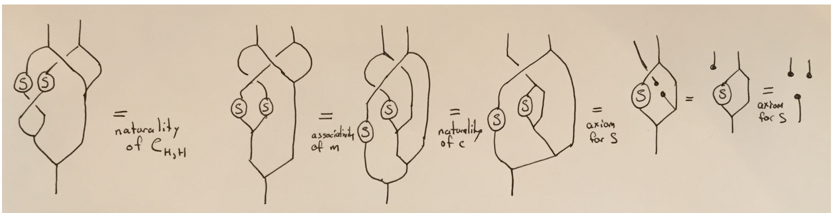
and

$$H \otimes H \xrightarrow{S \otimes S} H \otimes H \xrightarrow{c_{H,H}} H \otimes H \xrightarrow{m} H \tag{18}$$

are convolution-inverses of m, and therefore they are equal. To see that (125) is a convolution inverse for m, compute using string diagrams:



And symmetrically, $m * (S \circ m) = j \circ (\varepsilon \otimes \varepsilon)$, which is the unit of $\mathbf{V}(H \otimes H, H)$. To see that (18) is a convolution inverse for m, compute using string diagrams:



And symmetrically, $m * (m \circ (S \otimes S) \circ c_{H,H}) = j \circ (\varepsilon \otimes \varepsilon)$.

So both (125) and (18) are convolution inverses for m, so they are equal. This in particular tells us that S is a morphism of monoids $(H, j, m) \to (H, j, m \circ c_{H,H})$, and therefore an antimorphism of H. □

Example 122.

(1) Suppose that $\mathbf{C}$ has finite products. Then $\otimes = \times$ is the product and $I = 1$ is the terminal object. Then $\mathbf{C}$ is braided (in fact symmetric), with $c_{X,Y}$ determined by the unique map

$$\begin{tikzcd} Y \arrow[d, "1"] & X \times Y \arrow[l, "p_2"'] \arrow[r, "p_1"] \arrow[d, "\exists! c_{X,Y}"] & X \arrow[d, "1"] \\ Y & Y \times X \arrow[l, "p_1"'] \arrow[r, "p_2"] & X \end{tikzcd}$$

Any object X has a unique comonoid structure. The counit ε is the unique map $: X \to 1$, and $\delta \colon X \to X \times X$ is the unique map constructed from 1_X and 1_X.

$$\begin{tikzcd} & X \arrow[dl, "1"'] \arrow[d, "\delta"] \arrow[dr, "1"] & \\ X & X \times X \arrow[l, "p_2 = \varepsilon \times 1"] \arrow[r, "p_1 = 1 \times \varepsilon"'] & X \end{tikzcd}$$

Therefore, $\delta = \Delta$ is the diagonal map. $c_{X,X}\Delta = \Delta$ implies that X is cocommutative.

Furthermore, any $f \colon X \to Y$ in $\mathbf{C}$ is a morphism of comonoids. So a bimonoid in $\mathbf{C}$ is just a monoid in $\mathbf{C}$, because every object is a comonoid and the comonoid structure is unique.

(2) If $\mathbf{C} = \mathbf{Set}$, and G is a monoid in **Set**, then G is Hopf if and only if G is a group. For $x \in G$, we have that

$$S(x)x = m((S \otimes 1)(\Delta(x))) = j(\varepsilon(x)) = 1_G$$

and symmetrically, $xS(x) = 1_G$, so S defines an inverse for elements in the monoid G.

(3) The same can be done in any category $\mathbf{C}$ with finite products instead of **Set**, and replacing groups by internal groups.

(4) If $\mathbf{A}$ is the category of finitely presentable, reduced k-algebras. The objects of this category are algebras that are quotients of polynomial rings in finitely many variables by a finitely generated ideal, with no nontrivial nilpotent elements. Then $\mathbf{A}^{\mathrm{op}} = \mathbf{Aff}_k$ is the category of affine algebraic varieties over k.

Notice that $\mathbf{A}$ has coproducts

$$\begin{array}{ccccc} A & \longrightarrow & A \otimes B & \longleftarrow & B \\ a & \longmapsto & a \otimes 1 & & \\ & & 1 \otimes b & \longleftarrow\!\shortmid & b \end{array}$$

(for this, we need commutativity). The initial object is k. So $\mathbf{Aff}_k$ has all finite products, as the opposite category. A monoid in $\mathbf{Aff}_k$ is an affine monoid, and a Hopf monoid is an affine group.

Lecture 14 — 15 February 2016

Last time, we started seeing examples of Hopf algebras. We'll give lots more examples this time.

Example 123.

(1) If G is a finite group, then the functions from G to k, denoted k^G, is a commutative algebra with $(\alpha\beta)(g) = \alpha(g)\beta(g)$ for $g \in G$. This has a coalgebra structure. The comultiplication $\delta(\alpha) \in k^G \otimes k^G \cong k^{G\times G}$ is given by $\delta(\alpha)(g,h) = \alpha(gh)$, and the counit $\varepsilon\colon k^G \to k$ is given by $\varepsilon(\alpha) = \alpha(1)$.

If G is a group, then k^G is Hopf with antipode $S(\alpha)(g) = \alpha(g^{-1})$.

(2) In another point of view, if $\mathbf{Set}_f$ is the category of finite sets, then there is a strong monoidal functor $k^{(-)}\colon \mathbf{Set}_f^{\mathrm{op}} \to \mathbf{Vect}$ between monoidal categories $(\mathbf{Set}_f^{\mathrm{op}}, 1, \times) \to (\mathbf{Vect}, k, \otimes)$. This functor is also faithful. From here, it is easy to see that if k^G is Hopf, then G is a group, for G a monoid. (See the second examples sheet).

(3) When G is a monoid and k is a field, then the **monoid algebra** kG is the free vector space on G with multiplication defined on the basis as in G and extended linearly. The unit is $1_G \in G \subset kG$.

There is an adjunction

$$\begin{tikzcd} \textbf{Set} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \textbf{Vect} \arrow[l, bend left, "U"] \end{tikzcd}$$

where $F(X) = kX$ is a strong monoidal functor and U is the forgetful functor. Therefore, $kX \otimes kY \cong k(X \times Y)$ and $k \cong k1$.

This makes kG into a cocommutative bialgebra with $\delta\colon kG \to kG \otimes kG \cong k(G \times G)$ by $\delta(g) = (g, g)$ and $\varepsilon\colon kG \to k$ by $\varepsilon(g) = 1$ for all $g \in G$.

If G is now a group, then kG is Hopf with antipode $S(g) = g^{-1}$.

Remark 124. In all of these examples, $S^2 = 1$. This isn't always true: for example, in Taft's Hopf algebra, S has order 4.

Proposition 125. Suppose that H is a bialgebra in k-**Mod** for k a commutative ring. If H is generated by $X \subseteq H$ as an algebra, then a morphism of algebras $S\colon H \to H^{\mathrm{op}}$ is an antipode for H if it satisfies the antipode conditions on the generators.

What exactly does this mean? Usually, we have that the following diagram commutes if H is a Hopf algebra

$$\begin{tikzcd} H \arrow[r, "\delta"] \arrow[dr, "\varepsilon"] \arrow[dd, "\delta"'] & H \otimes H \arrow[r, "S\otimes 1"] & H \otimes H \arrow[dd, "m"] \\ & I \arrow[dr, "j"] & \\ H \otimes H \arrow[r, "1\otimes S"] & H \otimes H \arrow[r, "m"] & H \end{tikzcd}$$

But to say that this diagram commutes on the generators is to say that X equalizes the three maps $H \to H$ in the above diagram.

$$\begin{tikzcd} X \arrow[r, hook] & H \arrow[r, "\delta"] \arrow[dr, "\varepsilon"] \arrow[dd, "\delta"'] & H \otimes H \arrow[r, "S\otimes 1"] & H \otimes H \arrow[dd, "m"] \\ & & I \arrow[dr, "j"] & \\ & H \otimes H \arrow[r, "1\otimes S"] & H \otimes H \arrow[r, "m"] & H \end{tikzcd}$$

In Sweedler's notation, this means that for $x \in X$,

$$\sum S(x_1)x_2 = \varepsilon(x)1 = \sum x_1 S(x_2).$$

Proof of Proposition 125. It suffices to show that if the antipode conditions hold on $x, y \in H$, they hold on $xy \in H$, since then we can extend the result from the generators. Therefore,

$$
\begin{aligned}
\sum S((xy)_1)(xy)_2 &= \sum S(x_1y_1)x_2y_2 && H \text{ is a bialgebra} \\
&= \sum S(y_1)S(x_1)x_2y_2 && S \text{ is an antimorphism} \\
&= \sum S(y_1)\varepsilon(x)y_2 && \text{hypothesis} \\
&= \varepsilon(x)\sum S(y_1)y_2 && \text{rearrange} \\
&= \varepsilon(x)\varepsilon(y)1_H && \text{hypothesis} \\
&= \varepsilon(xy)1_H && \varepsilon \text{ is an algebra morphism.}
\end{aligned}
$$

The other antipode axiom is verified symmetrically. □

Exercise 126. Write down this proof with string diagrams or commutative diagrams.

Recall that the category $\mathbf{Vect}_{\mathbb{N}}$ of $\mathbb{N}$-graded vector spaces has tensor product

$$(V \otimes W)_n = \sum_{i=0}^{n} V_i \otimes W_{n-i}.$$

and unit

$$(I)_n = \begin{cases} k & n = 0 \\ 0 & \text{otherwise} \end{cases}.$$

If B is an $\mathbb{N}$-graded bimonoid in this situation, then the product respects gradings

$$B_n \otimes B_m \to B_{n+m}$$

and also the coproduct respects gradings

$$B_n \xrightarrow{\delta_n} \bigoplus_{i=0}^{n} B_i \otimes B_{n-i}.$$

Proposition 127. An $\mathbb{N}$-graded bialgebra B over a field k (a bimonoid in the monoidal category $\mathbf{Vect}_{\mathbb{N}}$ of $\mathbb{N}$-graded vector spaces) admits an antipode provided that $B_0 = k$.

Proof. Note that $\varepsilon\colon B \to I$ is graded, so $\varepsilon(B_i) = 0$ for $i > 0$ since $I_i = 0$. Write $B = \bigoplus_{n \geqslant 0} B_n$. We will define $S\colon B \to B$ inductively on each B_n.

For $n = 0$, define $S|_{B_0}\colon B_0 \hookrightarrow B$ as the inclusion of the degree 0 component.

For $n > 0$, suppose we have defined S on B_i for $0 \leqslant i \leqslant n$. Let $x \in B_n$. Then

$$\delta(x) = \sum_{i=0}^{n} x_i \otimes x'_{n-i} \in \bigoplus_{i=0}^{n} B_i \otimes B_{n-i}$$

for $x_i, x'_i \in B_i$. Now

$$x = (\varepsilon \otimes 1)(\delta(x)) = \sum_{i=1}^{n} \varepsilon(x_i)x'_{n-i} = \varepsilon(x_0)x'_n,$$

where the last equality holds because $\varepsilon(x_i) = 0$ for $i \neq 0$. Putting these together, we see that

$$\delta(x) = \sum_{i=1}^{n} x_i \otimes x'_{n-i} + 1 \otimes x$$

We want to have the antipode axiom $m((1 \otimes S)(\delta(x))) = \varepsilon(x)1_B = 0$. (We have $\varepsilon(x) = 0$ since $x \in B_n$). So substituting the last expression for $\delta(x)$ into antipode axiom, we see that

$$0 = \varepsilon(x)1_B = m(1 \otimes S)(\delta(x)) = \sum_{i=1}^{n} x_i S(x'_{n-i}) + S(x)$$

This implies that we may define

$$S(x) = -\sum_{i=1}^{n} x_i S(x'_{n-i})$$

where $\deg(x'_{n-i}) = n - i < n$. From this inductive definition, it is automatic that S is an antipode. □

Definition 128. An $\mathbb{N}$-graded bialgebra B over a commutative ring k is **connected** if $B_0 = k$.

Example 129. If V is a vector space, $T(V)$ is the tensor algebra. This algebra is graded $T(V) = \bigoplus_{i=0}^{\infty} V^{\otimes n}$ with multiplication $x \cdot y = x \otimes y$ for $x \in V^{\otimes n}$ and $y \in V^{\otimes m}$. There is a graded bialgebra structure on V with $\delta \colon T(V) \to T(V) \otimes T(V)$ the unique algebra morphism that on $v \in V$ is

$$\delta(v) = 1 \otimes v + v \otimes 1 \in k \otimes V \oplus V \otimes k.$$

This is a Hopf algebra, with $S \colon T(V) \to T(V)$ the unique morphism of algebras $T(V) \to T(V)^{\mathrm{op}}$ such that $S(v) = -v$ for $v \in V$.

Enveloping algebras of Lie algebras

Definition 130. A **Lie algebra** over a field k is a vector space $\mathfrak{g}$ with a **Lie bracket** $[-,-] \colon g \times \mathfrak{g} \to \mathfrak{g}$ that satisfies

- **antisymmetry** $[x, x] = 0$, and
- the **Jacobi identity**$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$.

Example 131. An associative algebra A has an associated Lie algebra A_{Lie} with Lie bracket $[a, b] = ab - ba$.

Lecture 15 17 February 2016

We'll continue with definitions of universal enveloping algebras of Lie algebras so that we can have some cool examples of Hopf algebras. Universal enveloping algebras are one of the most important examples of Hopf algebras, next to Group algebras.

Definition 132. A **morphism of Lie algebras** $f\colon \mathfrak{g} \to \mathfrak{h}$ is a vector space morphism $\mathfrak{g} \to \mathfrak{h}$ such that $[f(x), f(y)] = f([x,y])$.

Definition 133. **Lie** is the category of Lie algebras, and $\mathbf{Alg}_k = \mathbf{Mon}(\mathbf{Vect}_k)$ is the category of algebras over the field k.

Definition 134. There is a functor $(-)_{\mathbf{Lie}}\colon \mathbf{Alg} \to \mathbf{Lie}$ that takes an algebra A to the Lie algebra $A_{\mathbf{Lie}}$ with bracket $[x,y] = xy - yx$.

There is an adjunction $U \dashv (-)_{\mathbf{Lie}}$. $U(\mathfrak{g})$ can be constructed as $T(\mathfrak{g})/I$ where I is the two-sided ideal generated by $[x,y] - (x \otimes y - y \otimes x)$ for $x, y \in \mathfrak{g}$.

The unit $\eta_{\mathfrak{g}}\colon \mathfrak{g} \to (U(\mathfrak{g}))_{\mathbf{Lie}}$ is given by first mapping $x \in \mathfrak{g}$ to the element $x \in T(\mathfrak{g})$ of degree 1, and then to its image under the quotient, which we denote $\overline{x}$. $\eta_{\mathfrak{g}}$ is a morphism of Lie algebras since

$$\overline{[x,y]} = \overline{xy} - \overline{yx} = \overline{x \otimes y} - \overline{y \otimes x}.$$

Definition 135. $U(\mathfrak{g})$ is called the **universal enveloping algebra** of $\mathfrak{g}$.

The next theorem is a very important example of a Hopf algebra.

Theorem 136. $U(\mathfrak{g})$ is Hopf algebra.

Proof. We are going to show that $U(\mathfrak{g})$ is a Hopf algebra in a series of steps.

(a) Two morphisms of Lie algebras

$$\mathfrak{g} \xrightarrow{f} \mathcal{U} \xleftarrow{g} \mathfrak{h} \tag{19}$$

commute when $[f(x), g(y)] = 0$ for all $x \in \mathfrak{g}, y \in \mathfrak{h}$. The pair

$$\begin{array}{ccccc} \mathfrak{g} & \longrightarrow & \mathfrak{g} \times \mathfrak{h} & \longleftarrow & \mathfrak{h} \\ x & \longmapsto & (x, 0) & & \\ & & (0, y) & \longleftarrow\!\shortmid & y \end{array}$$

is the universal pair that commutes. This means that given any other pair as in (19), there is a unique $t\colon \mathfrak{g} \times \mathfrak{h} \to U$ such that the following commutes.

$$\begin{tikzcd} \mathfrak{g} \arrow[r] \arrow[dr, "f"'] & \mathcal{U} \arrow[d, dashed, "t"] & \mathfrak{h} \arrow[l] \arrow[dl, "g"] \\ & U & \end{tikzcd}$$

Proof of (a). Define $t(x,y) = f(x) + g(y)$. This makes the triangles commute, and it's a morphism of Lie algebras, and easily checked to be unique. $\square$

(b) If A, B are algebras, then

$$\begin{array}{ccccc} A & \longrightarrow & A \otimes B & \longleftarrow & B \\ a & \longmapsto & a \otimes 1 & & \\ & & 1 \otimes b & \longleftarrow\!\shortmid & b \end{array}$$

induces a map $\phi_{A,B}$ in the category **Lie**

$$\begin{tikzcd} A_{\mathbf{Lie}} \arrow[r] \arrow[dr] & A_{\mathbf{Lie}} \times B_{\mathbf{Lie}} \arrow[d, dashed, "\phi_{A,B}"] & B_{\mathbf{Lie}} \arrow[l] \arrow[dl] \\ & (A \otimes B)_{\mathbf{Lie}} & \end{tikzcd}$$

Proof of (b). Use (a), with $\phi_{A,B}(a, b) = a \otimes 1 + 1 \otimes b \in A \otimes B$. □

(c) Given (19) where f and g commute, then

$$U(\mathfrak{g}) \xrightarrow{U(f)} U(\mathcal{U}) \xleftarrow{U(g)} U(\mathfrak{h})$$

is a pair of commutative morphisms in **Alg**.

Proof of (c). It is enough to verify on the generators of $U(\mathfrak{g})$ and $U(\mathfrak{h})$, which are simply the elements of $\mathfrak{g}$ and $\mathfrak{h}$, respectively. So given $x \in \mathfrak{g}$, $y \in \mathfrak{h}$,

$$\begin{aligned} (Uf)(x) &= \overline{f(x)} \\ (Ug)(y) &= \overline{g(y)} \end{aligned}$$

and therefore

$$(Uf)(x)(Ug)(y) - (Ug)(y)(Uf)(x) = \overline{f(x)g(y)} - \overline{g(y)f(x)} = \overline{[f(x), g(y)]} = 0,$$

so $U(f)$ and $U(g)$ commute. □

(d) There is a natural transformation $\psi_{\mathfrak{g},\mathfrak{h}} \colon U(\mathfrak{g}) \otimes U(\mathfrak{h}) \to U(\mathfrak{g} \times \mathfrak{h})$ which with $\psi_0 \colon k \cong U(0)$ are a monoidal structure on U.

Proof of (d). From part (c), we have a that maps $U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{h}) \leftarrow U(\mathfrak{h})$ commute. Therefore, by question 1 on the first examples sheet, there is a unique ψ as in the diagram below.

$$\begin{tikzcd} U(\mathfrak{g}) \arrow[r] \arrow[dr] & U(\mathfrak{g}) \otimes U(\mathfrak{h}) \arrow[d, dashed, "\psi_{\mathfrak{g},\mathfrak{h}}"] & U(\mathfrak{h}) \arrow[l] \arrow[dl] \\ & U(\mathfrak{g} \times \mathfrak{h}) & \end{tikzcd}$$

□

(e) The unit $\eta\colon 1_{\mathbf{Lie}} \implies (-)_{\mathbf{Lie}}U$ is a monoidal natural transformation. The counit $\varepsilon\colon U(-)_{\mathbf{Lie}} \implies 1_{\mathbf{Alg}}$ is a monoidal natural transformation.

$$
\begin{tikzcd}
\mathfrak{g}\times\mathfrak{h} \arrow[r, "\eta_{\mathfrak{g}}\times\eta_{\mathfrak{h}}"] \arrow[rdd, bend right, "\eta_{\mathfrak{g}\times\mathfrak{h}}"'] & U(\mathfrak{g})_{\mathbf{Lie}}\times U(\mathfrak{h})_{\mathbf{Lie}} \arrow[d, "\phi_{\mathfrak{g},\mathfrak{h}}"] \\
 & (U(\mathfrak{g})\otimes U(\mathfrak{h}))_{\mathbf{Lie}} \arrow[d, "\psi_{\mathfrak{g},\mathfrak{h}}"] \\
 & U(\mathfrak{g}\times\mathfrak{h})_{\mathbf{Lie}}
\end{tikzcd}
\qquad
\begin{tikzcd}
U(A_{\mathbf{Lie}})\otimes U(B_{\mathbf{Lie}}) \arrow[d, "\psi_{A,B}"] \arrow[rdd, bend left, "\varepsilon_A\otimes\varepsilon_B"] & \\
U(A_{\mathbf{Lie}}\times B_{\mathbf{Lie}}) \arrow[d, "U(\phi_{A,B})"] & \\
U((A\otimes B)_{\mathbf{Lie}}) \arrow[r, "\varepsilon_{A\otimes B}"] & A\otimes B
\end{tikzcd}
\tag{20}
$$

Proof of (e). We check the commutativity on elements $(x,0)$ and $(0,y)$ to verify that η is monoidal, that is, the diagram on the left of (20) commutes.

$$
\begin{tikzcd}
(x,0) \arrow[r, mapsto] \arrow[rdd, mapsto] & (\overline{x},0) \arrow[d, mapsto] \\
 & \overline{x}\otimes 1 \arrow[d, mapsto] \\
 & \overline{(x,0)}
\end{tikzcd}
\qquad
\begin{tikzcd}
(0,y) \arrow[r, mapsto] \arrow[rdd, mapsto] & (0,\overline{y}) \arrow[d, mapsto] \\
 & 1\otimes\overline{y} \arrow[d, mapsto] \\
 & \overline{(y,0)}
\end{tikzcd}
$$

Similarly, we can check that ε is monoidal by checking commutativity of the right diagram of (20) for elements $(\overline{a}\otimes 1)$ and $(1\otimes\overline{b})$ for $a\in A$, $b\in B$. ☐

(f) (U,ψ,ψ_0) is strong monoidal.

Proof of (f). $U \dashv (-)_{\mathbf{Lie}}$ is a monoidal adjunction with unit η and counit ε, so invoking the Doctrinal Adjunction Theorem, we have that ψ,ψ_0 are isomorphisms. ☐

(g) Therefore, $(U,\psi^{-1},\psi_0^{-1})$ is an opmonoidal functor, where

$$
\begin{aligned}
&\psi^{-1}_{\mathfrak{g},\mathfrak{h}}\colon U(\mathfrak{g}\times\mathfrak{h})\to U(\mathfrak{g})\otimes U(\mathfrak{h})\\
&\psi_0^{-1}\colon U(0)\cong k.
\end{aligned}
$$

So this functor U sends comonoids to comonoids.

Any $\mathfrak{g}$ has a unique comonoid structure given by $\Delta\colon\mathfrak{g}\to\mathfrak{g}\times\mathfrak{g}$ and $0\colon\mathfrak{g}\to 0$. Hence, after applying U, we have a comonoid structure

$$
\begin{aligned}
\delta\colon U(\mathfrak{g}) &\xrightarrow{U(\Delta)} U(\mathfrak{g}\times\mathfrak{g}) \xrightarrow{\psi^{-1}_{\mathfrak{g},\mathfrak{g}}} U(\mathfrak{g})\otimes U(\mathfrak{g})\\
\varepsilon\colon U(\mathfrak{g}) &\xrightarrow{U(0)} U(0) \xrightarrow{\psi_0^1} k
\end{aligned}
$$

Notice that $U(\mathfrak{g})$ is commutative, because $\mathfrak{g}\xrightarrow{\Delta}\mathfrak{g}\times\mathfrak{g}$ is cocommutative.

So far, we have shown that $U(\mathfrak{g})$ is a comonoid in $(\mathbf{Alg},k,\otimes)$, that is, $U(\mathfrak{g})$ is a bimonoid.

(h) $U(\mathfrak{g})$ is Hopf.

Proof of (h). • We can define $S\colon U(\mathfrak{g}) \to U(\mathfrak{g})$ by $S(\overline{x}) = -\overline{x}$.

• Alternatively, denote by $\mathfrak{g}^{\mathrm{op}}$ the Lie algebra with bracket $\langle x, y\rangle = [x, y]$. Then there is a morphism

$$\begin{array}{ccc} \mathfrak{g}^{\mathrm{op}} & \longrightarrow & ((U(\mathfrak{g}))^{\mathrm{op}})_{\mathbf{Lie}} \\ x & \longmapsto & \overline{x} \end{array}$$

By the adjunction, there is a map $U(\mathfrak{g}^{\mathrm{op}}) \to U(g)^{\mathrm{op}}$ in **Alg**. So we get a map $S\colon U(\mathfrak{g}^{\mathrm{op}}) \to (U(\mathfrak{g}))^{\mathrm{op}}$ in **Alg**, where $U(\mathfrak{g}) \to U(\mathfrak{g}^{\mathrm{op}})$ is induced by $\mathfrak{g} \to \mathfrak{g}^{\mathrm{op}}\colon x \mapsto -x$.

So now it remains to verify the antipode axioms for S. We'll check one of them, the other one is similar.

$$\begin{aligned} m(S\otimes 1)(\delta(\overline{x})) &= m(S\otimes 1)(\overline{x}\otimes 1 + 1\otimes \overline{x}) \\ &= m(-\overline{x}\otimes 1 + 1\otimes\overline{x}) \\ &= -\overline{x} + \overline{x} \\ &= 0 \\ &= \varepsilon(\overline{x})1 \end{aligned}$$ □

This concludes the proof that $U(\mathfrak{g})$ is a Hopf algebra. □

Lecture 16 — 19 February 2016

Last time, we defined the **universal enveloping algebra** $U(\mathfrak{g})$ for a Lie algebra $\mathfrak{g}$. It has the following universal property: for any algebra A and morphism of Lie algebras $f\colon \mathfrak{g} \to A_{\mathbf{Lie}}$, there is a unique $h\colon U(\mathfrak{g}) \to A$ such that $h_{\mathbf{Lie}}$ makes the following diagram commute.

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\eta_{\mathfrak{g}}"] \arrow[dr, "f"'] & (U(\mathfrak{g}))_{\mathbf{Lie}} \arrow[d, dashed, "h_{\mathbf{Lie}}"] \\ & A_{\mathbf{Lie}} \end{tikzcd}$$

Definition 137. A $\mathfrak{g}$**-module** is a space V with a morphism of Lie algebras $\mathfrak{g} \to \mathrm{End}(V)_{\mathbf{Lie}}$.

Remark 138. The category of $\mathfrak{g}$-modules/$\mathfrak{g}$-representations $\mathfrak{g}$**-Mod** is isomorphic to $U(\mathfrak{g})$**-Mod**. If $\mathfrak{g} \to \mathrm{End}(V)_{\mathbf{Lie}}$ is a $\mathfrak{g}$-module, then the associated morphism of algebras $U(\mathfrak{g}) \to \mathrm{End}(V)$ is a $U(\mathfrak{g})$-module.

$$\begin{tikzcd} \mathfrak{g} \arrow[r] \arrow[dr] & U(\mathfrak{g})_{\mathbf{Lie}} \arrow[d] \\ & \mathrm{End}(V)_{\mathbf{Lie}} \end{tikzcd}$$

Modules over a Hopf monoid

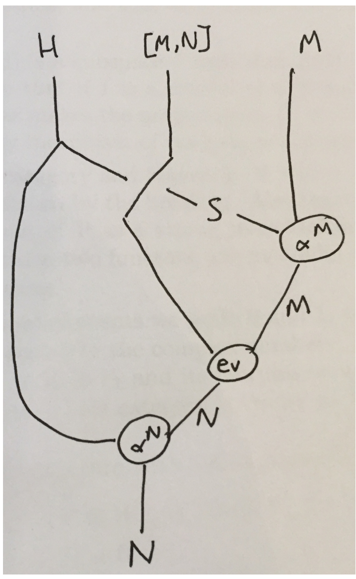
Theorem 139. Suppose that **V** is braided monoidal closed and H is a Hopf monoid in **V**. Then the category of left H-modules H-**Mod** is left closed with left internal Hom given by $\langle M, N\rangle_\ell = [M, N]$. The evaluation and coevaluation are those of **V** (equivalently, ev and coev are morphisms of H-modules).

Proof. Let $H = (H, j, m, \varepsilon, \delta, S)$. Let M, N be H-modules with structure maps $H \otimes M \xrightarrow{\alpha^M} M$ and $H \otimes N \xrightarrow{\alpha^N} N$.

The first thing we have to do is give module structure to $[M, N]$, that is, a map $\alpha\colon H \otimes [M, N] \to [M, N]$. Define it as the morphism corresponding to $\widehat{\alpha}\colon H \otimes [M, N] \otimes M \to N$ under the adjunction, where $\widehat{\alpha}$ is the composite

$$\begin{tikzcd}
H \otimes [M,N] \otimes M \arrow[r, "\delta\otimes 1\otimes 1"] \arrow[ddddrr, "\widehat{\alpha}"] & H \otimes H \otimes [M,N] \otimes M \arrow[r, "1\otimes c\otimes 1"] & H \otimes [M,N] \otimes H \otimes M \arrow[d, "1\otimes 1\otimes S\otimes 1"] \\
 & & H \otimes [M,N] \otimes H \otimes M \arrow[d, "1\otimes 1\otimes \alpha^M"] \\
 & & H \otimes [M,N] \otimes M \arrow[d, "1\otimes \mathrm{ev}"] \\
 & & H \otimes N \arrow[d, "\alpha^N"] \\
 & & N
\end{tikzcd}$$

Call this map $\widehat{\alpha}$. In terms of string diagrams, this is



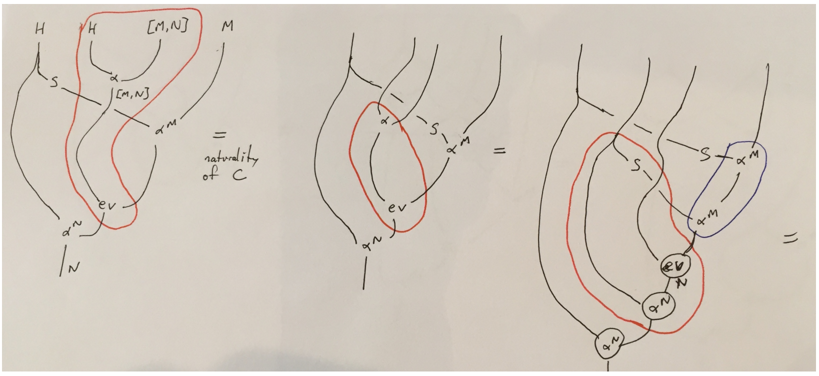
We first show that this is an action of H. That means that we want the following to commute:

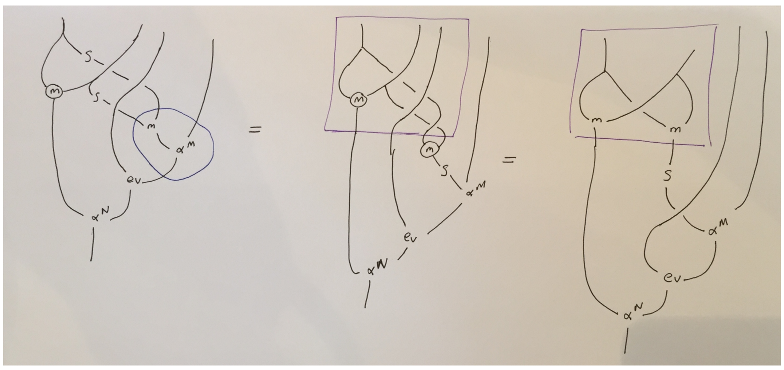
$$
\begin{tikzcd}
H\otimes H\otimes[M,N] \arrow[r, "1\otimes\alpha"] \arrow[d, "m\otimes 1"] & H\otimes[M,N] \arrow[d, "\alpha"] \\
H\otimes[M,N] \arrow[r, "\alpha"] & {[M,N]}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[M,N]} \arrow[d, "j\otimes 1"] \arrow[dr, "1"] & \\
H\otimes[M,N] \arrow[r, "\alpha"] & {[M,N]}
\end{tikzcd}
\tag{21}
$$

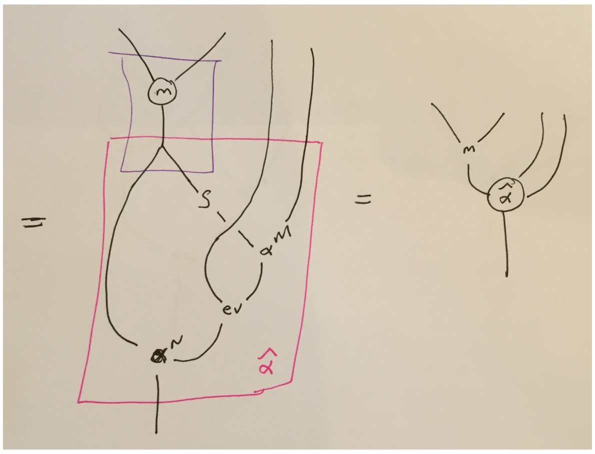
The left diagram in (21) is equivalent to the following diagram commuting

$$
\begin{tikzcd}
H\otimes H\otimes[M,N]\otimes M \arrow[r, "1\otimes\alpha\otimes 1"] \arrow[d, "m\otimes 1\otimes 1"] & H\otimes[M,N]\otimes M \arrow[d, "\alpha\otimes 1"] \arrow[ddr, bend left, "\widehat{\alpha}"] & \\
H\otimes[M,N]\otimes M \arrow[r, "\alpha\otimes 1"] \arrow[drr, bend right, "\widehat{\alpha}"'] & {[M,N]\otimes M} \arrow[dr, "\mathrm{ev}"] & \\
& & N
\end{tikzcd}
$$

We will prove this with string diagrams.



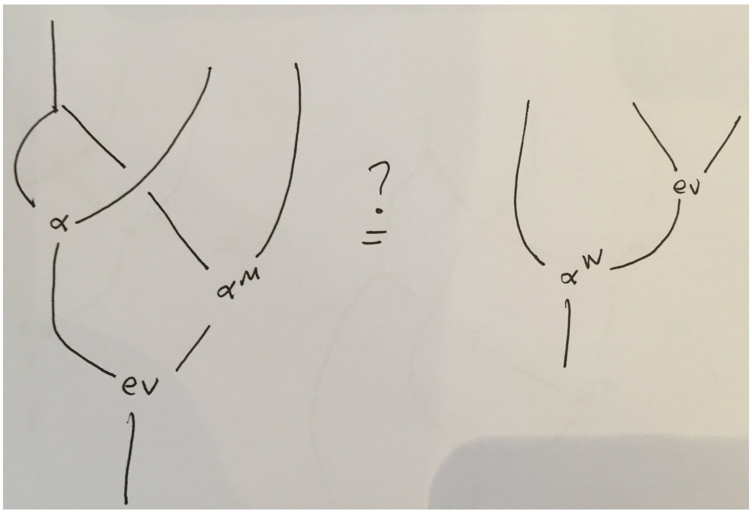




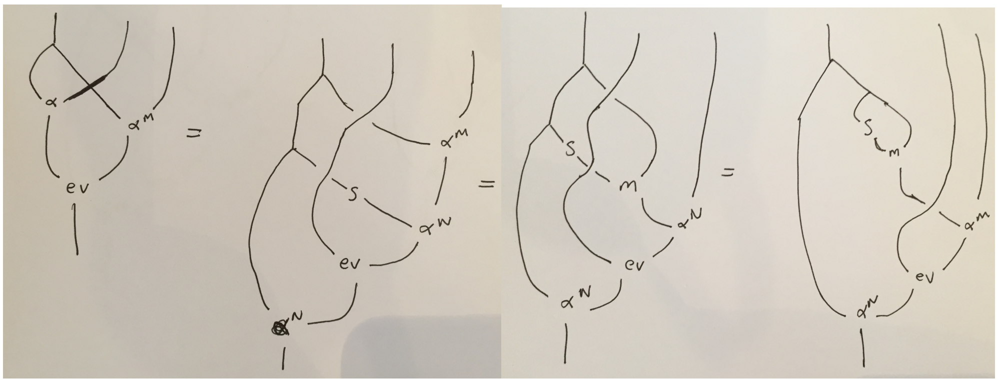
Next, we need to show that $\mathrm{ev}\colon [M,N]\otimes M \to N$ is a morphism of H-modules. That is, we need to show that the following commutes:

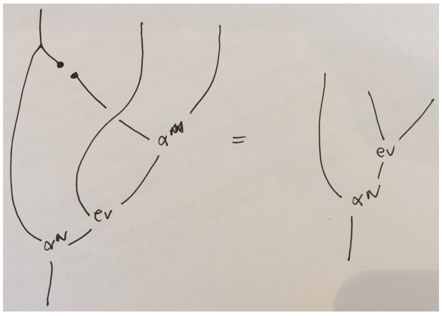
$$
\begin{tikzcd}
H\otimes[M,N]\otimes M \arrow[r, "1\otimes\mathrm{ev}"] \arrow[d] & H\otimes N \arrow[d, "\alpha^N"] \\
{[M,N]\otimes M} \arrow[r, "\mathrm{ev}"] & N
\end{tikzcd}
$$

In terms of string diagrams, this means we need to show the following:



We can prove this with the following equations in string diagrams





It remains to show that $\mathrm{coev}\colon M \to [N, M \otimes N]$ is a morphism of H-modules, or that the following diagram commutes:

$$\begin{tikzcd}
Ht\ M \arrow[r, "1\otimes\mathrm{coev}"] \arrow[d, "\alpha^N"] & H \otimes [N, M \otimes N] \arrow[d, "\alpha"] \\
M \arrow[r, "\mathrm{coev}"] & {[N, M \otimes N]}
\end{tikzcd}$$

Transposing under $- \otimes N \dashv [N, -]$ gives the following equivalent diagram

$$\begin{tikzcd}
H \otimes M \otimes N \arrow[r, "1\otimes\mathrm{coev}\otimes 1"] \arrow[ddrr, bend right, "\alpha^M \otimes 1"'] & H \otimes [N \otimes M \otimes N] \otimes N \arrow[d, "\alpha \otimes 1"] \arrow[ddr, bend left, "\hat{\alpha}"] & \\
& {[N \otimes M \otimes N] \otimes N} \arrow[dr, "\mathrm{ev}"] & \\
& & M \otimes N
\end{tikzcd}$$

The verification that this diagram commutes is left as an exercise. $\square$

Lecture 17 22 February 2016

Last time, we proved Theorem 139, which says that if H is a Hopf monoid, then H-**Mod** is left-closed. We give $[M,N]$ the structure of a H-module via

$$\alpha^N \circ (1\otimes \mathrm{ev}) \circ (1\otimes 1\otimes \alpha^M) \circ (1\otimes c\otimes 1) \circ (1\otimes S\otimes 1\otimes 1) \circ (\delta\otimes 1\otimes 1) \colon H\otimes [M,N]\otimes M \to M$$

Example 140. In the case that $\mathbf{V} = \mathbf{Vect}$, then for $x \in H$, $f \in [M,N] = \mathrm{Hom}_k(M,N)$, $m \in M$, we can write the successive applications of this modules structure in Sweedler notation as follows

$$\begin{array}{c}
x\otimes f\otimes m \\
\Big\downarrow{\scriptstyle \delta\otimes 1\otimes 1} \\
(\sum x_1\otimes x_2)\otimes f\otimes m \\
\Big\downarrow{\scriptstyle 1\otimes S\otimes 1\otimes 1} \\
(\sum x_1\otimes S(x_2))\otimes f\otimes m \\
\Big\downarrow{\scriptstyle 1\otimes c\otimes 1} \\
\sum x_1\otimes f\otimes S(x_2)\otimes m \\
\Big\downarrow{\scriptstyle 1\otimes 1\otimes \alpha^M} \\
\sum x_1\otimes f\otimes S(x_2)m \\
\Big\downarrow{\scriptstyle 1\otimes \mathrm{ev}} \\
\sum x_1\otimes f(S(x_2)m) \\
\Big\downarrow{\scriptstyle \alpha^N} \\
\sum x_1 f(S(x_2)m)
\end{array}$$

This means that if $(x\cdot f)(m) = \sum_{i=1}^n x_i\otimes x_i'$, then

$$(x\cdot f)(m) = \sum_i x_i f(S(x_i')m).$$

Example 141. When $H = kG$, for G a group, then $\delta(x) = x\otimes x$ if $x\in G$. So for G-modules M,N, $\mathrm{Hom}_k(M,N)$ is a G-module with $(x\cdot f)(m) = x\cdot f(x^{-1}m)$. for $x\in G$, $f\in \mathrm{Hom}_k(M,N)$, and $m\in M$.

Example 142. If $\mathfrak{g}$ is a Lie algebra, and M,N are $\mathfrak{g}$-modules, then $\mathrm{Hom}_k(M,N)$ is also a $\mathfrak{g}$-module. Here we interpret $\mathfrak{g}$-module as $U(\mathfrak{g})$-module. Then if the map $\mathfrak{g}\to U(\mathfrak{g})_{\mathbf{Lie}}$ is denoted by $x\mapsto \overline{x}$, the $\mathfrak{g}$-module structure on $\mathrm{Hom}_k(M,N)$ is given by

$$(x\cdot f)(m) = x\cdot f(m) - f(x\cdot m)$$

Proof. In Theorem 139 we proved a statement about left internal Homs. If instead we want H-**Mod** to be right closed, we need $S\colon H\to H$ to be invertible as a map in $\mathbf{V}$ and to use S^{-1} in the formula for the action, together with a braiding. □

Corollary 143. Assume the same conditions as in Theorem 139. Then a left H-module M has a left dual in H-**Mod** if and only if it has a dual in **V**.

Proof. ($\Longrightarrow$). Assume M has a left dual in H-**Mod**. Since the forgetful functor H-**Mod** $\to$ **V** is strict monoidal, we get that $U(M)$ has a left dual in **V**. (Strong monoidal functors preserve duals).

($\Longleftarrow$). It suffices to show that $N \otimes [M, I] \to [M, N]$ is an isomorphism H-**Mod** when M has a dual in **V**, by question 9 on the first examples sheet. But it is invertible in **V**, because M has a dual in **V**, and U reflects isomorphisms. □

Example 144. When $\mathbf{V} = \mathbf{Vect}$, finite-dimensional H-modules have a left dual in H-**Mod**, which is the dual vector space.

Comodules over Hopf algebras

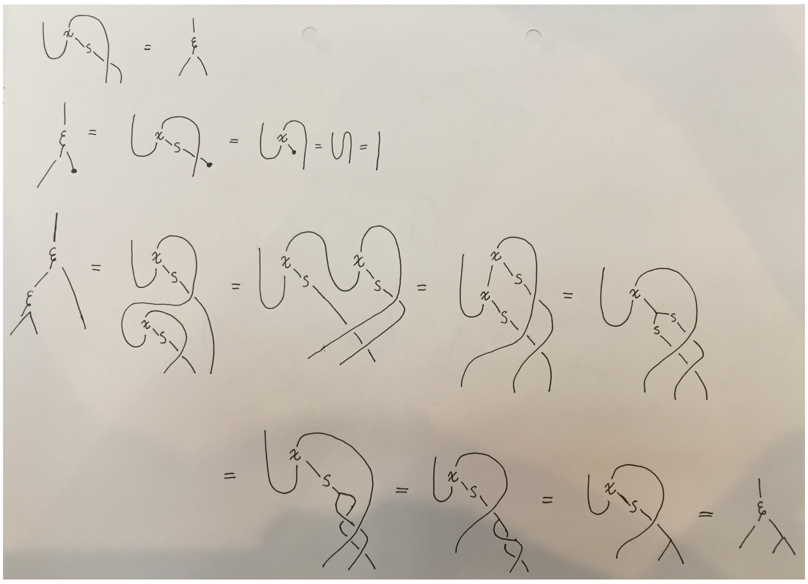
In the definition of modules, we stared with the assumption that **V** was braided and closed. We found a right adjoint $- \otimes X \dashv [X, -]$. Comodules are the duals of modules, so why don't we reverse the arrows and ask for a left adjoint $L \dashv - \otimes X$. This can be done, but turns out to be not so interesting for examples; most comodules come up when there are duals involved, so it's really more useful to assume we have duals.

Theorem 145. Let H be a Hopf monoid in a braided category **V**. Let $\chi \colon M \to M \otimes H$ a right comodule. If M has a (left) dual ${}^*M$ in **V**, then ${}^*M$ carries an H-comodule structure that makes $\mathrm{ev} \colon {}^*M \otimes M \to I$ and $\mathrm{coev} \colon I \to M \otimes {}^*M$ morphisms in $\mathbf{Comod}(H)$.

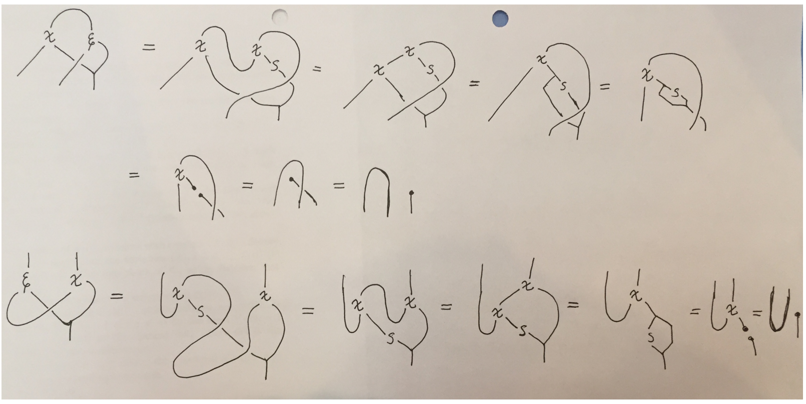
Proof. Define $\xi \colon {}^*M \to {}^*M \otimes H$ by

$$\begin{tikzcd}
{}^*M \arrow[r, "1\otimes \mathrm{coev}"] & {}^*M \otimes M \otimes {}^*M \arrow[r, "1\otimes\chi\otimes 1"] & {}^*M \otimes M \otimes H \otimes {}^*M \arrow[r, "1\otimes 1\otimes S\otimes 1"] & {}^*M \otimes M \otimes H \otimes {}^*M \arrow[d, "\mathrm{ev}\otimes c_{H,{}^*M}"] \\
 & & & {}^*M \otimes H
\end{tikzcd}$$

We have to show that ξ is a coaction. This means that we have to show both coassociativity $\xi \otimes 1 \circ \xi = 1 \otimes \delta \circ \xi$ and the counit law $1 \otimes \varepsilon \circ \xi = 1 = \varepsilon \otimes 1 \circ \xi$.



Next, we need to show that coevaluation $\mathrm{coev}\colon M \otimes {}^*M$ is a morphism in **Comod**(H).



□

Lecture 18 — 24 February 2016

2-Categories

Definition 146. A **2-category** $\mathfrak{K}$ consists of

- objects $A, B, C, \ldots$
- morphisms $f\colon A \to B$,

- **2-cells** $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$

- The objects and arrows form a category as usual.

- The 2-cells have domain and codomain which are parallel morphisms. (For example, in $\mathfrak{K} = \mathbf{Cat}$, α is a natural transformation).

- For each pair of objects, X, Y and each pair of morphisms $f, g \colon X \to Y$, and two cells between them have the structure of a category $\mathfrak{K}(X, Y)$ with composition

$$\begin{tikzcd} X \arrow[r, bend left=60, "f"] \arrow[r, "g" description] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & Y \end{tikzcd} \longmapsto \begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\beta\alpha"] & Y \end{tikzcd}$$

(For example, in **Cat**, this is "vertical" composition of natural transformations).

- There are identity natural transformations $1_f \colon f \to f$ such that $(1_X \cdot -) = 1$ and $(- \cdot 1_X) = 1$.

- For each $f \colon X \to Y$ and $g \colon Z \to W$ there are functors

$$\mathfrak{K}(Y, Z) \xrightarrow{-\cdot f} \mathfrak{K}(X, Z)$$

$$\mathfrak{K}(Y, Z) \xrightarrow{g\cdot -} \mathfrak{K}(Y, W)$$

(For example, in **Cat**,

$$\begin{tikzcd} \bullet \arrow[r, "f"] & \bullet \arrow[r, bend left, "h"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\beta"] & \bullet \arrow[r, "g"] & \bullet \end{tikzcd}$$

then $(\beta \cdot f)_x = \beta_{f(x)}$ and $(g \cdot \beta)_y = g(\beta_y)$.)

- Also, $\mathrm{dom}(\alpha \cdot f) = (\mathrm{dom}\,\alpha) \cdot f$ and $\mathrm{cod}(\alpha \cdot f) = (\mathrm{cod}\,\alpha) \cdot f$.

- If $X' \xrightarrow{f'} X \xrightarrow{f} Y$ and $Z \xrightarrow{g} W \xrightarrow{g'} W'$ then the following commute

$$\begin{tikzcd} \mathfrak{K}(Y, W) \arrow[r, "-\cdot f"] \arrow[dr, "-\cdot(ff')"'] & \mathfrak{K}(X, W) \arrow[d, "-\cdot f'"] \\ & \mathfrak{K}(X', W) \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{K}(Y, Z) \arrow[r, "g\cdot -"] \arrow[dr, "(g'g)\cdot -"'] & \mathfrak{K}(Y, W) \arrow[d, "g'\cdot -"] \\ & \mathfrak{K}(Y, W') \end{tikzcd}$$

- Given

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\beta"] & Z \end{tikzcd}$$

then there are two-cells $h \cdot \alpha \colon hf \implies hg$ and $\beta \cdot g \colon hg \implies kg$ such that

$$(\beta g)(h\alpha) = (k\alpha)(\beta f)$$

And similarly, there are $\beta f \colon hf \implies kf$ and $k\alpha \colon kf \implies kg$.

Definition 147. Given two 2-categories $\mathfrak{K}$ and $\mathfrak{L}$, a **2-functor** $\mathfrak{K} \xrightarrow{F} \mathfrak{L}$ is an assignment that sends objects to objects, morphisms to morphisms, 2-cells to 2-cells, and that preserves all the domains and codomains and the two types of composition.

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y \end{tikzcd} \mapsto \begin{tikzcd} FX \arrow[r, bend left, "Ff"] \arrow[r, bend right, "Fg"'] \arrow[r, phantom, "\Downarrow F\alpha"] & FY \end{tikzcd}$$

2-Category of Comonoids

Given a monoidal category $\mathbf{V}$, and comonoids C and D, we dan define a category $\mathbf{Comon}(\mathbf{V})(C, D)$ with

- objects are comonoid morphisms $C \to D$
- arrows

$$\begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & D \end{tikzcd}$$

are maps $\alpha \colon C \to I$ in $\mathbf{V}$ such that

$$\left(C \xrightarrow{\delta} C \otimes C \xrightarrow{\alpha \otimes f} D\right) = \left(C \xrightarrow{\delta} C \otimes C \xrightarrow{g \otimes \alpha} D\right)$$

We write $\alpha \rightharpoonup f = \alpha \otimes f \circ \delta$ and similarly, $g \leftharpoonup \alpha = g \otimes \alpha \circ \delta$ in the above.

- Composition $f \overset{\alpha}{\Rightarrow} g \overset{\beta}{\Rightarrow} h$ is convolution $\beta * \alpha = \beta \otimes \alpha \circ \delta$.
- The identity $f \overset{1_f}{\Rightarrow} f$ is $\varepsilon \colon C \to I$, which is a convolution identity by the counit laws.
- One can check (for example, with string diagrams) that

$$(\beta * \alpha) \rightharpoonup f = \beta \rightharpoonup (\alpha \rightharpoonup f) = \beta \rightharpoonup (g \leftharpoonup \alpha) = (\beta \rightharpoonup g) \leftharpoonup \alpha = (h \leftharpoonup \beta) \leftharpoonup \alpha = h \leftharpoonup (\beta * \alpha)$$

- Given

$$\begin{tikzcd} C \arrow[r, "f"] & D \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\beta"] & E \arrow[r, "g"] & F \end{tikzcd}$$

  define $\beta \cdot f : hf \implies kf$ as $\beta \circ f : C \to D \to I$ and define $g\beta : gh \implies gk$ as $\beta : D \to I$.

- We can check that $- \cdot f$ preserves composition and that $(\beta * \gamma)f = (\beta f) * (\gamma f)$. Furthermore, $(- \cdot f)$ preserves identities and $(g \cdot -)$ is an isofunctor, since $g \cdot (\beta * \gamma) = \beta * \gamma$ and $g \cdot 1_f = \varepsilon = 1$.

- It remains to check that given

$$\begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & D \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\beta"] & E \end{tikzcd}$$

  that we have the identity

$$(\beta \cdot g)(h \cdot \alpha) = (k \cdot \alpha)(\beta \cdot f). \tag{22}$$

  But $(\beta \cdot g)$ is the morphism in $\mathbf{V}$ given by $C \xrightarrow{g} D \xrightarrow{\beta} I$, and $(h \cdot \alpha)$ is the morphism in $\mathbf{V}$ given by $C \xrightarrow{\alpha} I$. Therefore, the left hand side of (22) is $(\beta g) * \alpha$ and on the right hand side $k\alpha$ is $\alpha : C \to I$ and βf is $C \xrightarrow{f} D \xrightarrow{\beta} I$. So the right hand side is $\alpha * (\beta f)$. Then

$$\text{LHS} = (\beta g) * \alpha = \beta \circ (g \otimes \alpha) \circ \delta = \beta \circ (\alpha \otimes f) \circ \delta = \alpha * (\beta f) = \text{RHS}$$

Definition 148. The 2-category we defined via the above is denoted by **Comon**(**V**).

Remark 149. If C is a comonoid, then $\mathbf{V}(C, I) \times \mathbf{V}(C, X) \to \mathbf{V}(C, X)$ given by $(\alpha, f) \mapsto \alpha \rightharpoonup f$ is an action of the monoid $(\mathbf{V}(C, I), \varepsilon, *)$ on the set $\mathbf{V}(C, X)$. Similarly, there is an action on the right given by $(f, \alpha) \mapsto f \leftharpoonup \alpha$.

These two actions make $\mathbf{V}(C, X)$ a bimodule over $\mathbf{V}(C, I)$, for example, $(\alpha \rightharpoonup f) \leftharpoonup \beta = \alpha \rightharpoonup (f \leftharpoonup \beta)$.

We know that when $\mathbf{V}$ is braided, we can tensor comonoids. Can we also tensor 2-cells? Given

$$\begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & D \end{tikzcd} \qquad \begin{tikzcd} C' \arrow[r, bend left, "f'"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\alpha'"] & D' \end{tikzcd}$$

in **Comon**(**V**), we can define

$$\begin{tikzcd} C \otimes C' \arrow[rr, bend left, "f \otimes f'"] \arrow[rr, bend right, "g \otimes g'"'] \arrow[rr, phantom, "\Downarrow \alpha\otimes\alpha'"] & & D \otimes D' \end{tikzcd}$$

where $\alpha \otimes \alpha' \colon C \otimes C' \to I$ is defined as the tensor product in $\mathbf{V}$.

We know that between categories.

Lemma 150. The functor $\mathbf{Comon}(\mathbf{V}) \times \mathbf{Comon}(\mathbf{V}) \xrightarrow{\otimes} \mathbf{Comon}(\mathbf{V})$ is a 2-functor.

Proof. One checks that if

$$\begin{tikzcd} C \arrow[r, bend left=60, "f", ""{name=F, below}] \arrow[r, "g" description, ""{name=G}] \arrow[r, bend right=60, "h"', ""{name=H, above}] & D \arrow[Rightarrow, from=F, to=G, "\alpha"] \arrow[Rightarrow, from=G, to=H, "\beta"] \end{tikzcd} \qquad \begin{tikzcd} C' \arrow[r, bend left=60, "f'", ""{name=F, below}] \arrow[r, "g'" description, ""{name=G}] \arrow[r, bend right=60, "h'"', ""{name=H, above}] & D' \arrow[Rightarrow, from=F, to=G, "\alpha'"] \arrow[Rightarrow, from=G, to=H, "\beta'"] \end{tikzcd}$$

then $(\beta\alpha) \otimes (\beta'\alpha') = (\beta \otimes \beta')(\alpha \otimes \alpha')$ and also that $1 \otimes 1 = 1$. □

Remark 151. If $\mathbf{V}$ is a braided monoidal category, then the point of all of this is to define a braiding on the category $\mathbf{Comod}(C)$ over a comonoid C. We will define a 2-functor $\mathbf{Comon}(\mathbf{V}) \to \mathbf{Cat}$ sending $C \mapsto \mathbf{Comod}(C)$. Then a 2-cell on $\mathbf{Comon}(\mathbf{V})$ will give a braiding on $\mathbf{Comod}(C)$.

Lecture 19 — 26 February 2016

Lemma 152. Let $\mathbf{V}$ be a braided category. Then the monoidal category $\mathbf{Comon}(\mathbf{V})$ (resp. $\mathbf{Mon}(\mathbf{V})$) is braided and the forgetful functor into $\mathbf{V}$ is braided if the braiding of $\mathbf{V}$ is a symmetry. Moreover, if the forgetful functor is braided, then $c^{-1}_{A,B} = c_{A,B}$ if A admits a monoid structure.

In other words, this lemma says that the braiding $c_{C,D} \colon C \otimes D \to D \otimes C$ is a morphism of comonoids for all C, D if and only if c is a symmetry.

Exercise 153. Prove Lemma 152.

Comod as a 2-functor

Definition 154. Let $\mathbf{V}$ be a monoidal category. We can define a 2-functor $\mathbf{Comod} \colon \mathbf{Comon}(\mathbf{V}) \to \mathbf{Cat}$. This functor sends

- a comonoid C to the category $\mathbf{Comod}(C)$ of right C-comodules;
- a morphism $f \colon C \to D$ to a functor $f_* \colon \mathbf{Comod}(C) \to \mathbf{Comod}(D)$ defined by **corestriction of scalars:**

$$(M \xrightarrow{\chi} M \otimes C) \xmapsto{f_*} (M \xrightarrow{\chi} M \otimes C \xrightarrow{1 \otimes f} M \otimes D)$$

  and f_* is the identity on morphisms.

- a 2-cell $\begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & D \end{tikzcd}$ (i.e. $\alpha\colon C \to I$ in **V** such that $\alpha \otimes f \circ \delta = g \otimes \alpha \circ \delta$) to the natural transformation

$$\begin{tikzcd} \mathbf{Comod}(C) \arrow[r, bend left, "f_*"] \arrow[r, bend right, "g_*"'] \arrow[r, phantom, "\Downarrow\alpha_*"] & \mathbf{Comod}(D) \end{tikzcd}$$

with component at (M, χ) given by $\alpha_{(M,\chi)}\colon M \xrightarrow{\chi} M \otimes C \xrightarrow{1\otimes\alpha} M$.

Exercise 155. Check that α_* is natural in $(M, \chi) \in \mathbf{Comod}(C)$.

We can check that this is indeed a 2-functor. Given $g, f\colon C \to D$, $(gf)_* = g_* f_*$ because both $g_* f_*(M, \chi)$ and $(gf)_*(M, \chi)$ are M with the coaction

$$M \xrightarrow{\chi} M \otimes C \xrightarrow{1\otimes f} M \otimes D \xrightarrow{1\otimes g} M \otimes E$$

and also $(1_C)_* = 1$.

Given the composition of two-cells,

$$\begin{tikzcd} C \arrow[r, bend left=60, "f"] \arrow[r, "g" description] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & D \end{tikzcd}$$

we have that $(\beta * \alpha)_* = \beta_* \alpha_*$ because, taking the component at $(M, \chi) \in \mathbf{Comod}(C)$,

$$(\beta * \alpha)_{(M,\chi)} = 1 \otimes (\beta \otimes \alpha) \circ 1 \otimes \delta \circ \chi = 1 \otimes \beta \circ \chi \circ 1 \otimes \alpha \circ \chi = (\beta_*)_{(M,\chi)} (\alpha_*)_{(M,\chi)}.$$

by the axioms of $\mathbf{Comon}(\mathbf{V})$. Finally, $\varepsilon_* = 1$.

Definition 156. A limit or colimit is **absolute** if it is preserved by any functor whatsoever.

Lemma 157. Suppose that **V** is (finitely) cocomplete. If $f\colon C \to D$ is a morphism of comonoids then $f_*\colon \mathbf{Comod}(C) \to \mathbf{Comod}(D)$ preserves (finite) colimits and U^C-absolute (U^C-split) equalizers.

$$\begin{tikzcd} \mathbf{Comod}(C) \arrow[rr, "f_*"] \arrow[dr, "U^C"'] & & \mathbf{Comod}(D) \arrow[dl, "U^D"] \\ & \mathbf{V} & \end{tikzcd}$$

Proof. U^D creates colimits, and U^D creates equalizers of U^D-split, and indeed U^D-absolute, pairs. Then if

$$E \xrightarrow{e} A \underset{s}{\overset{r}{\rightrightarrows}} B$$

is a diagram in **Comod**(D) and

$$U^D E \xrightarrow{U^D e} U^D A \underset{U^D s}{\overset{U^D r}{\rightrightarrows}} U^D B$$

is a split equalizer, then the original diagram is an equalizer. So if

$$E' \xrightarrow{e'} A' \underset{s'}{\overset{r'}{\rightrightarrows}} B'$$

is an U^C-split equalizer, then it is a $U^D f_*$-split equalizer. Then

$$f_* E' \xrightarrow{f_* e'} f_* A' \underset{f_* s'}{\overset{f_* r'}{\rightrightarrows}} f_* B'$$

is an equalizer in **Comod**(D). $\square$

Definition 158. Let V be a vector space with basis $(e_i)_{i \in I}$ and let C be a coalgebra. Then the **cofree comodule** over V is the vector space $V \otimes C$ with the coaction $V \otimes C \xrightarrow{1 \otimes \delta} V \otimes C \otimes C$.

Remark 159. The cofree comodule $V \otimes C$ is isomorphic to the coproduct $\bigoplus_{i \in I} C$ because

$$\bigoplus_{i \in I} C \cong \bigoplus_{i \in I} (k \otimes C) \cong \left(\bigoplus_{i \in I} k \right) \otimes C \cong V \otimes C$$

Lemma 160. Let $\mathbf{C}$ be the full subcategory of **Comod**(C) consisting of cofree comodules. Then suppose we have a diagram

$$\mathbf{C} \xrightarrow{J} \mathbf{Comod}(C) \underset{T}{\overset{S}{\rightrightarrows}} \mathbf{Comod}(D)$$

where S and T are k-linear functors that preserve coproducts and U^C-split equalizers. Then any $\alpha \colon SJ \implies TJ$ extends to a unique $\beta \colon S \implies T$.

Exercise 161. Let (M, χ) be a C-comodule. Why is M the equalizer of

$$M \otimes C \underset{1 \otimes \delta}{\overset{\chi \otimes 1}{\rightrightarrows}} M \otimes C \otimes C?$$

Proof of Lemma 160. Given $(M, \chi) \in$ **Comod**(C), we have a U^C-split equalizer

$$M \rightarrowtail M \otimes C \underset{1 \otimes \delta}{\overset{\chi \otimes 1}{\rightrightarrows}} M \otimes C \otimes C. \tag{23}$$

Here, $M \otimes C$ is cofree over M, that is, the coaction is $M \otimes C \xrightarrow{1 \otimes \delta} M \otimes C \otimes C$. Likewise, $M \otimes C \otimes C$ is cofree over $M \otimes C$, with coaction given by the map $M \otimes C \otimes C \xrightarrow{1 \otimes 1 \otimes \delta} M \otimes C \otimes C \otimes C$. This means that $M \otimes C$ is the coproduct of $(\dim M)$-many copies of C. Therefore, since S preserves coproducts,

$$S(M \otimes C) \cong S\left(\bigoplus_{i=1}^{\dim M} C\right) \cong \bigoplus_{i=1}^{\dim M} S(C) \cong M \otimes S(C).$$

Now applying S to the U^C-split equalizer (23), we have a commutative diagram

$$\begin{tikzcd}
S(M) \arrow[r, tail] & S(M \otimes C) \arrow[r, shift left, "S(\chi\otimes 1)"] \arrow[r, shift right, "S(1\otimes\delta)"'] \arrow[d] & S(M \otimes C \otimes C) \arrow[d] \\
& M \otimes S(C) \arrow[r, "\chi\otimes 1"] & M \otimes C \otimes S(C)
\end{tikzcd}$$

because S preserves coproducts. Then we may define a morphism d such that

$$\begin{tikzcd}
S(M \otimes C) \arrow[r, "S(1\otimes\delta)"] \arrow[d, "\cong"] & S(M \otimes C \otimes C) \arrow[d, "\cong"] \\
M \otimes S(C) \arrow[r, "1\otimes S(\delta)"] \arrow[dr, "d"'] & M \otimes S(C \otimes C) \arrow[d, "\cong"] \\
& M \otimes C \otimes S(C)
\end{tikzcd}$$

and this makes the diagram

$$\begin{tikzcd}
S(M) \arrow[r, tail] & S(M \otimes C) \arrow[r, shift left, "S(\chi\otimes 1)"] \arrow[r, shift right, "S(1\otimes\delta)"'] \arrow[d] & S(M \otimes C \otimes C) \arrow[d] \\
& M \otimes S(C) \arrow[r, shift left, "\chi\otimes 1"] \arrow[r, shift right, "d"'] & M \otimes C \otimes S(C)
\end{tikzcd}$$

commute. Similarly, we may construct a d' such that

$$\begin{tikzcd}
& M \otimes T(C) \arrow[r, shift left, "\chi\otimes 1"] \arrow[r, shift right, "d'"'] \arrow[d, "\cong"] & M \otimes C \otimes T(C) \arrow[d, "\cong"] \\
T(M) \arrow[r, tail] & T(M \otimes C) \arrow[r, shift left, "T(\chi\otimes 1)"] \arrow[r, shift right, "T(1\otimes\delta)"'] & T(M \otimes C \otimes C)
\end{tikzcd}$$

commutes. Then define β_M by the universal property of the equalizer, via the diagram

$$\begin{tikzcd}
S(M) \arrow[r, tail] \arrow[ddd, dashed, "\beta_M"] & S(M\otimes C) \arrow[r, shift left, "S(\chi\otimes 1)"] \arrow[r, shift right, "S(1\otimes\delta)"'] \arrow[d, "\cong"] & S(M\otimes C\otimes C) \arrow[d, "\cong"] \\
& M\otimes S(C) \arrow[r, shift left, "\chi\otimes 1"] \arrow[r, shift right, "d"'] \arrow[d, "1\otimes\alpha_C"] & M\otimes C\otimes S(C) \arrow[d, "1\otimes 1\otimes\alpha_C"] \\
& M\otimes T(C) \arrow[r, shift left, "\chi\otimes 1"] \arrow[r, shift right, "d'"'] \arrow[d, "\cong"] & M\otimes C\otimes T(C) \arrow[d, "\cong"] \\
T(M) \arrow[r, tail] & T(M\otimes C) \arrow[r, shift left, "T(\chi\otimes 1)"] \arrow[r, shift right, "T(1\otimes\delta)"'] & T(M\otimes C\otimes C)
\end{tikzcd}$$

One can check this is natural in (M, χ) by construction, and that $\beta_C = \alpha_C$ (notice that if M is already a cofree comodule, then α_M would fit in the diagram above for the dashed arrow). $\square$

Lemma 162. Let $\mathbf{V} = \mathbf{Vect}_k$, and let C, D be coalgebras with maps $f, g\colon C \to D$. Then each natural transformation $\tau\colon f_* \implies g_*$ is of the form $\tau = \alpha_*$ for a unique $\alpha\colon f \implies g$ in $\mathbf{Comon}(V)$.

Proof. Consider (C, δ) as an object of $\mathbf{Comod}(C)$. Then let $t = \tau_{(C,\delta)}\colon f_*(C, \delta) \to g_*(C, \delta)$. This is a right D-comodule homomorphism, so the following diagram commutes

$$\begin{tikzcd}
C \arrow[r, "\delta"] \arrow[d, "t"] & C\otimes C \arrow[r, "1\otimes f"] & C\otimes D \arrow[d, "t\otimes 1"] \\
C \arrow[r, "\delta"] & C\otimes C \arrow[r, "1\otimes g"] & C\otimes D
\end{tikzcd}$$

Or in equations,

$$t \otimes f \circ \delta = 1 \otimes g \circ \delta \circ t. \tag{24}$$

Moreover, we know that τ is natural, so for any right C-comodule homomorphism $s\colon C \to C$, the following diagram commutes

$$\begin{tikzcd}
f_*(C,\delta) \arrow[r, "f_*(s)"] \arrow[d, "t"] & f_*(C,\delta) \arrow[d, "t"] \\
g_*(C,\delta) \arrow[r, "g_*(s)"] & g_*(C,\delta)
\end{tikzcd}$$

But since $g_*(C, \delta) = f_*(C, \delta) = C$ as a vector space, and f_*, g_* are the identity on arrows, this diagram reduces to

$$\begin{tikzcd}
C \arrow[r, "s"] \arrow[d, "t"] & C \arrow[d, "t"] \\
C \arrow[r, "s"] & C
\end{tikzcd} \tag{25}$$

or in other words, $s \circ t = t \circ s$ for any right C-comodule homomorphism $s\colon C \to C$. By choosing s cleverly, we can show that t is a morphism of left C-comodules.

Note that there is an adjunction $U^C \dashv F$, where $F\colon \mathbf{Vect}_k \to \mathbf{Comod}(C)$ is the cofree-comodule functor $F(V) = (V \otimes C, 1 \otimes \delta)$. This adjunction gives us the isomorphism

$$\mathbf{Comod}(C)(C, F(k)) \cong \mathbf{Comod}(C)(C, C) \cong \mathrm{Hom}_k(C, k) \cong \mathrm{Hom}_k(U^C(C), k)$$

between hom-sets that in particular tells us for any C-comodule homomorphism $s\colon C \to C$, there is a unique $\beta\colon C \to k$ such that $s = \beta \otimes 1_C \circ \delta$. Applying (25) to this gives us that

$$\beta \otimes t \circ \delta = \beta \otimes 1_C \circ \delta \circ t, \tag{26}$$

and this holds for all linear functionals $\beta\colon C \to k$.

In particular, since we're working over the category of k-vector spaces, we know that any two points of C can be separated by some functional $\beta\colon C \to k$. Therefore, (26) becomes

$$1_C \otimes t \circ \delta = \delta \circ t \tag{27}$$

Now define $\alpha\colon C \to k$ by $\alpha = t \circ \varepsilon$. We will use all of the preceding to show that

α is a 2-morphism $f \implies g$ in $\mathbf{Comon}(\mathbf{Vect}_k)$.

$$\begin{aligned}
\alpha \rightharpoonup f &= \alpha \otimes f \circ \delta \\
&= (\varepsilon \circ t) \otimes f \circ \delta \\
&= (\varepsilon \otimes 1) \circ (t \otimes f) \circ \delta \\
&= (\varepsilon \otimes 1) \circ (1 \otimes g \circ \delta \circ t) && \text{by (24)} \\
&= \varepsilon \otimes g \circ \delta \circ t \\
&= g \circ t \\
&= g \otimes \varepsilon \circ \delta \circ t \\
&= g \otimes \varepsilon \circ (1_C \otimes t \circ \delta) && \text{by (27)} \\
&= g \otimes (\varepsilon \circ t) \circ \delta \\
&= g \otimes \alpha \circ \delta \\
&= g \leftharpoonup \alpha
\end{aligned}$$

So now we know that $\alpha\colon f \implies g$ in $\mathbf{Comon}(\mathbf{V})$. It remains to show that $\alpha_* = \tau$. By definition,

$$(\alpha_*)_{(C,\delta)}\colon f_*(C, \delta) \to g_*(C, \delta)$$

is given by the map $1 \otimes \alpha \circ \delta$. We have that

$$\begin{aligned}
1 \otimes \alpha \circ \delta &= 1 \otimes (\varepsilon \circ t) \circ \delta \\
&= 1 \otimes \varepsilon \circ 1 \otimes t \circ \delta \\
&= 1 \otimes \varepsilon \circ \delta \circ t && \text{by (27)} \\
&= t = \tau_{(C,\delta)}
\end{aligned}$$

so α_* and τ agree at component (C, δ). This is enough, since by Lemma 160 it suffices to show that they are the same for only the cofree comodules, and cofree comodules are coproducts of copies of C. $\square$

Exercise 163. What are the unit and the counit for the adjunction $U^C \dashv F$, where $F\colon \mathbf{Vect}_k \to \mathbf{Comod}(C)$ is the cofree comodule functor, defined on objects by $F(V) = (V \otimes C, 1 \otimes \delta)$ and on arrows by $F(f) = f \otimes 1$.

Lecture 20 — 29 February 2016

Definition 164. There is a functor $\mathbf{Comod}(C) \times \mathbf{Comod}(D) \xrightarrow{\otimes} \mathbf{Comod}(C \otimes D)$ for comonoids C and D in the braided category $\mathbf{V}$. On objects this is given by

$$(M, \chi), (N, \nu) \longmapsto (M \otimes N, 1 \otimes c \otimes 1 \circ \chi \otimes \nu).$$

On morphisms it's just given by $\otimes$.

If H is a comonoid, then $U^H\colon \mathbf{Comod}(H) \to \mathbf{Vect}$ is the forgetful functor.

Lemma 165. Let C, D, E be k-coalgebras. Let S, T be functors as in the diagram that preserve $U^{C\otimes D}$-split equalizers and filtered colimits.

$$\mathbf{Comod}(C) \times \mathbf{Comod}(D) \xrightarrow{\otimes} \mathbf{Comod}(C \times D) \underset{T}{\overset{S}{\rightrightarrows}} \mathbf{Comod}(E)$$

Then any natural transformation $\alpha\colon S\otimes \implies T\otimes$, there is a unique $\beta\colon S \implies T$ such that $\beta\otimes = \alpha$.

Proof. Warning! This is either completely wrong or incomplete. See Ignacio's errata. $\otimes$ preserves $(U^C \times U^D)$-split equalizers in $\mathbf{Comod}(C) \times \mathbf{Comod}(D)$ since

$$\begin{tikzcd}
\mathbf{Comod}(C) \times \mathbf{Comod}(D) \arrow[r, "\otimes"] \arrow[d, "U^C \times U^D"] & \mathbf{Comod}(C \otimes D) \arrow[d, "U^{C\otimes D}"] \\
\mathbf{Vect} \times \mathbf{Vect} \arrow[r, "\otimes"] & \mathbf{Vect}
\end{tikzcd}$$

commutes.

So $S\otimes$ and $T\otimes$ preserve $(U^C \times U^D)$-split equalizers. Then $\alpha_{(C,D)}\colon S(C \otimes D) \to T(C \otimes D)$ defines $\beta\colon S \implies T$ by the previous lecture, and in addition $\beta_{M\otimes N} = \alpha_{(M,N)}$. ∎

Co-quasi-triangular or braided bimonoids

We can motivate this definition by thinking of the braiding axioms in a different way. If $(\mathbf{C}, I, \otimes)$ is a monoidal category, we can think of a braiding on $\mathbf{C}$ as a natural transformation between $\otimes \circ \mathrm{sw}\colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ and $\otimes\colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, where $\mathrm{sw}\colon \mathbf{C} \times \mathbf{C}$ is the "swap" functor $(A, B) \mapsto (B, A)$.

$$\begin{tikzcd}
\mathbf{C} \times \mathbf{C} \arrow[rr, "\otimes"] \arrow[dr, "\mathrm{sw}"'] & \arrow[d, phantom, "\Downarrow c"] & \mathbf{C} \\
& \mathbf{C} \times \mathbf{C} \arrow[ur, "\otimes"'] &
\end{tikzcd} \tag{28}$$

Then we can write one of the axioms of a braiding as the following equality of 2-cells:

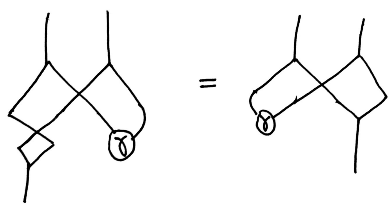
$$
\begin{tikzcd}
\mathbf{C}^3 \arrow[rr, "\mathrm{sw}\times 1"] \arrow[dr, "\otimes\times 1"'] & \arrow[d, phantom, "\overset{c\times 1}{\Longrightarrow}"] & \mathbf{C}^3 \arrow[rr, "1\times\mathrm{sw}"] \arrow[dl, "\otimes\times 1"] \arrow[dr, "1\times\otimes"'] & \arrow[d, phantom, "\overset{1\times c}{\Longrightarrow}"] & \mathbf{C}^3 \arrow[dl, "1\times\otimes"] \\
& \mathbf{C}^2 \arrow[rr, Rightarrow, "\alpha"] \arrow[dr, "\otimes"'] & & \mathbf{C}^2 \arrow[dl, "\otimes"] & \\
& & \mathbf{C} & &
\end{tikzcd}
=
\begin{tikzcd}
& \mathbf{C}^3 \arrow[rr, "\mathrm{sw}_{\mathbf{C},\mathbf{C}^2}"] \arrow[dl, "\otimes\times 1"'] \arrow[d, "1\times\otimes"] & & \mathbf{C}^3 \arrow[d, "\otimes\times 1"] \arrow[dr, bend left, "1\times\otimes"] & \\
\mathbf{C}^2 \arrow[r, Rightarrow, "\alpha"] \arrow[drr, bend right, "\otimes"'] & \mathbf{C}^2 \arrow[rr, "\mathrm{sw}"] \arrow[dr, "\otimes"'] & \arrow[d, phantom, "\overset{c}{\Longrightarrow}"] & \mathbf{C}^2 \arrow[r, Rightarrow, "\alpha"] \arrow[dl, "\otimes"] & \mathbf{C}^2 \arrow[dll, bend left, "\otimes"] \\
& & \mathbf{C} & &
\end{tikzcd}
\tag{29}
$$

Now suppose that H is a bimonoid in a symmetric monoidal category $\mathbf{V}$ with symmetry s. In the 2-category $\mathbf{Comon}(\mathbf{V})$, mimic the two cell (28) with the symmetry s taking the place of the swap functor and m taking the place of tensor.

$$
\begin{tikzcd}
H^{\otimes 2} \arrow[rr, "m"] \arrow[dr, "s_{H,H}"'] & \arrow[d, phantom, "\Downarrow\gamma"] & H \\
& H^{\otimes 2} \arrow[ur, "m"'] &
\end{tikzcd}
$$

The map γ is a 2-cell in $\mathbf{Comon}(\mathbf{V})$, that is, $\gamma\colon H\otimes H \to I$ in $\mathbf{V}$ such that

$$\gamma \rightharpoonup m = \gamma\otimes m\circ 1\otimes s\otimes 1\circ\delta\otimes\delta = \gamma\otimes m\circ 1\otimes 1\otimes s\circ 1\otimes s\otimes 1\circ\delta\otimes\delta = (m\circ s_{H,H}) \leftharpoonup \gamma.$$



By analogy to the diagram (29) we drew for the braiding, we have the following diagram.

$$
\begin{tikzcd}
H^{\otimes 3} \arrow[rr, "s_{H,H}\otimes 1"] \arrow[dr, "m\otimes 1"'] & \arrow[d, phantom, "\overset{\gamma\otimes 1}{\Longrightarrow}"] & H^{\otimes 3} \arrow[rr, "1\otimes s_{H,H}"] \arrow[dl, "m\otimes 1"] \arrow[dr, "1\otimes m"'] & \arrow[d, phantom, "\overset{1\otimes\gamma}{\Longrightarrow}"] & H^{\otimes 3} \arrow[dl, "1\otimes m"] \\
& H^{\otimes 2} \arrow[dr, "m"'] & & H^{\otimes 2} \arrow[dl, "m"] & \\
& & H & &
\end{tikzcd}
=
\begin{tikzcd}
H^{\otimes 3} \arrow[rr, "s_{H,H^{\otimes 2}}"] \arrow[d, "1\otimes m"] & & H^{\otimes 3} \arrow[d, "m\otimes 1"] \\
H^{\otimes 2} \arrow[rr, "s_{H,H}"] \arrow[dr, "m"'] & \arrow[d, phantom, "\overset{\gamma}{\Longrightarrow}"] & H^{\otimes 2} \arrow[dl, "m"] \\
& H &
\end{tikzcd}
\tag{30}
$$

Note that the α's from (29) disappear because they are identities here. In terms of morphisms in $\mathbf{V}$, we have the following equality of 2-cells

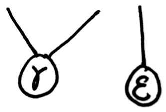
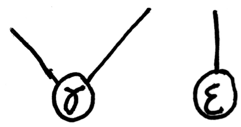
$$(m\cdot(1\otimes\gamma)\cdot(s\otimes 1)) * (m\cdot(\gamma\otimes 1)) = \gamma\cdot(1\otimes m)$$

What does it mean to take $1\otimes\gamma$? This is a tensor product of 2-cells in $\mathbf{Comon}(\mathbf{V})$, which is

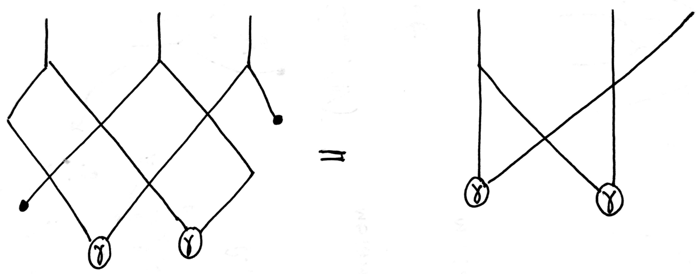
$$\left(\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd} \right) \otimes \left(\begin{tikzcd} C \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\beta"] & D \end{tikzcd} \right) = \begin{tikzcd} A \otimes C \arrow[r, bend left, "f\otimes h"] \arrow[r, bend right, "g\otimes k"'] \arrow[r, phantom, "\Downarrow\alpha\otimes\beta"] & B \otimes D \end{tikzcd}$$

where $\alpha \otimes \beta$ is given by literally tensoring the maps $\alpha \colon A \to I$ and $\beta \colon B \to I$.

And what is the identity 2-cell 1_{1_H}? This is the convolution identity in $\mathbf{V}(H, I)$, which is just $\varepsilon \colon H \to I$. Therefore, $\gamma \otimes 1 \colon m \otimes 1 \implies (m \circ s) \otimes 1$ is $\gamma \otimes \varepsilon \colon H^{\otimes 3} \to I$. Hence,

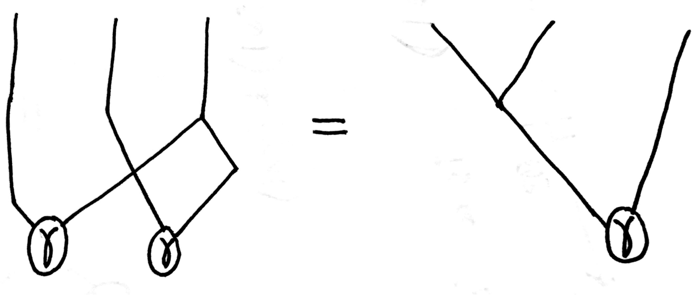
$m \cdot (\gamma \otimes 1) =$  $m \cdot (1 \otimes \gamma) =$ 

So the left hand side of (30) is their convolution



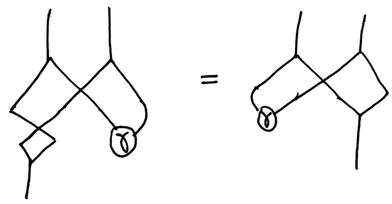
And the right hand side is

Above, we only went through the exposition for a single axiom of the braiding. We can do the same thing for the other braiding axiom, and recover another, similar equation to get the axiom

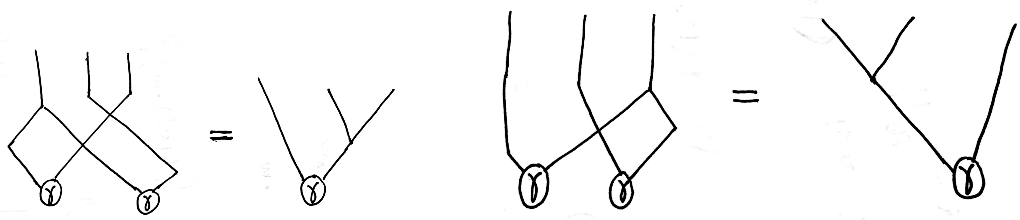


Definition 166. Let H be a bimonoid in the symmetric category $(\mathbf{V}, s)$. A **co-quasi triangular structure** on H (or **braiding**) is a $\gamma \colon H \to I$ that satisfies

- $\gamma \colon m \implies m \circ s_{H,H}$ is a 2-morphism in $\mathbf{Comon}(\mathbf{V})(H \otimes H, H)$.



- γ is invertible in the convolution monoid $\mathbf{V}(H \otimes H, I)$.
- the analogues of the braid axioms.

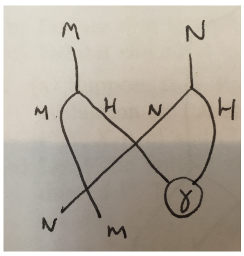


The idea is that $\mathbf{Comod} \colon \mathbf{Comon}(\mathbf{V}) \to \mathbf{Cat}$ will be monoidal in the appropriate sense and therefore send $\gamma \mapsto \gamma_*$ making γ_* into a braiding. We won't prove this because we don't quite have the time, and it involves many concepts that are mostly irrelevant for the rest of the course.

But here's another motivation for this definition.

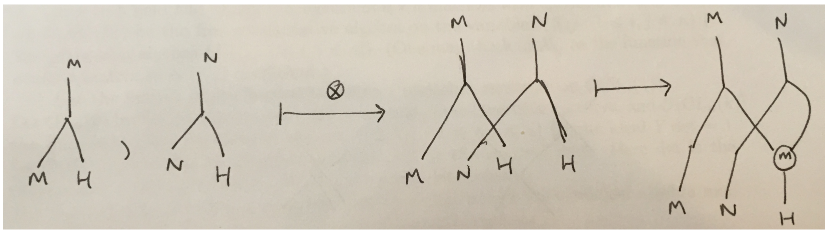
Theorem 167. If (H, γ) is a co-quasi triangular bimonoid in the symmetric monoidal category $\mathbf{V}$, then the monoidal category $\mathbf{Comod}(H)$ admits a braiding with components

$$c^{\gamma}_{M,N} \colon M \otimes N \longrightarrow M \otimes N \otimes H \otimes H \xrightarrow{1 \otimes s \otimes 1} N \otimes M \otimes H \otimes H \xrightarrow{s \otimes \gamma} N \otimes M$$



Proof. Observe that $\mathbf{Comod}(H) \times \mathbf{Comod}(H) \xrightarrow{\otimes} \mathbf{Comod}(H \otimes H) \xrightarrow{m_*} \mathbf{Comod}(H)$

is the usual tensor product of $\mathbf{Comod}(H)$.



Note also that $(m \circ s_{H,H})_* \circ \otimes$ is naturally isomorphic to $\otimes \circ \mathrm{sw} \colon \mathbf{Comod}(H) \otimes \mathbf{Comod}(H) \to \mathbf{Comod}(H)$.

Now $\gamma \colon m \implies (m \circ s_{H,H})$ is a 2-cell in $\mathbf{Comon}(\mathbf{V})$, so

$$\gamma_* \colon m_* \implies (m \circ s_{H,H})_*$$

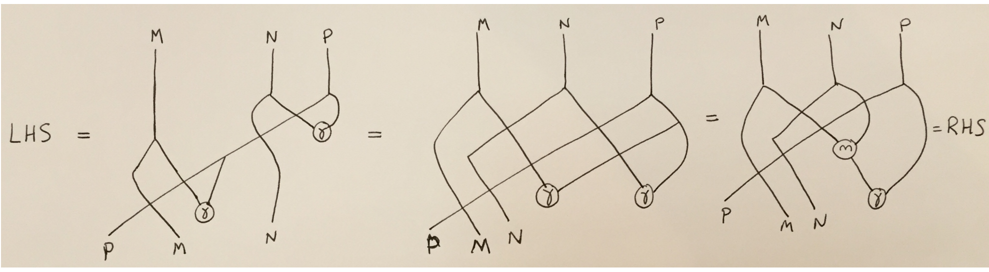
is a natural transformation. So the following diagram defines the natural transformation $c^\gamma \colon \otimes \implies \otimes \circ \mathrm{sw}$.

$$\begin{tikzcd}
\mathbf{Comod}(H)^2 \arrow[r, "\otimes"] \arrow[d, "\mathrm{sw}"'] \arrow[rr, bend left=40, "\otimes"] & \mathbf{Comod}(H \otimes H) \arrow[r, bend left, "m_*", ""{name=U, below}] \arrow[r, bend right, "(m \circ s_{H,H})_*"', ""{name=D, above}] & \mathbf{Comod}(H) \\
\mathbf{Comod}(H)^2 \arrow[urr, bend right=30, "\otimes"'] & &
\arrow[Rightarrow, from=U, to=D, "\gamma"]
\end{tikzcd}$$

This gives a natural transformation $\otimes \overset{c^\gamma}{\Longrightarrow} \otimes \circ \mathrm{sw}$ in $\mathbf{Comod}(H)$ with components $c^\gamma_{M,N}$ as in the statement. In particular, the components are morphisms of H-comodules.

We can now verify the braiding axioms: omitting the α's, one of the braid axioms is given by

$$(1_N \otimes c^\gamma_{M,P})(c^\gamma_{M,N} \otimes 1_P) = c^\gamma_{M,N\otimes P}.$$



Similarly, use the other axiom for γ to show that

$$(c^\gamma_{M,N} \otimes 1_P)(1_M \otimes c^\gamma_{N,P}) = c^\gamma_{M\otimes N,P}.$$

□

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Lemma 168. Suppose given $\alpha\colon f \implies g$ in $\mathbf{Comon}(\mathbf{V})$, for $f, g\colon C \to D$. Then if $\alpha_*\colon f_* \implies g_*$ is invertible as a natural transformation, it is convolution invertible in $\mathbf{V}(C, I)$.

Proof. By Lemma 162, the inverse to α_* must be of the form β_* for some $\beta\colon C \to I$. I claim that β is the convolution inverse of α. To see this, note that $(\beta_*)(\alpha_*) = 1_{f_*}$, so we have that $(\beta_*)(\alpha_*)_M = 1_{f_* M}$. This in particular means that the map

$$M \xrightarrow{\chi} M \otimes C \xrightarrow{1\otimes\alpha} M \xrightarrow{\chi} M \otimes C \xrightarrow{1\otimes\beta} M$$

is the identity map. Hence, we see by using coassociativity

$$1 = (1 \otimes \beta) \circ \chi \circ (1 \otimes \alpha) \circ \chi = 1 \otimes (\alpha \otimes \beta) \circ (1 \otimes \delta) \circ \chi = 1 \otimes (\alpha * \beta) \circ \chi$$

Hence, $\alpha * \beta = \varepsilon$, which is the convolution identity on $\mathbf{V}(C, I)$. Similarly, $\beta * \alpha = \varepsilon$. $\square$

Remark 169. This is a consequence of Lemma 165. Let $\mathbf{V} = \mathbf{Vect}_k$, and let H be a bialgebra. Then given any $\alpha\colon f_*\otimes \implies g_*\otimes$ as below, there is a unique $\widetilde{\alpha}\colon f_* \implies g_*$ such that $\widetilde{\alpha} \circ \otimes = \alpha$.

$$\begin{tikzcd}
\mathbf{Comod}(H) \times \mathbf{Comod}(H) \arrow[r, "\otimes"] & \mathbf{Comod}(H \otimes H) \arrow[r, bend left, "f_*"] \arrow[r, bend right, "g_*"'] \arrow[r, phantom, "\Downarrow\widetilde{\alpha}"] & \mathbf{Comod}(H)
\end{tikzcd}$$

If α is 1, then $\widetilde{\alpha}$ is 1 from $f_*\otimes$ to itself. If we have $f_*\otimes \overset{\alpha}{\Rightarrow} g_*\otimes \overset{\beta}{\Rightarrow} h_*\otimes$ then $f_* \overset{\widetilde{\alpha}}{\Rightarrow} g_* \overset{\widetilde{\beta}}{\Rightarrow} h_*$, so $\widetilde{\beta}\widetilde{\alpha} = \widetilde{\beta\alpha}$. So if α is invertible, then $\widetilde{\alpha}^{-1} = \widetilde{\alpha^{-1}}$.

Theorem 170. Let H be a bialgebra in the category of k-vector spaces. Then there is a bijection between coquasi triangular structures on H and braidings on $\mathbf{Comod}(H)$ given by $\gamma \mapsto c^\gamma$.

$$\begin{tikzcd}
\mathbf{V}^H \times \mathbf{V}^H \arrow[rr, "\otimes"] \arrow[dr, "\mathrm{sw}"'] & \Downarrow c & \mathbf{V}^H \\
& \mathbf{V}^H \times \mathbf{V}^H \arrow[ur, "\otimes"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
H \otimes H \arrow[rr, "m"] \arrow[dr, "s"'] & \Downarrow \gamma & H \\
& H \otimes H \arrow[ur, "m"'] &
\end{tikzcd}$$

Proof. By Theorem 167 we know that there is a braiding c^γ constructed from γ for each co-quasi-triangular structure on H. So we have to do the converse: construct γ given a braiding c.

Write $\mathbf{V}^H$ for $\mathbf{Comod}(H)$ and s for the symmetry of $\mathbf{V} = \mathbf{Vect}_k$.

Given a braiding c on $\mathbf{Comod}(H)$, and $M, N \in \mathbf{Comod}(H)$, consider the natural transformation $\tau\colon m_*\otimes \implies (m \circ s_{H,H})_*\otimes$ with components

$$\tau_{M,N}\colon m_*(M \otimes N) \xrightarrow{c_{M,N}} m_*(N \otimes M) \xrightarrow{s_{N,M}} (m \circ s_{H,H})_*(M \otimes N) \tag{31}$$

This natural transformation τ is of the form $\tau = \gamma_*\otimes$ for a unique

$$\begin{tikzcd} H \otimes H \arrow[r, bend left=50, "m"] \arrow[r, bend right=50, "m \circ s_{H,H}"'] \arrow[r, phantom, "\Downarrow\gamma"] & H \end{tikzcd}$$

in $\mathbf{Comon}(\mathbf{Vect})$ by a Lemma 165 and Lemma 162.

$$\begin{tikzcd} \mathbf{V}^H \times \mathbf{V}^H \arrow[r, "\otimes"] & \mathbf{V}^{H\otimes H} \arrow[r, bend left=50, "m_*"] \arrow[r, bend right=50, "(m \circ s_{H,H})_*"'] \arrow[r, phantom, "\Downarrow\gamma_*"] & \mathbf{V}^H \end{tikzcd}$$

The original natural transformation τ is invertible with inverse given by $\tau^{-1}_{M,N} = \tau_{M,N}$; this follows because s is a symmetry.

This implies that γ_* is invertible by Remark 169. Now since γ_* is invertible, γ is an invertible 2-cell in $\mathbf{Comon}(\mathbf{Vect})$ by Lemma 168. Hence, γ is convolution invertible.

Note that

$$c^{\gamma}_{M,N} = s_{M,N} \circ (\gamma_*)_{M \otimes N}. \tag{32}$$

But by the definition of τ (31) and the fact $\tau = \gamma_*\otimes$, we have

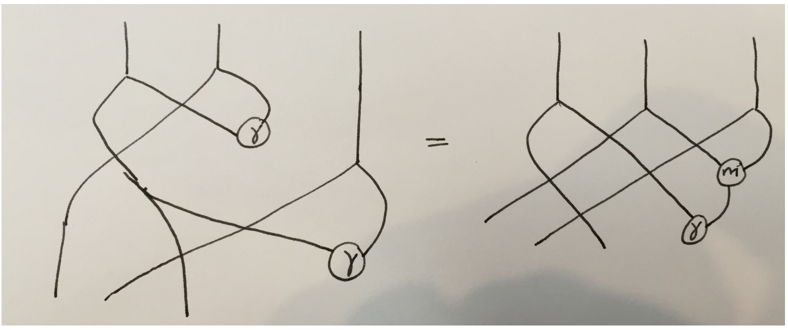
$$(\gamma_*)_{M\otimes N} = (\gamma_*\otimes)_{M,N} = \tau_{M,N} = s_{N,M} \circ c_{M,N}.$$

Substituting this into (32) gives

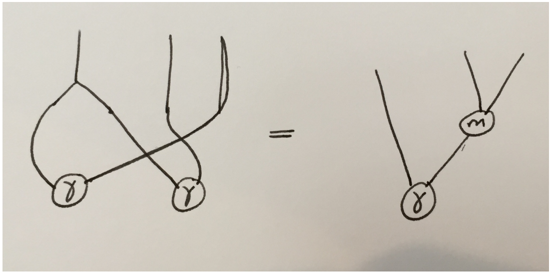
$$c^{\gamma}_{M,N} = s_{M,N} \circ (s_{N,M} \circ c_{M,N}) = c_{M,N}.$$

This establishes the desired bijection, so long as γ is a coquasi-triangular structure.

So it remains to check the axioms of a coquasi triangular structure for γ. Omitting the associativity constraint α in $\mathbf{Vect}_k$, one of the braid axioms gives us $(1_H \otimes c_{H,H})(c_{H,H} \otimes 1_H) = c_{H,H\otimes H}$.



then putting counits on the ends of the strings in the above diagram, we recover one of the axioms of a coquasi triangular structure.



The other axiom follows similarly from the other braid axiom. □

Coends

"Coends are some sort of colimity thing."

Definition 171. Let $T\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$ be a functor. Let $D \in \mathbf{D}$. A **dinatural** or **extraordinary natural** transformation $\tau\colon T \implies D$ is a family of morphisms $\tau_X\colon T(X,X) \to D$ such that for all $f\colon X \to Y$ in $\mathbf{C}$,

$$\begin{tikzcd}
T(Y,X) \arrow[r, "T(f,1)"] \arrow[d, "T(1,f)"] & T(X,X) \arrow[d, "\tau_X"] \\
T(Y,Y) \arrow[r, "\tau_Y"] & D
\end{tikzcd}$$

commutes.

Example 172. If $\mathbf{V}$ is monoidal closed and $X \otimes [X,Z] \xrightarrow{\mathrm{ev}_{X,Z}} Z$ is dinatural in X.

Definition 173. A **coend** of T is a universal dinatural $\tau\colon T \implies D$. That is, τ is dinatural and for every other $\beta\colon T(X,X) \to D'$, there is a unique $f\colon D \to D'$ such that

$$\begin{tikzcd}
T(X,X) \arrow[r, "\tau_X"] \arrow[dr, "\beta_X"'] & D \arrow[d, "f"] \\
& D'
\end{tikzcd}$$

Usually, D is denoted by $D = \int^X T(X,X)$.

Remark 174. If $\mathbf{D}$ is cocomplete and $\mathbf{C}$ is small, then $\int^X T(X,X)$ exists, and is the coequalizer depicted below

$$\coprod_{f \in \mathrm{mor}\,\mathbf{C}} T(\operatorname{cod} f, \operatorname{dom} f) \underset{\psi}{\overset{\phi}{\rightrightarrows}} \coprod_X T(X,X) \twoheadrightarrow \int^X T(X,X),$$

where ϕ is defined by

$$\begin{tikzcd} T(Y,X) \arrow[r, "T(f,1)"] \arrow[d, "i_f"] & T(X,X) \arrow[d, "i_X"] \\ \coprod_f T(\operatorname{cod} f, \operatorname{dom} f) \arrow[r, "\phi"] & \coprod_X T(X,X) \end{tikzcd}$$

and ψ is defined by

$$\begin{tikzcd} T(Y,X) \arrow[r, "T(1,f)"] \arrow[d, "i_f"] & T(X,X) \arrow[d, "i_X"] \\ \coprod_f T(\operatorname{cod} f, \operatorname{dom} f) \arrow[r, "\psi"] & \coprod_X T(X,X) \end{tikzcd}$$

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One more word on coends: let

$$T\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \times \mathbf{D}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{E}.$$

Then a family $\tau_{X,Y}\colon T(X,X,Y,Y) \to E$ is dinatural if and only if for fixed X, $\tau_{X,Y}$ is dinatural in Y and for fixed Y, $\tau_{X,Y}$ is dinatural in X.

Proposition 175. Suppose given a functor $T\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \times \mathbf{D}^{\mathrm{op}} \times \mathbf{D} \to \mathbf{E}$, and a family of morphisms $\tau_{X,Y}\colon T(X,X,Y,Y) \to E$ for some $E \in \mathbf{E}$. Then the following are equivalent:

(a) τ is dinatural considering as a functor $T\colon (\mathbf{C} \times \mathbf{D})^{\mathrm{op}} \times (\mathbf{C} \times \mathbf{D}) \to \mathbf{E}$.

(b) $\tau_{X,-}$ is dinatural for all $X \in \mathbf{C}$, and $\tau_{-,Y}$ is dinatural for all $Y \in \mathbf{D}$.

Proof. $(a) \implies (b)$. Since $\tau_{X,Y}$ is dinatural, for all $(f,g)\colon (X,Y) \to (X',Y')$ in $\mathbf{C} \times \mathbf{D}$, the following square commutes.

$$\begin{tikzcd} & T(X,X,Y,Y) \arrow[dr, "\tau_{X,Y}"] & \\ T(X',X,Y',Y) \arrow[ur, "T(f,X,g,Y)"] \arrow[dr, "T(X,f,Y,g)"'] & & E \\ & T(X',X',Y',Y') \arrow[ur, "\tau_{X',Y'}"'] & \end{tikzcd}$$

Simply take $f = 1_X$ or $g = 1_Y$ to see that $\tau_{X,Y}$ is dinatural in X for fixed Y and dinatural in Y for fixed X.

$(b) \implies (a)$. The following square commutes by composing all of the other commutative squares, each of which follows either from functoriality of T or by dinaturality of $\tau_{X,-}$ or $\tau_{-,Y'}$.

$$\begin{tikzcd}
 & T(X,X,Y,Y) \arrow[rrdd, bend left, "\tau_{X,Y}"] & & \\
 & T(X,X,Y',Y) \arrow[u, "T(1,1,g,1)"] \arrow[rd, "T(1,1,1,g)"] & & \\
T(X',X,Y',Y) \arrow[ruu, bend left, "T(f,1,g,1)"] \arrow[ru, "T(f,1,1,1)"'] \arrow[rd, "T(1,1,1,g)"] \arrow[rdd, bend right, "T(1,f,1,g)"'] & & T(X,X,Y',Y') \arrow[r, "\tau_{X,Y'}"] & E \\
 & T(X',X,Y',Y') \arrow[ru, "T(f,1,1,1)"'] \arrow[d, "T(1,f,1,1)"] & & \\
 & T(X',X',Y',Y') \arrow[rruu, bend right, "\tau_{X',Y'}"'] & &
\end{tikzcd}$$

□

For T and τ as above, write $\int^X T(X,X,Y,Y')$ for the coend object of $T(X,X,-,-)$. If all of these coends exist, then this is a functor $\int^X T(X,X,-,-)\colon \mathbf{D}^{\mathrm{op}} \times \mathbf{D} \to \mathbf{E}$. We can define the coend of this functor as well, denoted $\int^Y \int^X T(X,X,Y,Y)$. Similarly, we get $\int^X \int^Y T(X,X,Y,Y)$. The following theorem relates these two objects to the coend object $\int^{(X,Y)} T(X,X,Y,Y)$ of T.

Theorem 176 (Fubini's Theorem)**.** Given $T\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \times \mathbf{D}^{\mathrm{op}} \times \mathbf{D} \to \mathbf{E}$ and a dinatural transformation $\tau_{X,Y}\colon T(X,X,Y,Y) \to E$, then

$$\int^Y \int^X T(X,X,Y,Y) \cong \int^{(X,Y)} T(X,X,Y,Y) \cong \int^X \int^Y T(X,X,Y,Y).$$

provided the appropriate coends exist.

Reconstruction

Consider the full subcategory $\mathbf{Cat}/_d\mathbf{V} \hookrightarrow \mathbf{Cat}/\mathbf{V}$ with objects $\mathbf{C} \xrightarrow{U} \mathbf{V}$ such that $U(X)$ has a dual in $\mathbf{V}$. Here $\mathbf{V}$ is cocomplete, symmetric monoidal and $V \otimes -$ is cocontinuous for all $V \in \mathbf{V}$.

Define a functor $\mathbf{Comod}_d\colon \mathbf{Comon}(\mathbf{V}) \to \mathbf{Cat}/_d\mathbf{V}$. $\mathbf{Comod}_d(C)$ is the full subcategory of $\mathbf{Comod}(C)$ of those M such that $U^C(M)$ has a dual, and so we have

$$\mathbf{Comod}_d(C) \xrightarrow{U^C} \mathbf{V}.$$

On morphisms, given $f\colon C \to D$,

$$\begin{tikzcd}
\mathbf{Comod}_d(C) \arrow[rr, "f_*"] \arrow[rd, "U^C"'] & & \mathbf{Comod}_d(D) \arrow[ld, "U^D"] \\
 & \mathbf{V} &
\end{tikzcd}$$

where $f_*(M,\chi) = (M, 1\otimes f\circ\chi)$ for a comodule $M \xrightarrow{\chi} M\otimes C$. That is, $f_*(M,\chi)$ is the comodule with coaction

$$M \xrightarrow{\chi} M\otimes C \xrightarrow{1\otimes f} M\otimes D.$$

Definition 177. If $\mathbf{C} \xrightarrow{U} \mathbf{V}$ is in $\mathbf{Cat}/_d V$, then define $E(\mathbf{C},U) \in \mathbf{Comon}(\mathbf{V})$ as a representation of

$$\begin{array}{ccc} \mathbf{Comon}(\mathbf{V}) & \longrightarrow & \mathbf{Set} \\ D & \longmapsto & \mathbf{Cat}/_d\mathbf{V}((\mathbf{C},U),(\mathbf{Comod}_d(D),U^D)) \end{array}$$

That is,

$$\mathbf{Comon}(V)(E(\mathbf{C},U),D) \cong \mathbf{Cat}/_d\mathbf{V}((\mathbf{C},U),\mathbf{Comod}_d(D)).$$

Remark 178. If $E(U)$ always exists, then $E \dashv \mathbf{Comod}_d$.

Lemma 179. *$E(\mathbf{C},U)$ exists if $\int^X {}^*(U(X))\otimes U(X)$ exists.*

Proof. Write C for this coend. Define a comonoid structure on C as follows

$$\begin{tikzcd} {}^*U(X)\otimes U(X) \arrow[r, "1\otimes \mathrm{coev}_{U(Y)}\otimes 1"] \arrow[dd, "i_X"] & {}^*U(X)\otimes {}^*U(Y)\otimes U(Y)\otimes U(X) \arrow[d, "\cong"] \\ & {}^*U(X)\otimes U(X)\otimes {}^*U(Y)\otimes U(Y) \arrow[d, "i_X\otimes i_Y"] \\ \int^X {}^*U(X)\otimes U(X) \arrow[r, "\delta"] & \int^X {}^*U(X)\otimes U(X)\otimes \int^Y {}^*U(Y)\otimes U(Y) \end{tikzcd}$$

where i_X is the universal dinatural transformation into C. One checks that the top-right leg of the diagram is dinatural in X, and therefore it defines δ.

The counit is defined by

$$\begin{tikzcd} {}^*U(X)\otimes U(X) \arrow[d, "i_X"] \arrow[dr, "\mathrm{ev}_{U(X)}"] & \\ \int^X {}^*U(X)\otimes U(X) \arrow[r, "\varepsilon"'] & I \end{tikzcd}$$

One can check that (C,δ,ε) is a comonoid. For this one needs $C\otimes -$ to preserve colimits, and therefore coends.

A morphism (of comonoids)

$$\int^X {}^*U(X)\otimes U(X) \xrightarrow{f} D$$

is the same as a dinatural transformation

$${}^*U(X)\otimes U(X) \xrightarrow{\beta_X} D$$

which is the same as the D-comodule structure α_X being natural in X.

$$U(X) \xrightarrow{\alpha_X} U(X) \otimes D$$

So, to give f is to endow each $U(X)$ with a D-comodule structure, naturally in X. This is the same as a functor $F(X) = (U(X), \alpha_X)$.

$$\begin{tikzcd} \mathbf{C} \arrow[rr, "F"] \arrow[dr, "U"'] & & \mathbf{Comod}_d(D) \arrow[dl, "U^D"] \\ & \mathbf{V} & \end{tikzcd}$$

□

Remark 180. The unit of $E \dashv \mathbf{Comod}_d$

$$\begin{tikzcd} \mathbf{C} \arrow[rr, "N"] \arrow[dr, "U"'] & & \mathbf{Comod}_d(E(U)) \arrow[dl, "U^{E(U)}"] \\ & \mathbf{V} & \end{tikzcd}$$

$N(X)$ is $U(X)$ with the coaction

$$U(X) \xrightarrow{1\otimes \mathrm{coev}} U(X) \otimes {}^*U(X) \otimes U(X) \xrightarrow{1\otimes i_X} U(X) \otimes \int^Y {}^*U(Y) \otimes U(Y)$$

Two Questions

- What is the essential image of $\mathbf{Comod}_d \colon \mathbf{Comon}(\mathbf{V}) \to \mathbf{Cat}/_d\mathbf{V}$? This is the **Representation** or **Recognition Theorem**.
- Is it true that $E(\mathbf{Comod}_d(C), U^C) \xrightarrow{\text{counit}} C$ is an iso? This is the **Reconstruction Theorem**.

We will only prove the second of these theorems and just state the first.

Theorem 181 (Representation Theorem)**.** If $\mathbf{C}$ is an abelian k-linear category over a field k, with finite dimensional homs ($\mathbf{C}(X, Y) < \infty$ for all X, Y), $U \colon \mathbf{C} \to \mathbf{Vect}$ is faithful and exact, with values in finite-dimensional vector spaces. Then $N \colon \mathbf{C} \to \mathbf{Comod}_d(E(U))$ is an equivalence.

The proof of this theorem, which we won't worry about, relies very heavily on the fact that we are working with vector spaces.

Theorem 182 (Reconstruction Theorem)**.** For $\mathbf{V} = \mathbf{Vect}$, the coalgebra $E(U^C)$ exists and the counit $E(U^C) \to C$ is an isomorphism.

Remark 183. The Reconstruction Theorem means that $\mathbf{Comod}_d(C) \xrightarrow{U^C} \mathbf{Vect}$ has all the information to reconstruct the coalgebra C.

Exercise 184. If $F \dashv G$, then G is fully faithful if and only if $FG \overset{\varepsilon}{\Rightarrow} 1$ is an isomorphism.

Proof of Reconstruction Theorem. To show that the counit is an iso, it is equivalent to show that the right adjoint is fully faithful. So we have to show that

$$\mathbf{Comon}(\mathbf{Vect})(C, D) \longrightarrow \mathbf{Cat}/_d\mathbf{V}(\mathbf{Comod}_d(C), \mathbf{Comod}_d(D))$$

is an isomorphism.

Suppose given

$$\begin{tikzcd} \mathbf{Comod}_d(C) \arrow[rr, "T"] \arrow[dr] & & \mathbf{Comod}_d(D) \arrow[dl] \\ & \mathbf{Vect} & \end{tikzcd}$$

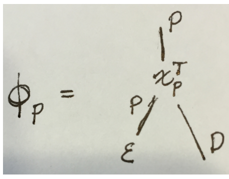
If (M, χ) is a C-comodule, then $T(M, \chi) = (M, \chi^T)$ for some coaction χ^T.

Given a finite dimensional subcoalgebra $P \xhookrightarrow{i} C$, regard P as a C-comodule. Then $T(P, \chi_P) = (P, \chi_P^T)$, where χ_P is the composite

$$\chi_P \colon P \xrightarrow{\delta_P} P \otimes P \xrightarrow{1 \otimes i} P \otimes C.$$

Define

$$\phi_P \colon P \xrightarrow{\chi_P^T} P \otimes D \xrightarrow{\varepsilon_P \otimes 1} D.$$



Note that for all $\gamma \colon P \to k$ linear, the map

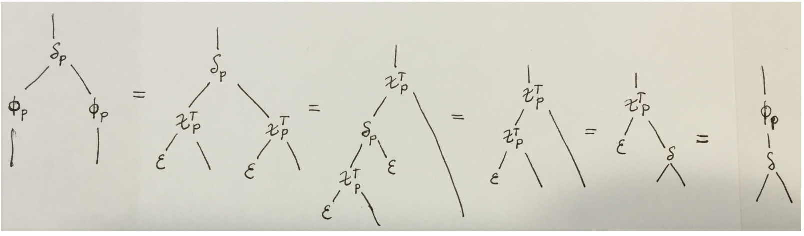
$$P \xrightarrow{\delta_P} P \otimes P \xrightarrow{\gamma \otimes 1} P$$

is a morphisms of P-comodules (this follows easily by associativity of δ_P). Therefore, $(\gamma \otimes 1_P) \circ \delta_p \colon P \to P$ is a morphism of C-comodules from (P, χ_P) to (P, χ_P). Applying T to this morphism gives a morphism of D-comodules. $(P, \chi_P^T) \longrightarrow (P, \chi_P^T)$, and the following commutes

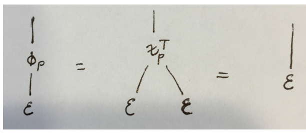
$$\begin{tikzcd} P \arrow[r, "\delta_P"] \arrow[d, "\chi_P^T"'] & P \otimes P \arrow[r, "\gamma \otimes 1"] & P \arrow[d, "\chi_P^T"] \\ P \otimes D \arrow[r, "\delta_P \otimes 1"] & P \otimes P \otimes D \arrow[r, "\gamma \otimes 1 \otimes 1"] & P \otimes D \end{tikzcd}$$

This holds for all $\gamma \colon P \to k$, which implies that

$$(1_P \otimes \chi_P^T) \circ \delta_P = (\delta_P \otimes 1_D) \circ \chi_P^T.$$



This shows that ϕ_P is compatible with comultiplication. Also, ϕ_p is compatible with the counits:



Next, we show that if $P \subseteq Q \subseteq C$ are finite-dimensional subcoalgebras, then

$$\begin{tikzcd} P \arrow[rr, hook, "u"] \arrow[dr, "\phi_P"'] & & Q \arrow[dl, "\phi_Q"] \\ & D & \end{tikzcd}$$

Note that u is a morphism of C-comodules, $(P, \chi_P) \xrightarrow{u} (Q, \chi_Q)$ by the definitions of χ_P, χ_Q. Therefore, because T is the identity on arrows,

$$(P, \chi_P^T) = T(P, \chi_P) \xrightarrow{\ u\ } T(Q, \chi_Q) = (Q, \chi_Q^T)$$

and the following commutes

$$\begin{tikzcd} P \arrow[r, "\chi_P^T"] \arrow[d, "u"] \arrow[rr, bend left=50, "\phi_P"] & P \otimes D \arrow[d, "u\otimes 1"] \arrow[dr, "\varepsilon\otimes 1"] & \\ Q \arrow[r, "\chi_Q^T"] \arrow[rr, bend right=50, "\phi_Q"'] & Q \otimes D \arrow[r, "\varepsilon\otimes 1"] & D \end{tikzcd}$$

Then, since C is a filtered union of finite-dimensional subcoalgebras, there is a unique $\phi \colon C \to D$ in **Comon**(**Vect**) such that

$$\begin{tikzcd} P \arrow[r, hook, "i"] \arrow[dr, "\phi_P"'] & C \arrow[d, "\phi"] \\ & D \end{tikzcd}$$

commutes.

It remains to show that $\phi_* = T$. Let $(M, \chi_M) \in \mathbf{Comod}_d(C)$ and $P \subseteq C$ be a finite dimensional subcoalgebra such that

$$\begin{tikzcd}
M \arrow[rr, "\chi_M"] \arrow[dr, "\chi'_M"'] & & M \otimes C \\
& M \otimes P \arrow[ur, "1 \otimes i"'] &
\end{tikzcd}$$

$$\phi_*(M, \chi_M) = \left(\begin{tikzcd}
M \arrow[r, "\chi_M"] \arrow[dr, "\chi'_M"'] & M \otimes C \arrow[r, "1 \otimes \phi"] & M \otimes D \\
& M \otimes P \arrow[u, dashed, "1 \otimes i"] \arrow[ur, "1 \otimes \phi_D"'] &
\end{tikzcd} \right) = (\phi_P)_*(M, \chi'_M)$$

If we write $\mathbf{C}(P) \overset{J}{\hookrightarrow} \mathbf{Comod}_d(\mathbf{C})$ as the full subcategory of those (M, χ_M) such that χ_M factors through $M \otimes P$. If $M \otimes P$ is the cofree P-comodule on P ($M \otimes P \cong P^{\dim M}$), then $TJ(M \otimes P) = (M \otimes P, 1_M \otimes \chi_P^T)$ since J and T preserve direct sums ($M \otimes TJ(P, \chi_P) = M \otimes (P, \chi_P)$).

$\chi'_M \colon M \to M \otimes P$ is a morphism of P-comodules, so we have $TJ(M, \chi'_M) \xrightarrow{\chi'_M} TJ(M \otimes P, 1 \otimes \chi_P)$ and the following diagram commutes

$$\begin{tikzcd}
M \arrow[r, "\chi'_M"] \arrow[d, "\chi_M^T"] & M \otimes P \arrow[d, "1 \otimes \chi_P^T"] \\
M \otimes D \arrow[r, "\chi'_M \otimes 1"] & M \otimes P \otimes D
\end{tikzcd}$$

We can extend this diagram as follows

$$\begin{tikzcd}
M \arrow[r, "\chi'_M"] \arrow[d, "\chi_M^T"] & M \otimes P \arrow[r, "1 \otimes \delta_P"] \arrow[d, "1 \otimes \chi_P^T"] & M \otimes P \otimes P \arrow[r, "1 \otimes \varepsilon \otimes 1"] \arrow[d, "1 \otimes 1 \otimes \phi_P"] & M \otimes P \arrow[d, "1 \otimes \phi_P"] \\
M \otimes D \arrow[r, "\chi'_M \otimes 1"] & M \otimes P \otimes D \arrow[r, equal] & Mt\ P \otimes D \arrow[r, "1 \otimes \varepsilon \otimes 1"] & M \otimes D
\end{tikzcd}$$

the middle square commutes by the definition of ϕ_P and a comodule axiom, and it's clear that the right diagram commutes. This means we can add some identities to this diagram

$$\begin{tikzcd}
M \arrow[r, "\chi'_M"] \arrow[d, "\chi_M^T"] & M \otimes P \arrow[r, "1 \otimes \delta_P"] \arrow[d, "1 \otimes \chi_P^T"] \arrow[rr, bend left, "1"] & M \otimes P \otimes P \arrow[r, "1 \otimes \varepsilon \otimes 1"] \arrow[d, "1 \otimes 1 \otimes \phi_P"] & M \otimes P \arrow[d, "1 \otimes \phi_P"] \\
M \otimes D \arrow[r, "\chi'_M \otimes 1"] \arrow[rrr, bend right, "1"'] & M \otimes P \otimes D \arrow[r, equal] & Mt\ P \otimes D \arrow[r, "1 \otimes \varepsilon \otimes 1"] & M \otimes D
\end{tikzcd}$$

Then this gives us the following identity

$$\chi_M^T = (1_M \otimes \phi_P) \circ \chi'_M$$

that is,

$$T(M, \chi_M) = (\phi_P)_*(M, \chi'_M) = \phi_*(M, \chi_M)$$

This completes the proof that $E(\mathbf{Comod}_d(C), U^C) \xrightarrow{e} C$ is an isomorphism. $\square$

Lecture 23 7 March 2016

Reconstruction of Bimonoids

Suppose now given

$$\mathbf{C} \xrightarrow{U} \mathbf{Vect} \xleftarrow{V} \mathbf{D}$$

where $U(X)$ and $V(Y)$ are dualizable for all X, Y. Then

$$U \otimes V \colon \mathbf{C} \times \mathbf{D} \xrightarrow{U \times V} \mathbf{Vect} \times \mathbf{Vect} \xrightarrow{\otimes} \mathbf{Vect}$$

$$\int^{X,Y} {}^*(U(X) \otimes V(Y)) \otimes (U(X) \otimes V(Y)) = E(U \otimes V) \xrightarrow{f} E(U) \otimes E(V) = \int^X {}^*U(X) \otimes U(X) \otimes \int^Y {}^*V(Y) \otimes V(Y)$$

the arrow labelled f is an isomorphism.

Remark 185. One can prove that (as on the fourth examples sheet)

$$\int^X {}^*U(X) \otimes U(X) \otimes \int^Y {}^*V(Y) \otimes V(Y) = \int^X \int^Y {}^*U(X) \otimes U(X) \otimes {}^*V(Y) \otimes V(Y) = \int^{X,Y} {}^*U(X) \otimes U(X) \otimes {}^*V(Y) \otimes V(Y)$$

Lemma 186. Let $\mathbf{V}$ be a cocomplete symmetric monoidal category such that $\otimes$ is cocontinuous in each variable. If $\mathbf{C} \xrightarrow{U} \mathbf{V} \xleftarrow{V} \mathbf{D}$, and $\mathbf{C}, \mathbf{D}$ are essentially small, $U(C)$ and $V(D)$ are dualizable for all $C \in \mathbf{C}$, $D \in \mathbf{D}$. Then the following commutes.

$$\begin{tikzcd}
\mathbf{C} \times \mathbf{D} \arrow[r, "N"] \arrow[d, "N \times N"] & \mathbf{Comod}_d(E(U \otimes V)) \arrow[d, "f_*", "\cong"'] \\
\mathbf{Comod}_d(E(U)) \times \mathbf{Comod}_d(E(N)) \arrow[r, "\otimes"] & \mathbf{Comod}_d(E(U) \otimes E(V))
\end{tikzcd}$$

Exercise 187. Prove Lemma 186. It's just a bunch of coends.

Corollary 188. If C, D are coalgebras over a field, then $E(U^C) \otimes E(U^D) \xrightarrow{u} C \otimes D$

$$\begin{tikzcd}
\mathbf{Comod}_d(C) \times \mathbf{Comod}_d(D) \arrow[r, "N"] \arrow[dr, "\otimes"'] & \mathbf{Comod}_d(E(U^C \otimes U^D)) \arrow[d, "u_*"] \\
& \mathbf{Comod}_d(C \otimes D)
\end{tikzcd}$$

Proof. Call $\mathbf{C}(C) = \mathbf{Comod}_d(C)$ and $e \colon E\mathbf{Comod}_d \implies 1$ the counit. Then by the lemma, we have that

$$\begin{tikzcd}
\mathbf{C}(C) \times \mathbf{C}(C) \arrow[r, "N"] \arrow[d, "N \times N"] & \mathbf{C}E(U^C \otimes U^D) \arrow[d, "f_*", "\cong"'] \\
\mathbf{C}(U^C) \times \mathbf{C}(U^D) \arrow[r, "\otimes"] & \mathbf{C}(E(U^C) \otimes E(U^D))
\end{tikzcd}$$

commutes. Therefore, the larger diagram also commutes.

$$\begin{tikzcd}
\mathbf{C}(C) \times \mathbf{C}(D) \arrow[r, "N"] \arrow[d, "N \times N"] \arrow[dd, bend right=60, "1"'] & \mathbf{C}E(U^C \otimes U^D) \arrow[d, "f_*", "\cong"'] \\
\mathbf{C}(U^C) \times \mathbf{C}(U^D) \arrow[r, "\otimes"] \arrow[d, "e_* \times e_*"] & \mathbf{C}(E(U^C) \otimes E(U^D)) \arrow[d, "C((e \otimes e)_*)"] \\
\mathbf{C}(C) \times \mathbf{C}(D) \arrow[r, "\otimes"] & \mathbf{C}(C \otimes D)
\end{tikzcd}$$

□

Theorem 189. Let C be a coalgebra over a field. Then there are bijections between

(i) Monoidal structures on C that make it into a bialgebra

(ii) Monoidal structures on $\mathbf{Comod}(C)$ that make the forgetful functor strict monoidal

(iii) Monoidal structures on $\mathbf{Comod}_d(C)$ that make the forgetful functor strict monoidal

Proof. $(i) \to (ii)$. ???

$(ii) \to (iii)$. Any structure as in (ii) restricts to finite-dimensional comodules.

$(iii) \to (i)$.

$$\begin{tikzcd}
\mathbf{Comod}_d(C) \times \mathbf{Comod}_d(D) \arrow[r, "\diamond"] \arrow[d, "U \times U"] & \mathbf{Comod}_d(C) \arrow[d, "U"] \\
\mathbf{Vect} \times \mathbf{Vect} \arrow[r, "\otimes"] & \mathbf{Vect}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{1} \arrow[rr, "J"] \arrow[dr, "I"'] & & \mathbf{Comod}_d(C) \arrow[dl, "U"] \\
& \mathbf{Vect} &
\end{tikzcd}$$

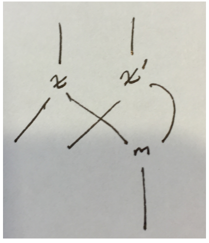
Corollary 188 tells us that $\mathbf{Comod}_d(C) \times \mathbf{Comod}_d(C) \xrightarrow{\otimes} \mathbf{Comod}_d(C \otimes C)$ has the universal property of N. Recall that for any $V \colon \mathbf{D} \to \mathbf{Vect}$ where $\mathbf{D}$ is small, and $F \colon \mathbf{D} \to \mathbf{Comod}_d(D)$, there is a unique $u \colon E(V) \to D$ such that

$$\begin{tikzcd}
\mathbf{D} \arrow[r, "N"] \arrow[dr, "F"'] & \mathbf{Comod}_d(E(V)) \arrow[d, dashed, "u_*"] \\
& \mathbf{Comod}_d(D)
\end{tikzcd}$$

There is a unique $m \colon C \otimes C \to C$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathbf{Comod}_d(C) \times \mathbf{Comod}_d(C) \arrow[r] \arrow[dr, "\diamond"'] & \mathbf{Comod}_d(C \otimes C) \arrow[d, "m_*"] \\
& \mathbf{Comod}_d(C)
\end{tikzcd}$$

We have that $(M,\chi) \diamond (M',\chi')$ is the usual tensor product of modules, with product given by



Similarly, there is some $j\colon k \to C$.

To prove the associativity of m, we ned to use a variation of Corollary 188 for three coalgebras. Given C, D, F coalgebras,

$$\begin{tikzcd}
\mathbf{Comod}_d(C) \times \mathbf{Comod}_d(D) \times \mathbf{Comod}_d(F) \arrow[r, "N"] \arrow[dr, "\otimes(\otimes\times 1)\cong\otimes(1\times\otimes)"'] & \mathbf{Comod}_d(E(U^C \otimes U^D \otimes U^F)) \arrow[d, "u_*"] \\
 & \mathbf{Comod}_d(C \otimes D \otimes F)
\end{tikzcd}$$

$u\colon E(U^C \otimes U^D \otimes U^F) \cong C \otimes D \otimes F$. So

$$\begin{tikzcd}
\mathbf{Comod}_d(C)^3 \arrow[r] \arrow[dr, "\diamond(1\times\diamond)=\diamond(\diamond\times 1)"'] & \mathbf{Comod}_d(C \otimes C \otimes C) \arrow[d] \\
 & \mathbf{Comod}_d(C \otimes C \otimes C)
\end{tikzcd}$$

This proves associativity of m.

Prove the unit laws for yourself. $\square$

Lecture 24 — 9 March 2016

Remark 190. Note that $\mathbf{Vect}_d$ means the category of dualizable vector spaces, which necessarily implies that these vector spaces are finite dimensional.

Theorem 191. *Let H be a bialgebra in $\mathbf{Vect}$. Then $\mathbf{Comod}_d(H)$ has left duals if and only if H is Hopf.*

Proof. ($\Longleftarrow$). We already know this!

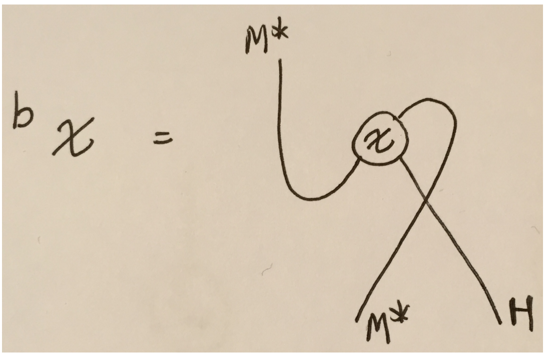
($\Longrightarrow$). We have a functor

$$\begin{tikzcd}
\mathbf{Comod}_d(H)^{\mathrm{op}} \arrow[r, "{}^*(-)"] \arrow[d] & \mathbf{Comod}_d(H) \arrow[d, "U^H"] \\
\mathbf{Vect}_d^{\mathrm{op}} \arrow[r, "{}^*(-)"] & \mathbf{Vect}_d
\end{tikzcd}$$

Then U^H is strict monoidal $\implies$ U preserves duals (see Exercise 192).

$$\begin{tikzcd}
\mathbf{Comod}_d(H^{\mathrm{cop}}) \arrow[r, "{}^*(-)"] \arrow[d] & \mathbf{Comod}_d(H)^{\mathrm{op}} \arrow[d, "U^H"] \\
\mathbf{Vect}_d^{\mathrm{op}} \arrow[r, "(-)^*"] & \mathbf{Vect}_d
\end{tikzcd}$$

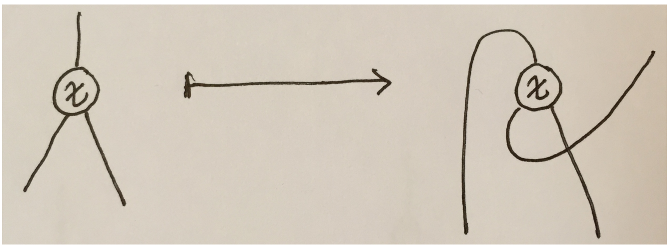
The dual of a module (M, χ) is $M^* \in \mathbf{Vect}$ with coaction ${}^\flat\chi$



It's easy to check that $(M^*, {}^\flat\chi)$ is an H-comodule if $(M, \chi) \in \mathbf{Comod}_d(H^{\mathrm{cop}})$.

Then we can check that the two functors $(-)^*$ and ${}^*(-)$ are equal,

$$\begin{tikzcd}
\mathbf{Comod}_d(H^{\mathrm{cop}}) \arrow[r, shift left, "(-)^*"] & \mathbf{Comod}_d(H) \arrow[l, shift left, "{}^*(-)"]
\end{tikzcd}$$



and this comes from the fact that $\mathbf{Vect}_d$ is a symmetric category.

Moreover, we have that $({}^*(-))^* \cong 1$ via the usual $M^{**} \cong M$ in $\mathbf{Vect}$. So we get a functor T such that

$$\begin{tikzcd}
\mathbf{Comod}_d(H^{\mathrm{cop}}) \arrow[r, "{}^*(-)"] \arrow[d] \arrow[rr, bend left, "T"] & \mathbf{Comod}_d(H)^{\mathrm{op}} \arrow[r, "{}^*(-)"] \arrow[d] & \mathbf{Comod}_d(H) \arrow[d] \\
\mathbf{Vect}_d \arrow[r, "(-)^*"] \arrow[rr, bend right, "1"'] & \mathbf{Vect}_d^{\mathrm{op}} \arrow[r, "(-)^*"] & \mathbf{Vect}_d
\end{tikzcd}$$

commutes. $T(M,\chi)$ is M with the unique comodule structure that makes $T(M) \cong {}^*({}^*M)$ a morphism of H-comodules, that is,

$$\begin{tikzcd} U^H T(M) \arrow[r, "\cong"] \arrow[d, equal] & U^H({}^*({}^*M)) \arrow[d, "\cong"] \\ M \arrow[r, "\cong"] & M^{**} \end{tikzcd}$$

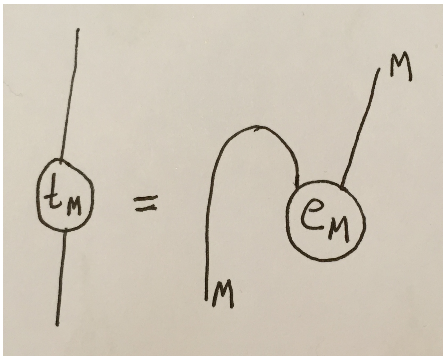
in $\mathbf{Comod}_d(H)$. Then $T = S_*$ for some $S\colon H^{\mathrm{cop}} \to H$ a morphism of coalgebras, and we have the following

$$\begin{tikzcd} \mathbf{Comod}_d(H)^{\mathrm{op}} \arrow[r, "(-)^*"] \arrow[rr, bend right, "{}^*(-)"'] & \mathbf{Comod}_d(H^{\mathrm{cop}}) \arrow[r, "S_*"] & \mathbf{Comod}_d(H) \end{tikzcd}$$

Then $S_*(-)^*$ gives left duals. So now we're almost there, because we have something that looks like an antipode. Denote by

$$e_M\colon (S_*M^*) \otimes M \to k$$
$$n_M\colon k \to M \otimes (S_*M^*)$$

the evaluation and coevaluation. Notice that e_M is dinatural in $M \in \mathbf{Comod}_d(H)$. Define $t_M\colon M \to M$ by

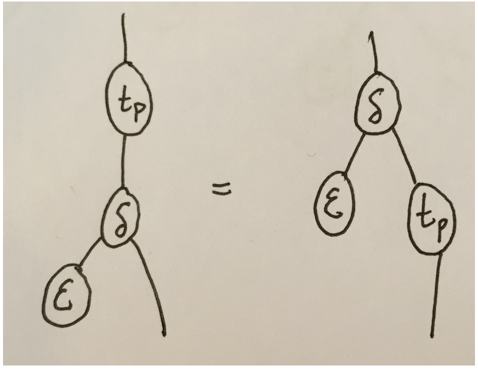


and t_M is natural in $M \in \mathbf{Comod}_d(H)$. If $P \subseteq H$ is a finite dimensional subcoalgebra, then P is an H-comodule. For all $\omega\colon P \to k$, then

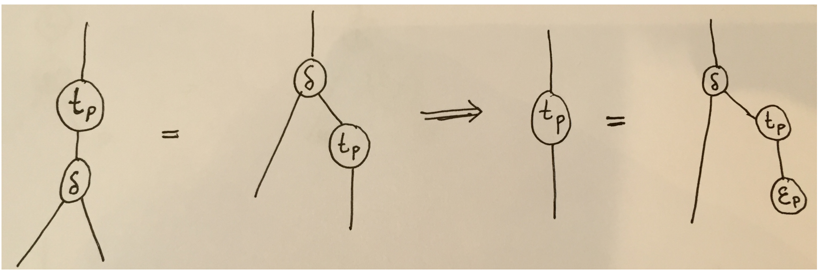
$$P \xrightarrow{\delta} P \otimes P \xrightarrow{\omega\otimes 1} P$$

is a morphism of right P-comodules $\implies$ a morphism of right H-comodules.

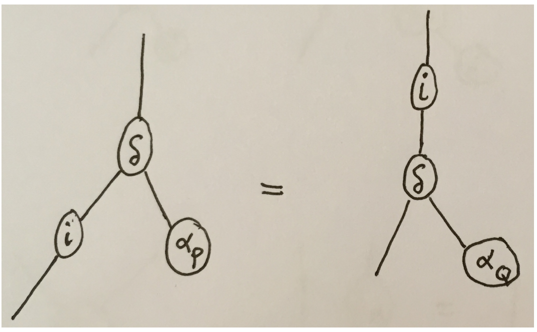
So we have that



by naturality of t_M. So define α_P as $\alpha_P = \varepsilon \circ t_P$. Then we can reinterpret the previous equation as

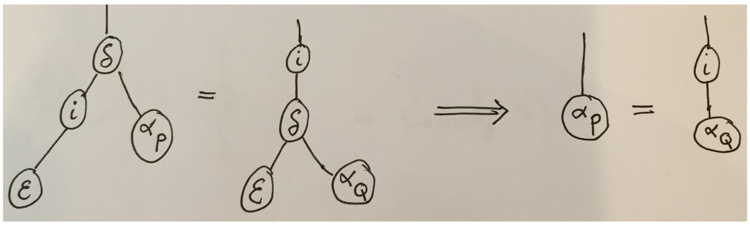


Now suppose that $P \xhookrightarrow{i} Q \subseteq H$ for P, Q finite dimensional subcomodules. Then by naturality of t, $i \circ t_P = t_Q \circ i$ (note that $i \colon P \to Q$ is a morphism of right H-comodules). Then



applying ε gives that $\alpha_P = \alpha_Q \circ i$, that is $\alpha_P = \alpha_Q|_P$. This means that there is some $\alpha \colon H \to k$ such that $\alpha|_P = \alpha_P$ for each finite dimensional subcoalgebra

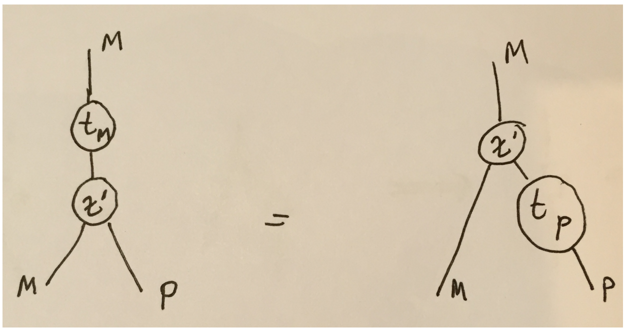
$P \subseteq H$.



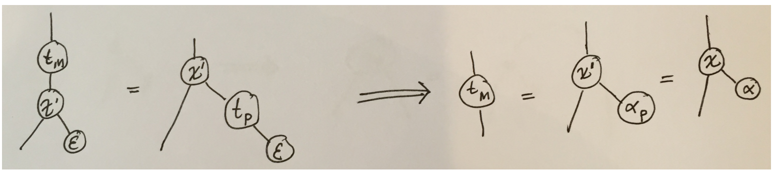
Recall that the coaction $M \xrightarrow{\chi} M \otimes H$ factors through some $M \otimes P$ for $P \subseteq H$ finite dimensional. Write $M \xrightarrow{\chi'} M \otimes P$ for the factorization. This is a morphism of right H-comodules. By naturality of t, we know that the following commutes.

$$\begin{tikzcd}
M \arrow[r, "t_M"] \arrow[d, "\chi'"] & M \arrow[d, "\chi'"] \\
M \otimes P \arrow[r, "t_{M \otimes P}"] & M \otimes P
\end{tikzcd}$$

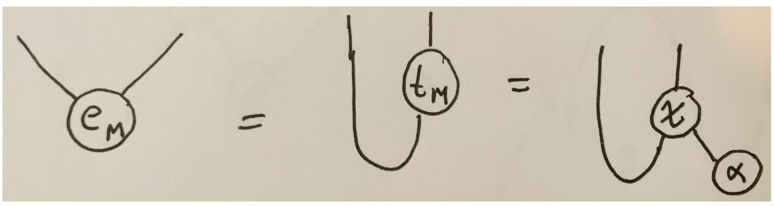
Notice that $M \otimes P$ is the direct sum of $\dim M$-many copies of P, so we can write $t_{M \otimes P}$ as $1_M \otimes t_P$. So we can rewrite the diagram above in terms of string diagrams as



Apply ε on the right-hand dangling string labelled P to deduce that $t_M = 1_M \otimes \alpha_P \circ \chi' = 1_M \otimes \alpha \circ \chi$, the last equality holding because α_P is the restriction of α to P.



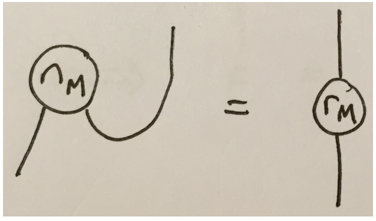
We can recover e_M from the definition of t_M as follows:



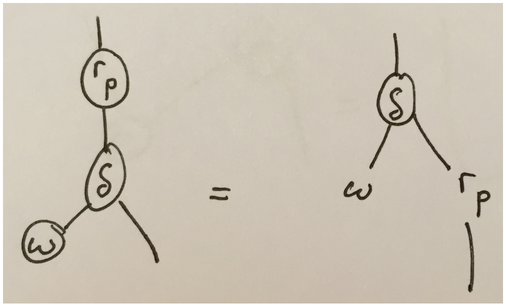
The same can be done for $n\colon k \to M \otimes S_*(M)^*$.

strings

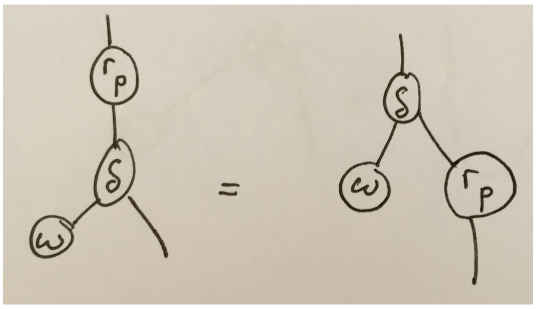
Proof Sketch. We define $t_M\colon M \to M$ such that



and then show that



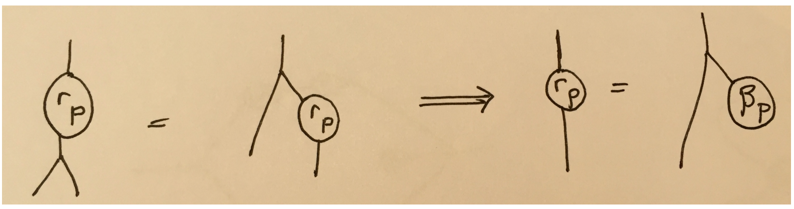
for all $\omega\colon P \to K$ in the same way as we did for t_P. Therefore,



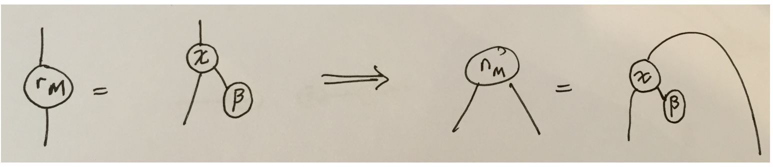
for $\beta_P = \varepsilon \circ r_P$. Now show that given $P \subseteq Q \subseteq H$ finite-dimensional subcoalgebras,

$$\begin{tikzcd} P \arrow[rr, hook] \arrow[dr, "\beta_P"'] & & Q \arrow[dl, "\beta_Q"] \\ & k & \end{tikzcd}$$

as we did for α_P and α_Q. In this way we get $\beta\colon H \to k$ with

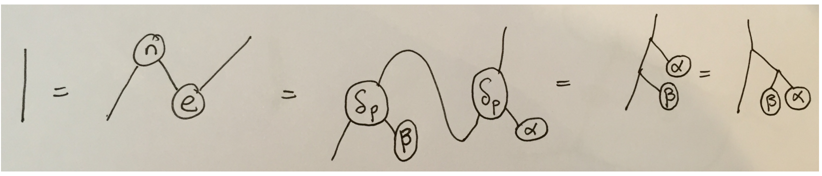


Use the naturality of $t_M : M \to M$ to recover n from r_M as

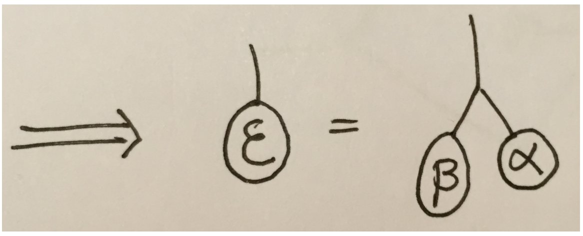


and this is the same proof as before. $\square$

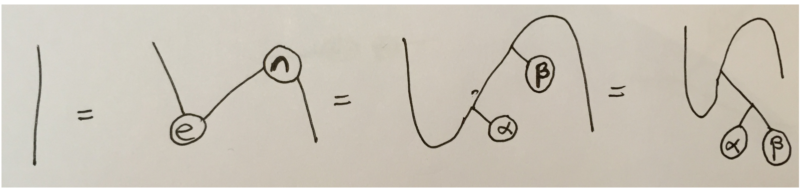
Now having expressions for n and e lets us rewrite the triangular expressions for e and n. Then substituting into the first triangular law $M = P \subseteq H$ a finite-dimensional subcoalgebra, we see that



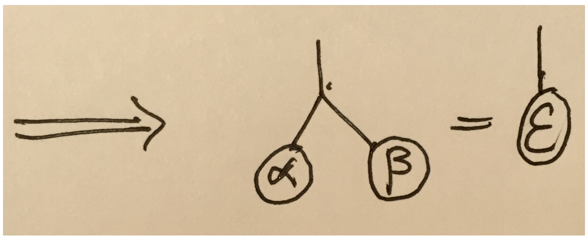
So $\varepsilon = \beta * \alpha$ (the convolution product).



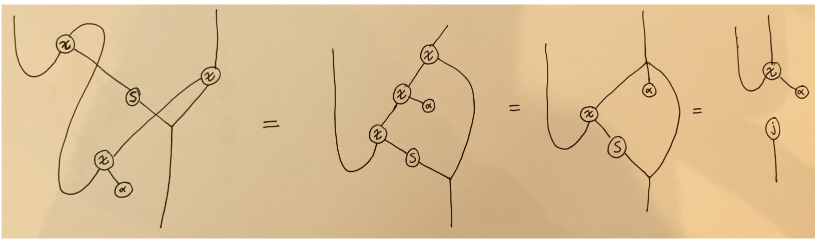
Substituting into the other triangular law, we get



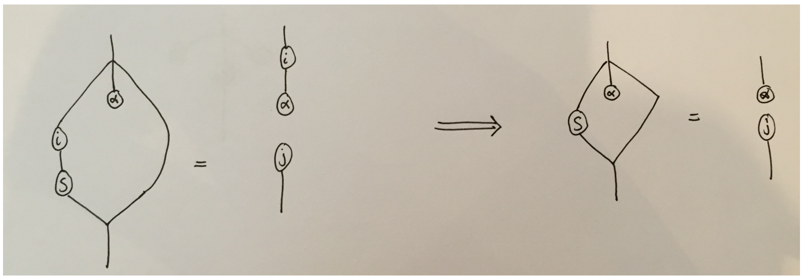
So similarly, $\varepsilon = \alpha * \beta$.



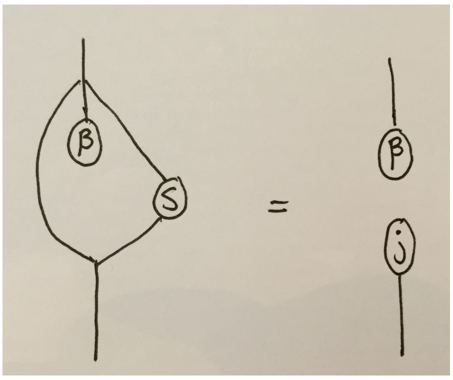
On the other hand, e is a morphism of comodules. So we get the really gross string diagram



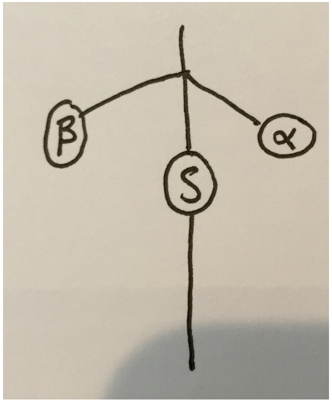
This holds for all H-comodules M. In particular, it's true for all finite dimensional subcomodules, $P \xhookrightarrow{i} H$. Therefore, we get something that looks kind of like the antipode laws, but not quite.



The fact that n is a morphism of H-comodules similarly implies another fake antipode law.



Then finally, if we define the following weird thing to be the antipode.



Then we can check that H is a hopf algebra. $\square$

Exercise 192. If $F\colon \mathbf{V} \to \mathbf{W}$ is strong monoidal then F preserves dual pairs.