# 2-Kac-Moody Algebras

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## 0.1 Introduction

Category theory is a useful language for understanding mathematics. In exchange for the initial overhead cost of some abstraction, the return is greater efficiencies in understanding and learning new mathematical concepts as they are put into a categorical framework. There has been, in recent years, interest in *categorification* – the process of taking an object and finding a categorical structure that enhances it. For example, we can categorify a natural number $n$ by a vector space $V$ of dimension $n$ – the category of vector spaces over some field categorifies the natural numbers. An example that perhaps better reveals the importance of categorification is the passage from Betti numbers to homology groups in algebraic topology.

The connection between Kac-Moody algebras and categorification begins with *higher categorical actions* – representing elements of the Kac-Moody algebra by functors between categories rather than linear maps between weight spaces. Similarity in structure between many higher categorical actions on geometric spaces hinted that there was a categorification of Kac-Moody algebras that controlled these actions. This conjecture was realized by Khovanov-Lauda and independently by Rouquier to define 2-Kac-Moody algebras. A 2-Kac-Moody algebra is an additive 2-category $\dot{\mathscr{U}}_q(\mathfrak{g})$ that categorifies a variant of the quantum group $U_q(\mathfrak{g})$ associated to the Kac-Moody algebra $\mathfrak{g}$.

The purpose of this essay is to define the 2-Kac-Moody algebra $\dot{\mathscr{U}}_q(\mathfrak{g})$ and prove that this definition categorifies the integral idempotent quantum group ${}_{\mathcal{A}}\dot{U}_q(\mathfrak{g})$. The first task is much more work than it seems – the definition isn't one that can be given concisely on a single page. Along the way, I will introduce quantum groups, 2-categories, idempotent completions, and explain what it means for 1-cells to be adjoint in a 2-category.

In the process of showing that the 2-Kac-Moody algebra $\dot{\mathscr{U}}_q(\mathfrak{g})$ categorifies $\dot{U}_q(\mathfrak{g})$, I will demonstrate how each of the defining relations of $\dot{U}_q(\mathfrak{g})$ are categorified as 2-isomorphisms between 1-morphisms as in $\dot{\mathscr{U}}_q(\mathfrak{g})$. This is then used to prove that the existence of a homomorphism between the integral form of $\dot{U}_q(\mathfrak{g})$ and the Grothendieck group of $\dot{\mathscr{U}}_q(\mathfrak{g})$. Finally, I outline a proof that this homomorphism is actually an isomorphism.

The essay is divided into three sections: In chapter 1, I quickly define quantum groups and their idempotent and integral forms. In chapter 2, I define the 2-Kac-Moody algebra $\dot{\mathscr{U}}_q(\mathfrak{g})$ as well as give some background on 2-categories. Finally, in chapter 3, I explain how the 2-Kac-Moody algebra categorifies the idempotent form of the quantum group.

**Remark 0.1.1.** If you're reading this essay far in the future because you're interested in the topic (i.e. you're not giving me a mark on it), let me suggest a few references. The paper that this essay references most often is [4], although this paper is the third in a series that includes [2, 3]. For the most part, it is possible to get a the general idea of [4] without reading its prequels, but the proofs rely on these prequels heavily. The paper [4] is actually a generalization of [7], which dealt only with the $\mathfrak{sl}(2)$ case. [6] is a good expository paper that expands on [7] and gives plenty of motivation. If you're new to the subject, I would recommend starting with [6].

Rouquier independently defined 2-Kac-Moody algebras (he was the first to use the term, actually) in [10] and [11]. Both of these papers are quite dense, however, so I would recommend reading the Khovanov-Lauda paper first. Cautis and Lauda proved in [9] that Rouquier's 2-Kac-Moody algebras and Khovanov-Lauda 2-Kac-Moody algebras have the same higher representation theory anyway, so they are functionally equivalent categorifications insofar as applications go.

# Chapter 1

# Pre-categorification

In this chapter, we give the preliminaries necessary to define the 2-Kac-Moody algebra. For the purposes of this essay, I won't need much of the theory of these objects, so this section will be mostly definitions. A good reference is for the theory of quantum groups and their idempotent forms is either [5] or [8].

It's generally a terrible idea to blast through definitions at the beginning of a talk or paper, because that means that people don't have time to absorb it. Unfortunately, listing unmotivated definitions is an efficient way to get to other, more interesting and/or meaningful mathematics. While that shouldn't excuse me from what I'm about to do, it will hopefully serve as an apology for doing it anyway.

## 1.1 Set-up

This section exists purely to set up some notation that will be used constantly. For the rest of this essay, we will use the following definitions and notation.

- $\mathfrak{g}$ is a Kac-Moody algebra.
- $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$.
- $\alpha_1, \ldots, \alpha_n$ are the simple roots of $\mathfrak{g}$.
- $A = (a_{ij})$ is the (generalized) Cartan matrix of $\mathfrak{g}$.
- $I = \{1, \ldots, n\}$.
- $\Lambda_W$ is the weight lattice of $\mathfrak{g}$.
- $(\lambda, \mu)$ denotes the standard invariant form on $\lambda, \mu \in \Lambda_W$.
- $\langle \lambda, \mu \rangle = 2\frac{(\lambda,\mu)}{(\mu,\mu)}$ for $\lambda, \mu \in \Lambda_W$.

## 1.2 Quantum Groups

Although the term "Quantum Group" comes from physics, they have worked their way into the standard canon of Lie theory and representation theory. Quantum groups were introduced and first studied by Drinfeld and Jimbo, who used the term for a certain class of special Hopf algebras that are nontrivial deformations of the universal enveloping algebras of Lie algebras or, more generally, Kac-Moody algebras. The form used here is sometimes referred to as Drinfeld-Jimbo quantum groups, but the reader should be aware that the term "quantum group" has no concrete definition, but rather depends on the author and usually refers to a class of Hopf algebras related to deformations of universal enveloping algebras.

The Drinfeld-Jimbo quantum group is denoted $U_q(\mathfrak{g})$, similarly to the universal enveloping algebra $U(\mathfrak{g})$. The subscript $q$ is the parameter that controls the deformation of $U(\mathfrak{g})$; the idea is that as $q \to 1$, the original enveloping algebra is recovered from its deformation, although this is neither rigorous nor precise in our treatment. For a treatment of quantum groups as topological Hopf algebras, see [5, Chapters XVI, XVII].

To define $U_q(\mathfrak{g})$, we first need to set up some notation. The following definitions occur inside the field $\mathbb{Q}(q)$.

**Definition 1.2.1.** The **$q$-integer** or **quantum integer** $[n]_q$ is defined for any positive integer $n$ as

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \ldots + q^{-(n-1)}$$

The **$q$-factorial** is defined as

$$[0]_q! := 1 \qquad [n]_q! := [n]_q[n-1]_q \cdots [1]_q$$

The **$q$-binomial coefficient** is defined as

$$\begin{bmatrix} n \\ m \end{bmatrix}_q := \frac{[n]_q!}{[n-m]_q![m]_q!}$$

The idea behind this definition is that the quantum integers behave in many ways similar to ordinary integers. As $q \to 1$, $[n]_q \to n$ and $q$-integers become integers, $q$-factorials become factorials, and $q$-binomial coefficients become ordinary binomial coefficients.

With this notation out of the way, let's define quantum groups.

**Definition 1.2.2.** Let $\mathfrak{g}$ be a Kac-Moody algebra, with Cartan matrix $A = (a_{ij})$ and simple roots $\alpha_1, \ldots, \alpha_n$ and weight lattice $\Lambda_W$. Let $q$ be a variable.

The **Drinfeld-Jimbo Quantum Group** $U_q(\mathfrak{g})$ is the noncommutative, unital, associative $\mathbb{Q}(q)$-algebra generated by $K_\lambda$ for $\lambda \in \Lambda_W$, $E_{+1}, \dots, E_{+n}$, and $E_{-1}, \dots, E_{-n}$, subject to the relations

$$K_0 = 1, \qquad K_\lambda K_\mu = K_\mu K_\lambda = K_{\lambda+\mu}$$

$$K_\lambda E_{\pm i} = q^{\pm(\lambda,\alpha_i)} E_{\pm i} K_\lambda$$

$$[E_{+i}, E_{-i}] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \tag{1.2.1}$$

where $K_i = K_{\lambda_i}$ for $\lambda_i = \frac{(\alpha_i,\alpha_i)}{2}\alpha_i$ and $q_i = q^{(\alpha_i,\alpha_i)/2}$. We also impose the **quantum Serre relations** for $i \neq j$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} E_{\pm i}^k E_{\pm j} E_{\pm i}^{(1-a_{ij})-k} = 0 \tag{1.2.2}$$

**Remark 1.2.3** (Nonrigorous)**.** This definition looks quite similar to the Chevalley-Serre presentation of the universal enveloping algebra of a semisimple Lie algebra $U(\mathfrak{g})$, but with generators $K_i$ instead of elements $H_i$ of the Cartan subalgebra, and with a few extra $q$'s thrown in here and there. Morally, (but certainly *not* rigorously), $K_i$ should be thought of as $q_i^{H_i}$ and $K_i^{-1}$ as $q_i^{-H_i}$, so that we have the following (certainly *not* rigorous) intuitive limit as $q \to 1$

$$\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \text{ ``}=\text{'' } \frac{q_i^{H_i} - q_i^{-H_i}}{q_i - q_i^{-1}} \xrightarrow[q \to 1]{} H_i$$

So in the (intuitive but not rigorous) limit, the commutator relation (1.2.1) reduces to the usual one $[E_{+i}, E_{-j}] = \delta_{ij} H_i$ for $U(\mathfrak{g})$. In this sense, $U_q(\mathfrak{g})$ is a deformation of $U(\mathfrak{g})$. For details, refer to [5, Chapter XVII].

To get a handle on how these quantum groups behave, let's take a look at the quantum deformation of the primordial Lie algebra $\mathfrak{sl}(2)$.

**Example 1.2.4.** $\mathfrak{sl}(2)$ has a $1 \times 1$ Cartan matrix $(2)$, and a single simple root $\alpha_1 = 2$ with $(\alpha_1, \alpha_1) = 2$. We have that $q_1 = q^{(\alpha_1,\alpha_1)/2} = q$. The weight lattice of $\mathfrak{sl}(2)$ is $\Lambda_W = \mathbb{Z}$, so the generators of $U_q(\mathfrak{sl}(2))$ are $E = E_{+1}, F = E_{-1}$ and $K_n$ for each $n \in \mathbb{Z}$. But wait! By the equation $K_\lambda K_\mu = K_{\lambda+\mu}$, we can reduce this list of generators since $K^n = K_1^n = K_n$ for any $n \in \mathbb{Z}$. In particular, $U_q(\mathfrak{sl}(2))$ is generated by just $K = K_1$, $K^{-1}$ $E$ and $F$, with relations

$$K^{-1}K = KK^{-1} = 1 \qquad KE = q^2 EK \qquad KF = q^{-2} EK \qquad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

The list of generators for $U_q(\mathfrak{g})$ may seem initially a bit overwhelming, given that there is one generator $K_\lambda$ for each element of $\lambda$ of the weight lattice $\Lambda_W$. But it turns out that the quantum group $U_q(\mathfrak{g})$ is finitely generated because the Cartan subalgebra $\mathfrak{h}$ (and therefore $\mathfrak{h}^*$) is always finite-dimensional for any Kac-Moody algebra – the list of generators $K_\lambda$ reduces to just one generator for each basis element of $\mathfrak{h}$.

## 1.3 The Lusztig Algebra

Quantum groups are nice objects to study because they have analogues of many of the properties of enveloping algebras. In particular, they share the triangular decomposition of $U(\mathfrak{g})$.

**Definition 1.3.1.** $U_q(\mathfrak{g})^+$ is the subalgebra of $U_q(\mathfrak{g})$ generated by the $E_{+i}$. Likewise, $U_q^-(\mathfrak{g})$ is the subalgebra of $U_q(\mathfrak{g})$ generated by the $E_{-i}$.

In literature, such as [1, 2, 3, 4], these are often realized as the Lusztig algebra $\mathbf{f} \cong U_q(\mathfrak{g})^+ \cong U_q(\mathfrak{g})^-$.

**Definition 1.3.2.** The **Lusztig algebra f** is the quotient of the free associative $\mathbb{Q}(q)$-algebra generated by $\theta_i$, $i = 1, \ldots, n$ modulo the ideal $\mathfrak{J}$ generated by the elements,

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} \theta_i^k \theta_j \theta_i^{1-a_{ij}}$$

for all $i \neq j$, $i, j = 1, \ldots, n$.

Both $U_q(\mathfrak{g})^\pm$ are isomorphic to the **Lusztig algebra f** by an isomorphism sending $E_{\pm i}$ to $\theta_i$; that this is an isomorphism is clear because the only relation imposed between $E_{\pm i}$ and $E_{\pm j}$ is the quantum Serre relation (1.2.2).

**Definition 1.3.3.** The **integral form** of the Lusztig algebra is the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $\mathbf{f}$ generated by elements $\theta_i^a/[a]_{q_i}!$ for all $a \in \mathbb{N}$ and $i \in I$. This is often denoted ${}_{\mathcal{A}}\mathbf{f}$ in literature, but don't ask me why. I think a left subscript is generally terrible notation, in addition to being annoying to typeset.

We often also denote by ${}_{\mathcal{A}}U_q(\mathfrak{g})^\pm$ the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $U_q(\mathfrak{g})^\pm$ generated by elements $E_{\pm i}^a/[a]_{q_i}!$; this is isomorphic to ${}_{\mathcal{A}}\mathbf{f}$.

The integral form of $\mathbf{f}$ is important because the Grothendieck group of the 2-Kac-Moody algebra $\mathscr{U}_q(\mathfrak{g})$ is a $\mathbb{Z}[q, q^{-1}]$-algebra that will be isomorphic to ${}_{\mathcal{A}}\dot{U}_q(\mathfrak{g})$ – see chapter 3.

## 1.4 The Idempotented form of a Quantum Group

The categorification that produces 2-Kac-Moody algebras from quantum groups proceeds not from $U_q(\mathfrak{g})$, but from the idempotented modification $\dot{U}_q(\mathfrak{g})$. The reasons for studying $\dot{U}_q(\mathfrak{g})$ are outlined in the introduction of [4]. They mostly come down to the fact that modules over $\dot{U}_q(\mathfrak{g})$ are the same as modules over $U_q(\mathfrak{g})$ that have integral weight decompositions. These modules are important in representation theory of quantum groups, just as they are important in representation theory of semisimple Lie algebras. Moreover, all representations of quantum groups that have been categorified have integral weight decompositions, according to [4]; in such a categorification, such a weight space decomposition corresponds to categorifications of the idempotents of $\dot{U}_q(\mathfrak{g})$ as projection functors. Moreover, $\dot{U}_q(\mathfrak{g})$ is natural to study because it has a Hopf algebra structure, with comultiplication and antipode analogous to those of the quantum group $U_q(\mathfrak{g})$. It is also a bimodule over $U_q(\mathfrak{g})$.

**Definition 1.4.1.** The idempotented form $\dot{U}_q(\mathfrak{g})$ is the nonunital, associative $\mathbb{Q}(q)$ algebra obtained from $U_q(\mathfrak{g})$ by replacing the unit element by a set of orthogonal idempotents $1_\lambda$, one for each $\lambda \in \Lambda_W$, such that

$$1_\lambda 1_\mu = \delta_{\lambda\mu} 1_\mu$$

$$K_\lambda 1_\mu = 1_\mu K_\lambda = q^{\langle \mu,\lambda\rangle} 1_\lambda \tag{1.4.1}$$
$$E_{+i} 1_\lambda = 1_{\lambda+\alpha_i} E_{+i} \qquad E_{-i} 1_\lambda = 1_{\lambda-\alpha_i} E_{-i}.$$

This also defines the $U_q(\mathfrak{g})$-bialgebra structure of $\dot{U}_q(\mathfrak{g})$.

The fact that the weight lattice $\Lambda_W$ is infinite ruins any hope that $\dot{U}_q(\mathfrak{g})$ is unital. The unit should be the sum of all of the idempotents, but there is no way to make sense of the infinite sum $\sum_{\lambda\in\Lambda_W} 1_\lambda$ of all the idempotents as an element of $\dot{U}_q(\mathfrak{g})$. Nevertheless the idempotented modification decomposes as a direct sum of weight spaces

$$\dot{U}_q(\mathfrak{g}) = \bigoplus_{\lambda,\mu\in\Lambda_W} 1_\mu \dot{U}_q(\mathfrak{g}) 1_\lambda,$$

as occurs with any associative algebra with an orthogonal set of idempotents.

It is inconvenient that $\dot{U}_q(\mathfrak{g})$ doesn't have a unit, but this is compensated for by the following remark.

**Remark 1.4.2.** For any associative ring $A$ with a collection of mutually orthogonal idempotents, there is an additive category $\mathscr{A}$. The data of the ring $A$ together with its collection of idempotents is equivalent to the data of $\mathscr{A}$.

Indeed, given $A$ and a collection of idempotents $\{e_i \mid i \in I\}$, define the additive category $\mathscr{A}$ as follows. The objects of $\mathscr{A}$ are the idempotents $e_i$ of $A$; and the arrows $\mathrm{Hom}_{\mathscr{A}}(e_i, e_j)$ are $e_j A e_i$.

It's not too hard to see that $\mathscr{A}$ is a category. The identity morphism $1_{e_i}$ is simply $e_i$. This behaves as an identity, since any $f\colon e_i \to e_j$ is an element of $e_j A e_i$, and therefore may be written $f = e_j a e_i$, so

$$f \circ 1_{e_i} = f e_i = e_j a e_i e_i = e_j a e_i = f,$$

and similarly $e_j f = f$. Composition of $f\colon e_i \to e_j$ and $g\colon e_j \to e_k$ is their product $gf$. This is associative since multiplication in $A$ is associative. This shows that $\mathscr{A}$ is a category.

Moreover, $\mathscr{A}$ is additive. The homs of $\mathscr{A}$ are abelian groups under addition of the ring $A$, and composition (multiplication in $A$) distributes over addition (addition in $A$), so $\mathscr{A}$ is additive.

Finally, given $\mathscr{A}$, we can recover $A$ as

$$A = \bigoplus_{X,Y \in \mathscr{A}} \mathrm{Hom}_{\mathscr{A}}(X, Y);$$

this coincides with the usual decomposition of $A$ as $A = \bigoplus_{i,j \in I} e_j A e_i$.

Following this remark, we may regard $\dot{U}_q(\mathfrak{g})$ as a category. Morally, this is not a categorification of $\dot{U}_q(\mathfrak{g})$. It doesn't replace elements of $\dot{U}_q(\mathfrak{g})$ by higher analogues; the previous proposition shows that the category $\dot{U}_q(\mathfrak{g})$ contains no more data that $\dot{U}_q(\mathfrak{g})$ the algebra. So instead we are just moving horizontally along the $n$-category ladder.

Nevertheless, it is often convenient to think of $\dot{U}_q(\mathfrak{g})$ in this way. Through this lens, we can expect that a categorification of the idempotented form $\dot{U}_q(\mathfrak{g})$ is a 2-category instead of an ordinary 1-category, since $\dot{U}_q(\mathfrak{g})$ is itself a category.

**Example 1.4.3.** Recall quantum $\mathfrak{sl}(2)$ from Example 1.2.4. The weight lattice of $\mathfrak{sl}(2)$ is $\mathbb{Z}$, so the idempotented form $\dot{U}_q(\mathfrak{sl}(2))$ replaces the unit of $U_q(\mathfrak{sl}(2))$ by idempotents $1_m$ for $m \in \mathbb{Z}$. Alternatively, this is the $\mathbb{Q}(q)$-linear category with objects $n \in \mathbb{Z}$ and morphisms

$$E_{\pm} 1_n\colon n \to n \pm 2$$

Notice that there are no more $K$'s; this is because $K 1_n = q^n 1_n$ by (1.4.1). To simplify the notation, we will write $E_{\pm} 1_n = E_n^{\pm}$. These morphisms satisfy the relations

$$E_{n-2}^{+} E_n^{-} - E_{n+2}^{-} E_n^{+} = \frac{q^n - q^{-n}}{q - q^{-1}} 1_n = [n]_q 1_n. \tag{1.4.2}$$

The last relation is the analogue of the commutator relations.

We can draw a picture of this category

$$\cdots \rightleftarrows (n-2) \underset{E_n^-}{\overset{E_{n-2}^+}{\rightleftarrows}} n \underset{E_{n+2}^-}{\overset{E_n^+}{\rightleftarrows}} (n+2) \rightleftarrows \cdots$$

This looks eerily similar to a picture you might draw for a representation of $\mathfrak{sl}(2)$ with $E_\pm$ actions.

The picture of the category $\dot{U}_q(\mathfrak{sl}(2))$ reveals one of the primary reasons that thinking about the idempotented form is useful. It makes it more straightforward, in some sense, to see how a categorification of $U_q(\mathfrak{g})$ might be built. It also reveals the connection between $\dot{U}_q(\mathfrak{g})$ modules and modules of $U_q(\mathfrak{g})$ that have integral weight decompositions, as mentioned earlier.

**Definition 1.4.4.** By analogy to Definition 1.3.3, denote by ${}_{\mathcal{A}}U_q(\mathfrak{g})^\pm$ the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $U_q(\mathfrak{g})^\pm$ generated by elements $E_{\pm i}^a 1_\lambda / [a]_{q_i}!$.

## 1.5 An analogue of the Killing Form

The semilinear form $\langle -, - \rangle$ is the analogue of the Killing form on the idempotented form of the quantum group. The definition can be found in [4, Definition 2.3], but the definition is quite complicated and we only need a few properties of the form anyway, recorded below.

**Definition 1.5.1** ([4, Definition 2.3]). There is a semilinear form $\langle -, - \rangle \colon \dot{U}_q(\mathfrak{g}) \times \dot{U}_q(\mathfrak{g}) \to \mathbb{Q}(q)$ with properties

$$\langle 1_\lambda, 1_\mu \rangle = \delta_{\lambda,\mu} 1_\lambda$$

$$\langle E_{\pm i} 1_\lambda, E_{\pm j} 1_\mu \rangle = \delta_{i,j} \delta_{\lambda,\mu} \frac{1}{(1 - q_i^2)}.$$

Semilinearity means that for $f(q) \in \mathbb{Q}(q)$,

$$\langle f(q) X, Y \rangle = f(q^{-1}) \langle X, Y \rangle$$
$$\langle X, f(q) Y \rangle = f(q) \langle X, Y \rangle$$

**Proposition 1.5.2** ([4, Proposition 2.5]). *The form $\langle -, - \rangle$ is nondegenerate on $\dot{U}_q(\mathfrak{g})$, and restricts to a pairing ${}_{\mathcal{A}}\dot{U}_q(\mathfrak{g}) \times {}_{\mathcal{A}}\dot{U}_q(\mathfrak{g}) \to \mathbb{Z}[q, q^{-1}]$. In particular, this means that if $X \in \dot{U}_q(\mathfrak{g})$ such that $\langle X, Y \rangle = 0$ for all $Y \in \dot{U}_q(\mathfrak{g})$, then $X = 0$.*

# Chapter 2

# Categorification

In this chapter, I define the 2-Kac-Moody algebra $\dot{\mathscr{U}}_q(\mathfrak{g})$. Along the way, I introduce some 2-categorical preliminaries, idempotent completions, and demonstrate some relations between 2-morphisms in $\mathscr{U}_q(\mathfrak{g})$.

## 2.1 2-Categories

Given that $\dot{U}_q(\mathfrak{g})$ is a category (at least from one angle), it's categorification will move up the hierarchy of $n$-categories into the wacky and wonderful world of 2-categories, just as categorifying a set promotes it to a 1-category from it's prior lowly status.

So what is a 2-category? Well, here's the standard definition given by category theorists which is simultaneously beautifully concise and yet entirely uneducational.

**Definition 2.1.1.** A 2**-category** is a category enriched in categories.

What does that mean? This first requires the notion of an enriched category. If $(\mathscr{V}, \otimes, I)$ is a symmetric monoidal category, then a **category** $\mathscr{C}$ **enriched in** $\mathscr{V}$ is the same as an ordinary category, except that the homs $\mathscr{C}(A, B)$ are objects in $\mathscr{V}$, and composition and unit maps are replaced by morphisms in $\mathscr{V}$

$$\mathscr{C}(B, C) \otimes \mathscr{C}(A, B) \xrightarrow{\circ} \mathscr{C}(A, C), \qquad I \xrightarrow{1_A} \mathscr{C}(A, A).$$

A 2-category $\mathfrak{C}$ is a category enriched in the symmetric monoidal category $(\mathbf{Cat}, \times, \mathbf{1})$.

So if $\mathfrak{C}$ is a 2-category, this means that for objects $A, B \in \mathfrak{C}$, the set of morphisms $\mathfrak{C}(A, B)$ itself forms a category, and moreover the composition $\mathfrak{C}(B, C) \times \mathfrak{C}(A, B) \to \mathfrak{C}(A, C)$ and identity $\mathbf{1} \xrightarrow{1_A} \mathfrak{C}(A, A)$ are morphisms of categories, that is, a functors. So we can unwrap Definition 2.1.1 to find a more workable definition of 2-categories.

**Definition 2.1.2** (Definition 2.1.1 revised)**.** A **2-category** $\mathfrak{C}$ consists of

- objects, also called 0**-cells**, written $A, B, C, \ldots$
- morphisms, also called 1**-cells** or 1**-morphisms**, written $A \xrightarrow{f} B$,
- morphisms between morphisms, also called 2-cells or 2-morphisms and written $f \overset{\alpha}{\Rightarrow} g$ or diagrammatically,

$$\begin{tikzcd} A \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & B \arrow[from=U, to=D, Rightarrow, "\alpha"] \end{tikzcd}$$

These data are subject to the following rules which generalize the usual notion of a category.

- the objects and morphisms of $\mathfrak{C}$ form a category, as usual; composition is associative and there are identities $1_A$ for each object $A \in \mathfrak{C}$. In Definition 2.1.1, this is because $\mathfrak{C}$ is a category before we enrich it.
- Each $\mathfrak{C}(A, B)$ is itself a category, meaning that we can compose morphisms $\alpha\colon f \implies g$ and $\beta\colon g \implies h$ in an associative manner. This is called **vertical composition**, and is depicted

$$\begin{tikzcd} A \arrow[r, bend left=60, "f"{name=F}] \arrow[r, "g"{name=G, description}] \arrow[r, bend right=60, "h"'{name=H}] & B \arrow[from=F, to=G, Rightarrow, "\alpha"] \arrow[from=G, to=H, Rightarrow, "\beta"] \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & B \arrow[from=U, to=D, Rightarrow, "\beta\alpha"] \end{tikzcd} .$$

  Moreover, for each $f\colon A \to B$ in $\mathfrak{C}(A, B)$, there is an identity morphism $1_f$ such that $\alpha 1_f = \alpha$ and $1_f \beta = \beta$ for all $\alpha\colon f \implies g$ and $\beta\colon h \implies f$ and any $h, g \in \mathfrak{C}(A, B)$.

  In Definition 2.1.1, this corresponds to the fact that $\mathfrak{C}$ is enriched in categories; each $\mathfrak{C}(A, B)$ is itself a category.

- There is another notion of composition of 2-cells, called **horizontal composition.** Given morphisms $f, g\colon A \to B$ and $h, k\colon B \to C$ in $\mathfrak{C}$, suppose given 2-cells $\alpha\colon f \implies g$ and $\beta\colon h \implies k$. Their horizontal composition is

$\beta \cdot \alpha \colon hf \implies kg$. This is depicted diagramatically as

$$
\begin{tikzcd}
A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\beta"] & C
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[rr, bend left, "hf"] \arrow[rr, bend right, "kg"'] \arrow[rr, phantom, "\Downarrow\beta\cdot\alpha"] & & B
\end{tikzcd}
$$

Moreover, this composition is associative.

In Definition 2.1.1, this corresponds to the fact that composition is a functor, and therefore gives a two-cell of $\mathfrak{C}(A, C)$ for each pair of two-cells in $\mathfrak{C}(B, C) \times \mathfrak{C}(A, B)$.

- Horizontal composition of identity 2-cells $1_f \colon f \implies f$ must respect composition of 1-cells. That is, $1_f \cdot 1_g = 1_{fg}$.

  Moreover, we must also have

$$
\begin{tikzcd}
A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, bend left, "1_B"] \arrow[r, bend right, "1_B"'] \arrow[r, phantom, "\Downarrow 1_{1_B}"] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B
\end{tikzcd}
$$

  These laws are imposed because Definition 2.1.1 requires that $\mathfrak{C}$ is enriched in categories, which in turn requires that the identity map $\to \mathfrak{C}(A, A)$ is a functor.

- We have **interchange law**, which essentially states that the composition functor $\mathfrak{C}(B, C) \times \mathfrak{C}(A, B) \to \mathfrak{C}(A, C)$ respects (vertical) composition of 2-cells, as functors are required to do. The interchange law states that $(\delta\gamma) \cdot (\beta\alpha) = (\delta \cdot \beta)(\gamma \cdot \alpha)$. Diagrammatically, this might make more sense:

$$
\begin{tikzcd}
A \arrow[r] \arrow[r, bend left=80, phantom, "\Downarrow\alpha"] \arrow[r, bend right=80, phantom, "\Downarrow\beta"] \arrow[r, bend left=80] \arrow[r, bend right=80] & B
\end{tikzcd}
\cdot
\begin{tikzcd}
B \arrow[r] \arrow[r, bend left=80, phantom, "\Downarrow\gamma"] \arrow[r, bend right=80, phantom, "\Downarrow\delta"] \arrow[r, bend left=80] \arrow[r, bend right=80] & C
\end{tikzcd}
=
\begin{array}{c}
\begin{tikzcd}
A \arrow[r] \arrow[r, bend left=80] \arrow[r, bend left=80, phantom, "\Downarrow\alpha"] & B \arrow[r] \arrow[r, bend left=80] \arrow[r, bend left=80, phantom, "\Downarrow\gamma"] & C
\end{tikzcd} \\
\circ \\
\begin{tikzcd}
A \arrow[r] \arrow[r, bend right=80] \arrow[r, bend right=80, phantom, "\Downarrow\beta"] & B \arrow[r] \arrow[r, bend right=80] \arrow[r, bend right=80, phantom, "\Downarrow\delta"] & C
\end{tikzcd}
\end{array}
$$

  When drawing diagrams, we ignore the distinction between these two compositions and simply draw

$$
\begin{tikzcd}
A \arrow[r] \arrow[r, bend left=80] \arrow[r, bend right=80] \arrow[r, bend left=80, phantom, "\Downarrow\alpha"] \arrow[r, bend right=80, phantom, "\Downarrow\beta"] & B \arrow[r] \arrow[r, bend left=80] \arrow[r, bend right=80] \arrow[r, bend left=80, phantom, "\Downarrow\gamma"] \arrow[r, bend right=80, phantom, "\Downarrow\delta"] & C
\end{tikzcd}
$$

**Remark 2.1.3.** Notice that there are no 2-cells between 1-cells $f$ and $g$ unless the domain and codomain of $f$ and $g$ agree.

**Example 2.1.4.** Perhaps the most obvious, and indeed the prototypical, example of a 2-category is **Cat**, the category of small categories and functors between them. The two-cells are given by natural transformations, and horizontal composition of $\alpha \colon F \implies G$ and $\beta \colon H \implies K$ is the natural transformation $\beta \cdot \alpha$ with components

$$(\beta \cdot \alpha)_A = \beta_{GA} \circ H\alpha_A = K\alpha_A \circ \beta_{FA}$$

**Example 2.1.5.** Another, more abstract, example of a 2-category is any strict monoidal category. In fact, a strict monoidal category $\mathscr{V}$ can be viewed as a 2-category with a single 0-cell $*$, 1-cells coming from objects of $\mathscr{V}$, and two cells coming from morphisms of $\mathscr{V}$. Composition of 1-cells is the tensor product of objects, and horizontal composition of morphisms is tensor product of morphisms. Vertical composition of 2-cells is the usual composition of morphisms in $\mathscr{V}$.

We can also define functors between 2-categories; they are exactly what you expect (that is, if you're sufficiently familiar with category theory to expect anything at all).

**Definition 2.1.6.** A **2-functor** $\mathfrak{F} \colon \mathfrak{C} \to \mathfrak{D}$ between two-categories is a map that sends 0-cells to 0-cells, 1-cells to 1-cells, and 2-cells to 2-cells, such that, on the 0-cells and 1-cells, $\mathfrak{F}$ is a usual functor, and moreover $\mathfrak{F}$ respects both horizontal and vertical composition of 2-cells, and all forms of identities.

- $\mathfrak{F}(1_A) = 1_{\mathfrak{F}A}$ for all 0-cells $A \in \mathfrak{C}$,
- $\mathfrak{F}(1_f) = 1_{\mathfrak{F}f}$ for all 1-cells $f$ in $\mathfrak{C}$,
- $\mathfrak{F}(gf) = \mathfrak{F}(g)\mathfrak{F}(f)$ for 1-cells $f, g$ in $\mathfrak{C}$,
- $\mathfrak{F}(\alpha \cdot \beta) = \mathfrak{F}(\alpha) \cdot \mathfrak{F}(\beta)$ for all two-cells $\alpha, \beta$ in $\mathfrak{C}$,
- $\mathfrak{F}(\beta\alpha) = \mathfrak{F}(\beta)\mathfrak{F}(\alpha)$ for all two-cells $\alpha, \beta$ in $\mathfrak{C}$.

The 2-categories that we will discuss are categorifications of algebras. This means that they have more structure than just a category – they are additive 2-categories.

**Definition 2.1.7.** An **additive** 2**-category** $\mathfrak{C}$ is a category enriched in additive categories.

Just as we did with the definition of a 2-category, we can unravel this definition to be more workable. This definition tells us that $\mathfrak{C}$ is a 2-category in which the hom-categories $\mathfrak{C}(A, B)$ are additive for all 0-cells $A$ and $B$, and moreover the composition functor is additive. In particular, the following properties will be essential for us.

- There is a notion of direct sum of 1-cells, so long as they have the same domain and codomain.

- Composition of 1-cells distributes over direct sum: $(f \oplus g)h = fh \oplus gh$ and $k(f \oplus g) = kf \oplus kg$.

Similarly, we may define $k$-linear 2-categories. The categorification of $\dot{U}_q(\mathfrak{g})$ will be $k$-linear for $k = \mathbb{Q}(q)$.

**Definition 2.1.8.** A **$k$-linear 2-category** is a category enriched in $k$-linear categories. Recall that a $k$-linear category is a category enriched in $k$-vector spaces.

**Remark 2.1.9.** In general, if `[adjective]` is used to describe a certain type of categories, then `[adjective]` 2-categories are usually defined as 2-categories enriched in `[adjective]` categories.

## 2.2 The 2-Category $\mathcal{U}_q(\mathfrak{g})$

Now comes the exciting part. The 2-category $\mathcal{U}_q(\mathfrak{g})$, as we will see in chapter 3, is the categorification of the idempotented form $\dot{U}_q(\mathfrak{g})$. As discussed in the previous sections, this categorification is a 2-category because we can think of the idempotented modification as a 1-category.

To define a 2-category, we need to specify the objects (0-cells), morphisms (1-cells) and the morphisms between morphisms (2-cells). While the objects and morphisms of $\mathcal{U}_q(\mathfrak{g})$ are quite straightforward to define, and indeed look quite similar to those in $\dot{U}_q(\mathfrak{g})$, the 2-cells require quite a bit of explanation. This is because the 2-cells are defined in terms of a specific type of planar diagram, not unlike a braid diagram, but for which each strand has a direction and a label and admits some number of dots. In addition to this, certain carefully chosen relations are imposed on these diagrams. The use of diagrams allows us to organize the tremendous amount of data contained in $\mathcal{U}_q(\mathfrak{g})$; the diagrams are a useful tool for managing combinatorial complexity.

The definition given here follows [4], where they defined this category $\mathcal{U}_q(\mathfrak{g})$ using string-like diagrams to define the 2-cells. A similar category was defined by Rouquier in the papers [10, 11] using a long list of generators and relations.[†] These two constructions are related in [9, Theorem 1.1]; it turns out that the 2-representations of the Khovanov-Lauda version and the Rouquier version are equivalent, and therefore both provide appropriate categorifications of $\dot{U}_q(\mathfrak{g})$. However, there are some advantages to using the definition given in [4]: string diagrams are more intuitive than Rouquier's generators and relations [10, §3.3.3].

[†]The video lecutures of these two papers given by Rouquier are quite elucidating, if a bit dry. To find them, search https://www.newton.ac.uk/webseminars for "Rouquier."

Moreover, Rouquier outsources proofs of several important relations within $\mathscr{U}_q(\mathfrak{g})$ by citing [3, 4]; these proofs are apparently better done with the diagrams.

The definition of $\mathscr{U}_q(\mathfrak{g})$ is summarized in the appendices; the generators are listed in Appendix A and the relations between them are listed in Appendix B. I would recommend tearing out those pages for reference while reading the rest of this essay; they are quite handy and significantly more concise than the section below.

### 2.2.1 Set-up

This section is a list of definitions for notational convenience later. It's quite boring, but their utility will become apparent in the definition of $\mathscr{U}_q(\mathfrak{g})$. Also recall the conventions from section 1.1, which hold throughout the essay. This notational convenience is borrowed from [4].

**Definition 2.2.1** ([4, Section 2.1.4])**.** A **signed sequence** $\mathbf{i} = (\varepsilon_1 i_1, \varepsilon_2 i_2, \ldots, \varepsilon_m i_m)$ of length $m$ is a sequence of $m$-many elements $i_k \in I$, each tagged by a sign $\varepsilon_k \in \{+, -\}$.

If $\mathbf{i}$ and $\mathbf{j}$ are two signed sequences, $\mathbf{ij}$ denotes their concatenation.

We may also use signed sequences to write weights and elements of $\dot{U}_q(\mathfrak{g})$ in a concise form.

**Definition 2.2.2.** Let $\mathbf{i} = (\varepsilon_1 i_1, \varepsilon_2 i_2, \ldots, \varepsilon_m i_m)$ be a signed sequence. Then define

$$\alpha_{\mathbf{i}} := (\varepsilon_1 \alpha_{i_1}) + (\varepsilon_2 \alpha_{i_2}) + \ldots + (\varepsilon_m \alpha_{i_m}).$$

**Definition 2.2.3.** For a signed sequence $\mathbf{i} = (\varepsilon_1 i_1, \varepsilon_2 i_2, \ldots, \varepsilon_m i_m)$, define

$$E_{\mathbf{i}} 1_\lambda := E_{\varepsilon_1 i_1} E_{\varepsilon_2 i_2} \cdots E_{\varepsilon_m i_m} 1_\lambda = 1_{\lambda + \alpha_{\mathbf{i}}} E_{\mathbf{i}} 1_\lambda \in \dot{U}_q(\mathfrak{g})$$

**Example 2.2.4.** For example, $\mathbf{i} = (+2, -1, +1, +3, -2, +1)$ is a signed sequence with elements in $\{1, 2, 3\}$, and $\alpha_{\mathbf{i}}$ is the weight

$$\alpha_{\mathbf{i}} = \alpha_2 - \alpha_1 + \alpha_1 + \alpha_3 - \alpha_2 + \alpha_1 = \alpha_1 + \alpha_3.$$

and $E_{\mathbf{i}} 1_\lambda$ is the element of $\dot{U}_q(\mathfrak{g})$

$$E_{\mathbf{i}} 1_\lambda = E_{+2} E_{-1} E_{+1} E_{+3} E_{-2} E_{+1} 1_\lambda.$$

This last definition is really only used in Lemma 3.3.5.

**Definition 2.2.5.** Given an element $\nu$ of the root lattice, $\nu = \sum_{i \in I} \nu_i \alpha_i$, let $\mathrm{Seq}(\nu)$ be the set of all signed sequences $\mathbf{i}$ such that $\alpha_{\mathbf{i}} = \nu$. The **length** of $\nu$ is $\sum_{i \in I} \nu_i$.

We also make the following convention, which is essential for making the definition of relations between 2-cells in $\mathscr{U}_q(\mathfrak{g})$ concise.

**Definition 2.2.6.** If there is a summation $\sum_{a=0}^{x}$ with $x < 0$, then we take the convention that the summation is empty and therefore vanishes.

### 2.2.2 Objects of $\mathscr{U}_q(\mathfrak{g})$

There is one object $\lambda$ for each $\lambda \in \Lambda_W$, where $\Lambda_W$ is the weight lattice of $\mathfrak{g}$. There's nothing complicated going on here: the objects of this 2-category are the same as those in the 1-category $\dot{U}_q(\mathfrak{g})$.

### 2.2.3 Morphisms of $\mathscr{U}_q(\mathfrak{g})$

For any two objects $\lambda$ of $\mathscr{U}_q(\mathfrak{g})$ and any simple root $\alpha_i$, we define morphisms

$$\mathcal{E}_{+i}1_\lambda : \lambda \to \lambda + \alpha_i \qquad \text{and} \qquad \mathcal{E}_{-i}1_\lambda : \lambda \to \lambda - \alpha_i.$$

This notation is already suggestive of how this categorifies $\dot{U}_q(\mathfrak{g})$. We can expect that morphisms $\mathcal{E}_{\pm i}1_\lambda$ correspond to elements $E_{\pm i}1_\lambda$ in $\dot{U}_q(\mathfrak{g})$.

On these morphisms, we include an artificial degree shift, for any $t \in \mathbb{Z}$, denoted by appending a $\{t\}$ to the end of a morphism as $\mathcal{E}_{\pm i}1_\lambda\{t\}$. If the degree shift notation is not included, we mean the degree shift by zero. This artificial degree shift will correspond to multiplication by $q^t$ in the Grothendieck group of $\mathscr{U}_q(\mathfrak{g})$; modules over $\mathbb{Z}[q, q^{-1}]$ are often categorified by graded algebraic structures, and their Grothendieck groups have $\mathbb{Z}[q, q^{-1}]$-module structure, see chapter 3.

**Remark 2.2.7.** Different authors use different conventions for the degree shift notation. [9] uses $\langle t \rangle$, and [4] use $\{t\}$. While $\langle t \rangle$ seems to be more appropriate, as it is often used for degree shifts in graded rings, the notation $\{t\}$ clashes less with the plethora of $\langle$'s and $\rangle$'s already floating around in this essay. Later we'll have degree shifts by $\langle \lambda, \alpha_i \rangle$, and then you'll hopefully agree that curly braces are a better choice of notation.

So the basic morphisms with domain $\lambda$ are

$$\mathcal{E}_{+i}1_\lambda\{t\} : \lambda \to \lambda + \alpha_i \qquad \text{and} \qquad \mathcal{E}_{-i}1_\lambda\{t\} : \lambda \to \lambda - \alpha_i$$

for any $i \in I$, $t \in \mathbb{Z}$ and $\lambda \in \Lambda_W$.

The composition of two morphisms, say $\mathcal{E}_{+i}1_\lambda\{t\}$ and $\mathcal{E}_{+j}1_{\lambda+\alpha_i}\{s\}$, we denote by

$$\mathcal{E}_{+i}\mathcal{E}_{+j}1_\lambda\{s + t\} = \mathcal{E}_{+j}1_{\lambda+\alpha_i}\{s\} \circ \mathcal{E}_{+i}1_\lambda\{t\}.$$

This is a morphism in degree $s + t$. Composition of three morphisms may be denoted $\mathcal{E}_{+i}\mathcal{E}_{-j}\mathcal{E}_{-k}1_\mu\langle s + t + r\rangle$, for example, and so on. Note that composition looks not too dissimilar from multiplication of elements in $\dot{U}_q(\mathfrak{g})$, which will be vital to realizing $\mathscr{U}_q(\mathfrak{g})$ as it's categorification.

Alternatively, since the notation for composition might quickly become cumbersome, we can use signed sequences. We may denote an $m$-fold composition of basic morphisms by

$$\mathcal{E}_{\mathbf{i}}1_\lambda\{t\} : \lambda \to \lambda + \alpha_{\mathbf{i}}$$

where $\mathbf{i} = (\varepsilon_1 i_1, \ldots, \varepsilon_m i_m)$ is a signed sequence of length $m$, and

$$\mathcal{E}_{\mathbf{i}} 1_\lambda\{t\} = \mathcal{E}_{\varepsilon_1 i_1} \mathcal{E}_{\varepsilon_2 i_2} \cdots \mathcal{E}_{\varepsilon_m i_m} 1_\lambda\{t\}$$

To compose two morphisms $\mathcal{E}_{\mathbf{i}} 1_\lambda\langle t\rangle$ and $\mathcal{E}_{\mathbf{j}} 1_\mu\langle s\rangle$, we simply concatenate sequences $\mathbf{i}$ and $\mathbf{j}$ and add degrees, provided that the domains and codomains match:

$$\mathcal{E}_{\mathbf{ji}} 1_\lambda\{s+t\} = \mathcal{E}_{\mathbf{j}} 1_{\lambda+\alpha_{\mathbf{i}}}\{s\} \circ \mathcal{E}_{\mathbf{i}} 1_\lambda\{t\}.$$

We also allow formal finite direct sums of morphisms $\lambda \to \mu$ as morphisms $\lambda \to \mu$. This is analogous to taking sums of elements of $\dot{U}_q(\mathfrak{g})$.

**Definition 2.2.8.** The set of 1-morphisms between $\lambda$ and $\mu$ in $\mathscr{U}_q(\mathfrak{g})$ altogether consists of formal finite direct sums of morphisms of the form

$$\mathcal{E}_{\mathbf{i}} 1_\lambda\{t\} \colon \lambda \to \mu,$$

where $\mathbf{i}$ is a signed sequence of length $m$ such that $\mu = \lambda + \alpha_{\mathbf{i}}$.

**Example 2.2.9.** For $\mathfrak{g} = \mathfrak{sl}(2)$, such a morphism might look like

$$\mathcal{E}_{+} 1_n\{t\} \oplus \mathcal{E}_{+-+} 1_n\{s\},$$

where $+$ and $+-+$ are shorthand for the signed sequences $(+1)$ and $(+1, -1, +1)$, respectively. Since there's only one simple root, this shorthand has no ambiguity.

Although this business with signed sequences and degree shifts might look intimidating, in reality the 1-morphisms are quite simple. A 1-morphism either adds and subtracts some number of simple roots (encoded in the signed sequence) to a given weight and have been artificially given degrees. Then we allow formal finite direct sums of these.

### 2.2.4 2-cells of $\mathscr{U}$

For any two morphisms $\mathcal{E}_{\mathbf{i}} 1_\lambda\{t\}$ and $\mathcal{E}_{\mathbf{j}} 1_\mu\{s\}$, the set of 2-morphisms between them consists of $\mathbb{Q}(q)$-linear combinations of a certain type of diagrams of degree $t - s$. These diagrams are built from both horizontal (literally sticking diagrams next to each other) and vertical (literally stacking diagrams bottom to top) compositions of some generators, modulo some carefully chosen relations. Each generator is given a degree, and the degree of a diagram in total will be the sum of the degrees of the generators that compose the diagram, whether they are composed horizontally or vertically. To describe all of this structure, we will first describe in subsection 2.2.5 what a diagram looks like and what does and doesn't compose a diagram. Then we will describe in subsection 2.2.6 the generators of the set of 2-morphisms. Finally, we will explain the relations and motivate where each of the relations comes from in subsection 2.2.7. The list of generators can be found in Appendix A, and the list of relations can be found in Appendix B. If you haven't done so already, now would be a great time to tear out these appendices for quick reference.

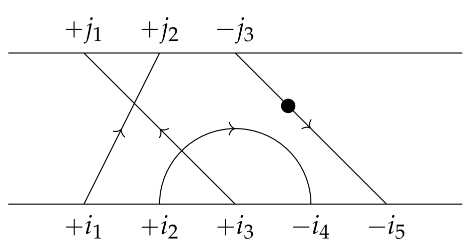
### 2.2.5 Diagrams representing 2-cells

The motivation for these diagrams is given by flipping around the diagrams representing 2-cells in a 2-category. Usually, 0-cells are points in the plane, 1-cells are lines and 2-cells are regions. To define the diagrams for $\mathcal{U}_q(\mathfrak{g})$, we turn this around and represent 0-cells by regions in the plane, 1-cells by lines, and 2-cells by points. Of course, it quickly becomes much more complicated than that.

Any one of these diagrams that represents a 2-morphism $\mathcal{E}_{\mathbf{i}}1_\lambda\{t\} \to \mathcal{E}_{\mathbf{j}}1_\mu\{s\}$ is drawn as an immersed, oriented one-manifold in the strip $\mathbb{R} \times [0,1]$ of the $xy$-plane. Additionally, we label each component by a simple root and place dots on the components. On the upper boundary $\mathbb{R} \times \{1\}$, place the signed sequence $\mathbf{j}$ with $\varepsilon_k j_k$ at the coordinate $(k,1)$. Similarly place the signed sequence $\mathbf{i}$ on the lower boundary of the strip $\mathbb{R} \times \{0\}$. Lines are drawn between $\varepsilon_a i_a$ and $\varepsilon_b j_b$ if $i_a = j_b$. Orientations are given to these lines corresponding to the signs $\varepsilon_a$ and $\varepsilon_b$. A minus sign means that the strand is oriented down, and a plus sign means that the strand is oriented up. We also permit U-turns such that a line may go from $+i_a$ to $-i_b$, or $+j_a$ to $-j_b$. Occasionally, we refer to the lines as "strands."

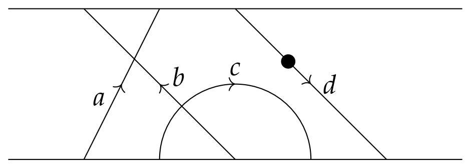
The lines carry some number of dots on them, which may freely slide along a line so long as they don't meet the ends. This is best illustrated by example.

**Example 2.2.10.** For example, if $\mathbf{i} = (+i_1, +i_2, +i_3, -i_4, -i_5)$ and $\mathbf{j} = (+j_1, +j_2, -j_3)$, then one possible such diagram is



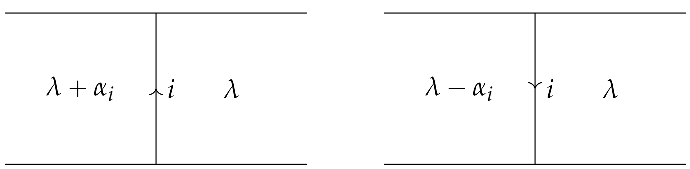

We consider two diagrams equivalent up to boundary-preserving homotopies. Since each line connects two numbers in a sequence which are the same, we may instead just label each line in a diagram by the number on it's endpoint. Then knowing this and the orientation of each strand, we may recover the sequences $\mathbf{i}$ and $\mathbf{j}$.

**Example 2.2.11.** Continuing Example 2.2.10, let's suppose that $i_1 = j_2 = a$, $i_2 = i_4 = b$, $i_3 = j_1 = c$, $i_5 = j_3 = d$. Then we redraw the diagram with labels only on the strands, not the endpoints.



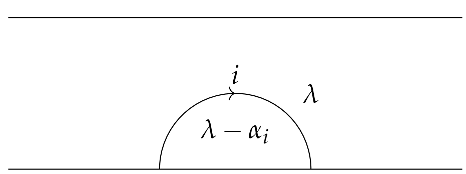

Then we can recover the sequence $(+b, +a, -d) = \mathbf{j}$ by reading off the labels of the strands in the order they meet the upper boundary $\mathbb{R} \times \{1\}$. To find the signs, look at the orientations of the strands. Up means a sign of $+$, and down means a sign of $-$. We can recover similarly the sequence $\mathbf{i}$= $(+a, +c, +b, -c, -d)$ by reading off the labels on the bottom boundary $\mathbb{R} \times \{0\}$.

Additionally, we label each region of the strip $\mathbb{R} \times [0, 1]$ cut out by the diagram by an object of $\mathscr{U}_q(\mathfrak{g})$, that is, an element of the weight lattice $\Lambda_W$, such that the rightmost region is labelled by $\lambda$ (remember that we are talking about morphisms $\mathcal{E}_i 1_\lambda\{t\} \implies \mathcal{E}_j 1_\mu\{s\}$), and the two regions separated by a strand labelled $i$ differ by $\alpha_i$, with $\lambda + \alpha_i$ on the region to the left if the strand is oriented up, or $\lambda - \alpha_i$ on the left if the strand is oriented down. This rule holds regardless of the number of dots on a strand.



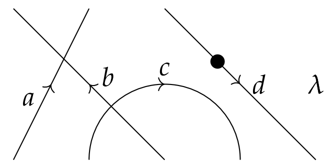

(2.2.1)

Note that this labelling convention is consistent with the U-turns, since the region inside the U-turn will differ from the label outside by $\alpha_i$ if labelled by $i$, and crossing the strand twice both increments and decrements this label by $\alpha_i$, leaving the outside label unchanged.

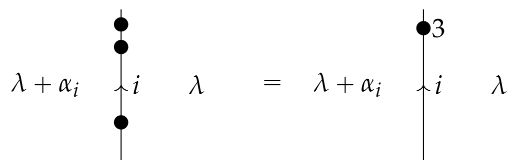



We further simplify the notation by omitting the lower and upper boundary of the strip, and also only drawing the rightmost region label. The other region labels may be recovered from this and the rule (2.2.1).

**Example 2.2.12.** A fully simplified diagram may be drawn, for example,




A final simplification that we might make is, if there are multiple dots on one line in the region between intersections, we draw only a single dot and label this dot by the number of dots on the strand.

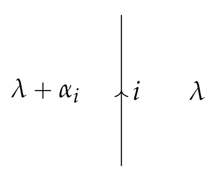


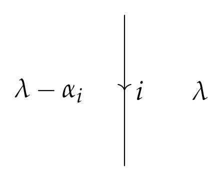

So now that there is a notion of a diagram that represents a 2-morphism, let's write down the generators and relations of the 2-morphisms.

### 2.2.6 Generators of 2-cells

This section is summarized in Appendix A. In case it wasn't clear in the previous section, diagrams are read bottom to top. The signed sequence on the bottom of the diagram comes from the domain of the 2-morphism, and the signed sequence on the top comes from the codomain.

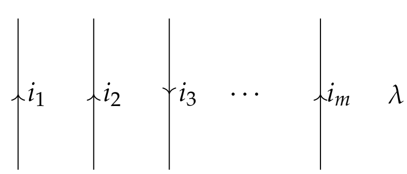
We start with the identity morphisms of a 1-cell $\mathcal{E}_{\pm i}1_\lambda\{t\}$. These are depicted by

$\mathrm{id}_{\mathcal{E}_{+i}1_\lambda\{t\}} =$ 

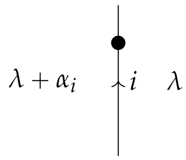
$\mathrm{id}_{\mathcal{E}_{-i}1_\lambda\{t\}} =$ 


(2.2.2)

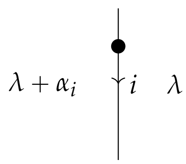
Both of these morphisms have degree shift zero. Horizontal composition of these gives the identity 2-morphism of $\mathcal{E}_{\mathbf{i}}1_\lambda\{t\}$

$\mathrm{id}_{\mathcal{E}_{\mathbf{i}}1_\lambda\{t\}} =$ 

The mysterious dots from the previous section represent degree shift morphisms, shifting the degree down by $(\alpha_i, \alpha_i)$.

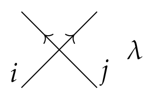



$$: \mathcal{E}_{+i}1_\lambda\{t\} \implies \mathcal{E}_{+i}1_\lambda\{t - (\alpha_i, \alpha_i)\} \tag{2.2.3}$$

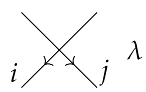



$$: \mathcal{E}_{-i}1_\lambda\{t\} \implies \mathcal{E}_{-i}1_\lambda\{t - (\alpha_i, \alpha_i)\} \tag{2.2.4}$$

There are also the crossover morphisms that shift degree by $-(\alpha_i, \alpha_j)$.

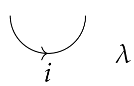



$$: \mathcal{E}_{+i+j}1_\lambda\{t\} \implies \mathcal{E}_{+j+i}1_\lambda\{t + (\alpha_i, \alpha_j)\} \tag{2.2.5}$$

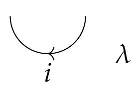



$$: \mathcal{E}_{-i-j}1_\lambda\{t\} \implies \mathcal{E}_{-j-i}1_\lambda\{t + (\alpha_i, \alpha_j)\} \tag{2.2.6}$$

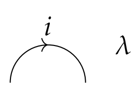
The final generators are the U-turns (these represent units and counits of adjunctions modulo degree shift between $\mathcal{E}_{+i}1_\lambda$ and $\mathcal{E}_{-i}1_{\lambda+\alpha_i}$ – see subsection 2.3.2). These have degree $c_{\pm i,\lambda} = \frac{1}{2}(\alpha_i, \alpha_i)(1 \pm \langle \alpha_i, \lambda \rangle)$.



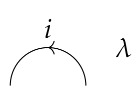

$$: 1_\lambda\{t\} \implies \mathcal{E}_{-i+i}1_\lambda\{t - c_{+i,\lambda}\} \tag{2.2.7}$$




$$: 1_\lambda\{t\} \implies \mathcal{E}_{+i-i}1_\lambda\{t - c_{-i,\lambda}\} \tag{2.2.8}$$




$$: \mathcal{E}_{+i-i}1_\lambda\{t\} \implies 1_\lambda\{t - c_{-i,\lambda}\} \tag{2.2.9}$$

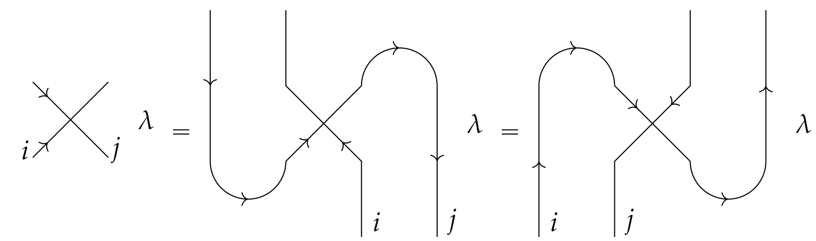



$$: \mathcal{E}_{-i+i}1_\lambda\{t\} \implies 1_\lambda\{t - c_{+i,\lambda}\} \tag{2.2.10}$$

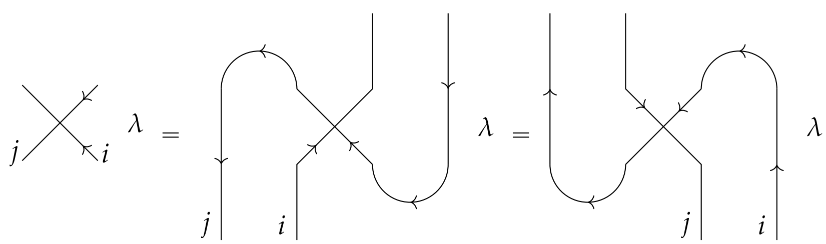
To remember whether or not the degree of one of these U-turns is $c_{+i,\lambda}$ or $c_{-i,\lambda}$, notice that the sign is a plus when the orientation is counterclockwise, and the sign is a minus when the orientation is clockwise.

This completes the list of generators of the 2-cells of $\mathcal{U}_q(\mathfrak{g})$, but it is notationally convenient to introduce another two crossing morphisms with the lines oriented in opposite directions. These are composites of other 2-cells already described, and will be (very!) convenient later.

**Definition 2.2.13** ([4, equations 3.11, 3.12])**.**



(2.2.11)

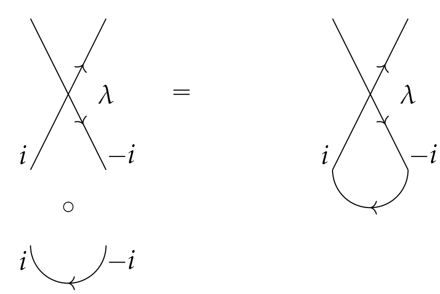


(2.2.12)

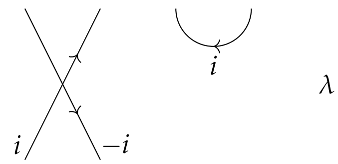
In each of the equations, the first equality defines the crossing and the second ones follow from relation (2.2.17) and (2.2.13), (2.2.14), below.

**Definition 2.2.14.** An element of $\mathcal{U}_q(\mathfrak{g})(\mathcal{E}_{\mathbf{i}}1_\lambda\{t\}, \mathcal{E}_{\mathbf{j}}1_\mu\{s\})$ is a 2-morphism of degree $s - t$ built from vertical and horizontal composition of the generators (2.2.2)-(2.2.10), subject to the diagrammatic relations of subsection 2.2.7. Vertical composition of these 2-morphisms is conjoining them vertically, stacking bottom to top. Horizontal composition is putting two diagrams next to each other.

**Example 2.2.15.** For example, the vertical composition of a U-turn and a crossover looks like



and their horizontal composition looks like

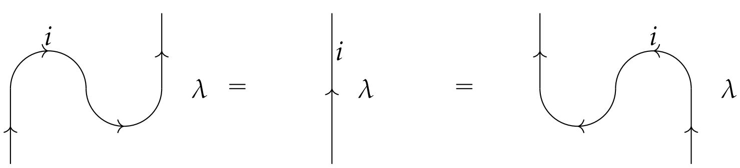



This isn't a complete description of 2-cells in $\mathscr{U}_q(\mathfrak{g})$, however. The 1-cells $\mathcal{E}_{\mathbf{i}}1_\lambda\{t\}$ are not all of the 1-cells in $\mathscr{U}_q(\mathfrak{g})$; in general, they are direct sums of 1-cells of this form. In any additive category, a morphism $\bigoplus_{i=1}^n A_i \to \bigoplus_{j=1}^m B_m$ in represented by an $m \times n$ matrix $F = (f_{ij})$ with $f_{ij} = \pi_i F \nu_j \colon A_j \to B_i$. Therefore, a general 2-cell in $\mathscr{U}_q(\mathfrak{g})$ is a actually matrix of the diagrams described in Definition 2.2.14 (just when you thought it couldn't get any worse).

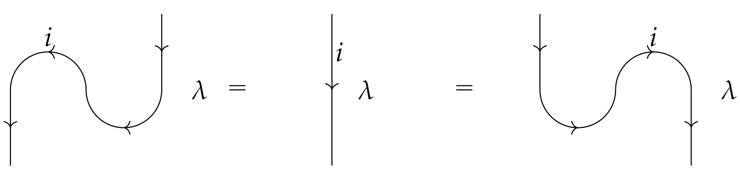
### 2.2.7 Relations between 2-cells

This section is summarized in Appendix B

This first set of relations expresses the fact that the 1-morphisms $\mathcal{E}_{+i}1_\lambda\{t\}$ and $\mathcal{E}_{-i}1_{\lambda+\alpha_i}\{s\}$ are biadjoint up to degree shifts – see subsection 2.3.2. These relations mirror the string diagrams for the triangular identities of an adjunction.



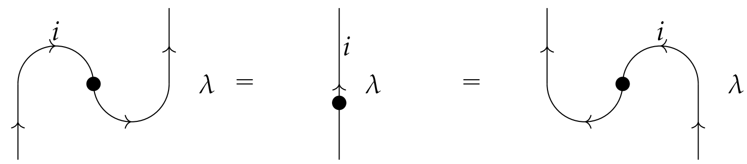

(2.2.13)



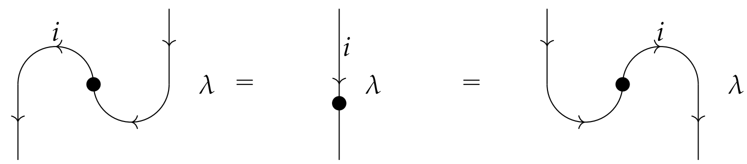

(2.2.14)

If for the moment we ignore degrees, biadjoint means that the two morphisms are both left and right adjoints of each other (see definition (2.3.8)). That is, $\mathcal{E}_i 1_\lambda \dashv \mathcal{E}_{-i}1_{\lambda+\alpha_i}$ and $\mathcal{E}_{-i}1_{\lambda+\alpha_i} \dashv \mathcal{E}_i 1_\lambda$, up to degree. Worrying about degrees, however, the biadjointness property fails because of degree shifts. Instead, we have a property that Khovanov and Lauda [4, Section 3.3.1] termed **almost biadjointness** – see subsection 2.3.2.

The relations (2.2.13) and (2.2.14) both hold also when there are dots on the strands.

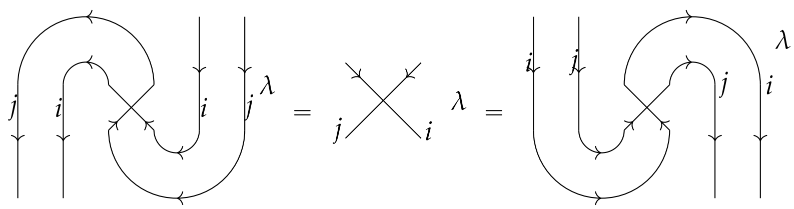


(2.2.15)



(2.2.16)

The following relation gives another use for the U-turn generators: turning around a crossing.

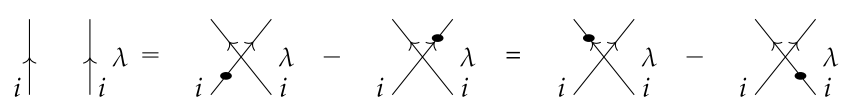


(2.2.17)

This relation, along with (2.2.15) and (2.2.16) is an expression that all 2-morphisms are "cyclic with respect to the biadjoint structure," see [6]. Essentially, this ensures that all diagrams related by planar isotopy represent the same 2-morphism in $\mathscr{U}_q(\mathfrak{g})$ [6, line following equation 3.68].

In [2, 3], Khovanov and Lauda defined algebras which they called **Quiver Hecke Algebras** and proved that the category of finite-dimensional projective modules over these algebras categorify ${}_{\mathcal{A}}\mathbf{f}$, or equivalently, ${}_{\mathcal{A}}U_q(\mathfrak{g})^{\pm}$. The quiver Hecke algebras are defined using diagrams not too dissimilar to the ones below, expressing relations between sequences of elements of $I$, but without orientations on the strands. Here, because the quiver Hecke algebras categorify half of the quantum group $\dot{U}_q(\mathfrak{g})$, we include the quiver Hecke algebra relations and give all strands the same orientations.

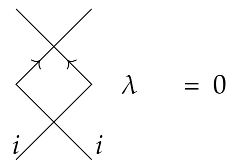
We have the dot slide relations.



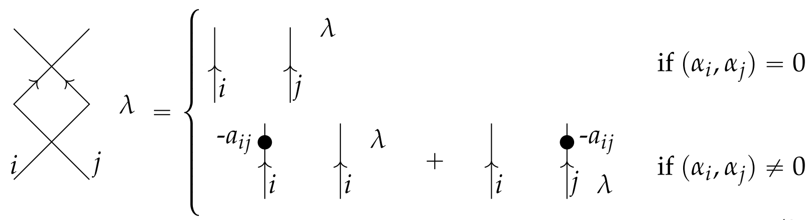
(2.2.18)


(2.2.19)

Various relations for untangling strands.

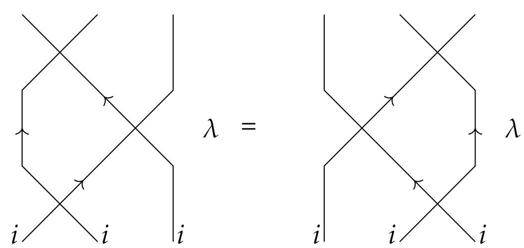



(2.2.20)



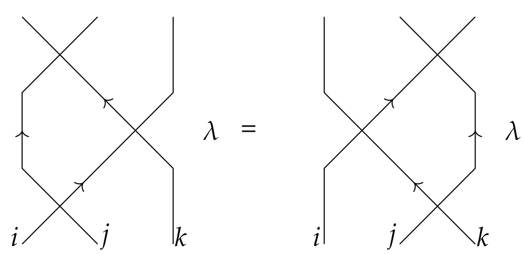

(2.2.21)

And various Reidemeister-move-like relations.



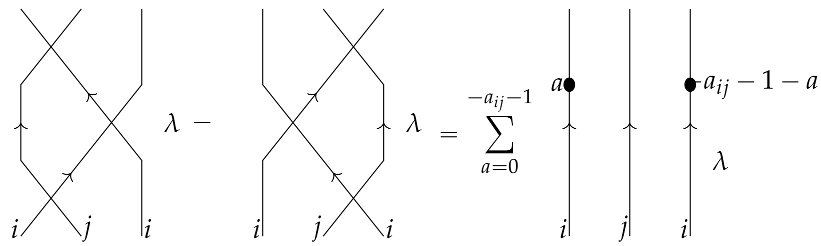

(2.2.22)

Unless both $i = k$ and $(\alpha_i, \alpha_j) \neq 0$.



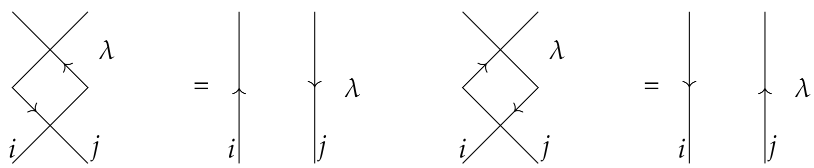

(2.2.23)

Otherwise, when $(\alpha_i, \alpha_j) \neq 0$,




(2.2.24)

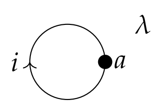
The next relations aren't quite quiver Hecke algebra relations because the strands aren't oriented in the same direction, but they express the fact that the two new crossover morphisms (2.2.11) and (2.2.12) are isomorphisms. This is crucially used to prove that $\dot{\mathscr{U}}_q(\mathfrak{g})$ categorifies $\dot{U}_q(\mathfrak{g})$ in Lemma 3.2.2.




(2.2.25)

The first nonintuitive relation that we impose is to say that all dotted bubbles of negative degree are just zero morphisms.

**Definition 2.2.16.** A **dotted bubble** is a composite of U-turns and degree shift morphisms that forms a circle, oriented either counterclockwise or clockwise, with some number of dots on it (remember, by relations (2.2.15) and (2.2.16) it doesn't matter where we put the dots on a strand, so we let them freely slide around the bubble).



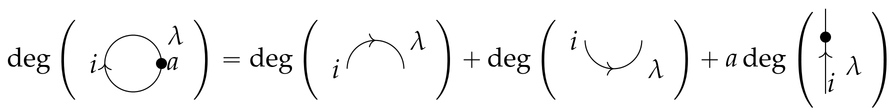

(2.2.26)

This composite represents a morphism from $1_\lambda$ to itself, possibly with some degree shift.

We can very precisely determine when a dotted bubble has negative degree. The proof of this proposition is easy, but it is included to give an idea of how to work with degrees of 2-cells in $\mathscr{U}_q(\mathfrak{g})$.

**Proposition 2.2.17.** *A clockwise dotted bubble as in* (2.2.26) *has negative degree precisely when* $a < \langle \alpha_i, \lambda \rangle - 1$. *A counterclockwise dotted bubble has negative degree precisely when* $a < -\langle \alpha_i, \lambda \rangle - 1$.

*Proof.* Let's start with the clockwise dotted bubble pictured in (2.2.26). We count the degree of this 2-cell as follows (recall that the degree is the sum of the degrees of the generators that compose the bubble).



$$
\begin{aligned}
&= \tfrac{(\alpha_i,\alpha_i)}{2}\,(1-\langle\alpha_i,\lambda\rangle) + \tfrac{(\alpha_i,\alpha_i)}{2}\,(1-\langle\alpha_i,\lambda\rangle) + a(\alpha_i,\alpha_i) \\
&= (\alpha_i,\alpha_i)(1-\langle\alpha_i,\lambda\rangle + a)
\end{aligned}
$$

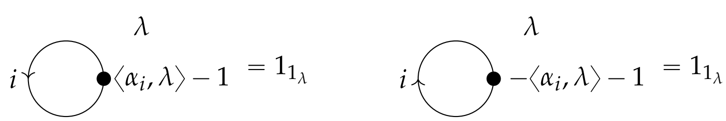
Now $(\alpha_i, \alpha_i)$ is always positive, because the $\alpha_i$'s are simple roots of $\mathfrak{g}$. Hence, the degree of this dotted bubble is negative only when $1 - \langle\alpha_i, \lambda\rangle + a < 0$, or when $a < \langle\alpha_i, \lambda\rangle - 1$.

The case where the bubble is instead oriented clockwise is similar, but instead there is a $1 + \langle\alpha_i, \lambda\rangle$ term instead of $1 - \langle\alpha_i, \lambda\rangle$. □

In light of this proposition, we impose the following relations, which demand that dotted bubbles of negative degree are zero.

(2.2.27)

We also demand that the dotted bubbles of degree zero are identity morphisms $1_\lambda \implies 1_\lambda$. Proposition 2.2.17 tells us how many dots we must put on the bubble to make this the case (in more sophisticated language, it tells us the exact degree shift needed so that the morphism has overall degree zero).

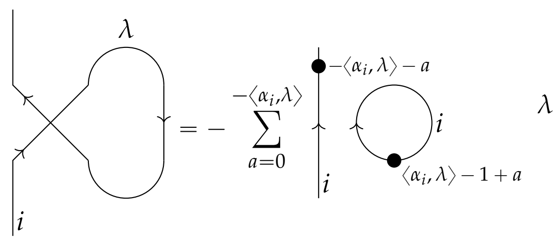


(2.2.28)

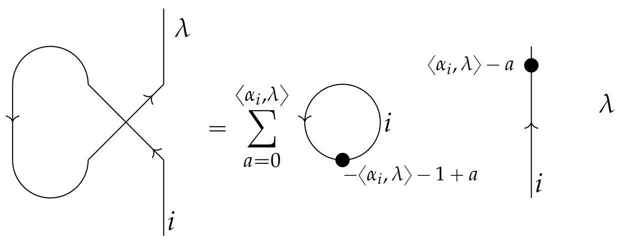
According to [6], this condition on negative degree bubbles is imposed for two reasons. First, it is enforced so that the dimension of the space of 2-cells in each degree is finite-dimensional. Moreover, "this is further justified by the fact that negative degree bubbles act by zero in the action of $\mathcal{U}_q(\mathfrak{g})$ on cohomology rings of flag varieties" [6, two paragraphs preceding equation 3.38].

Now that we've gone through all of relations that are easy to motivate, we are left with these others below. (2.2.31) and (2.2.32) are crucial to showing that

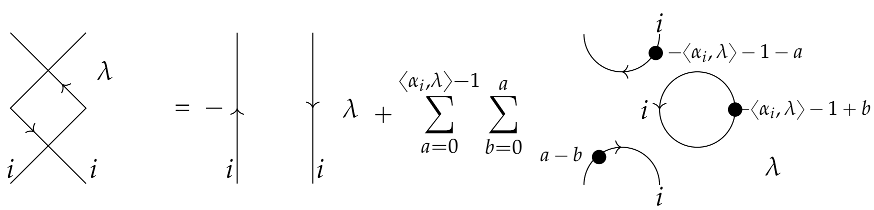
$\dot{\mathcal{U}}_q(\mathfrak{g})$ is a categorification of $\dot{U}_q(\mathfrak{g})$ in Lemma 3.2.3.



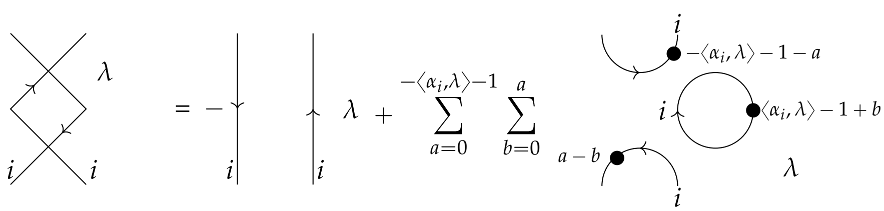

(2.2.29)




(2.2.30)




(2.2.31)




(2.2.32)

Note that the summations on the right hand side of the above four equations may vanish according to Definition 2.2.6. This is particularly important for the proof of Lemma 3.2.3.

These relations require a bit of explaining. They were originally developed by Lauda as part of the categorification of $\mathfrak{sl}(2)$ in [7], and have made their way into the categorification of $\mathfrak{g}$ as relations between the 1-morphisms associated to only a single simple root. The motivation for these relations comes from counting the graded dimensions of the spaces of 2-homs.

In the representation theory of finite groups, the inner product between characters is categorified by the hom-spaces between representations. There is an isomorphism between the representation ring of $G$ (the Grothendieck group of the category of representations of $G$) and the ring of class functions on $G$, under which the inner product between characters corresponds to the dimension of $\mathrm{Hom}_{CG}(V, W)$. In this sense, the inner product is categorified by hom-spaces in representation theory.

In the context of $\mathcal{U}_q(\mathfrak{g})$, the semilinear form on $\dot{U}_q(\mathfrak{g})$ will be categorified by the graded-hom between 2-cells.

**Definition 2.2.18.** For any morphisms $X, Y$ with the same domain and codomain, we define

$$\mathrm{grHom}(X, Y) = \bigoplus_{t \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{U}_q(\mathfrak{g})}(X\{t\}, Y).$$

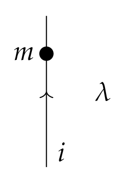
The **graded dimension** of one of these graded homs is the generating function for the dimensions of the direct summands.

$$\mathrm{grdim}\,\mathrm{grHom}(X, Y) = \sum_{t \in \mathbb{Z}} \dim \mathrm{Hom}_{\mathcal{U}_q(\mathfrak{g})}(X\{t\}, Y) \cdot q^t$$

By analogy to the situation of representations of finite groups, this graded dimension should match the value of the $\dot{U}_q(\mathfrak{g})$ semilinear form on $X$ and $Y$; this is what Lauda was trying to achieve when constructing the categorification of $\mathfrak{sl}(2)$ in [7]. The relation (2.2.29) depicts an equality of 2-cells $\mathcal{E}_{+i}1_\lambda\{t\} \implies \mathcal{E}_{+i}1_\lambda\{t-d\}$, where $d = (\alpha_i, \alpha_i)(2 - \langle \alpha_i, \lambda \rangle)$. Therefore, this is a relation within $\mathrm{grHom}(\mathcal{E}_{+i}1_\lambda, \mathcal{E}_{+i}1_\lambda)$. The graded dimension of this graded-hom should match the value of the semilinear form on $E_{+i}1_\lambda$, $E_{+i}1_\lambda$

$$\langle E_{+i}1_\lambda, E_{+i}1_\lambda \rangle = \frac{1}{(1-q_i)^2} = 1 + q_i^2 + q_i^4 \ldots = 1 + q^{(\alpha_i, \alpha_i)} + q^{2(\alpha_i, \alpha_i)} + \ldots \tag{2.2.33}$$

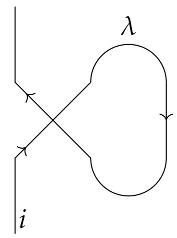
This means that the dimension of $\mathcal{U}_q(\mathfrak{g})(\mathcal{E}_{+i}1_\lambda\{t\}, \mathcal{E}_{+i}1_\lambda)$ should be 1 if $t = m(\alpha_i, \alpha_i)$ for some nonnegative integer $m$, or zero otherwise. We already have one 2-cell in this space: the degree shift morphism




(2.2.34)

spans $\mathcal{U}_q(\mathfrak{g})(\mathcal{E}_{+i}1_\lambda\{m(\alpha_i, \alpha_i)\}, \mathcal{E}_{+i}1_\lambda)$.

But this leaves us with a problem. According to the value of the semilinear form (2.2.33), we have identified all of the morphisms $\mathcal{E}_{+i}1_\lambda\{t\} \implies \mathcal{E}_{+i}1_\lambda$. Yet there is another morphism, namely the left hand side of (2.2.29).




(2.2.35)

Therefore, we need a relation that writes this morphism as a linear combination of the degree shift morphisms (2.2.34). Of course, we may also horizontally compose these 2-cells with 2-cells $1_\lambda \implies 1_\lambda$. The exact choice of the relations (2.2.29) and (2.2.30) was derived using a categorical action of $\mathfrak{sl}(2)$ on the cohomology of partial flag varieties and then generalized to $\mathcal{U}_q(\mathfrak{g})$ [6, Section 3.4]. Similarly, the need for relations (2.2.31) and (2.2.32) was discovered using the same dimension counting argument, and they are exactly what we need for the categorification homomorphism to respect the commutator relation – see Lemma 3.2.3.

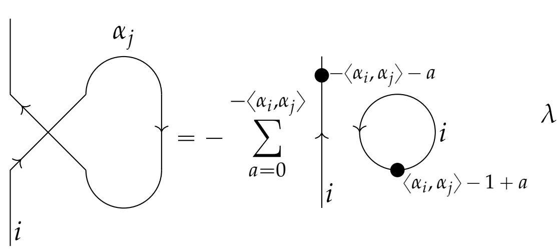
**Remark 2.2.19.** If we were to ignore the grading on 1-morphisms and omit degree shifts of the 2-morphisms, then we obtain a categorification of the non-quantum idempotented universal enveloping algebra $\dot{U}(\mathfrak{g})$. As with most categorifications, the addition of a grading corresponds to moving from a categorification of algebras over $k$ to algebras over $k(q)$.

## 2.3 Working with $\mathcal{U}_q(\mathfrak{g})$

In this section, we record properties of $\mathcal{U}_q(\mathfrak{g})$ and some more relations among the 2-cells that will be used to prove that $\dot{\mathcal{U}}_q(\mathfrak{g})$ categorifies $\dot{U}_q(\mathfrak{g})$.

### 2.3.1 Fake bubbles

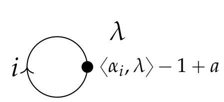
There is something suspicious in the relations (2.2.29), (2.2.30), (2.2.31), (2.2.32). For example, take a look at (2.2.30) for $\lambda = \alpha_j$ for some $j \neq i$.



Then all terms, for any value of $a = 0, \ldots, -\langle \alpha_i, \alpha_j \rangle$ have a dotted bubble with a negative number on it's dot. Recall that a dot labelled with the number $m$ means the $m$-fold composition of a degree shift 2-cell (2.2.3) or (2.2.4). So a negative number on the dot means composing a degree shift morphism with itself a negative number of times.

The same problem occurs in relations (2.2.29), (2.2.31) and (2.2.32).

Nevertheless, these are the relations that Lauda needed to impose to make sense of the categorical action of $\mathfrak{sl}(2)$ in a geometrical context [6, Section 3.4], so we will try to work with them as best as we can. Notice that the overall degree of these negative dotted bubbles is positive in this case, namely,

deg



$$= (\alpha_i, \alpha_i)\,(1 - \langle \alpha_i, \lambda \rangle) + (\langle \alpha_i, \lambda \rangle - 1 + a)(\alpha_i, \alpha_i)$$
$$= a(\alpha_i, \alpha_i) \geqslant 0$$

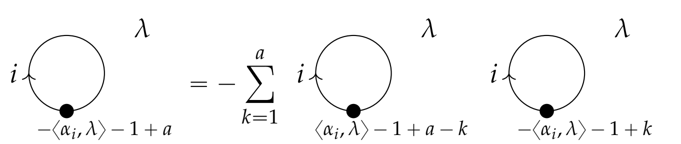
since both $a \geqslant 0$, $(\alpha_i, \alpha_i) > 0$.

The overall degree of these bubbles is similarly positive in (2.2.29), (2.2.31) and (2.2.32) as well. This tells us that the existence of these weird specimens doesn't contradict the relation (2.2.27) that tells us dotted bubbles of negative degree are zero.

To overcome the weirdness of this situation, we add these negative dotted bubbles to the category as formal symbols in $\mathrm{grHom}(1_\lambda, 1_\lambda)$, and call them **fake bubbles**.

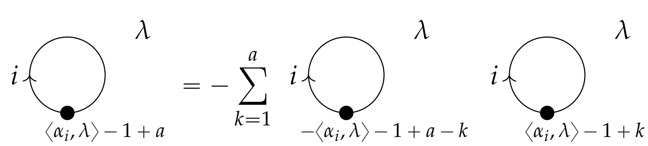
**Definition 2.3.1.** The **fake bubbles** are formal symbols in $\mathrm{grHom}(1_\lambda, 1_\lambda)$ defined inductively by the following rules

- Fake bubbles of degree zero are 1, as in the relation (2.2.28).
- if $\langle \alpha_i, \lambda \rangle \geqslant 0$, then




(2.3.1)

- if $\langle \alpha_i, \lambda \rangle < 0$, then

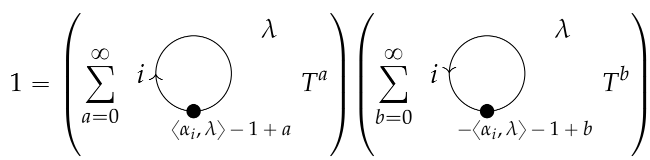


All of the orientations in these formulas above may be reversed to define the clockwise-oriented fake bubbles.

In the above definition, notice that the bubbles on the right hand side of (2.3.1) are not all fake.

This definition of fake bubbles can be encapsulated in the following nifty equation, called the **infinite Grassman relation**, because it appears as the limit of the defining equation of the cohomology ring of Grassmannians [6, equation 3.62]. In fact, this relation holds for any bubbles, not just fake ones; see Proposition 2.3.17. This is where the inductive definition of the fake bubbles comes from.

**Proposition 2.3.2.** *The fake bubbles are uniquely determined by the equation in the formal variable $T$ (that is, an equation in the power series ring $\mathscr{U}_q(\mathfrak{g})(1_\lambda, 1_\lambda)[[T]]$.)*




(2.3.2)

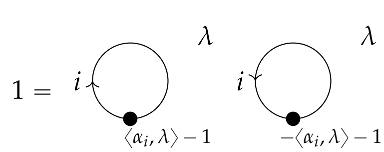
*and the relation* (2.2.28). *Note that the orientation of the bubbles is clockwise in the left hand term and counterclockwise in the right hand term.*

*Proof.* To establish this claim, we have to reproduce Definition 2.3.1 from the above equation. Notice that if $\langle \alpha_i, \lambda \rangle \geqslant 0$, then the left hand term in (2.3.10) consists of entirely fake bubbles, while the right hand term is the "real" bubbles. Otherwise, if $\langle \alpha_i, \lambda \rangle < 0$, then the right hand side is the fake bubbles and the left hand side is the real ones. The two cases are symmetric, so let's assueme that $\langle \alpha_i, \lambda \rangle \geqslant 0$ to match the equation (2.3.1) we wrote above.

In the case that $\langle \alpha_i, \lambda \rangle = 0$, the coefficient bubbles of $T^a$ and $T^b$ for $a, b \geqslant 1$ have negative degree, and therefore vanish by (2.2.27). The two coefficients of $T^0$ both have degree zero, and therefore are identities and (2.3.10) holds.

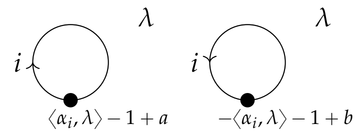
So it remains to show the proposition for $\langle \alpha_i, \lambda \rangle > 0$. We compare coefficients of $T^k$ on both sides of the infinite Grassman equation.

For $k = 0$, we have the equation

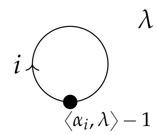



which holds by (2.2.28) because both of these dotted bubbles have degree zero.

For $k > 0$, we have the equation

$\sum_{a+b=k}$  $= 0.$ (2.3.3)

But for the term $a = 0$, the bubble



has degree zero, and hence is equal to 1. So we may move the remainder of the terms in (2.3.3) to the other side of the equation to recover (2.3.1). □

### 2.3.2 Almost biadjoints

In Example 1.4.3, we drew a picture of the category $\dot{U}_q(\mathfrak{sl}(2))$ in which $E^+1_n$ and $E^-1_n$ are morphisms between the elements of the weight lattice. A categorical action of $\mathfrak{sl}(2)$ replaces each weight of $\mathfrak{sl}(2)$ by a $k$-linear category $\mathscr{V}_n$ and each $E^{\pm}1_n$ by a functor between these categories. In the categorical actions coming from geometry, it is often the case that these two functors are both left and right adjoints of each other. Therefore, to define $\mathscr{U}_q(\mathfrak{g})$ such that 2-functors $\mathscr{U}_q(\mathfrak{g}) \to \mathbf{Cat}$ correspond to these higher representations of $\mathfrak{g}$, we need a notion of what it means for 1-morphisms in 2-categories to be adjoint to each other.

Since the prototypical example of a 2-category is the category **Cat** of small categories, it makes sense to generalize from **Cat** to arbitrary 2-categories. One of the ideas that is particularly important for categorification of quantum groups is the notion of adjunctions internal to a 2-category. The idea is to replace categories by 0-cells, functors by 1-cells, and natural transformations by 2-cells in the definition of an adjunction. In most cases, facts about adjunctions in **Cat** are also true for adjunctions internal to 2-categories.

**Definition 2.3.3.** An **adjunction internal to a 2-category** $\mathfrak{C}$, written $f \dashv g$, is a pair of 1-cells $f\colon A \to B$ and $g\colon B \to A$, together with 2-cells $\eta\colon 1_A \implies gf$ and $\varepsilon\colon fg \implies 1_B$, called the **unit** and **counit** of the adjunction, respectively. The unit and counit must satisfy the triangular identities below.

$$
\begin{tikzcd}
A \arrow[rr, bend left=60, "f"] \arrow[rr, bend right=60, "f"'] \arrow[rr, "fgf" description] \arrow[rr, phantom, shift left=3ex, "\Downarrow 1_f\cdot\eta"] \arrow[rr, phantom, shift right=3ex, "\Downarrow \varepsilon\cdot 1_f"] & & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow 1_f"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[rr, bend left=60, "g"] \arrow[rr, bend right=60, "g"'] \arrow[rr, "gfg" description] \arrow[rr, phantom, shift left=3ex, "\Downarrow \eta\cdot 1_g"] \arrow[rr, phantom, shift right=3ex, "\Downarrow 1_g\cdot\varepsilon"] & & A
\end{tikzcd}
=
\begin{tikzcd}
B \arrow[r, bend left, "g"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow 1_g"] & A
\end{tikzcd}
\tag{2.3.4}
$$

This definition makes it obvious that adjunctions internal to a 2-category are preserved by any 2-functor $\mathfrak{F}\colon \mathfrak{C} \to \mathfrak{D}$, because the definition is an equation of 2-cells, each of which is preserved by a 2-functor.

When we talk about an adjunction $F \dashv G$ in **Cat**, it is often useful to talk about the natural bijection between hom-sets. However, since the 0-cells in $\mathfrak{C}$ have no notion of either objects or arrows, we must instead reason about the "generalized objects" of the 0-cells. This is similar to the idea of a "generalized element" in the definition of a monomorphism $f$. A monomorphism is a generalization of an injective function, and the definitions look very similar. A function $f\colon X \to Y$ is injective if $f(x) = f(x') \implies x = x'$ for all $x, x' \in X$; a morphism $f\colon X \to Y$ is monic if $fx = fx' \implies x = x'$ for all $x, x'\colon Z \to X$. We do the same thing for generalized objects. An object $X$ of a category $\mathscr{A}$ can be realized as a functor $\xrightarrow{X} \mathscr{A}$; a generalized object of the 0-cell $A$ inside $\mathfrak{C}$ is then described as any 1-cell $X \to A$ for any other 0-cell $X$. This idea of generalized elements is used to formulate a definition of an adjunction internal to $\mathfrak{C}$ analogous to the definition in **Cat** using a natural bijection between hom-sets.

**Lemma 2.3.4.** *If $f \dashv g$, is an adjunction internal to $\mathfrak{C}$ between $f\colon A \to B$ and $g\colon B \to A$, then there is a bijection between 2-cells $fa \implies b$ and $a \implies gb$ for any $a\colon X \to A$ and $b\colon X \to B$.*

*Proof.* Given a 2-cell $\alpha\colon fa \implies b$, construct $\beta\colon a \implies gb$ by

$$\beta = (1_g \cdot \alpha)(\eta \cdot 1_a).$$

Conversely, given a 2-cell $\beta\colon a \implies gb$, construct $\alpha\colon fa \implies b$ by

$$\alpha = (\varepsilon \cdot 1_b)(1_f \cdot \beta).$$

We can check that these constructions are inverse using the triangle laws. Since the constructions are dual, we need only check that they are inverse in one direction; the other will follow.

Given $\alpha\colon fa \implies b$, we first construct $\beta = (1_g \cdot \alpha)(\eta \cdot 1_a)$. We want to show that the 2-cell $(\varepsilon \cdot 1_b)(1_f \cdot \beta)$ constructed from this $\beta$ is equal to $\alpha$.

$$\begin{aligned}
(\varepsilon \cdot 1_b)(1_f \cdot \beta) &= (\varepsilon \cdot 1_b)(1_f \cdot ((1_g \cdot \alpha)(\eta \cdot 1_a))) \\
&= (\varepsilon \cdot 1_b)(1_f \cdot 1_g \cdot \alpha)(1_f \cdot \eta \cdot 1_a) \\
&= (\varepsilon \cdot 1_b)(1_{fg} \cdot \alpha)(1_f \cdot \eta \cdot 1_a) \\
&= (\alpha)(\varepsilon \cdot 1_{fa})(1_f \cdot \eta \cdot 1_a) \qquad (2.3.5) \\
&= (\alpha)((\varepsilon \cdot 1_f)(1_f \cdot \eta) \cdot 1_a) \\
&= \alpha \cdot (1_f \cdot 1_a) \\
&= \alpha \cdot (1_{fa}) \\
&= \alpha
\end{aligned}$$

In each step, we used an application of the interchange law to distribute 2-cells. The particularly tricky step is (2.3.5), where we used the following lemma, which follows from the interchange law. □

**Lemma 2.3.5.** *Given any diagram*

$$\begin{tikzcd}
A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\phi"] & B \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\psi"] & C
\end{tikzcd}$$

*we have that* $(\psi \cdot 1_g)(1_h \cdot \phi) = (1_k \cdot \phi)(\psi \cdot 1_f)$.

In fact, this is equivalent to Definition 2.3.3

**Lemma 2.3.6.** *Suppose that* $f \colon A \to B$ *and* $g \colon B \to A$ *have the property that, for each pair of* 1*-cells* $a \colon X \to A$ *and* $b \colon X \to B$*, there is a bijection* $\Phi_{a,b}$ *between* 2*-cells* $fa \implies b$ *and* $a \implies gb$*.*

*Assume further that* $\Phi$ *is natural in both* $a$ *and* $b$*, in the sense that*

$$\Phi_{a,b}(\chi \circ (1_f \cdot \alpha)) = \Phi_{a',b}(\chi) \circ \alpha \qquad (1_g \cdot \beta) \circ \Phi_{a,b}(\theta) = \Phi_{a,b'}(\beta \circ \theta)$$

*for any* $\chi \colon fa' \implies b$ *and* $\alpha \colon a \implies a'$*, and for any* $\theta \colon fa \implies b$ *and* $\beta \colon b \implies b'$*.*

*Then* $f \dashv g$*.*

*Proof.* We need to produce a unit and a counit for the adjunction. To that end, define the unit $\eta = \Phi_{1_A,f}(1_f) \colon 1_A \implies gf$ and $\varepsilon = \Phi^{-1}_{g,1_B}(1_g) \colon fg \implies 1_B$. To show that $f \dashv g$, we need to check that the proposed unit and counit satisfy the triangular identities (2.3.4).

We need only check one of these identities; checking the other triangular identity is dual. We have that

$$\begin{aligned}(\varepsilon \cdot 1_f) \circ (1_f \cdot \eta_A) &= (\Phi^{-1}_{g,1_A}(1_g) \cdot 1_f) \circ (1_f \cdot \Phi_{1_A,f}(1_f)) \\ &= \Phi^{-1}_{gf,f}(1_{gf}) \circ (1_f \cdot \eta) \\ &= \Phi^{-1}_{1_A,f}(\eta \circ 1_{gf}) \\ &= \Phi^{-1}_{1_A,f}(\eta) \\ &= \Phi^{-1}_{1_A,f}(\Phi_{1_A,f}(1_f)) \\ &= 1_f\end{aligned}$$

And this is one of the triangular identities. □

**Proposition 2.3.7.** *Let $f\colon A \to B$ and $g\colon B \to A$ be 1-cells in $\mathfrak{C}$. Then the following are equivalent:*

*(a) an adjunction $f \dashv g$ with unit $\eta$ and counit $\varepsilon$;*

*(b) a bijection between 2-cells $fa \implies g$ and $a \implies gb$, natural in both $a\colon X \to A$ and $b\colon X \to B$.*

*In either case, $f \dashv g$.*

*Proof.* $(a) \implies (b)$ is Lemma 2.3.4. $(b) \implies (a)$ is Lemma 2.3.6. □

**Definition 2.3.8.** In a 2-category $\mathfrak{C}$, morphisms $f\colon A \to B$ and $g\colon B \to A$ are **biadjoint** if both $f \dashv g$ and $g \dashv f$.

In $\mathscr{U}_q(\mathfrak{g})$, we have an adjunction

$$\mathcal{E}_{+i}1_\lambda\{t\} \dashv \mathcal{E}_{-i}1_{\lambda+\alpha_i}\{t - c_{-i,\lambda}\}$$

with unit (2.2.7) and counit (2.2.9), and similarly an adjunction

$$\mathcal{E}_{-i}1_{\lambda+\alpha_i}\{-t + c_{+i,\lambda}\} \dashv \mathcal{E}_{+i}1_\lambda\{t\}$$

with unit (2.2.8) and counit (2.2.10). The triangular identities of the adjunction are expressed by (2.2.13) and (2.2.14). Notice that the 2-morphisms that express the triangular laws are degree-preserving. Therefore, they preserve the degrees of both the right and left adjoints.

Although the $\mathcal{E}_{+i}1_\lambda$ and $\mathcal{E}_{-i}1_{\lambda+\alpha_i}$ morphisms are not quite biadjoint to each other, they are biadjoint up to degree shifts. Khovanov and Lauda coined the term **almost biadjoints** in [4] to describe this situation.

Although they are not biadjoint, we do have a chain of adjunctions of the degree shifted morphisms $\mathcal{E}_{+i}1_\lambda$ for any $i$, $\lambda$, and $t \in \mathbb{Z}$.

$$\cdots \dashv \mathcal{E}_{-i}1_{\lambda+\alpha_i}\{-t+c_{+i,\lambda}\} \dashv \mathcal{E}_i 1_\lambda\{t\} \dashv \mathcal{E}_{-i}1_{\lambda+\alpha_i}\{-c_{+i,\lambda}\} \dashv \mathcal{E}_{+i}1_\lambda\{2c_{+i,\lambda} + t\} \dashv \cdots$$

This then begs the question: does any 1-morphism of the form $\mathcal{E}_{\mathbf{i}}1_\lambda$ have a left and right adjoint, for any signed sequence $\mathbf{i}$? The answer is yes, and we can determine both a left and right adjoint using the following.

**Lemma 2.3.9.** *If we have the following commutative diagram of objects and* 1*-morphisms in a* 2*-category* $\mathfrak{C}$

$$\begin{tikzcd} A \arrow[r, shift left, "f"] & B \arrow[l, shift left, "g"] \arrow[r, shift left, "h"] & C \arrow[l, shift left, "k"] \end{tikzcd}$$

*such that* $f \dashv g$ *and* $h \dashv k$*, then* $hf \dashv gk$*.*

*Proof.* By Proposition 2.3.7, there are bijections between 2-cells $fa \implies b$ and $a \implies gb$ for $a\colon X \to A$ and $b\colon X \to B$, and similarly, between 2-cells $hb \implies c$ and $b \implies kc$ for $c\colon X \to C$. We can compose them to get a bijection between 2-cells $hfa \implies c$ and $a \implies gkc$. This defines the adjunction $hf \dashv gk$. □

**Lemma 2.3.10.** *Any* 1*-morphism* $\mathcal{E}_{\mathbf{i}}1_\lambda\{t\}$ *has both a left and right adjoint*

*Proof.* The 1-morphism $\mathcal{E}_{\mathbf{i}}1_\lambda\{t\}$ is the composition of finitely many morphisms, each of the form $\mathcal{E}_{\pm i_j}1_{\lambda_j}\{t_j\}$. We know that each of these morphisms has both a left and right adjoint, so the left and right adjoints for $\mathcal{E}_{\mathbf{i}}1_\lambda$ can be constructed using Lemma 2.3.9. □

A general 1-morphism in $\mathcal{U}_q(\mathfrak{g})$, however, is a formal finite direct sum of morphisms of the form $\mathcal{E}_{\mathbf{i}}1_\lambda\{t\}$. Knowing the following lemma, we can say that any 1-morphism in $\mathcal{U}_q(\mathfrak{g})$ has both left and right adjoints.

**Lemma 2.3.11.** *Let* $\mathfrak{C}$ *be an additive* 2*-category. Let* $h, f\colon A \to B$ *and* $g, k\colon B \to A$ *such that* $f \dashv g$ *and* $h \dashv k$*. Then* $f \oplus h \dashv g \oplus k$*.*

*Proof.* Let $a\colon X \implies A$ an $b\colon X \implies b$ be 2-cells. Denote by $\mathrm{Hom}_{\mathfrak{C}}(x, y)$ the collection of 2-cells between 1-cells $x$ and $y$. The fact that $f \dashv g$ gives a bijection $\mathrm{Hom}_{\mathfrak{C}}(fa, b) \cong \mathrm{Hom}_{\mathfrak{C}}(a, gb)$, and similarly, $h \dashv k$ gives a bijection $\mathrm{Hom}_{\mathfrak{C}}(ha, b) \cong \mathrm{Hom}_{\mathfrak{C}}(a, kb)$. Then we may compose natural bijections to get a natural bijection

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{C}}((f \oplus h)a, b) &\cong \mathrm{Hom}_{\mathfrak{C}}(fa \oplus ha, b) \\ &\cong \mathrm{Hom}_{\mathfrak{C}}(fa, b) \oplus \mathrm{Hom}_{\mathfrak{C}}(ha, b) \\ &\cong \mathrm{Hom}_{\mathfrak{C}}(a, gb) \oplus \mathrm{Hom}_{\mathfrak{C}}(a, kb) \\ &\cong \mathrm{Hom}_{\mathfrak{C}}(a, gb \oplus kb) \\ &\cong \mathrm{Hom}_{\mathfrak{C}}(a, (g \oplus k)b). \end{aligned}$$

This defines, by Proposition 2.3.7, an adjunction $f \oplus h \dashv g \oplus k$. □

**Proposition 2.3.12.** *Every* 1*-morphism in* $\mathcal{U}_q(\mathfrak{g})$ *has both a left adjoint and right adjoint.*

*Proof.* This immediately follows from Lemma 2.3.9, Lemma 2.3.10 and Lemma 2.3.11. $\square$

### 2.3.3 Identities, isomorphisms, and more relations

In this section, we collect a few more facts about the 2-category $\mathcal{U}_q(\mathfrak{g})$ which will either prove useful later or illustrate how to work with the objects in this 2-category.

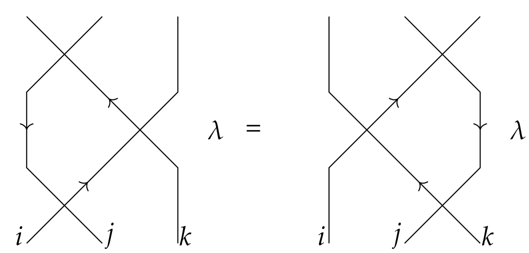
First, notice that the relations defining $\mathcal{U}_q(\mathfrak{g})$ give relations between certain configurations only when these 2-cells have all strands oriented up. We can use relations (2.2.17) and (2.2.15), (2.2.16) to turn over the crossings and define the same relations upside-down. This is recorded in the following proposition.

**Proposition 2.3.13.** *If a relation holds between two* 2*-cells in* $\mathcal{U}_q(\mathfrak{g})$*, then it also holds between the same diagrams when turned upside-down.*

But what if the strands of the diagram are oriented in different directions? Can we deduce, for instance, a relation like (2.2.23) except where the middle strand is oriented in the opposite direction? This would match our geometric intuition, where we can pull one strand over a crossing regardless of orientation.

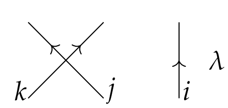
**Remark 2.3.14.** The purpose of including the following proposition is to correct or expand on the proof given in [4, Proposition 3.5]. The proof of this that appeared there is either wrong or incomplete; the version below hopefully clarifies the proof they attempted to describe.

**Proposition 2.3.15** ([4, Proposition 3.5])**.** *If* $i, j, k$ *are not all the same, then*

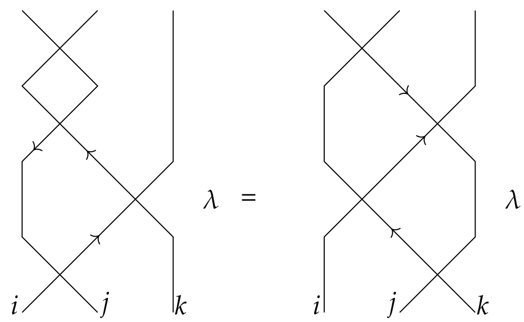



(2.3.6)

*Proof.* First assume that $j \neq k$. Then postcompose the both sides of (2.3.6) with the isomorphism

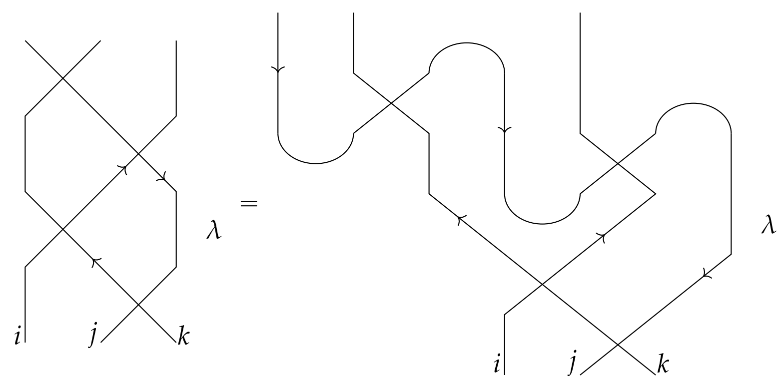


We know that this is an isomorphism by (2.2.25). So it suffices to show that

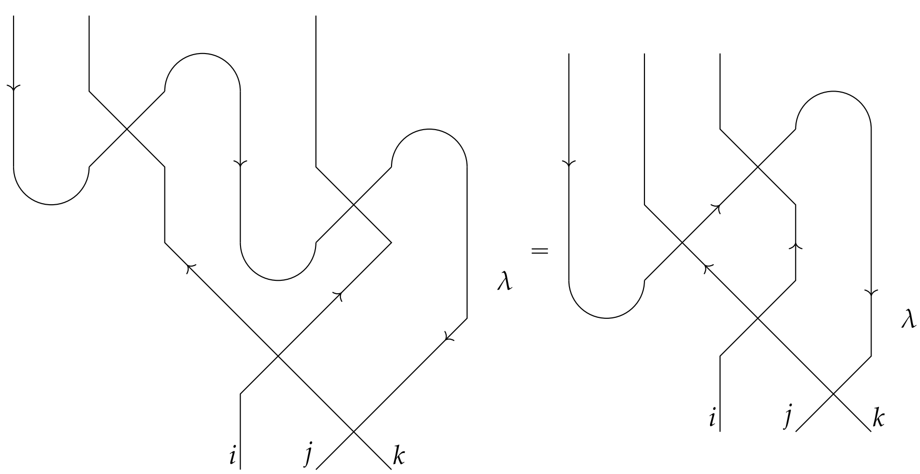


(2.3.7)

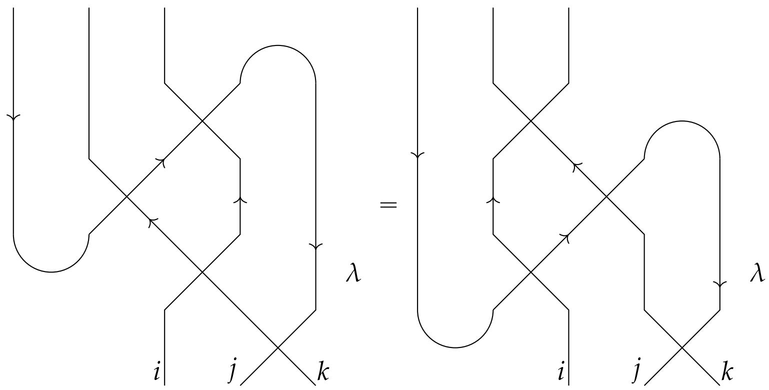
To that end, the first step is to take the right hand side of the equation and use Definition 2.2.13 to orient all of the strands in the same direction across crossings.



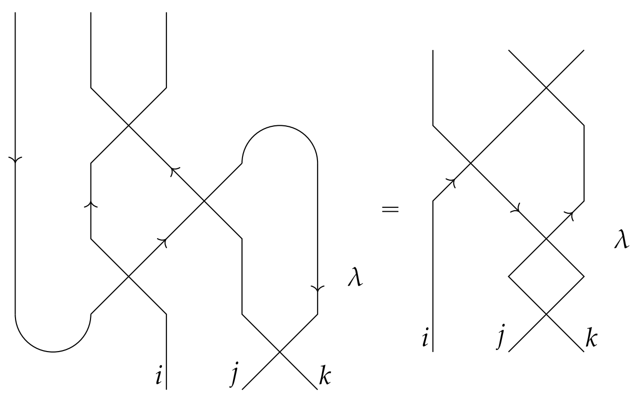
Then we can use (2.2.13) to straighten out the kink in the $j$-strand.



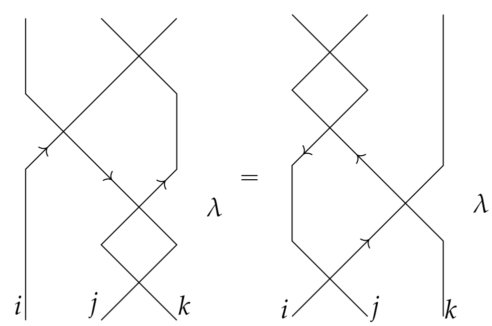
Then we can use (2.2.23) to cross the $i$-strand over the crossing of the $j$ and $k$ strands.




Then again use the definition Definition 2.2.13 to remove the U-turns.

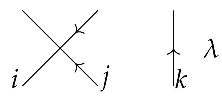



Finally, apply (2.2.25) twice, to the $j$ and $k$ stands at the top to cross them over twice and to the $j$ and $k$ strands at the bottom to untangle them. (2.2.25) is applicable since $j \neq k$. t

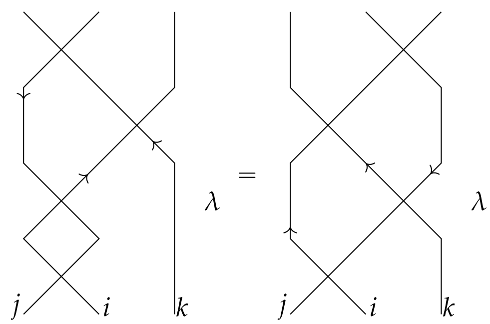


This is the left hand side of (2.3.7), so we have shown (2.3.6) in the case that $j \neq k$.

The other case is if $i \neq j$. In this case, pre-compose both sides with the isomorphism

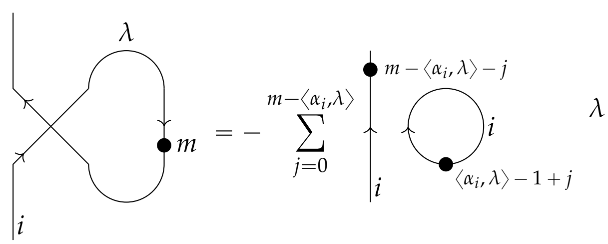



Again, this is indeed an isomorphism by (2.2.25). Then it suffices to show that

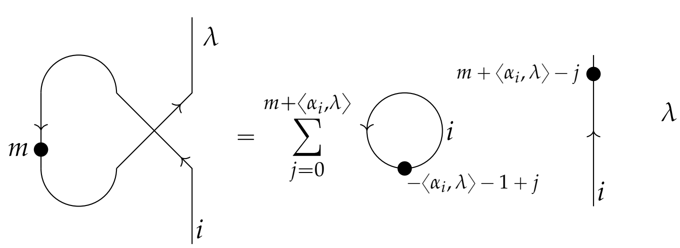



This equation can be shown in a similar way to (2.3.7). ☐

**Proposition 2.3.16** ([7, Proposition 5.4]). *The following equations hold in $\mathcal{U}_q(\mathfrak{g})$ for any $\lambda$ and $i$.*




(2.3.8)



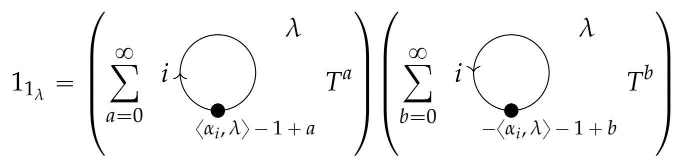

(2.3.9)

*Notice that the right-hand-side of these equations might vanish according to Definition 2.2.6.*

*Proof.* The proof of these propositions is an easy induction argument, using (2.2.18) to move the dots to the other side of the crossover one at a time and then (2.2.29) or (2.2.30) when all dots are on the other side of the crossover. $\square$

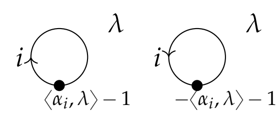
In Proposition 2.3.2, we showed that the definition of the fake bubbles was encapsulated in the infinite Grassman relation (2.3.10). In fact, the infinite Grassman equation holds for all bubbles in $\mathcal{U}_q(\mathfrak{g})$, not just the fake ones.

**Proposition 2.3.17** ([4, Equation 3.7], [7, Proposition 5.5])**.** *For any $\lambda \in \Lambda_W$, the following equation holds in the power series ring $\mathcal{U}_q(\mathfrak{g})(1_\lambda, 1_\lambda)[[T]]$.*



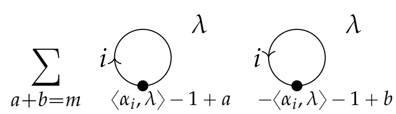
(2.3.10)

*Proof.* It suffices to compare coefficients on both sides of the power series. The coefficient of $T^0$ is



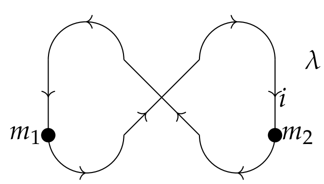

Both of these bubbles are identities by (2.2.28), so match the coefficient of $T^0$ in the power series on the left hand side.

So it remains to show that for $m > 0$ that the coefficient of $T^m$ vanishes. This coefficient is



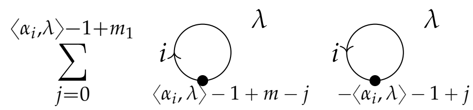
(2.3.11)

To show that this vanishes, consider two ways of decomposing the diagram below for $m_1 + m_2 = m$.



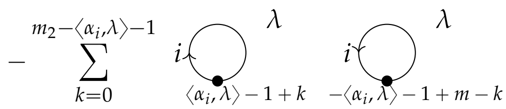
(2.3.12)

On one hand, we can use (2.3.8) to decompose (2.3.12) as



(2.3.13)

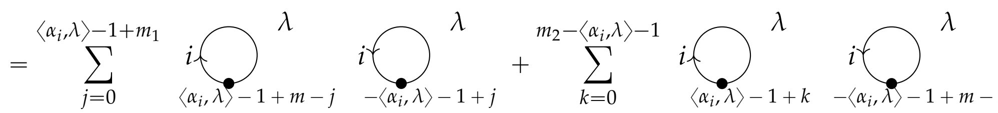
On the other hand, we can use (2.3.9) to decompose (2.3.12) as



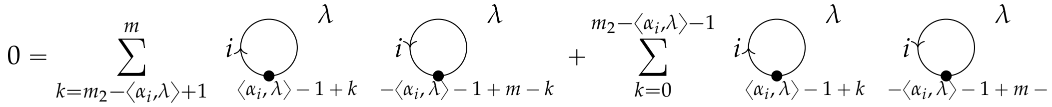

(2.3.14)

Since these are both equal to (2.3.12), then they are equal, so their difference is zero.

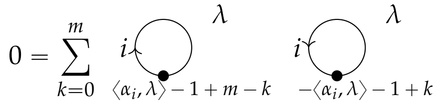
0 = (2.3.13) − (2.3.14)




Into the first term of the above, make the substitution $j = m - k$. So when $j = 0$, $k = m$ and when $j = m_1 + \langle\alpha_i, \lambda\rangle - 1$, $k = m_2 - \langle\alpha_i, \lambda\rangle + 1$. So we have




Now combine terms:




The right hand side is equal to (2.3.11), which is the coefficient of $T^m$. So we have shown that the coefficient of $T^m$ vanishes for $m \geqslant 1$, and therefore we have proved the infinite Grassman relation. □

## 2.4 Idempotent Completions

The importance of idempotents has already been seen in the construction of $\dot{U}_q(\mathfrak{g})$ from the quantum group $U_q(\mathfrak{g})$. Moving from $U_q(\mathfrak{g})$ to $\dot{U}_q(\mathfrak{g})$, we replace the unit with a collection of orthogonal idempotents, one for each element of $\Lambda_W$. A categorification of $U_q(\mathfrak{g})$ would be a monoidal category $\mathscr{V}$, which is a 2-category with a single object (see Example 2.1.5). The analogy of replacing the unit of $U_q(\mathfrak{g})$ by a system of idempotents is to replace the single object in the monoidal category $\mathscr{V}$ by a collection of objects, one for each element of $\Lambda_W$.

The next step in constructing our categorification of $\dot{U}_q(\mathfrak{g})$ is to take the idempotent completion of the 2-category $\mathscr{U}_q(\mathfrak{g})$. The main reason that we take the idempotent completion of $\mathscr{U}_q(\mathfrak{g})$ is to get a 2-category that appropriately categorifies $\dot{U}_q(\mathfrak{g})$; without the idempotent completion of $\mathscr{U}_q(\mathfrak{g})$, the map between the Grothendieck group of $\mathscr{U}_q(\mathfrak{g})$ and $\dot{U}_q(\mathfrak{g})$ wouldn't even be a homomorphism!

**Definition 2.4.1.** An **idempotent** $e\colon A \to A$ in $\mathscr{C}$ is a morphism of $\mathscr{C}$ such that $ee = e$. An idempotent **splits** if there are $f\colon A \to B$ and $g\colon B \to A$ such that $e = gf$ and $fg = 1_B$.

What does it mean to idempotent complete a category $\mathscr{C}$? A category is called **idempotent complete** if every idempotent splits. The idempotent completion of $\mathscr{C}$ is an idempotent complete category containing $\mathscr{C}$, universal among such categories.

**Definition 2.4.2.** The **idempotent completion** (or **Karoubi envelope**) of a category $\mathscr{C}$ is the category $\dot{\mathscr{C}}$ (also denoted $\mathrm{Kar}(\mathscr{C})$), where

- objects are pairs $(A, e)$ where $A$ is an object of $\mathscr{C}$ and $e\colon A \to A$ is idempotent;
- morphisms $f\colon (A, e) \to (A', e')$ are morphisms $f\colon A \to B$ in $\mathscr{C}$ such that $e' f e = f$.

It is not immediately obvious (at least not to me) that this is a category, but this is easy to check.

**Proposition 2.4.3.** *$\dot{\mathscr{C}}$, as defined above, is a category with identity arrows given by $1_{(A,e)} = e\colon (A, e) \to (A, e)$ and composition inherited from composition in $\mathscr{C}$*

*Proof.* We need to check that composition is well-defined and associative, and moreover that the proposed identity morphisms are actually identities. Given $f\colon (A, e) \to (B, d)$ in $\dot{\mathscr{C}}$, we have

$$1_{(B,d)} f = df = ddfe = dfe = f \quad \text{and} \quad f1_{(A,e)} = fe = dfee = dfe = f.$$

If $(B, d) \to (C, c)$ is another morphism of $\dot{\mathscr{C}}$, then we want to show that composition of $f$ and $g$ is well-defined, or that $cgfe = gf$. But notice

$$cgfe = c(cgd)(dfe)e = (cgd)(dfe) = gf.$$

Finally, composition inherits associativity from $\mathscr{C}$. $\square$

We also want to know that $\dot{\mathscr{C}}$ is actually idempotent complete.

**Proposition 2.4.4.** *Every idempotent splits in $\dot{\mathscr{C}}$.*

*Proof.* Suppose $e\colon (A,a) \to (A,a)$ is an idempotent in $\dot{\mathscr{C}}$. Then we have that $aea = e$ and $ee = e$, $aa = a$. Furthermore, $e = 1_{(A,e)}e = ae$ and similarly, $e = ea$. Write $e = gf$ where $g = ea$ and $f = ae$. Then

$$gf = eaae = eae = ee = e \text{ and } fg = aeea = aea = ea = e = 1_{(A,e)}.$$

So $e$ splits in $\dot{\mathscr{C}}$. $\square$

Notice that every identity morphism in $\mathscr{C}$ is idempotent, therefore $(A, 1_A)$ is an element of $\dot{\mathscr{C}}$. In this manner, $\mathscr{C}$ sits inside $\dot{\mathscr{C}}$; there is a fully faithful functor $I\colon \mathscr{C} \to \dot{\mathscr{C}}$ defined on objects by $I(A) = (A, 1_A)$ and on morphisms by $I(f\colon A \to B) = f\colon (A, 1_A) \to (B, 1_B)$. The universal property of $\dot{\mathscr{C}}$ is expressed by factoring through this functor.

**Definition 2.4.5.** A functor $F\colon \mathscr{C} \to \mathscr{D}$ is said to **split idempotents of** $\mathscr{C}$ if for every idempotent $e$ of $\mathscr{C}$, $F(e)$ is a split idempotent in $\mathscr{D}$.

**Proposition 2.4.6.** *$\dot{\mathscr{C}}$ is universal among all categories $\mathscr{D}$ with a functor $F\colon \mathscr{C} \to \mathscr{D}$ that splits idempotents in $\mathscr{C}$.*

*More precisely, suppose $\mathscr{D}$ is a category with a functor $F\colon \mathscr{C} \to \mathscr{D}$ such that if $e\colon A \to A$ is an idempotent of $\mathscr{C}$, then there are $f_e, g_e \in \mathscr{D}$ such that $F(e) = g_e f_e$ and $f_e g_e$ is an identity. Then there is a unique functor $\dot{F}\colon \dot{\mathscr{C}} \to \mathscr{D}$ such that $F = \dot{F}I$.*

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "I"] \arrow[dr, "F"'] & \dot{\mathscr{C}} \arrow[d, dashed, "\dot{F}"] \\
& \mathscr{D}
\end{tikzcd}$$

*Proof.* Any idempotent $e\colon A \to A$ in $\mathscr{C}$ splits in $\mathscr{D}$ as $F(e) = g_e f_e$ for some $f_e\colon F(A) \to X_e$ and $g_e\colon X_e \to F(A)$, with $f_e g_e = 1_{X_e}$. Define $\dot{F}(A,e) = X_e$.

Note that if $e = 1_A$, then we may without loss choose $f_e = g_e = 1_{FA}$. In essence, we are making a choice of splitting for each idempotent $e\colon A \to A$, so we may choose that $1_{FA}$ splits like this.

On arrows $f\colon (A,a) \to (B,b)$, define $\dot{F}(f) = f_b F(f) g_a$. Let's check that $\dot{F}$ respects composition and identities.

$$\dot{F}(1_{(A,a)}) = \dot{F}(a) = f_a F(a) g_a = f_a g_a f_a g_a = 1_{X_e} = 1_{\dot{F}(A,e)}$$

Given $s\colon (A,a) \to (B,b)$ and $t\colon (B,b) \to (C,c)$, note that $t = t1_{(B,b)} = tb$. Then,

$$\begin{aligned}
\dot{F}(t)\dot{F}(s) &= f_c F(t) g_b f_b F(s) g_a \\
&= f_c F(t) F(b) F(s) g_a \\
&= f_c F(tbs) g_a \\
&= f_c F(ts) g_a \\
&= \dot{F}(ts).
\end{aligned}$$

Finally, $\dot{F}I = F$ by construction. $\square$

If $F, G\colon \mathscr{C} \to \mathscr{D}$ both split idempotents, there is also a correspondence between natural transformations $F \implies G$ and $\dot{F} \implies \dot{G}$.

**Proposition 2.4.7.** *If $F, G\colon \mathscr{C} \to \mathscr{D}$ are two functors that split idempotents in $\mathscr{C}$, and $\alpha\colon F \implies G$ is a natural transformation, then there is a natural transformation $\dot{\alpha}\colon \dot{F} \implies \dot{G}$ such that $\dot{\alpha}_{I(A)} = \alpha_A$ for all objects $A$ of $\mathscr{C}$.*

*Proof.* If $e$ is an idempotent in $\mathscr{C}$, write $F(e) = g_e f_e$ for the splitting of $F(e)$, and write $G(e) = k_e h_e$ for the splitting of $G(e)$.

$$F(e)\colon F(A) \xrightarrow{f_e} \dot{F}(A, e) \xrightarrow{g_e} F(A) \qquad G(e)\colon G(A) \xrightarrow{h_e} \dot{G}(A, e) \xrightarrow{k_e} G(A)$$

Define $\dot{\alpha}_{(A,e)} = h_e \circ \alpha_A \circ g_e$. This definition is indeed natural: given a morphism $f\colon (A, a) \to (B, b)$ in $\dot{\mathscr{C}}$, we want to show that the following commutes:

$$\begin{array}{ccc} \dot{F}(A,a) & \xrightarrow{\dot{F}(f)} & \dot{F}(B,b) \\ \big\downarrow{\scriptstyle \dot{\alpha}_{(A,a)}} & & \big\downarrow{\scriptstyle \dot{\alpha}_{(B,b)}} \\ \dot{G}(A,a) & \xrightarrow{\dot{G}(f)} & \dot{G}(B,b) \end{array}$$

But replacing each of $\dot{G}$, $\dot{\alpha}$ and $\dot{F}$ with its definition and using the naturality of $\alpha$, we see that

$$\begin{aligned} \dot{G}(f) \circ \dot{\alpha}_{(A,a)} &= h_b G(f) k_a h_a \alpha_A g_a \\ &= h_b G(f) G(a) \alpha_A g_a \\ &= h_b G(f 1_{(A,a)}) \alpha_A g_a \\ &= h_b G(f) \alpha_A g_a \\ &= h_b \alpha_B F(f) g_a \\ &= h_b \alpha_B F(1_{(B,b)} f) g_a \\ &= h_b \alpha_B F(b) F(f) g_a \\ &= h_b \alpha_B g_b f_b F(f) g_a \\ &= \dot{\alpha}_{(B,b)} \circ \dot{F}(f) \end{aligned}$$

Therefore, the square above commutes, so $\dot{\alpha}$ is natural.

Notice that when $e = 1_A$, then $g_e = 1_{FA}$ and $h_e = 1_{GA}$, so $\dot{\alpha}_{(A,1_A)} = \alpha_A$. $\square$

Ultimately, we want to take the idempotent completion of the 2-category $\mathscr{U}_q(\mathfrak{g})$, but so far we've only defined the idempotent completion of ordinary 1-categories. So we need a notion of idempotent completions of 2-categories. To define this, we just take idempotent completions of each of the hom-categories within a 2-category.

**Definition 2.4.8.** Let $\mathfrak{C}$ be a 2-category. The **idempotent completion** $\dot{\mathfrak{C}}$ of $\mathfrak{C}$ is the 2-category with 0-cells the same as in $\mathfrak{C}$, and hom-categories $\dot{\mathfrak{C}}(A,B) = \mathrm{Kar}(\mathfrak{C}(A,B))$, where $A, B$ are 0-cells of $\mathfrak{C}$.[‡] The composition functor $\circ\colon \dot{\mathfrak{C}}(B,C) \times \dot{\mathfrak{C}}(A,B) \to \dot{\mathfrak{C}}(A,C)$ comes from the composition functor for $\mathfrak{C}$ via the universal property of the idempotent completion:

$$\begin{tikzcd} \mathfrak{C}(B,C) \times \mathfrak{C}(A,B) \arrow[r, "\text{inclusions}"] \arrow[d, "\circ"] & \dot{\mathfrak{C}}(B,C) \times \dot{\mathfrak{C}}(A,B) \arrow[d, dashed, "\circ"] \\ \mathfrak{C}(A,C) \arrow[r, "\text{inclusion}"] & \dot{\mathfrak{C}}(A,C) \end{tikzcd} \tag{2.4.1}$$

The above definition implicitly uses the following easy fact. This fact is not hard to see, because idempotents in $\mathscr{C} \times \mathscr{D}$ are pairs $(e, f)$ where $e$ is idempotent in $\mathscr{C}$ and $f$ is idempotent in $\mathscr{D}$.

**Proposition 2.4.9.** *Let $\mathscr{C}, \mathscr{D}$ be categories. Then $\mathrm{Kar}(\mathscr{C} \times \mathscr{D}) \cong \mathrm{Kar}(\mathscr{C}) \times \mathrm{Kar}(\mathscr{D})$.*

In the idempotent completion of a 2-category, all of the idempotent 2-morphisms split, instead of the idempotent 1-morphisms. Moreover, by using the definition of idempotent completion of a 1-category to define idempotent completion of a 2-category, we can lift many properties about idempotent completions from the situation for 1-categories.

**Proposition 2.4.10.** *There is a 2-functor $\mathfrak{I}\colon \mathfrak{C} \to \dot{\mathfrak{C}}$ that is universal among 2-functors $\mathfrak{F}\colon \mathfrak{C} \to \mathfrak{D}$ such that all idempotent 2-morphisms split under $\mathfrak{F}$.*

*Proof.* Define $\mathfrak{I}\colon \mathfrak{C} \to \dot{\mathfrak{C}}$

- on 0-cells by $\mathfrak{I}(A) = A$;
- on 1-cells $f\colon A \to B$ by $\mathfrak{I}(f) = I_{A,B}(f)$, where $I_{A,B}\colon \mathfrak{C}(A,B) \to \dot{\mathfrak{C}}(A,B)$ is the inclusion functor of the idempotent completion of 1-categories.
- On a 2-cell $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$, define $\mathfrak{I}(\alpha)$ by $I_{A,B}(\alpha)$.

The fact that $\mathfrak{I}$ is a genuine 2-functor follows from the way that composition is defined in $\dot{\mathfrak{C}}$. Composition of 1-cells $f\colon A \to B$ and $g\colon B \to C$ is

$$\mathfrak{I}(g) \circ \mathfrak{I}(f) = I_{B,C}(g) \circ I_{A,B}(f) = I_{A,C}(gf) = \mathfrak{I}(gf)$$

by commutativity of (2.4.1). A similar equation holds for horizontal composition of 2-cells, and $\mathfrak{I}$ respects vertical composition of 1-cells since each $I_{A,B}$ is a functor and therefore respects composition.

[‡]Recall that $\mathrm{Kar}(\mathscr{C})$ also denotes the idempotent completion of $\mathscr{C}$; we use the alternative notation here because putting a dot above something with more than one letter looks dumb.

The 2-functor $\mathfrak{F}\colon \mathfrak{C} \to \mathfrak{D}$ defines functors $F_{A,B}\colon \mathfrak{C}(A,B) \to \mathfrak{D}(\mathfrak{F}(A), \mathfrak{F}(B))$ for all 0-cells $A, B$ of $\mathfrak{C}$. We may define a new 2-functor $\dot{\mathfrak{F}}\colon \dot{\mathfrak{C}} \to \mathfrak{D}$ that is the same as $\mathfrak{F}$ on 0-cells, but on $\dot{\mathfrak{C}}(A,B)$ is given by $\dot{F}_{A,B}$, where $\dot{F}_{A,B}$ arises from the universal property of the idempotent completion. To apply Proposition 2.4.6, we use the assumption that all idempotent 2-morphisms split under $\mathfrak{F}$.

To check that $\dot{\mathfrak{F}}$ factors $\mathfrak{I}$, we need to check commutativity of the below diagram of 2-categories and 2-functors

$$\begin{tikzcd} \mathfrak{C} \arrow[r, "\mathfrak{I}"] \arrow[dr, "\mathfrak{F}"'] & \dot{\mathfrak{C}} \arrow[d, "\dot{\mathfrak{F}}"] \\ & \mathfrak{D} \end{tikzcd}$$

but because $\dot{\mathfrak{F}}$ and $\mathfrak{F}$ agree on 0-cells, we need only check that the following diagrams of categories and functors commute for each pair $A, B$ of 0-cells of $\mathfrak{C}$.

$$\begin{tikzcd} \mathfrak{C}(A,B) \arrow[r, "I_{A,B}"] \arrow[dr, "F_{A,B}"'] & \dot{\mathfrak{C}}(A,B) \arrow[d, "\dot{F}_{A,B}"] \\ & \mathfrak{D}(\mathfrak{F}(A), \mathfrak{F}(B)) \end{tikzcd}$$

This commutes by Proposition 2.4.6. □

Finally, since in the case that we are concerned with, the category is additive, we need to know the following.

**Proposition 2.4.11.** *If $\mathscr{C}$ is an additive category, then $\dot{\mathscr{C}}$ is also additive and the inclusion $I\colon \mathscr{C} \to \dot{\mathscr{C}}$ is an additive functor.*

*Proof.* To show that $\dot{\mathscr{C}}$ is an additive category, we need to show that for each pair of objects $(A,a)$ and $(B,b)$, the morphisms $\dot{\mathscr{C}}((A,a),(B,b))$ forms an abelian group. We already know that $\mathscr{C}(A,B)$ has an abelian group structure, and moreover by Definition 2.4.2 that each arrow $(A,a) \to (B,b)$ is an arrow $A \to B$ in $\mathscr{C}$. So claim that $\dot{\mathscr{C}}((A,a),(B,b))$ is the subgroup of $\mathscr{C}(A,B)$ consisting of all morphisms $f\colon A \to B$ such that $bfa = f$. We just need to check that this is actually a subgroup.

This isn't hard. We have that $b0a = 0$, so $0 \in \dot{\mathscr{C}}((A,a),(B,b))$, and if $f, g \in \dot{\mathscr{C}}((A,a),(B,b))$, then $b(f-g)a = bfa - gfa = f - g$. So this is indeed a subgroup.

Since composition in $\dot{\mathscr{C}}$ is inherited from composition in $\mathscr{C}$, it respects the abelian group structure. Hence, $\dot{\mathscr{C}}$ is additive.

Finally, the inclusion functor $I$ is additive because the abelian group structures on homs in $\dot{\mathscr{C}}$ are subgroups of the abelian group structures on homs in $\mathscr{C}$. In particular, $I(f+g) = I(f) + I(g)$ and $I(0) = 0$. □

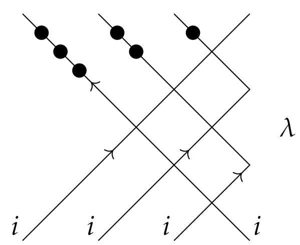
**Corollary 2.4.12.** *The previous proposition immediately tells us that the idempotent completion of an additive* 2*-category is additive as well, and moreover that the inclusion* 2*-functor is additive.*

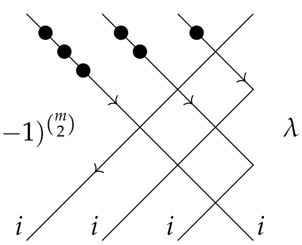
## 2.5 The 2-Kac-Moody Algebra

Finally, we can define the 2-Kac-Moody algebra. This is the object that categorifies $\dot{U}_q(\mathfrak{g})$.

**Definition 2.5.1.** The **2-Kac-Moody Algebra** $\dot{\mathscr{U}}_q(\mathfrak{g})$ is the idempotent completion of the 2-category $\mathscr{U}_q(\mathfrak{g})$.

**Example 2.5.2.** Here are some examples of idempotent 2-morphisms in $\mathscr{U}_q(\mathfrak{g})$.

$e_{+i,m} =$ 


$e_{+i,m} = (-1)^{\binom{m}{2}}$ 


The proof that these are actually idempotents is given in [3, Lemma 5], and the specific equality that we seek is in particular the left hand side of Equation 11 on the bottom of page 6 (our idempotents are mirrored left-to-right from the ones in [3]). The proof is not particularly hard, but it is a tedious manipulation of diagrams and I'm getting sick of typesetting diagrammatic proofs at this point.

# Chapter 3

# Decategorification

In this chapter, we justify how $\dot{\mathscr{U}}_q(\mathfrak{g})$ categorifies $\dot{U}_q(\mathfrak{g})$ by establishing an isomorphism from $\dot{U}_q(\mathfrak{g})$ to the Grothendieck group of $\dot{\mathscr{U}}_q(\mathfrak{g})$. This has been foreshadowed for a while now in the construction of $\mathscr{U}_q(\mathfrak{g})$. Namely, the objects of $\dot{\mathscr{U}}_q(\mathfrak{g})$ are the same as the objects of $\dot{U}_q(\mathfrak{g})$, when we consider the latter as a category. Likewise, the 1-cells of $\mathscr{U}_q(\mathfrak{g})$ are named in a suggestive manner: the 1-morphism $\mathcal{E}_{\pm i}1_\lambda\{t\}$ of $\mathscr{U}_q(\mathfrak{g})$ should clearly correspond to $E_{\pm i}1_\lambda \in \dot{U}_q(\mathfrak{g})$, but this leaves the question of what to do with the grading shift.

To deal with the grading shifts, we may consider the Grothendieck group of $\dot{\mathscr{U}}_q(\mathfrak{g})$ as a $\mathbb{Z}[q, q^{-1}]$-module where multiplication by $q^t$ corresponds to shifting the degree up by $t$. More precisely, we set

$$q^t[\mathcal{E}_{\mathbf{i}}1_\lambda\{s\}, e] = [\mathcal{E}_{\mathbf{i}}1_\lambda\{s+t\}, e]$$

Therefore, the element $q^t E_{\pm i}1_\lambda$ should correspond to $\mathcal{E}_{\pm i}1_\lambda\{t\}$ in $\dot{U}_q(\mathfrak{g})$. That this is well-defined and fully establishes a categorification of $\dot{U}_q(\mathfrak{g})$ can be made precise in the following two theorems.

**Theorem 3.0.1** ([4, Proposition 3.27])**.** *The assignment $E_{\mathbf{i}}1_\lambda \mapsto [\mathcal{E}_{\mathbf{i}}1_\lambda]$ extends to a $\mathbb{Z}[q, q^{-1}]$-algebra homomorphism $\gamma\colon {}_{\mathcal{A}}\dot{U}_q(\mathfrak{g}) \to K_0(\dot{\mathscr{U}}_q(\mathfrak{g}))$.*

**Theorem 3.0.2** ([4, Theorems 1.1, 1.2])**.** *The categorification homomorphism $\gamma$ is an isomorphism if the 2-morphisms of $\mathscr{U}_q(\mathfrak{g})$ satisfies a nondegeneracy condition.*

Note that even though $\dot{U}_q(\mathfrak{g})$ is a $\mathbb{Q}(q)$-algebra, $\dot{\mathscr{U}}_q(\mathfrak{g})$ is only a categorification of the integral form ${}_{\mathcal{A}}\dot{U}_q(\mathfrak{g})$. This is because it doesn't make sense to multiply an element of the Grothendieck group by an arbitrary rational function in $q$, that is, fractional degree shifts are nonsense.

The remainder of this section will be devoted to the proof of Theorem 3.0.1. The second theorem won't be proven here: to do so would likely require another

30 pages. However, we will outline a proof of Theorem 3.0.2 and tell you where to find all of the necessary pieces in the literature.

## 3.1 Grothendieck groups

**Definition 3.1.1.** The **(split) Grothendieck group** $K_0(\mathscr{A})$ of an additive category $\mathscr{A}$ is the abelian group generated by elements $[X]$ over all objects $X$ of $\mathscr{A}$, with relations $[X] = [X']$ if $X \cong X'$ and $[X \oplus Y] = [X] + [Y]$.

**Remark 3.1.2.** The definition of Grothendieck group we use here is usually called the split Grothendieck group. It is more common to define the Grothendieck group with relations $[X] = [A] + [B]$ whenever there is an exact sequence $0 \to A \to X \to B \to 0$; we demand that these sequences split. But it doesn't matter because we will be taking the Grothendieck groups of the hom-categories of $\dot{U}_q(\mathfrak{g})$ where the direct sum is formal and exact sequences don't exist.

As with most definitions that we make on 1-categories, we can lift the Grothendieck group construction up to the setting of 2-categories by taking Grothendieck groups of the hom-categories. However, instead of a Grothendieck *group*, we get a Grothendieck *additive category*.

**Definition 3.1.3.** If $\mathfrak{C}$ is an additive 2-category, then $K_0(\mathfrak{C})$ is the additive category with the same objects as $\mathfrak{C}$, and morphisms $A \to B$ given by $K_0(\mathfrak{C}(A, B))$.

**Remark 3.1.4.** $K_0(\mathfrak{C})$ is often called just the Grothendieck category, but a category is not the categorification of an abelian group, so saying just "Grothendieck category" is a bit like saying "Grothendieck set" for $K_0(\mathscr{C})$. Nevertheless, this is what appears in literature. I'll probably give in to laziness at some point and write just "Grothendieck category" but you should cringe when I do so.

In the case of $\dot{\mathscr{U}}_q(\mathfrak{g})$, the elements of the Grothendieck additive category are equivalence classes $[\mathcal{E}_{\mathbf{i}}1_\lambda\{t\}, e]$ where $e\colon \mathcal{E}_{\mathbf{i}}1_\lambda \to \mathcal{E}_{\mathbf{i}}1_\lambda$ is an idempotent. When the idempotent $e$ is just the identity morphism of $\mathcal{E}_{\mathbf{i}}1_\lambda\{t\}$, then we abuse notation to just write $[\mathcal{E}_{\mathbf{i}}1_\lambda]$, meaning $[\mathcal{E}_{\mathbf{i}}1_\lambda, 1_{\mathcal{E}_{\mathbf{i}}1_\lambda}]$.

In order for there to be any hope of $\dot{\mathscr{U}}_q(\mathfrak{g})$ categorifying $\dot{U}_q(\mathfrak{g})$, we have to have more structure than just that of an additive category on $K_0(\dot{\mathscr{U}}_q(\mathfrak{g}))$. Luckily, this is the case.

**Proposition 3.1.5.** *$K_0(\dot{\mathscr{U}}_q(\mathfrak{g}))$ has the structure of a $\mathbb{Z}[q, q^{-1}]$-algebra with a system of idempotents $[1_\lambda]$ for $\lambda \in \Lambda_W$.*

*Proof.* First, let's define a $\mathbb{Z}[q, q^{-1}]$-module structure on each $K_0(\dot{\mathscr{U}}_q(\mathfrak{g})(\lambda, \mu))$ for each $\lambda, \mu \in \Lambda_W$. These Grothendieck groups are already $\mathbb{Z}$-modules, so we need only define the effect of multiplying by $q^t$ for $t \in \mathbb{Z}$. This is degree-shifting.

$$q^t[\mathcal{E}_{\mathbf{i}}1_\lambda\{s\}, e] = [\mathcal{E}_{\mathbf{i}}1_\lambda\{s+t\}, e]$$

Now I need only to define a system of orthogonal idempotents for the $\mathbb{Z}[q, q^{-1}]$-module $\bigoplus_{\lambda,\mu \in \Lambda_W} K_0(\dot{\mathscr{U}}_q(\mathfrak{g})(\lambda, \mu))$. Then Remark 1.4.2 tells us that the multiplication on this module is given by composition if the two 1-cells can be composed, or zero otherwise.

$$[\mathcal{E}_{\mathbf{j}} 1_\mu \{s\}, e'][\mathcal{E}_{\mathbf{i}} 1_\lambda \{t\}, e] = \begin{cases} [\mathcal{E}_{\mathbf{ji}} 1_\lambda \{s+t\}, e' \cdot e] & \text{if } \lambda = \mu \\ 0 & \text{if } \lambda \neq \mu \end{cases}$$

The system of idempotents is given by $[1_\lambda]$ for $\lambda \in \Lambda_W$; the above definition implies $[1_\lambda][1_\mu] = \delta_{\lambda,\mu}[1_\lambda]$, so they are indeed orthogonal and idempotent. □

The theorems Theorem 3.0.1 and Theorem 3.0.2 describe relations between this $\mathbb{Z}[q, q^{-1}]$ structure and the integral form ${}_{\mathcal{A}}\dot{U}_q(\mathfrak{g})$.

**Remark 3.1.6.** There is an action of $\mathbb{Z}[q, q^{-1}]$ on the 1-morphisms $\mathscr{U}_q(\mathfrak{g})$ defined by $q^t \cdot X = X\{t\}$ and $m \cdot X = \bigoplus_{a=1}^{m} X$. For example, if $f(q) = 2q^3 + q^{-1} + 1$, then

$$f(q) \cdot \mathcal{E}_{\mathbf{i}} 1_\lambda \{t\} = \mathcal{E}_{\mathbf{i}} 1_\lambda \{t+3\} \oplus \mathcal{E}_{\mathbf{i}} 1_\lambda \{t+3\} \oplus \mathcal{E}_{\mathbf{i}} 1_\lambda \{t-1\} \oplus \mathcal{E}_{\mathbf{i}} 1_\lambda \{t\}$$

This extends to the idempotent completion $\dot{\mathscr{U}}_q(\mathfrak{g})$ by $q^t(X, e) = (X\{t\}, e)$.

## 3.2 Proof of Theorem 3.0.1

To define a homomorphism $\gamma \colon {}_{\mathcal{A}}\dot{U}_q(\mathfrak{g}) \to K_0(\dot{\mathscr{U}}_q(\mathfrak{g}))$, first define an assignment

$$E_{\mathbf{i}} 1_\lambda \mapsto [\mathcal{E}_{\mathbf{i}} 1_\lambda] \tag{3.2.1}$$

for all signed sequences $\mathbf{i}$. We want to check that this extends to a homomorphism of $\mathbb{Z}[q, q^{-1}]$-modules. To do that, it is enough to verify that the relations between elements of ${}_{\mathcal{A}}\dot{U}_q(\mathfrak{g})$ hold under the image of this map.

The following theorem makes are lives much easier.

**Theorem 3.2.1** ([4, Section 3.6])**.** *$K_0(\dot{\mathscr{U}}_q(\mathfrak{g}))$ is a free $\mathbb{Z}[q, q^{-1}]$-algebra.*

Therefore, it suffices to show that (3.2.1) extends to a homomorphism of $\mathbb{Q}(q)$-algebras $\dot{U}_q(\mathfrak{g}) \to K_0(\dot{\mathscr{U}}_q(\mathfrak{g})) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q)$.

The hard part is showing that each of the relations between elements of $\dot{U}_q(\mathfrak{g})$ holds in $K_0(\dot{\mathscr{U}}_q(\mathfrak{g})) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q)$ under the assignment (3.2.1). In the categorification, an equality between elements $X$ and $Y$ of $1_\mu \dot{U}_q(\mathfrak{g}) 1_\lambda$ is replaced by a 2-isomorphism between the corresponding elements of $\dot{\mathscr{U}}_q(\mathfrak{g})(\lambda, \mu)$. So to show relations between elements of the Grothendieck group $K_0(\dot{\mathscr{U}}_q(\mathfrak{g}))$, we have to establish 2-isomorphisms in $\dot{U}_q(\mathfrak{g})$.

This first lemma establishes the relation $[E_{+i}1_\lambda, E_{-j}1_\lambda] = 0$ for $i \neq j$. We must first rearrange the equation so that neither side has negative terms:

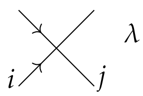
$$E_{+i}E_{-j}1_\lambda = E_{-j}E_{+i}1_\lambda.$$

This is the form that is amenable to categorification.

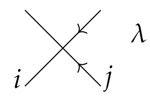
**Lemma 3.2.2** ([4, Proposition 3.26])**.** *For each $i, j \in \{1, \dots, n\}$ such that $i \neq j$, and for any object $\lambda$ of $\mathcal{U}_q(\mathfrak{g})$, there are 2-isomorphisms*

$$\mathcal{E}_{+i-j}1_\lambda \cong \mathcal{E}_{-j+i}1_\lambda$$

*Proof.* There are 2-morphisms



$: \mathcal{E}_{+i-j}1_\lambda \implies \mathcal{E}_{-j+i}1_\lambda$



$: \mathcal{E}_{-j+i}1_\lambda \implies \mathcal{E}_{+i-j}1_\lambda$

They are inverse to each other by (2.2.25). □

We also have the categorified version of the commutator relation.

$$[E_{+i}1_\lambda, E_{-i}1_\lambda] = \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}1_\lambda. \tag{3.2.2}$$

By Definition 1.4.1 of the action of $U_q(\mathfrak{g})$ on $\dot{U}_q(\mathfrak{g})$, the right hand side is equal to (recall that $K_i = K_{\frac{(\alpha_i,\alpha_i)}{2}\alpha_i}$)

$$\frac{1}{q_i - q_i^{-1}}(K_i 1_\lambda - K_i^{-1}1_\lambda) = \frac{q_i^{\langle \alpha_i, \lambda\rangle} - q_i^{-\langle \alpha_i, \lambda\rangle}}{q_i - q_i^{-1}}1_\lambda = [\langle \alpha_i, \lambda\rangle]_{q_i}1_\lambda$$

where $[\langle \alpha_i, \lambda\rangle]_{q_i}$ is the $q$-integer

$$[\langle \alpha_i, \lambda\rangle]_{q_i} = q_i^{\langle \alpha_i, \lambda\rangle - 1} + q_i^{\langle \alpha_i, \lambda\rangle - 3} + \ldots + q_i^{-(\langle \alpha_i, \lambda\rangle - 1)}.$$

So we can rewrite (3.2.2) in a form that is amenable to categorification.

$$E_{+i}E_{-i}1_\lambda = E_{-i}E_{+i}1_\lambda + \sum_{s=1}^{\langle \alpha_i, \lambda\rangle - 1} q^{-2s + \langle \alpha_i, \lambda\rangle - 1}1_\lambda$$

The next proposition expresses the categorification of this relation.

**Lemma 3.2.3** ([4, Proposition 3.25])**.** *For each $i \in \{1, \dots, n\}$, and for any object $\lambda$ of $\mathcal{U}_q(\mathfrak{g})$, there are 2-isomorphisms*

$$\mathcal{E}_{+i-i}1_\lambda \cong \mathcal{E}_{-i+i}1_\lambda \oplus \bigoplus_{s=0}^{\langle \alpha_i, \lambda\rangle - 1} 1_\lambda\{-2s + \langle \alpha_i, \lambda\rangle - 1\} \qquad \text{if } \langle \alpha_i, \lambda\rangle \geqslant 0$$

$$\mathcal{E}_{-i+i}1_\lambda \cong \mathcal{E}_{+i-i}1_\lambda \oplus \bigoplus_{s=0}^{\langle \alpha_i, \lambda\rangle - 1} 1_\lambda\{-2s + \langle \alpha_i, \lambda\rangle - 1\} \qquad \text{if } \langle \alpha_i, \lambda\rangle \leqslant 0$$

*Proof.* The two cases are symmetric; we only deal with the case that $\langle \alpha_i, \lambda \rangle \geqslant 0$. In this case, we are trying to establish the isomorphism

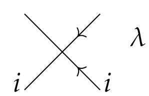
$$\mathcal{E}_{+i-i}1_\lambda \cong \mathcal{E}_{-i+i}1_\lambda \oplus \bigoplus_{s=0}^{\langle \alpha_i,\lambda\rangle - 1} 1_\lambda\{-2s + \langle \alpha_i, \lambda \rangle - 1\}.$$

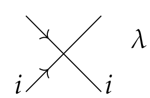
We will define two 2-cells

$$\alpha\colon \mathcal{E}_{+i-i}1_\lambda \implies \mathcal{E}_{-i+i}1_\lambda \oplus \bigoplus_{s=0}^{\langle \alpha_i,\lambda\rangle - 1} 1_\lambda\{-2s + \langle \alpha_i, \lambda \rangle - 1\}$$

$$\beta\colon \mathcal{E}_{-i+i}1_\lambda \oplus \bigoplus_{s=0}^{\langle \alpha_i,\lambda\rangle - 1} 1_\lambda\{-2s + \langle \alpha_i, \lambda \rangle - 1\} \implies \mathcal{E}_{+i-i}$$

as below. Recall that in any additive category, a morphism $\bigoplus_{i=1}^{n} A_i \to \bigoplus_{j=1}^{m} B_m$ in represented by an $m \times n$ matrix $F = (f_{ij})$ with $f_{ij}\colon A_i \to B_j$. In this case, $\alpha$ is a $\langle \alpha_i, \lambda \rangle \times 1$ matrix of 2-cells, and $\beta$ is a $1 \times \langle \alpha_i, \lambda \rangle$ matrix of 2-cells. So define

$\beta = \Big[$  $, \quad \beta_0, \quad \cdots \quad \beta_t, \quad \cdots \quad \beta_{\langle \alpha_i,\lambda\rangle-1} \Big]$

$\alpha = \Big[$ $-$ 

$\alpha_0$

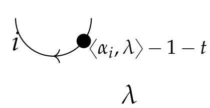
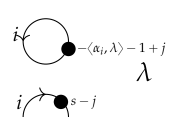
$\vdots$

$\alpha_s$

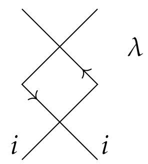
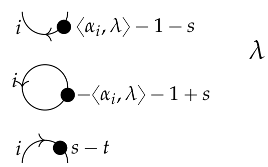
$\vdots$

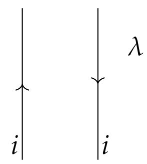
$\alpha_{\langle \alpha_i,\lambda\rangle-1} \Big]$

where

$\beta_t =$  $\qquad \alpha_s = \sum_{j=0}^{s}$ 

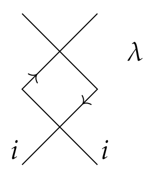
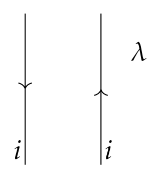
It is easy to show that $\beta$ is a left inverse for $\alpha$, since $\beta$ and $\alpha$ have been chosen so that $\beta\alpha$ reproduces the relation (2.2.31).

$\beta\alpha = -$  $+ \sum_{s=0}^{\langle\alpha_i,\lambda\rangle-1} \sum_{t=0}^{s}$ 

$=$  by (2.2.31)

$= \mathrm{id}_{\mathcal{E}_{+i-i}1_\lambda}$

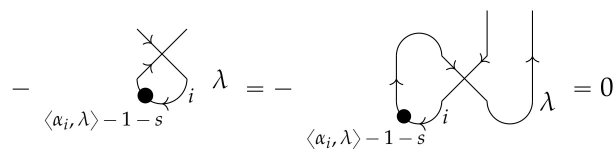
It is significantly harder to show that $\alpha\beta$ is the identity, not only because the product $\alpha\beta$ is a matrix of diagrams, but also because it's harder to deduce that the elements of this matrix are either zero or identities. To show that $\alpha\beta$ is the identity matrix, there are several things that we need to show, each of which has been outsourced to a lemma.

- $-$  $=$  . This is shown in Lemma 3.2.4.
- The off-diagonal terms in the first column vanish. This is shown in Lemma 3.2.6.
- The off-diagonal terms in the first row vanish. This is shown in Lemma 3.2.5.
- The remaining diagonal terms are identities: $\alpha_s\beta_s = 1_{1_\lambda}$. This is shown in Lemma 3.2.7.
- The remaining off-diagonal terms vanish: $\alpha_s\beta_t = 0$ for $s \neq t$. This is shown in Lemma 3.2.8.

**Lemma 3.2.4.** *Because we have assumed that $\langle\alpha_i,\lambda\rangle \geqslant 0$, the summation in* (2.2.32) *vanishes by Definition 2.2.6, so we have the desired equality.*

**Lemma 3.2.5.** *Aside from the element in the upper left corner of the matrix $\alpha\beta$, all of the elements of the first row of this matrix vanish.*

*Proof.* Consider the elements along the first row of the matrix $\alpha\beta$. Aside from the upper-left corner, these look like (for $0 \leqslant s \leqslant \langle \alpha_i, \lambda \rangle - 1$)



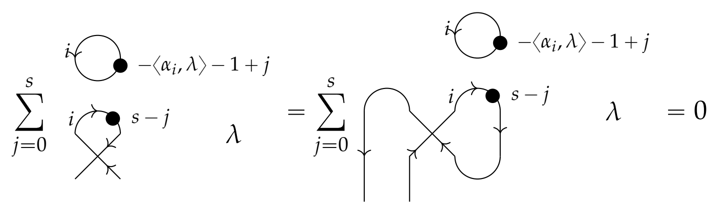

The first equality here follows from the Definition 2.2.13 and the second by Proposition 2.3.16. In particular, we apply (2.3.8) (but rotated upside down using Proposition 2.3.13) in the second equality; the upper limit of the summation on the right hand side of (2.3.8) is in this case

$$\langle \alpha_i, \lambda \rangle - 1 - s - \langle \alpha_i, \lambda \rangle = -s - 1 \leqslant -1 < 0$$

and therefore vanishes by our summation limits convention (Definition 2.2.6). $\square$

**Lemma 3.2.6.** *Aside from the element in the upper left corner of the matrix $\alpha\beta$, all of the elements of the first column of this matrix vanish.*

*Proof.* This is likely the least obvious of the five things we need to show. Consider the elements down the first column of the matrix $\alpha\beta$. Aside from the upper-left corner, these look like (for $0 \leqslant s \leqslant \langle \alpha_i, \lambda \rangle - 1$)



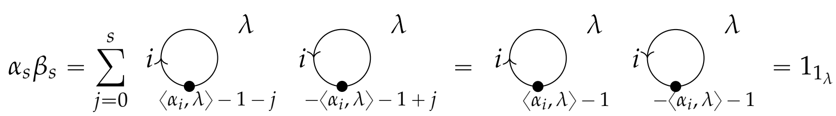

The first equality follows by Definition 2.2.13, the second by Proposition 2.3.16. In particular, we apply (2.3.8) in the second equality; the upper limit of the summation on the right hand side in (2.3.8) is in this case

$$(s - j) - \langle \alpha_i, \lambda \rangle \leqslant s - \langle \alpha_i, \lambda \rangle \leqslant -1 < 0$$

and therefore vanishes by Definition 2.2.6. $\square$

**Lemma 3.2.7.** $\alpha_s \beta_s = 1_{1_\lambda}$.

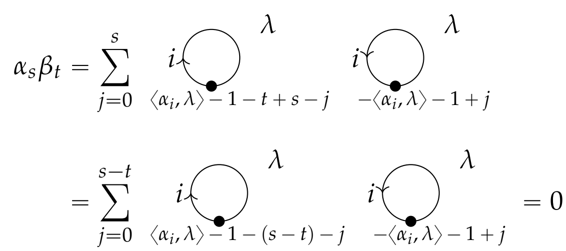
*Proof.*



The second equality follows because the bubble that is oriented counterclockwise in the above equation has degree $(\alpha_i, \alpha_i)(-j)$, and therefore vanishes when $j > 0$ by (2.2.27). So we are left with only the $j = 0$ term, in which case both bubbles have degree zero and are identity 2-morphisms by (2.2.28). $\square$

**Lemma 3.2.8.** *$\alpha_s \beta_t = 0$ for $s \neq t$.*

*Proof.*




The second equality follows because the counterclockwise oriented bubble has degree $(\alpha_i, \alpha_i)(-t + s - j)$, so vanishes when $j > s - t$ by (2.2.27). The third equality follows for $t > s$ by Definition 2.2.6, or for $s \leqslant t$ by Proposition 2.3.17. $\square$

This concludes the proof of Lemma 3.2.3, categorifying the commutator relation of ${}_{\mathcal{A}}\dot{U}_q(\mathfrak{g})$. $\square$

The last relation that we need to categorify is the quantum Serre relation (1.2.2) . This is where we need to work within the idempotent complete category $\dot{\mathcal{U}}_q(\mathfrak{g})$ as opposed to the category $\mathcal{U}_q(\mathfrak{g})$. As we did when categorifying the commutator relations, we need to massage the quantum Serre relations to be in a form more amenable to categorification. In $\dot{U}_q(\mathfrak{g})$, these relations take the form

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} E_{\pm i}^{k} E_{\pm j} E_{\pm i}^{1-a_{ij}-k} 1_\lambda = 0$$

By expanding the $q$-binomial coefficient and then dividing both sides by the $q$-factorial $[1 - a_{ij}]_{q_i}!$, we can rewrite the quantum Serre relations as

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \left( \frac{E_{\pm i}^{k}}{[k]_{q_i}!} \right) E_{\pm j} \left( \frac{E_{\pm i}^{1-a_{ij}-k}}{[1 - a_{ij} - k]_{q_i}!} \right) = 0$$

And finally, we rearrange both sides so that all of the terms on each side are positive.

$$\sum_{k=0}^{\left\lfloor \frac{1-a_{ij}}{2} \right\rfloor} \left( \frac{E_{\pm i}^{2k}}{[2k]_{q_i}!} \right) E_{\pm j} \left( \frac{E_{\pm i}^{1-a_{ij}-2k}}{[1-a_{ij}-2k]_{q_i}!} \right) = \sum_{k=0}^{\left\lfloor \frac{a_{ij}}{2} \right\rfloor} \left( \frac{E_{\pm i}^{2k+1}}{[2k+1]_{q_i}!} \right) E_{\pm j} \left( \frac{E_{\pm i}^{a_{ij}-2k}}{[a_{ij}-2k]_{q_i}!} \right)$$

This is the form of the quantum Serre relations that we will categorify.

The first step here is to find the appropriate element of $\dot{\mathscr{U}}_q(\mathfrak{g})$ that categorifies the elements

$$X = E_{\pm i}^{a}/[a]_{q_i}!.$$

To do so, rearrange the previous equation

$$[a]_{q_i}!X = E_{\pm i}E_{\pm i}\cdots E_{\pm i}.$$

Since multiplication by $q$ corresponds to degree shift, and multiplication is composition of morphisms, the equation we seek for $X$ is

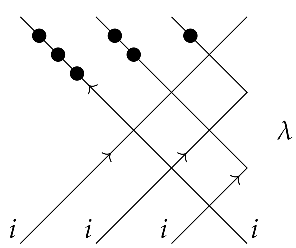
$$[a]_{q_i}! \cdot X = \mathcal{E}_{\pm i^m}1_\lambda$$

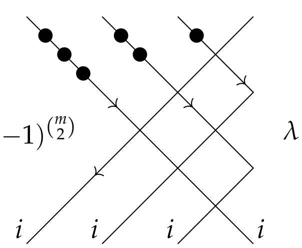
where $\pm i^m$ is the signed sequence $(\pm i, \pm i, \dots, \pm i)$ of length $a$ and $[a]_{q_i}! \cdot X$ is as in Remark 3.1.6. Fortunately, we don't have to search too hard for these elements $X$, because Khovanov and Lauda did the heavy lifting for us [4, Section 3.5].

**Definition 3.2.9.** Let $\pm i^m = (\pm i, \pm i, \dots, \pm i)$ be the signed sequence of length $m$ consisting of only $i$'s, each with the same sign. Then we define

$$X_{\pm i,m} = q^{-\binom{m}{2}\frac{(\alpha_i,\alpha_i)}{2}} \cdot (\mathcal{E}_{\pm i^m}1_\lambda, e_{\pm i,m})$$

where

$e_{+i,m} =$ 


$e_{-i,m} = (-1)^{\binom{m}{2}}$ 


The idempotence of $e_{+i,m}$ and $e_{-i,m}$ is explained in Example 2.5.2. Furthermore, according to [4], we have the desired property

$$[m]_{q_i}! \cdot X_{\pm i,m} = \mathcal{E}_{\pm i^m}1_\lambda,$$

but see the remark below.

**Remark 3.2.10.** In [4] in the (unnumbered) equation following equation (3.55), the claim is made that $[m]_{q_i}! \cdot X_{\pm i,m} = \mathcal{E}_{\pm i^m} 1_\lambda$. Unfortunately, this is baffling. The authors cite [2, 3], but I couldn't find this identity in either of those papers.

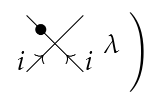
I couldn't figure out a proof of this either, although not for lack of trying. In the idempotent completion $\dot{\mathscr{C}}$ of an additive category $\mathscr{C}$, the direct sum $(A, a) \oplus (B, b)$ is the object $(A \oplus B, \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right))$. Therefore, when we take the direct sum $[m]_{q_i}! \cdot X_{\pm i,m}$, it appears that the idempotent associated to it is the diagonal matrix $E$ with the diagonal composed of idempotents $e_{\pm i,m}$. An isomorphism

$$[m]_{q_i}! \cdot X_{\pm i,m} = ([m]_{q_i}! \cdot \mathcal{E}_{\pm i^m} 1_\lambda, E) \xrightarrow{\sim} (\mathcal{E}_{\pm i^m}, 1_{\mathcal{E}_{\pm i^m}})$$

consists a row vector $\alpha$ and a column vector $\beta$ such that $\beta\alpha = E$. An obvious choice would be to take all elements of $\alpha$ and $\beta$ to be idempotents $e_{+i,m}$, but in that case it may not be that $\alpha\beta = 1_{\mathcal{E}_{\pm i^m} 1_\lambda}$; that is, some sum of these idempotents is equal to an identity morphism.

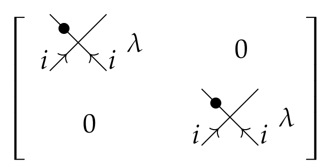
**Example 3.2.11.** Let's illustrate the difficulty with a simple example, the case of $X_{+i,2}$. We have

$$X_{+i,2} = \left( \mathcal{E}_{+i+i} 1_\lambda \left\{ -\tfrac{(\alpha_i, \alpha_i)}{2} \right\}, \right.$$

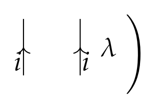


$$[2]_{q_i}! \cdot X_{+i,2} = (q_i^2 + q_i^{-2}) \cdot X_{+i,2} = X_{+i,2}\{(\alpha_i, \alpha_i)\} \oplus X_{+i,2}\{-(\alpha_i, \alpha_i)\}$$
$$= \left( \mathcal{E}_{+i,+i} 1_\lambda \left\{ \tfrac{(\alpha_i,\alpha_i)}{2} \right\} \oplus \mathcal{E}_{+i,+i} 1_\lambda \left\{ -\tfrac{3(\alpha_i,\alpha_i)}{2} \right\}, E \right)$$

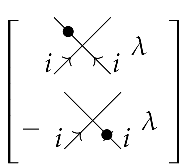
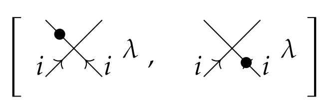
where $E$ is the idempotent matrix of diagrams

$E =$ 

We want to show that $[2]_{q_i}! \cdot X_{+i,2}$ is isomorphic to

$\left( \mathcal{E}_{+i+i} 1_\lambda, \right.$  $\left. \right).$

This means that we want to find $\alpha\colon (\mathcal{E}_{+i+i} 1_\lambda, 1_{\mathcal{E}_{+i+i} 1_\lambda}) \to [2]_{q_i}! \cdot X_{+i,2}$ and $\beta\colon [2]_{q_i}! \cdot X_{+i,2} \to (\mathcal{E}_{+i+i} 1_\lambda, 1_{\mathcal{E}_{+i+i} 1_\lambda})$ such that $\alpha\beta = E$ and $\beta\alpha = 1_{\mathcal{E}_{+i+i} 1_\lambda}$. Looking at the relation (2.2.18), one choice of $\alpha$ and $\beta$ might be

$\alpha =$  $\beta =$ 

so that $\beta\alpha = 1_{\mathcal{E}_{+i+i}1_\lambda}$. But in this case, $\alpha\beta \neq E$; and moreover, it seems that choosing $\alpha$ and $\beta$ such that $\alpha\beta = E$ ruins the other equality.

If you can figure this out please let me know!

Finally, we have the categorification of the quantum Serre relations.

**Lemma 3.2.12** ([4, Proposition 3.24])**.** *For any $\lambda \in \Lambda_W$ and for any distinct $i, j$, there are 2-isomorphisms*

$$\bigoplus_{k=0}^{\left\lfloor {1-a_{ij}}/{2} \right\rfloor} X_{\pm i,2k} \circ \mathcal{E}_{\pm j} 1_{\mu_k} \circ X_{\pm i,1-a_{ij}-2k} 1_\lambda \cong \bigoplus_{k=0}^{\left\lfloor \frac{a_{ij}}{2} \right\rfloor} X_{\pm i,2k+1} \circ \mathcal{E}_{\pm j} 1_{\nu_k} \circ X_{\pm i,a_{ij}-2k} 1_\lambda \tag{3.2.3}$$

*where $\mu_k = \lambda + (1 - a_{ij} - 2k)\alpha_i$ and $\nu_k = \lambda + (a_{ij} - 2k)\alpha_i$.*

*Proof sketch.* The proof in [4, Proposition 3.24] references [3, Proposition 6] and [2, Proposition 2.13]. Brundan proves a more general result in [1, Lemma 3.10] and deduces the categorified quantum Serre relations from it.

All of these proofs demonstrate the categorified quantum Serre relations in the context of the quiver Hecke algebras; the category of projective modules over quiver Hecke algebras categorify the positive (or negative) half $U_q(\mathfrak{g})^\pm$ of the quantum group $U_q(\mathfrak{g})$. The quiver Hecke algebras themselves have a diagrammatic algebra similar to the one for 2-morphisms of $\mathscr{U}_q(\mathfrak{g})$, but without any of the U-turns or orientations on the strands. The proof in [2, 3] is carried out with diagrams.

The quantum Serre relations are relations entirely within $U_q(\mathfrak{g})^\pm$, and therefore the proof using diagrams in [2, 3] can be translated to this case directly. The orientation of all strands will be the same for the isomorphisms (3.2.3), since the signed sequence on both sides have either entirely $+$ terms or entirely $-$ terms. So the lack of orientations on strands in the proofs given in [2, 3] doeesn't matter. ∎

Finally, we can prove the existence of the homomorphism $\gamma$.

*Proof of Theorem 3.0.1.* We need to check that the defining relations of $\dot{U}_q(\mathfrak{g})$ are respected by the assignment (3.2.1).

- The relations $K_\mu E_{\pm i} 1_\lambda = q^{\pm\langle \mu,\alpha_i\rangle} E_{\pm i} K_\mu 1_\lambda$ follow from $\mathbb{Z}[q, q^{-1}]$-linearity and the $U_q(\mathfrak{g})$-module structure of $\dot{U}_q(\mathfrak{g})$ via the equation

$$K_\mu E_{\pm i} 1_\lambda = q^{\langle \mu, \lambda \pm \alpha_i \rangle} E_{\pm i} 1_\lambda q^{\pm\langle \mu,\alpha_i\rangle} E_{\pm i} K_\mu 1_\lambda$$

- $[E_{+i}, E_{-j}] = 0$ is respected by Lemma 3.2.2.

- $[E_{+i} 1_\lambda, E_{-i} 1_\lambda] = \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} 1_\lambda$ is respected by Lemma 3.2.3.

- The quantum Serre relations are respected by Lemma 3.2.12.

Hence, (3.2.1) extends to a homomorphism

$$\gamma_{\mathbb{Q}(q)} \colon \dot{U}_q(\mathfrak{g}) \to K_0(\dot{\mathcal{U}}_q(\mathfrak{g})) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q),$$

and then this restricts to a homomorphism $\gamma \colon {}_{\mathcal{A}}\dot{U}_q(\mathfrak{g}) \to K_0(\dot{\mathcal{U}}_q(\mathfrak{g}))$. □

## 3.3 Proof sketch of Theorem 3.0.2

There is no trick in the proof of Theorem 3.0.1; we first show that $\gamma$ is injective, and then that $\gamma$ is surjective. As always, let's start with the easier thing to prove, which is in this case injectivity. The proof of injectivity depends crucially on the nondegeneracy condition (Definition 3.3.2), while the proof of surjectivity doesn't need it at all.

**Remark 3.3.1.** In [4, Section 6.4], it is shown that when $\mathfrak{g} = \mathfrak{sl}(n)$ the nondegeneracy condition holds. According to [6], Webster [12] showed that nondegeneracy in fact holds for any symmetrizable Kac-Moody algebra.

### 3.3.1 Injectivity of $\gamma$

The argument for injectivity of $\gamma$ is essentially a dimension counting argument to show that the graded dimension of the space of graded Homs in $\dot{\mathcal{U}}_q(\mathfrak{g})$ matches the value of the semilinear form on $\dot{U}_q(\mathfrak{g})$. Then nondegeneracy (Proposition 1.5.2) of the form on $\dot{U}_q(\mathfrak{g})$ can be used to show that $\gamma$ is injective over $\mathbb{Q}(q)$, and therefore injective over $\mathbb{Z}[q, q^{-1}]$.

The first step for any dimension-counting argument is to find a basis. Khovanov and Lauda [4, Section 3.2] defined a spanning set of $\mathrm{grHom}(\mathcal{E}_{\mathbf{i}}1_\lambda, \mathcal{E}_{\mathbf{j}}1_\lambda)$, for any two signed sequences $\mathbf{i}, \mathbf{j}$ and any weight $\lambda$. This spanning set consists of minimal 2-morphisms $B_{\mathbf{i},\mathbf{j},\lambda}$ from $\mathcal{E}_{\mathbf{i}}1_\lambda$ to $\mathcal{E}_{\mathbf{j}}1_\lambda$ composed horizontally with some number of bubbles; minimal meaning that the strands within such a diagram $B_{\mathbf{i},\mathbf{j},\lambda}$ have no self intersection and each pair of strands intersects only once.

The nondegeneracy condition says that:

**Definition 3.3.2** ([4, Definition 3.15]). $\dot{\mathcal{U}}_q(\mathfrak{g})$ is **nondegenerate** if the set $B_{\mathbf{i},\mathbf{j},\lambda}$ is a basis for $\mathrm{grHom}(\mathcal{E}_{\mathbf{i}}1_\lambda, \mathcal{E}_{\mathbf{j}}1_\lambda)$ for all $\mathbf{i}$, $\mathbf{j}$, $\lambda$.

The graded dimension of the graded homs between pairs 2-morphisms can be used to define a semilinear form on $K_0(\dot{\mathcal{U}}_q(\mathfrak{g})(\lambda, \mu))$

$$[\mathcal{E}_{\mathbf{i}}1_\lambda, e]\ ,\ [\mathcal{E}_{\mathbf{j}}1_\lambda, e'] \mapsto \mathrm{grdim}\,\mathrm{grHom}((\mathcal{E}_{\mathbf{i}}1_\lambda, e), (\mathcal{E}_{\mathbf{j}}1_\lambda, e'))$$

This can then be extended to all $K_0(\dot{\mathcal{U}}_q(\mathfrak{g}))$ by declaring that it takes the value zero on the pair $X, Y$ unless $X$ and $Y$ have the same domain and codomain. By imposing $\mathbb{Q}(q)$-semilinearity, it further extends to a $\mathbb{Q}(q)$-semilinear form on $K_0(\dot{\mathcal{U}}_q(\mathfrak{g})) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q)$.

The nondegeneracy assumption allows us to count the graded dimension of any graded hom of 2-morphisms, and in particular it can be shown [4, Corollary 3.14 and comment following] that

$$\operatorname{grdim} \operatorname{grHom}(\mathcal{E}_{\mathbf{i}} 1_\lambda, \mathcal{E}_{\mathbf{j}} 1_\lambda) = \pi \langle E_{\mathbf{i}} 1_\lambda, E_{\mathbf{j}} 1_\lambda \rangle \tag{3.3.1}$$

for $\pi \in \mathbb{Q}(q)$ that depends only on $\mathbf{i}, \mathbf{j}$, and is in particular nonzero.

**Lemma 3.3.3.** *$\gamma$ is injective if $\dot{\mathcal{U}}_q(\mathfrak{g})$ is nondegenerate.*

*Proof sketch.* We first extend $\gamma$ to a $\mathbb{Q}(q)$-morphism $\gamma\colon K_0(\dot{\mathcal{U}}_q(\mathfrak{g})) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q) \to U_q(\mathfrak{g})$. It suffices to show that $\gamma$ is injective over $\mathbb{Q}(q)$.

Assume that $X \in \dot{U}_q(\mathfrak{g})$ lies in the kernel of $\gamma$. Then

$$\operatorname{grdim} \operatorname{grHom}(\gamma(X), \mathcal{E}_{\mathbf{i}} 1_\lambda) = 0$$

for all 2-morphisms $\mathcal{E}_{\mathbf{i}} 1_\lambda$ such that this make sense.

But by (3.3.1), this means that

$$\langle X, E_{\mathbf{i}} 1_\lambda \rangle = 0$$

for all $\mathbf{i}$, $\lambda$. But elements of the form $E_{\mathbf{i}} 1_\lambda$ span $\dot{U}_q(\mathfrak{g})$, so this then implies that $\langle X, Y \rangle = 0$ for all $Y \in \dot{U}_q(\mathfrak{g})$. So by Proposition 1.5.2, $X = 0$. Hence, $\gamma$ is injective. ☐

### 3.3.2 Surjectivity of $\gamma$

The surjectivity of $\gamma$ is significantly more complicated to prove, but doesn't use the assumption that $\dot{\mathcal{U}}_q(\mathfrak{g})$ is nondegenerate. It is more complicated because it uses results from [2, 3] about the quiver Hecke algebras to decompose idempotent endomorphisms of $\mathcal{E}_{\mathbf{i}} 1_\lambda$ for any $\mathbf{i}$ and $\lambda$.

The first step is to establish an equivalence of categories between $\dot{\mathcal{U}}(\lambda, \mu)$ and the category proj-$R$ of right projective modules over a certain carefully chosen algebra $R$. This makes proving surjectivity much easier, since the Grothendieck groups of these two categories will be isomorphic, but proj-$R$ has a distinguished basis of the indecomposable projective modules. Then, to show surjectivity of $\gamma$, we need only show that each indecomposable projective module lies in the image of $\gamma$ because these indecomposable projectives form a basis for $K_0(\text{proj-}R)$.

**Lemma 3.3.4** ([4, Equation 3.85])**.** *There is an equivalence of categories $F\colon \dot{\mathcal{U}}(\lambda, \mu) \simeq$ proj-$R$, where $R$ is the ring*

$$R = \bigoplus_{\substack{\mathbf{i},\mathbf{j} \\ \alpha_{\mathbf{i}} = \alpha_{\mathbf{j}} = \mu - \lambda}} \operatorname{grHom}(\mathcal{E}_{\mathbf{i}} 1_\lambda, \mathcal{E}_{\mathbf{j}} 1_\lambda)$$

*given by*

$$F(\mathcal{E}_{\mathbf{i}}1_\lambda, e) = \bigoplus_{\substack{\mathbf{j} \\ \lambda+\alpha_{\mathbf{j}}=\mu}} \mathrm{grHom}((\mathcal{E}_{\mathbf{i}}1_\lambda, e), \mathcal{E}_{\mathbf{j}}1_\lambda).$$

The next step is to show that each of these indecomposable projectives that form a basis of $K_0(\text{proj-}R)$ (and therefore of $K_0(\dot{\mathcal{U}}(\lambda,\mu))$) is the direct summand of something with a nice form.

**Lemma 3.3.5** ([4, Lemma 3.38]). *If $P$ is an indecomposable projective right $R$-module, then $P$ is isomorphic to a direct summand of $F(\mathcal{E}_{\nu,-\nu'}1_\lambda, 1_{\mathcal{E}_{\nu,-\nu'}1_\lambda})$, for some $\mathcal{E}_{\nu,-\nu'}1_\lambda$ of the form*

$$\mathcal{E}_{\nu,-\nu'}1_\lambda = \bigoplus_{\mathbf{i}\in Seq(\nu)} \mathcal{E}_{\mathbf{i}}1_\lambda \oplus \bigoplus_{\mathbf{j}\in Seq(\nu')} \mathcal{E}_{\mathbf{j}}1_\lambda,$$

*such that the length of $\nu$ plus the length of $\nu'$ is equal to $m$. (Recall that $Seq(\nu)$ and the length of $\nu$ were defined in Definition 2.2.5.)*

Decomposing $F(\mathcal{E}_{\nu,-\nu'}1_\lambda, 1_{\mathcal{E}_{\nu,-\nu'}1_\lambda})$ into indecomposable projectives corresponds to decomposing the identity morphism $1_{\mathcal{E}_{\nu,-\nu'}1_\lambda}$ into orthogonal idempotents

$$1_{\mathcal{E}_{\nu,-\nu'}1_\lambda} = e_1 + e_2 + \ldots + e_r.$$

with $e_ie_j = \delta_{ij}e_i$. Therefore, $P$ is the image of $[\mathcal{E}_{\nu,-\nu'}1_\lambda, e_k]$ for some $k$. So it suffices to show that $[\mathcal{E}_{\nu,-\nu'}1_\lambda, e_k]$ is in the image of $\gamma$.

Showing this is made easier because $\mathcal{E}_{\nu,-\nu'}1_\lambda$ has such a nice form. Khovanov and Lauda exhibit a map

$$K_0(R(\nu)\otimes R(\nu')\otimes \Pi\text{- proj}) \to \mathrm{grHom}(\mathcal{E}_{\nu,-\nu'}, \mathcal{E}_{\nu,-\nu'}), \tag{3.3.2}$$

where $R(\nu)$ and $R(\nu')$ are quiver Hecke algebras and $\Pi$ is a polynomial algebra. They then show that $e_k$ is the image of $e'_k \otimes e''_k \otimes 1$ under this map, where $e'_k$ is a minimal idempotent of $R(\nu)$ and $e''_k$ is a minimal idempotent of $R(\nu')$. Then the following theorem takes us most of the way to the desired result.

**Theorem 3.3.6** ([2, Proposition 3.18]). *Projective modules over the quiver Hecke algebras categorify the positive/negative half of the quantum group. In particular, there are algebras $R(\nu)$ such that*

$${}_{\mathcal{A}}U_q(\mathfrak{g})^{\pm} \cong K_0(R(\nu)\text{- proj}). \tag{3.3.3}$$

Under the isomorphism (3.3.3), we have that $[\mathcal{E}_{\nu,-\nu'}, e_k]$ is in the image of ${}_{\mathcal{A}}U_q(\mathfrak{g})^+ \otimes {}_{\mathcal{A}}U_q(\mathfrak{g})^-$ under the composite map

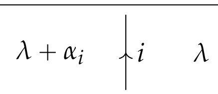
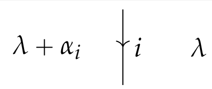
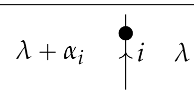
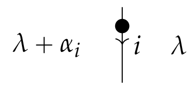
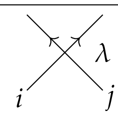
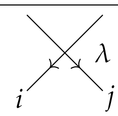
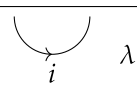
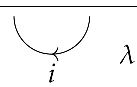
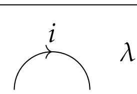
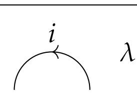
$${}_{\mathcal{A}}U_q(\mathfrak{g})^+ \otimes {}_{\mathcal{A}}U_q(\mathfrak{g})^- \hookrightarrow 1_\mu({}_{\mathcal{A}}\dot{U}_q(\mathfrak{g}))1_\lambda \xrightarrow{\gamma} K_0(\dot{\mathcal{U}}_q(\mathfrak{g})(\lambda,\mu)).$$

This outlines the proof of the following.

**Lemma 3.3.7** ([4, Section 3.8]). *$\gamma$ is surjective.*

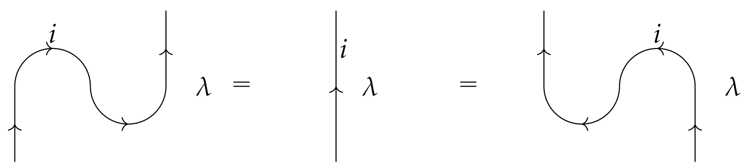
# Appendix A

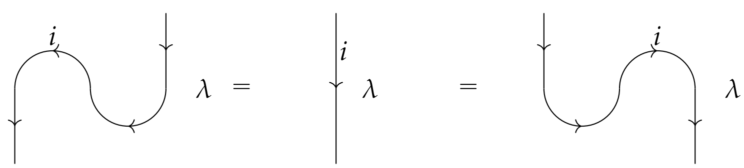
# $\mathscr{U}_q(\mathfrak{g})$ 2-morphism generators

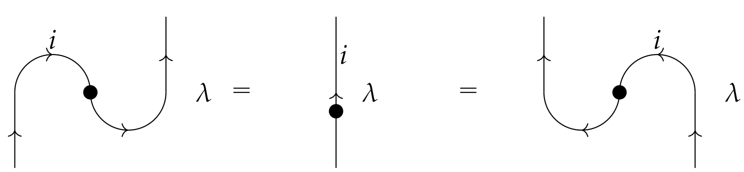
| Generator | Domain and Codomain | Degree Shift |
|---|---|---|
|  | $\mathcal{E}_{+i}1_\lambda\{t\} \implies \mathcal{E}_{+i}1_\lambda\{t\}$ | $0$ |
|  | $\mathcal{E}_{-i}1_\lambda\{t\} \implies \mathcal{E}_{-i}1_\lambda\{t\}$ | $0$ |
|  | $\mathcal{E}_{+i}1_\lambda\{t\} \implies \mathcal{E}_{+i}1_\lambda\{t-(\alpha_i,\alpha_i)\}$ | $(\alpha_i,\alpha_i)$ |
|  | $\mathcal{E}_{-i}1_\lambda\{t\} \implies \mathcal{E}_{-i}1_\lambda\{t-(\alpha_i,\alpha_i)\}$ | $(\alpha_i,\alpha_i)$ |
|  | $\mathcal{E}_{+i+j}1_\lambda\{t\} \implies \mathcal{E}_{+j+i}1_\lambda\{t+(\alpha_i,\alpha_j)\}$ | $-(\alpha_i,\alpha_j)$ |
|  | $\mathcal{E}_{-i-j}1_\lambda\{t\} \implies \mathcal{E}_{-j-i}1_\lambda\{t+(\alpha_i,\alpha_j)\}$ | $-(\alpha_i,\alpha_j)$ |
|  | $1_\lambda\{t\} \implies \mathcal{E}_{-i+i}1_\lambda\{t-c_{+i,\lambda}\}$ | $c_{+i,\lambda} = \frac{(\alpha_i,\alpha_i)}{2}(1+\langle\alpha_i,\lambda\rangle)$ |
|  | $1_\lambda\{t\} \implies \mathcal{E}_{+i-i}1_\lambda\{t-c_{-i,\lambda}\}$ | $c_{-i,\lambda} = \frac{(\alpha_i,\alpha_i)}{2}(1-\langle\alpha_i,\lambda\rangle)$ |
|  | $\mathcal{E}_{+i-i}1_\lambda\{t\} \implies 1_\lambda\{t-c_{+i,\lambda}\}$ | $c_{-i,\lambda} = \frac{(\alpha_i,\alpha_i)}{2}(1-\langle\alpha_i,\lambda\rangle)$ |
|  | $\mathcal{E}_{+i-i}1_\lambda\{t\} \implies 1_\lambda\{t-c_{-i,\lambda}\}$ | $c_{+i,\lambda} = \frac{(\alpha_i,\alpha_i)}{2}(1+\langle\alpha_i,\lambda\rangle)$ |

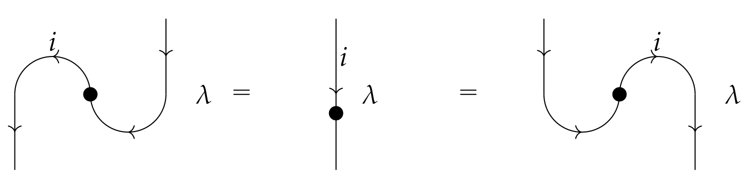
# Appendix B

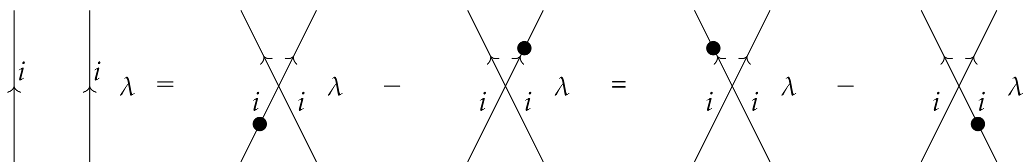
# $\mathscr{U}_q(\mathfrak{g})$ 2-morphism relations



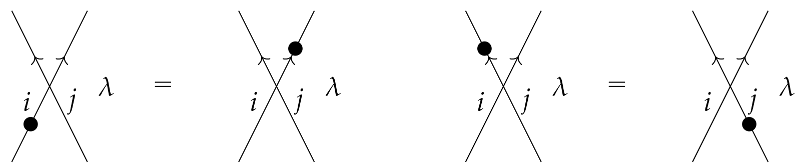


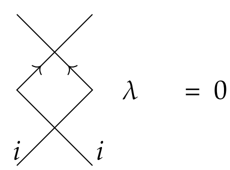




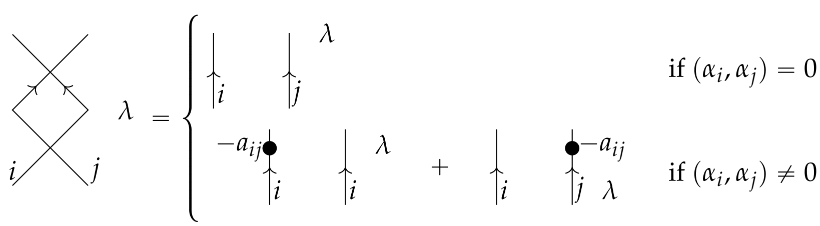



If $i \neq j$,

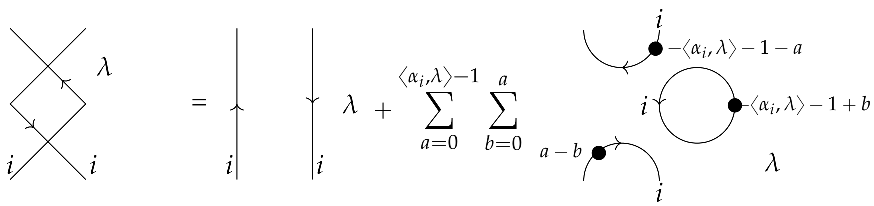


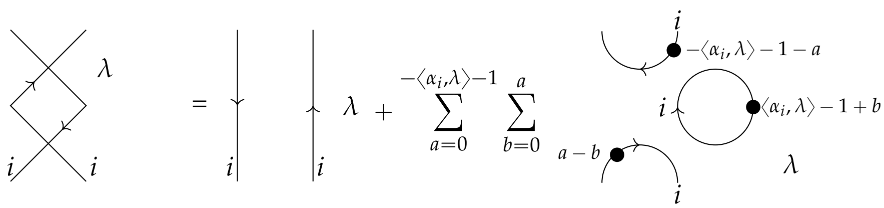



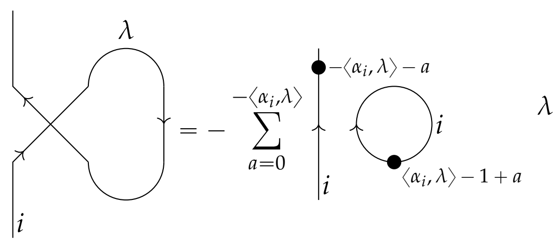

If $i \neq j$,

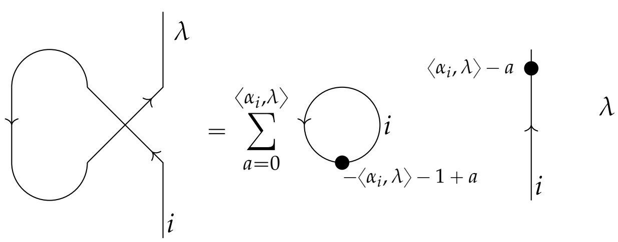



If $i \neq j$,

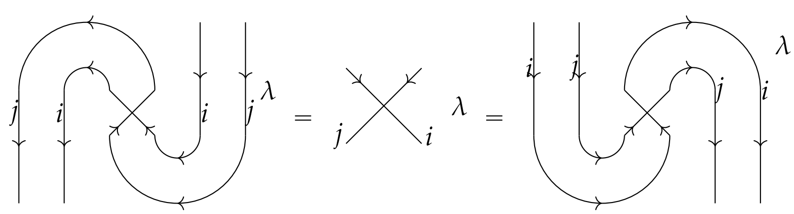

λ
=
λ +
⟨αi,λ⟩−1
a=0
a
b=0
−⟨αi, λ⟩ − 1 − a
⟨αi, λ⟩ − 1 + b
a − b
i
λ

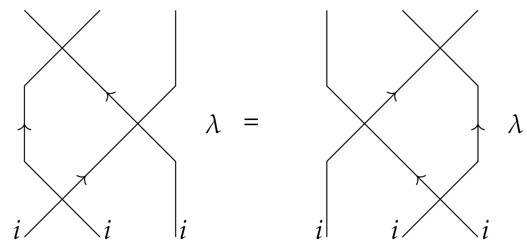

λ
=
λ +
−⟨αi,λ⟩−1
a=0
a
b=0
−⟨αi, λ⟩ − 1 − a
⟨αi, λ⟩ − 1 + b
a − b
i
λ


λ
= −
−⟨αi,λ⟩
a=0
−⟨αi, λ⟩ − a
⟨αi, λ⟩ − 1 + a
i
λ

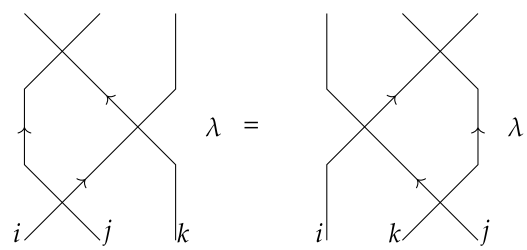

λ
=
⟨αi,λ⟩
a=0
⟨αi, λ⟩ − a
−⟨αi, λ⟩ − 1 + a
i
λ

If $i \neq j$,

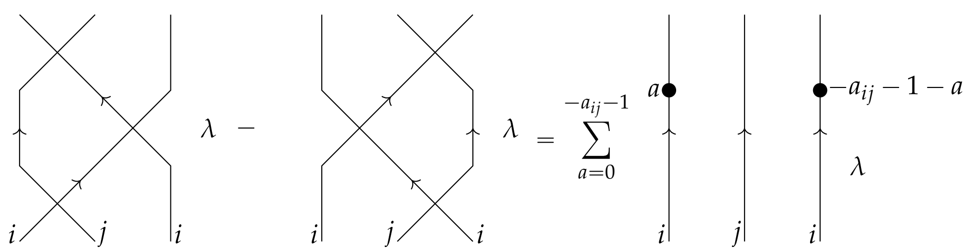

j
i
λ
=
λ
=
λ



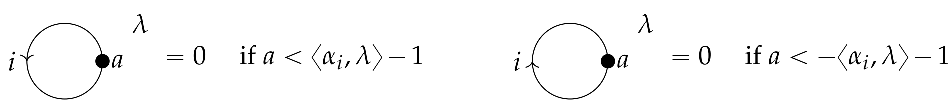

Unless $i = k$ and $(\alpha_i, \alpha_j) \neq 0$.



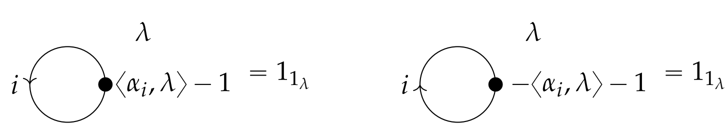

Otherwise,




Dotted bubbles of negative degree are zero.




Dotted bubbles of degree zero are identities.



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