The topological Hochschild homology of Z and Z/p

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$$F_m \to B\widehat{GL}(R[G(S^m)]) \to B\widehat{GL}(R)$$

$$F'_m \to B\widehat{GL}(R[G(S^m)])^+ \to B\widehat{GL}(R)^+$$

where the + denotes Quillens plus construction. By definition, the total space and the base of the last fibration are components of $K(R[G(S^m)])$ respectively $K(R)$.

The stable K-theory is so defined, that F' is an approximation to an m-fold delooping of $K^s(R)$. In particular, this makes K^s into a spectrum in a canonical way. We compute the stable homotopy of $K(R[G(S^m)])$ relative to K(R) in two different ways. First note that the fibrations have sections. Since the total space in the second fibration has a product structure, we have a homotopy equivalence

$$F' \times K(R) \cong K(R[G(S^m)])$$

For the relative stable homotopy we obtain

$$\pi^s_{i+m}(K(R[G(S^m)]), K(R)) \cong \pi^s_{i+m}(F' \wedge K(R)_+)$$

Since F' is m-connected, this equals the generalized homology of the space K(R) with coefficients in the spectrum $K^s(R)$, for small i. In the limit over m, we obtain equality.

But the spectrum $K^s(R)$ is a module spectrum over R, so it is a product of Eilenberg-MacLane spectra. The homology with coefficient in this spectrum is a sum of ordinary homology groups, with coefficients in the homotopy groups of $K^s(R)$.

We can compute the relative stable homotopy in a different way, noticing that since stable homotopy is a homology theory, it does not change under the plus construction. This means, that we can use the first fibration to compute it. We obtain a spectral sequence converging to the relative stable homotopy. In the limit over m, this spectral sequence collapses, and we obtain a formula

$$\pi^s_{i+m}(B\widehat{GL}(R[G(S^m)]), B\widehat{GL}(R)) \cong H_i(GL(R), M(R))$$

For details, see [5] , [11].

Combining our two calculations, we get

$$H_k(GL(R), M(R)) \cong \bigoplus_{i+j=k} H_i(K(R); \pi_j(K^s(R)))$$

In particular, assuming the conjecture that stable K-theory equals topological Hochschild homology, and recalling that by a computation of Quillen the higher homology of $GL(\mathbb{Z}/p)$ with coefficients in $\mathbb{Z}/p$ vanishes, we obtain

$$H_i(GL(\mathbb{Z}/p), M(\mathbb{Z}/p)) \cong \begin{cases} 0 & i \text{ odd} \\ \mathbb{Z}/p & i \text{ even} \end{cases}$$

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§1. We are going to determine the Hochschild homology of the ringfunctors given by $X \mapsto \mathbb{Z}[X]$ respectively $X \mapsto \mathbb{Z}/p[X]$.

Recall from [2] that if F(-) is a commutative ring functor, then we can define the topological Hochschild homology THH(F). This is a hyper-Γ-space in the sense of [10] and [15]. This means that in particular, that it has a ringstructure up to homotopy. We can also make a ringspace out of F. Let $\underline{F}$ denote the infinite loopspace $\lim \Omega^n F(S^n)$. This can be made into a ring up to homotopy, and there is a map $\underline{F} \to$ THH(F). In particular, the spectrum obtained from the infinite loopstructure associated to the additive structure in THH(F) is a module spectrum over $\underline{F}$.

It follows that if F is given as F(X)=R[X] for a commutative ring, then THH(F) is a product of Eilenberg-MacLane spectra. The argument is, that using the unit map

$$S^0 \to R[S^0]$$

we can construct a retraction of spectra

$$\underline{\underline{THH(R)}} \to \underline{\underline{R}} \wedge \underline{\underline{THH(R)}} \to \underline{\underline{THH(R)}}$$

Smash product of an Eilenberg-MacLane spectrum with any spectrum is a product of Eilenberg-MacLane spectra. It follows, that THH(R) is a retract of a product of Eilenberg-Maclane spectra. But then it is a product of Eilenberg-MacLane spectra itself.

Let K(M,n) denote the Eilenberg-MacLane spectrum of dimension n, which corresponds to the R-module M. Let $\prod'$ denote restricted product.

Theorem 1.1. a) $$\underline{\underline{THH(\mathbb{Z}/p)}} = \prod_{i=0}^{\infty}{}' K(\mathbb{Z}/p, 2i)$$

b) $$\underline{\underline{THH(\mathbb{Z})}} = K(\mathbb{Z},0) \times \prod_{i=1}^{\infty}{}' K(\mathbb{Z}/i, 2i-1)$$

c) The map THH($\mathbb{Z},\mathbb{Z}$) $\to$ THH($\mathbb{Z}/p$, $\mathbb{Z}/p$) is the product of the canonical map $K(\mathbb{Z},0) \to K(\mathbb{Z}/p, 0)$ with the Bockstein maps

$$K(\mathbb{Z}/pi, 2pi-1) \to K(\mathbb{Z}/p, 2pi)$$

d) We can choose the isomorphism in part a, so that if

$$\iota_{2i} \in H^{2i}(K(\mathbb{Z}/p, 2i), \mathbb{Z}/p) \subset H^{2i}(THH(\mathbb{Z}/p), \mathbb{Z}/p)$$

is the fundamental class, then the coproduct in cohomology , given by the product in THH, is computed by the following formula

$$\Delta\, \iota_{2i} = \sum_{j=0}^{i} \iota_{2j} \otimes \iota_{2(i-j)}$$

e) We can chose the isomorphism in part b, so that if

$$\iota_{2i-1} \in H^{2i-1}(K(\mathbb{Z}/i\ , 2i-1\ ), \mathbb{Z}/p) \subset H^{2i-1}\ (\ THH(\ \mathbb{Z}/p\ ), \mathbb{Z}/p)$$

is the fundamental class, then the coproduct is given by

$$\Delta\ \iota_{2pi-1} = 1 \otimes \iota_{2pi-1} + \iota_{2pi-1} \otimes 1 + \beta \left(\sum_{j=1}^{i-1} \iota_{2pj-1} \otimes \iota_{2p(i-j)-1} \right)$$

Here β denotes the Bockstein associated to p.

Remark: Part d simply asserts that the multiplicative structure is maximally nontrivial. This could also be formulated with homotopy groups (the ring of homotopy groups is a graded polynomial ring). Part e can also be formulated in terms of homotopy groups, but for this one needs homotopy groups with finite coefficients.

The proof of theorem 1.1 will occupy the rest of this section. We are going to compute the spectrum homology of the spectrum of the topological Hochschild homology. This will be done by spectral sequences. In order to compute the differentials, and to solve certain extension problems in these spectral sequences, we will need precise information about the homology of the spectrum. These computations will be done in §3.

The first remark to be done, is that as THH(R) is a product of Eilenberg-MacLane spectra, its homotopy type is determined by its homology. Actually, it is even determined by the homology with coefficients in $\mathbb{Z}/p$ for each p, together with complete knowledge of all higher Bockstein maps.

Each Eilenberg-MacLane spectrum $K(\mathbb{Z}/p^n , m)$ contributes two summands in the spectrum homology of the topological Hochschild homology, both isomorpic to $\mathcal{A} /(\beta)$, the dual of the Steenrod algebra at p modulo the Bockstein. One copy is shifted in dimension by m and one copy by m+1. The two classes are related by the higher Bockstein associated to p^n.

Fix a prime p. From now on, all homology groups are with coefficients in $\mathbb{Z}/p$. The simplicial structure of topological Hochschild homology provides us with a spectral sequence converging to its spectrum homology. Let us first consider the case THH($\mathbb{Z}/p$). The E^1 - term of this spectral sequence is given by

$$E^1_{p,*} \cong \mathcal{A}^{\otimes\ p+1} .$$

The first differential is given by the boundary maps of the simplicial object. These induce the boundary maps defining (ordinary) Hochschild homology $H(\mathcal{A})$ of $\mathcal{A}$ acting on itself.

It follows, that E^2 is isomorphic to Hochschild homology of $\mathcal{A}$ acting on itself. Recall from [4] that for a commutative ring S,

$$H(S) \cong \mathrm{Tor}_{S \otimes S} (S , S)$$

Recall from [11] , that

$$\mathcal{A} = \mathbb{Z}/2 [\xi_1 , \xi_2 , ...] \quad , \deg \xi_i = 2^i - 1 \qquad (p = 2)$$

$$\mathcal{A} = \mathbb{Z}/p [\xi_1 , \xi_2 , . . .] \otimes \mathbb{Z}/p [\tau_0 , \tau_1 , . . .] \ / \ \tau_1^2 = 0$$

$$\deg \xi_i = 2p^i - 2 \ , \ \deg \tau_i = 2p^i - 1 \qquad (p > 2)$$

The Künneth formula applied to the complex defining Tor says that if M_1 and M_2 are bimodules over the rings R_1 respectively R_2 then

$$\mathrm{Tor}_{R_1 \otimes R_2}(M_1 \otimes M_2 , M_1 \otimes M_2) \cong \mathrm{Tor}_{R_1} (M_1 , M_1) \otimes \mathrm{Tor}_{R_2}(M_2 , M_2)$$

Let $\mathcal{A}'$ be defined by the formula

$$\mathcal{A}' = \mathbb{Z}/2 [\xi_1 \otimes 1 - 1 \otimes \xi_1 , \xi_2 \otimes 1 - 1 \otimes \xi_2 , \ldots] \subset \mathcal{A} \otimes \mathcal{A} \qquad p = 2$$

$$\mathcal{A}' = \mathbb{Z}/p [\xi_1 \otimes 1 - 1 \otimes \xi_1 , \ldots] \otimes \mathbb{Z}/p [\tau_0 \otimes 1 - 1 \otimes \tau_0 , \ldots] \qquad p > 2.$$

Then the Künneth formula may be applied in the present situation, in view of the two maps given by the inclusion $\mathcal{A}' \to \mathcal{A} \otimes \mathcal{A}$ and the diagonal map $\mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, respectively. We obtain that

$\gamma \otimes 1$?

$$H (\mathcal{A} \otimes \mathcal{A}) \cong \mathcal{A} \otimes \mathrm{Tor}_{\mathcal{A}'} (\mathbb{Z}/p , \mathbb{Z}/p) .$$

Using the Künneth formula again, we can further decompose the Tor -factor in this tensor product. We obtain

$$H (\mathcal{A}) \cong \mathcal{A} [\lambda_1 , \lambda_2 , \ldots] / \lambda_i^2 = 0 \quad ; \ \deg \lambda_i = (1 , 2^i-1) \quad (p = 2)$$

$$H (\mathcal{A}) \cong \mathcal{A} [\lambda_1 , \lambda_2 , \ldots] / \lambda_i^2 = 0 \quad \overbrace{\otimes}^{\Gamma(\gamma_0)\otimes} \Gamma (\gamma_1) \otimes \Gamma (\gamma_2) \ldots$$

$$\deg \lambda_i = (1 , 2p^i - 2) ; \deg \gamma_i = (1 , 2p^i - 1) \qquad (p > 2) .$$

The class $\gamma_i^{(a)}$ is represented by $1 \otimes \tau_i \otimes \tau_i \otimes \ldots \otimes \tau_i$ (where the tensor product has a+1 factors), and λ_i by $1 \otimes \xi_i$.

The gamma-algebra $\Gamma (a)$ is defined as the vectorspace over $\mathbb{Z}/p$ with basis given by the symbols $a^{(i)}$, and equipped with a product given by $a^{(i)} a^{(j)} = \binom{i+j}{i} a^{(i+j)}$. An exercise in binomial coefficients shows that

$$\Gamma (a) = \mathbb{Z}/p [a^{(p^0)} , a^{(p^1)} , \ldots] / \left(a^{(p^i)} \right)^p = 0$$

The spectral sequence is slightly different in the cases p = 2 and p odd . In case p = 2 , the multiplicative generators are all in filtration 1 , so for dimensional reasions, all differentials vanish on them. Since the product is compatible with the simplicial filtration, this implies that all differentials vanish.

That is, $E^\infty = E^2$ in the spectral sequence, as a ring. Passing from E^∞ to the spectrum homology, we have an extension problem. This problem is resolved by the following lemma, which we are going to prove in §3.

Lemma 1.2. Let $\overline{\lambda}_i \in H_* (THH(\mathbb{Z}/2) ; \mathbb{Z}/2)$ represent the permanent cycle λ_i . Then

$$(\overline{\lambda}_i)^2 = \overline{\lambda}_{i+1}$$

up to a nonzero factor, and counted modulo decomposables.

The fact that there are no differentials in the spectral sequence, proves that the spectrum homology of THH($\mathbb{Z}/2$) is a free module over $\mathcal{A}$ with exactly one generator in each even degree. It follows that in the product of Eilenberg-MacLane spectra, homotopy equivalent to THH ($\mathbb{Z}/2$), there is exacly one copy of each of the spectra K($\mathbb{Z}/2$, 2i) , $i \geq 0$. That is, 1.1. a follows for p = 2.

1.1.d follows for p = 2 from lemma 1.2 . By changing the homotopy equivalence of THH($\mathbb{Z}/2$) to the product of Eilenberg-MacLane spectra, we can arrange that $(\bar{\lambda}_i)^2 = \bar{\lambda}_{i+1}$, not only modulo decomposables or up to a constant. 1.1. d follows now from dualization. In case p is odd, there are nontrivial differentials.

Lemma 1.3. For $1 < k < p-1$ the differential d_k is identically zero, and

$$d_{p-1}(\gamma_i^{(p^j)}) = \lambda_{i+1}\left(\gamma_i^{(p^{j-1})}\ \gamma_i^{(p^{j-2})}\ \dots\ \gamma_i^{(1)}\right)^{p-1} \quad ?$$

This will be proved in §3.

The ring E^2 is in this case generated by the classes λ_i and $\gamma_i^{(p^j)}$. Since the classes λ_i have filtration 1, all differentials d_i for $i > 1$ vanish on them. The first p-1 differentials are therefore determined by lemma 1.3.

We want to compute E^p.

We can write the E^{p-1} term as a tensor product:

$$E^{p-1} \cong A_0 \otimes A_1 \otimes \dots \otimes \mathcal{A}$$

where $A_i = \mathcal{A}[\lambda_{i+1}]/\lambda_{i+1}^2 \otimes \Gamma(\gamma_i)$.

The differential d_{p-1} maps A_i to itself, so we can consider the homology of A_i with respect to it.

A_i is the direct sum of two copies of $\mathcal{A} \otimes \Gamma(\gamma_i)$, indexed by 1 and λ_{i+1}. The differential maps one of the copies to the other. In each dimension congruent to 0 modulo 2p the ring $\mathcal{A} \otimes \Gamma(\gamma_i)$ has one copy of the vectorspace $\mathbb{Z}/p$. The differential decreases degree by 1. We claim that the differential is injective. This also proves, by dimension counting, that the kernel consists exactly of the elements of filtration less than p. To check the injectivity, note that it suffices to prove the nonvanishing of the differential on monomials in the symbols

$$\gamma_i^{(p^j)} .$$

This follows directly from the formula for the differential.

The homology of A_i with respect to d_{p-1} equals

$$B_i = \mathcal{A}[\gamma_i]/(\gamma_i)^p .$$

The Künneth formula shows that

$$E^p = B_0 \otimes B_1 \otimes \dots .$$

This ring has a set of generators in filtration less than or equal to 1. It follows that all higher differentials are zero. As in the case p = 2, this statement proves 1.1 a for odd p. Again, we have a multiplicative extension problem.

Lemma 1.4. Let $\bar{\gamma}_i \in H_*(\, THH(\mathbb{Z}/p)\, ;\, \mathbb{Z}/p\,)$ represent the permanent cycle γ_i. Then, up to a factor, and modulo decomposables

$$(\,\bar{\gamma}_i\,)^p \;=\; \bar{\gamma}_{i+1}\;.$$

The proof will be given in §3.

In the same way as for p = 2, this proves 1.1. d for odd p.

We now turn to the spectrum THH($\mathbb{Z}$). We fix a prime p. The argument will be different in the two cases p = 2 and p odd.

As before, we have a spectral sequence with

$$E^2 \;=\; H(\,\bar{\mathcal{A}}\,)$$

where $\bar{\mathcal{A}} = H_*(\,\mathbb{Z}\,;\,\mathbb{Z}/p\,)$ is the spectrum homology of the Eilenberg-Maclane spectrum of the ring $\mathbb{Z}$.

This is a spectral sequence of algebras over $\bar{\mathcal{A}}$, which are free as $\bar{\mathcal{A}}$ - modules. We first treat the case p = 2.

The ring structure of $\bar{\mathcal{A}}$ is known, see [9]. It is a polynomial algebra over $\mathbb{Z}/2$ on one generator η of degree 2, and generators $\bar{\xi}_i$ of degree $2^i - 1$ for each $i \geq 2$.

The ring map $\mathbb{Z} \to \mathbb{Z}/2$ induces a map $\bar{\mathcal{A}} \to \mathcal{A}$. This map is given by

$$\eta \mapsto \xi_1^2$$

$$\bar{\xi}_i \mapsto \xi_i\;.$$

There is a spectral sequence converging to the spectrum homology with coefficients in $\mathbb{Z}/2$ of THH($\mathbb{Z}$). Using the reformulation of Hochschild homology as a Tor, and the Künneth formula, we can compute that

$$E^2 \;=\; \bar{\mathcal{A}}\,[\, e_3\,,\, e_4\,,\, e_8\,,\, e_{16}\,\ldots\,]\;/\;(e_i)^2 = 0$$

The class e_3 is given by $1 \otimes \eta$, and the class e_{2^i} by $1 \otimes \xi_i$.

The classes e_i all have filtration 1 , so all differentials vanish, and E^∞ equals E^2.

We consider the multiplicative extensions in the E^∞. We claim, that we can choose representatives $\bar{e}_{2^i}$ in $H_*(\,THH(\mathbb{Z})\,)$ of the classes e_{2^i}, so that they are related by the extension

$$(\,\bar{e}_{2^i}\,)^2 \;=\; \bar{e}_{2^{i+1}}\;.$$

Under the map of spectral sequences induced by the simplicial map

$$THH(\mathbb{Z}) \rightarrow THH(\mathbb{Z}/2)$$

the class e_{2^i} represented by $1 \otimes \bar{\xi}_i$ this maps to $1 \otimes \xi_i$.

The map of E^2 terms

$$\bar{\mathcal{A}}[\ e_3\ , e_4\ , e_8\ , \ldots]/\ (\ e_i\ )^2 \quad \rightarrow \quad \bar{\mathcal{A}}[\ \lambda_2\ , \lambda_4 \ldots]\ /\ \ (\ \lambda_i\ )^2$$

sends e_3 represented by $1 \otimes \eta$ to zero, and e_{2^i} to λ_{2^i} .

We have already solved the extension problem in THH($\mathbb{Z}$/2). We know, that we can choose classes $\bar{\lambda}_i$ representing λ_i so that $(\bar{\lambda}_i)^2 = \bar{\lambda}_{2i}$. In particular, H_*(THH($\mathbb{Z}$/2)) is a polynomial algebra. The image of H_*(THH($\mathbb{Z}$)) is a subalgebra, containing the image of $\bar{\mathcal{A}}$ and the image of $\bar{e}_4$. Since $\bar{e}_4$ maps to $\bar{\lambda}_4$ modulo decomposables, the image of $\bar{e}_4$ is algebraically independent of $\mathcal{A}$, and so algebraically independent of the image of $\bar{\mathcal{A}}$. It follows, that the image of H_*(THH($\mathbb{Z}$)) in H_*(THH($\mathbb{Z}$/2) is a polynomial algebra, isomorphic to $\bar{\mathcal{A}}[\bar{e}_4]$. On the other hand, the square of $\bar{e}_3$ is either equal to $\eta\bar{e}_4$ or zero for dimensional reasons. The first possibility would contradict that THH($\mathbb{Z}$) is a product of Eilenberg-MacLane spectra. There would be a nontrivial k-invariant, since $Sq^1_*(\bar{e}_3)$ would have square $\bar{e}_4$.

It follows , that H_*(THH($\mathbb{Z}$)) contains $\mathcal{A} \otimes \mathbb{Z}/2[\ e_3\ , e_4\ ]\ /\ (\ e_3\ )^2$. Counting dimensions, we conclude that this is indeed all of the homology. In particular, we can choose $\bar{e}_{2^i}$ so that

$$(\ \bar{e}_{2^i}\ )^2 \ = \ \bar{e}_{2^{i+1}} \quad .$$

We noted above, that THH($\mathbb{Z}$) is homotopy equivalent to a product of Eilenberg-MacLane spectra. More precicely,

$$THH(\mathbb{Z}) \ = \ \mathbb{Z} \times \prod_i{}' \ K(\ G_i\ \ , i\ ) \quad .$$

where G_i are finite groups. If we only ask for a 2-primary equivalence, we can assume that the groups G_i are 2-groups.

The homology of the Eilenberg-MacLane spectrum K($\mathbb{Z}/2^r$, i) is a free module on two generators over $\bar{\mathcal{A}}$. The E^∞ - term above is also a free module over $\bar{\mathcal{A}}$. It has one generator for every dimension congruent to 0 or 1 modulo 4. Counting dimensions, we see that this can only be accounted for by a product

$$THH(\mathbb{Z})_{(2)} \ \tilde{=} \ \mathbb{Z}_{(2)} \times \prod_i{}' K(\ \mathbb{Z}/l_i\ , 4i\text{-}1\ ) \quad .$$

We have to determine the numbers $l_i \geq 2$.

We claim that if $i = 2^{i'} j$, with odd j , then l_i is at most $2^{i'+1}$. Actually, the homology of K($\mathbb{Z}/l_i$, 4i-1) occurs in E^∞ as the free module over $\bar{\mathcal{A}}$ generated by classes in dimensions 4i-1 and 4i . The generators are classes of the form $ae_3e_4 \ldots e_{2^{i'+1}}$, respectively $ae_{2^{i'+2}}$, where a is the unique product of multiplicative generators e_{2^r} of degree $2^{i'+2}$ (j-1). Let s be the number of generators occuring in this product. Then the two generators have filtration s+i'+1 respectively s+1.

This means that the fundamental homology class in $H_{4i-1}(K(\mathbb{Z}/l_i , 4i-1) , \mathbb{Z}/2)$ is in the image of the map

$$H_{4i-1}(F_{s+i'+1}) \to H_{4i-1}(THH(\mathbb{Z})) \quad .$$

Let β_{2^r} denote the higher Bockstein, defined inductively on the kernel of $\beta_{2^{r-1}}$. These are the differentials in a spectral sequence converging to the tensor product of $\mathbb{Z}/2$ with the 2-local homology modulo torsion. If ~~x~~ is a class in $H_*(\ , \mathbb{Z}/2)$, which is the reduction of a class in $H_*(\ , \mathbb{Z}/2^{r+1})$, then β_{2^r} vanishes on the class x.

The fundamental class is the image of a higher Bockstein. The nontrivial class in $H_{4i}(K(\mathbb{Z}/ l_i , 4i-1))$ maps through β_{l_i} to it. In particular, this shows that there is an element in $H_*(F_{s+i'+2})$ where this higher Bockstein is defined and nontrivial. This result can be improved, by noticing that the class of dimension 4i in the Eilenberg-MacLane spectrum has filtration s+1, so it is not in the image of (*)

$$H_{4i}(F_s) \to H_{4i} (THH(\mathbb{Z})) \ .$$

Combining these two statements, we see that the higher Bockstein β_{l_i} is defined on a nontrivial element of $H_*(F_{s+i'+1} / F_s)$. (?)

But the quotient $F_{s+i'+1} / F_s$ is the suspension of a disjoint union of smashproducts of Eilenberg- MacLane spectra, so relative to the suspension of the space of components , its homology with coefficients in the 2-primary localization of $\mathbb{Z}$ is 2-torsion. By induction, the torsion in the homology with $\mathbb{Z}_2$ - coefficients of $F_{s+i'+1} / F_s$ is at most $2^{i'+1}$-torsion . But then, the higher Bockstein operation $\beta_{2^{i'+1}}$ is only defined for the trivial element.

It follows that l_i divides $2^{i'+1}$, which is our claim. The next claim, is that have an equality $l_i = 2^{i'+1}$. This is equivalent to the statement 1.1 b for 2-primary torsion. Recall from [2] that we have a product of infinite loop spaces

$$\mu : \ E\mathbb{Z}/2 \overset{\mathbb{Z}/2}{\times} THH(\mathbb{Z})^2 \to THH(\mathbb{Z})$$

Let $C_* = C_*(THH(\mathbb{Z}))$ be the complex defining spectrum homology. Let W_* be the standard free resolution of $\mathbb{Z}$ over the groupring $\mathbb{Z}[\mathbb{Z}/2]$. This chaincomplex has one generator e_i in each dimension, as a chain complex over the groupring.

The map μ induces a map of chain complexes (for more details on this, see the discussion of Dyer-Lashof operations on spectrum homology in §2)

$$\mu_* \ : \ W_* \otimes C_* \otimes C_* \to C_*$$

This map is invariant under the action of $\mathbb{Z}/2$.

Let $\bar{x} \in C_*$ be a chain, representing a homology class with coefficients in

$\mathbb{Z}/2^r$. That is, there is a chain $\bar{y} \in C_*$, so that $\delta \bar{x} = 2^r \bar{y}$. We define the Pontryagin square (see [6])

$$P : H_n(C_* ; \mathbb{Z}/2^r) \to H_{2n}(C_* ; \mathbb{Z}/2^{r+1})$$

by the formula $P(x) = \mu_*(e_0 \otimes x \otimes x - 2^r e_1 \otimes y \otimes x)$.

Then, if red denotes reduction modulo 2^r, red(P(x)) = x^2.

Using this explicit chain representing x^2, we obtain the following statement about homology with coefficients in $\mathbb{Z}/2$ and the higher Bockstein operations:

$$\beta_{2^{r+1}} (x^2) = x \beta_{2^r} (x) \qquad r > 1$$

$$\beta_2 (x^2) = Q^n \beta_2 (x)$$

where $Q^n (y) = \mu_* (e_1 \otimes y \otimes y)$.

We want to apply this to the classes $\bar{e}_i$. In §3, we prove

Lemma 1.5. $\quad Q^4 (\bar{e}_3) = 0$.

Moreover, by the argument above, $l_1 \leq 2$, so it has to equal 2. It follows that $\beta_2 (\bar{e}_4) = \bar{e}_3$. By the formulas above, using the Cartan formula for $Q^8(\bar{e}_3\bar{e}_4)$, and by our choice of e_{2^i}, we obtain inductively that

$$\beta_{2^j} (\bar{e}_{2^i}) = 0 , \qquad j \leq i - 2 .$$

that is, $\beta_{2^{i-1}}$ is defined on $\bar{e}_{2^i}$. Using that the higher Bocksteins are derivations, we obtain that $\beta_{2^{i-1}}$ is defined on a class representing the generator in dimension $2^i j$. Our claim about l_i follows , finishing the proof of 1.1 b. for 2-primary torsion.

The coproduct formula 1.1. e.follows from our computation of the multiplicative structure in H_* (THH($\mathbb{Z}$)). Choose the isomorphism with the product of Eilenberg-MacLane spectra so, that the generators $\bar{e}_i$ chosen above maps trivially into all the factors except one. Then ι_{4i-1} is dual to $\bar{e}_3 (\bar{e}_4)^{i-1}$, and $\beta(\iota_{4i-1})$ is dual to $(\bar{e}_4)^i$. The formula follows on dualizing.

The case of odd torsion is similar, but involves differentials as an extra complication. In this case

$$\bar{\mathcal{A}} = \mathbb{Z}/p[\bar{\xi}_1 ,\bar{\xi}_2 , . . .] \otimes \mathbb{Z}/p [\bar{\tau}_1, \bar{\tau}_2 . .] / (\bar{\tau}_i)^2 .$$

and the map $\bar{\mathcal{A}} \to \mathcal{A}$ is given by $\bar{\xi}_i \to \xi_i$, $\bar{\tau}_1 \to \tau_i$.

The map of spectral sequences induced by the ring map $\mathbb{Z} \to \mathbb{Z}/p$ is in this case on the E^2-level an inclusion

$$\bar{\mathcal{A}}[\lambda_1,\lambda_2, ...] / (\lambda_i)^2 \otimes \Gamma(\gamma_1) \otimes .. \subset \mathcal{A}[\lambda_1,\lambda_2,...] / (\lambda_i)^2 \otimes \Gamma(\gamma_0) \otimes \Gamma(\gamma_1) \otimes$$

Since this map is injective, the first nontrivial differential in the spectral sequence of THH($\mathbb{Z}$) is determined by the first nontrivial differential in the spectral sequence of THH($\mathbb{Z}$/p). Recall from lemma 1.3 :

$$d_{p-1}(\gamma_i^{(p^j)}) = \lambda_{i+1}(\gamma_i^{(p^{j-1})}\gamma_i^{(p^{j-2})}\dots)^{p-1} .$$

In the same way as we did when we discussed the case THH($\mathbb{Z}$/2), we write

$$E^{p-1}(\mathrm{THH}(\mathbb{Z}/p)) = A_1 \otimes A_2 \dots$$

$$A_1 = \mathcal{A}[\lambda_1]/(\lambda_1)^2$$

$$A_i = \mathcal{A}[\lambda_i]/(\lambda_i)^2 \otimes \Gamma(\gamma_{i-1}) , \qquad i \geq 2 .$$

Using the Künneth formula, and the computation of the homology of A_i done above, we obtain that

$$E^p(\mathrm{THH}(\mathbb{Z}/p)) \cong B_1 \otimes B_2 \otimes \dots$$

where $B_1 = A_1$; $B_i \cong \mathcal{A}[\gamma_{i-1}]/(\gamma_{i-1})^p$.

All algebra generators of E^p have filtration 1, so there can be no further differentials. We now have to solve the extension problems.

In this case, the target of the map

$$H_*(\mathrm{THH}(\mathbb{Z})) \to H_*(\mathrm{THH}(\mathbb{Z}/p)) .$$

is the tensor product of a polynomial algebra with an exterior algebra. In particular, since the image of γ_1 has a nontrivial square, not contained in $\mathcal{A}$, it is algebraically independent of $\mathcal{A}$. Moreover, a class representing λ_1 has a trivial square for dimensional reasons. By the same argument as in the case p=2, it follows that

$$H_*(\mathrm{THH}(\mathbb{Z})) \cong \mathcal{A}[\bar{\lambda}_1, \bar{\gamma}_1]/(\bar{\lambda}_1)^2 .$$

Now the rest of the argument that we used in the case p=2 works. There is a p-primary equivalence

$$\mathrm{THH}(\mathbb{Z}) \cong \mathbb{Z} \times \prod_i{}' K(\mathbb{Z}/l_i, 2pi-1)$$

We have to determine the p-power l_i . By arguing with the filtration, we obtain that

$$p \leq l_i \leq p^{i'+1} .$$

where $i = jp^{i'}$ (j prime to p). To conclude the proof of 1.1. b, we have to show that

$$l_i = p^{i'+1} .$$

Let μ be the product

$$\mu_* : E\mathbb{Z}/p \times \mathrm{THH}(\mathbb{Z})^p \to \mathrm{THH}(\mathbb{Z}) .$$

Let W_* be the standard free resolution of $\mathbb{Z}$ over $\mathbb{Z}[\ \mathbb{Z}/p\ ]$, with one generator e_i in each dimension i . W_* is given by the formulas

$$\delta e_{2i+1} = (\ 1-g\ )\ e_{2i}$$

$$\delta e_{2i} = (\ 1 + g + \dots g^{p-1}\ )\ e_{2i-1}\ .$$

The Pontryagin p^{th} power

$$P : H_n\ (\ THH(\mathbb{Z});\ \mathbb{Z}/p^r\ ) \ \rightarrow\ H_{np}\ (\ THH(\mathbb{Z});\ \mathbb{Z}/p^{r+1}\ )$$

is given by

$$P(\ x\ ) = \ \mu_*\Big((\ e_0 \otimes x \dots \otimes x\ ) - p^r\ (\sum_{i=1}^{p-1} i\ (\ e_1 \otimes x \otimes x \dots \otimes y \otimes x \dots \ \otimes x\ )\ )\ .$$

In the term indexed by i in the sum, the factor y occurs at the ith place.

If red denotes reduction modulo p^r , we have that red (P(x)) = x^p . In this case, we obtain for homology with coefficients in $\mathbb{Z}/p$

$$\beta_{p^i}\ (\ P(x)) = x^{p-1}\ \beta_{p^{i-1}}\ (x)\ \ .$$

As in the case p=2, this implies $l\ =\ p^{i'+1}$.

§ 2. In this section, we start the proof of lemmas 1.3, 1.4 and 1.5. The method we use, is to examine the structure on the spectrum homology of THH(R) induced by the multiplicative structure on THH(R). We define Dyer-Lashof operations on this spectrum homology, which are related to the multiplicative structure. The evaluation of these , then gives information on the multiplicative structure of the space THH(R) . In particular we can compute products of homology classes, and certain Dyer-Lashof operations. In order to extend the definition of these to the spectrum homology, we need to specify certain extra data, as will be explained below.

In order to compute these operations, and also in order to prove the lemma on the differentials in the spectral sequence converging to the homology of THH(R) , we compare the topological Hochschild homology to the simplicial spectrum $S^1_+ \wedge R$ Recall from [2] that there is a map of simplicial spectra

$$\lambda : S^1_+ \wedge R \ \rightarrow\ THH(R)$$

Composing with the multiplication map of THH(R), we obtain a map of simplicial spectra

$$E\mathbb{Z}/p_+ \overset{\mathbb{Z}/p}{\wedge} (\ S^1_+ \wedge R\ )^{\wedge p} \rightarrow THH(R)$$

Of course, due to the usual problems with smash products of spectra, the last statement is not quite true. For our purposes, it is not necessary to pursue the question whether we can make it precisely true or not, because we can work with finite approximations.

Finally, we will also compute the differentials in the spectral sequence converging to spectrum homology. This computation will also depend on comparison with a simpler spectral sequence. The ingredient needed to link the two spectral sequences is again the map λ above.

Let X be a space with a basepoint. We will only be concerned with "nice" spaces, for instance geometric realizations of simplicial sets.

We first describe the homology of the power construction.

Let $\{ x_i \}$ be a basis of the homology of X with coefficients in $\mathbb{Z}/p$. We fix a free action of $\mathbb{Z}/p$ on S^{2r-1}, so that the inclusion $S^{2r-1} \subset S^{2r+1}$ is equivariant. Then [8] the homology of the quotient of the $\mathbb{Z}/p$-action on $S^{2r-1}_+ \wedge X^p$ has a basis consisting of the classes

$$(2.1)\quad \left\{ \begin{array}{ll} [x_{i_1}, x_{i_2}, \ldots x_{i_p}] & ;\ \deg = \sum \deg x_{ik} \\ e_i \otimes (x_j)^p = e_i \otimes x_j \otimes x_j \ldots \otimes x_j & ;\ 0 \underset{\leq}{<} i \leq 2r-2 ,\ \deg = i + p\deg(x_j) \\ \{ x_{i_1}, x_{i_2}, \ldots x_{i_p} \} & ;\ \deg = 2r-1 + \sum \deg x_{ik} \end{array} \right.$$

with the relations

$$[x_{i_1}, x_{i_2}, \ldots x_{i_p}] = [x_{i_2}, x_{i_3} \ldots x_{i_p}, x_{i_1}]$$

$$\{ x_{i_1}, x_{i_2}, \ldots x_{i_p} \} = \{ x_{i_2}, x_{i_3} \ldots x_{i_p}, x_{i_1} \} .$$

The inclusion $S^{2r-1} \subset S^{2r+1}$ preserves the first two types of classes, and maps

$$\{ x_{i_1}, x_{i_2}, \ldots x_{i_p} \}$$

to zero, unless $i_1 = i_2 = \ldots i_p$, in which case the class goes to

$$e_{2r-1} \otimes x_{i_1} \otimes \ldots x_{i_1}$$

We can form the direct limit of all spheres of odd dimension. As a limiting case we obtain, that

$$H_*(E\mathbb{Z}/p_+ \overset{\mathbb{Z}/p}{\wedge} X^{\wedge p} ; \mathbb{Z}/p)$$

has a basis consisting of classes

$$(2.2)\quad \left\{ \begin{array}{ll} [x_{i_1}, x_{i_2}, \ldots x_{i_p}] & \\ e_i \otimes (x_j)^{\otimes p} & i \geq 0 \end{array} \right.$$

with relations and degrees as in 2.1.

Now, assume that X is an infinite loop space. Then there is a structure map

$$(2.3)\qquad \mu : E\mathbb{Z}/p_+ \overset{\mathbb{Z}/p}{\wedge} X^p \to X$$

If $a \in H_r(X ; \mathbb{Z}/p)$, one defines the Dyer-Lashof operation (eg in [7]) as the image of the class

$$e_{i-(p-1)r} \otimes a^{\otimes p} \in H_{i+r} (E\mathbb{Z}/p_+ \overset{\mathbb{Z}/p}{\wedge} X^p ; \mathbb{Z}/p)$$

under the map of homology induced by μ .

Now assume that X has two product structures, which are commutative up to all higher homotopies. Assume that for each deloop B^nX we have structure maps

$$\mu_n : E\Sigma_{p+} \wedge (B^nX)^p \to \tilde{B}^{np}X$$

which are Σ_p-equivariant with respect to the action permuting factors on the left side, and permuting desuspension coordinates cyclically right on the right side.

Also assume that these map are related by commutative diagrams of Σ_p equivariant maps

$$\begin{array}{ccc} S^p \wedge E\Sigma_{p+} \wedge (B^nX)^{\wedge p} & \to & E\Sigma_{p+} \wedge (\tilde{B}^{n+1} X)^{\wedge p} \\ \downarrow \mathrm{id}\wedge\mu_n & & \downarrow \mathrm{id}\wedge\mu_{n+1} \\ S^p \wedge B^{np} X & \to & \tilde{B}^{(n+1)p} X \end{array}$$

For instance, this is possible if X arises as a hyper-Γ-space in the sense of [1S].

We now introduce a further structure. The action of Σ_p on S^{np} given by permutation of coordinates, defines a spherical fibration over $B\Sigma_p$. This fibration is not trivial, but it is conceivable that it becomes fiberhomotopy trivial after restricting to a skeleton of $B\Sigma_p$.

According to [1], this indeed occurs. Given a natural number m , if a sufficiently high power of p divides m , then the vectorbundle defined by cyclic permutation of the coordinates in $\mathbb{R}^{mp}$ is trivial as a vectorbundle on the ~~r-skeleton~~ $(B\Sigma_p)_r$. We choose a trivialization

$$t_n : (E\Sigma_{p+})_r \overset{\Sigma_p}{\wedge} S^{mp} \to S^{mp} .$$

This trivialization can be used to trivialize certain other relevant fibre bundles. Let $\tilde{B}^{mp}X$ be the (mp)-fold deloop of X , considered as a Σ_p-space using the permutation of the coordinates in groups of p. Similarily, let Σ_p act on the smashproduct $(B^mX)^{\wedge p}$ by permutation of the factors of the smash product.

The suspension map $S^1 \wedge B^mX \to B^{m+1}X$ induces equivariant maps

$$(2.4) \quad \begin{cases} S^{np} \wedge (B^mX) \to (B^{m+n}X)^{\wedge p} \\ S^{np} \wedge \tilde{B}^{mp} X \to \tilde{B}^{(m+n)p} X \end{cases}$$

These maps induce maps of fibre bundles over $B\Sigma_p$.
Now, let n be divisible by m .

We choose trivializations t'_m of the bundles over $(B\Sigma_p)_r$ given by $\tilde{B}^{mp}X$, so that the trivializations are compatible with the pairings above. For instance, we can rewrite $\tilde{B}^{mp}X$ as

$$\Omega^{Np} \tilde{B}^{(m+N)p} X$$

with the appropriate action, and then trivialize this bundle, using t_m .

These trivializations restrict to trivializations of $\mathbb{Z}/p$-bundles.

Combining these trivializations with the structure map, for each n divisible by m we obtain a map:

$$f_n \ : \ (E\mathbb{Z}/p_+)_r \overset{\mathbb{Z}/p}{\wedge} (B^n X)^{\wedge p} \to$$

$$\to ((E\mathbb{Z}/p_+)_r \overset{\mathbb{Z}/p}{\wedge} \tilde{B}^{np} X) \xrightarrow{t'_n} (B\mathbb{Z}/p_+)_r \wedge B^{np} X \to B^{np} X \ .$$

Using f_n we obtain a Dyer-Lashof operation

$$\tilde{Q}^i : H_{n+r}(B^n X \ ; \mathbb{Z}/p) \to H_{np+r+i}(B^{np} X \ ; \mathbb{Z}/p) \ .$$

by the formula

$$(2.5) \qquad \tilde{Q}^i (a) = f_{n\,*}(e_{i-r(p-1)} \otimes a^{\otimes p}) \ .$$

Since the trivializations are choosen to be compatible with the stabilization (2.4), the operations also commute with the homology suspension

$$\sigma_m : H_*(B^r X) \to H_{\cancel{*}}\cancel{(B}^{r+m} X) \ .$$

$*+m$

In particular, we can compare them to the usual Dyer-Lashof operations defined on X, without use of any trivializations. We obtain

(?) $$(2.6) \qquad \tilde{Q}^i \sigma_m (a) = \sigma_m \tilde{Q}^i (a) \ .$$

Finally we obtain an operation defined on

$$\lim H_*(B^{p^r} \ ; \mathbb{Z}/p)$$

$B^r X$??

by choosing a mutually compatible family of trivializations t_{pr} , one for each skeleton of $B\mathbb{Z}/p$.

We now consider the map of simplicial spectra, defined in [2]

$$\lambda : S^1_+ \wedge R \to THH(R)$$

We want to compute the map obtained from this using the multiplicative structure on THH(R)

$$E\Sigma_{n+} \overset{\Sigma_n}{\wedge} (S^1_+ \wedge B^m R)^{\wedge n} \to E\Sigma_{n+} \overset{\Sigma_n}{\wedge} THH(R)[m]^{\wedge n} \to THH(R)[mn]$$

on the E^2 - level of the associated spectral sequences. The symbol THH(R)[m] here means the m-fold delooping of the infinite loopspace THH(R) , corresponding to the infinite loopstructure obtained from the additive structure in THH(R) .

In order to do so, we first have to analyze the source of this map, that is, we have to analyze the simplicial set $(S^1)^n$. We consider this set as the diagonal of the multisimplicial set obtained by taking product of n copies of the standard simplicial S^1 , the one with two nondegenerate simplices. The symmetric group Σ_n acts on this simplicial set by permuting the simplicial directions.

Let $\mathcal{T}_n$ be the simplicial space

$$E\Sigma_{n+} \overset{\Sigma_n}{\wedge} \left((S^1)^n_+ \wedge (B^m R)^{\wedge n} \right)$$

The simplicial filtration on $T^n_+ = (S^1)^n_+$ lifts to a filtration

$$\mathcal{T}_0 \subset \mathcal{T}_1 \subset \; . \; . \; . \; \subset \mathcal{T}_n = \mathcal{T} .$$

This filtration induces a spectral sequence

$$E^2_{i,j}(\mathcal{T}) \Rightarrow H_{i+j} (\mathcal{T})$$

The space $\mathcal{T}_i / \mathcal{T}_{i-1}$ can be described in terms of the orbits under Σ_n of nondegenerate simplices in the torus T^n. It is a wedge of spaces to be described below, and the components of the wedge are indexed by such orbits.

The wedge component corresponding to a nondegenerate simplex σ of T^n is homeomorphic to

$$EH_+ \overset{H}{\wedge} S^r \wedge (B^m R)^{\wedge n} \quad .$$

where r is the dimension of the simplex, and H is its isotropy group.

In particular, if σ is in the unique orbit of nondegenerate n-cells, then the wedge component corresponding to σ is equal to

$$S^n \wedge (B^m R)^{\wedge n} \quad .$$

We also consider the more general situation, where σ is in the image of a torus, of dimension possibly less than n. Let

$$\varphi : [1,2,3, \, . \, . \, . \, n] \to [1,2,3, \, . \, . \, i]$$

be a surjection of sets. Then φ defines a diagonal map

$$\varphi_* : (S^1)^i \to (S^1)^n$$

by the formula $\varphi (x_1, x_2 , \, . \, . \, . \,) = (x_{\varphi(1)}, x_{\varphi(2)}, \, . \, . \,)$.

We can assume without loss of generality, that σ is monotone increasing.

The image of T^i will not be invariant under the action of Σ_n . Let

$$H = \Sigma_{\varphi^{-1}(1)} \times \Sigma_{\varphi^{-1}(2)} \times \, . \, . \, . \, \Sigma_{\varphi^{-1}(i)} \quad .$$

Then H is the isotropy group of σ. The normalizer N(H) of H in Σ_n leaves the torus T^i fixed as a set. It acts on the this torus through the map

$$W(H) = N(H)/H \to \Sigma_i \quad .$$

The image of this map is the group of permutations, which leave invariant the function $a \to$ cardinality (φ^{-1} (a)) , defined for $a \in [1,2, \, . \, . \, i]$.

We can now describe the map $\lambda : \mathcal{T}^i \to$ THH(R)[mn] on the quotients of the filtration induced by the simplicial structure.

Lemma 2.7. The map is given in dimension j on the wedgecomponent corresponding to the orbit of σ in $(S^1)^n$ as the following composite:

$$\begin{array}{c}
E\Sigma_H \overset{H}{\wedge} S^j \wedge B^mR \wedge B^mR \wedge \dots \wedge B^mR \\
\Big\downarrow \mathrm{id} \wedge \mu_{\varphi^{-1}(1)} \wedge \mu_{\varphi^{-1}(2)} \wedge \cdots \\
S^j \wedge E\Sigma_{\varphi^{-1}(1)} \overset{\Sigma_{\varphi^{-1}(1)}}{\wedge} B^m(R) \wedge \dots \wedge B^m(R) \wedge E\Sigma_{\varphi^{-1}(2)} \overset{\Sigma_{\varphi^{-1}(2)}}{\wedge} \dots \\
\Big\downarrow \\
S^j \wedge B^{|\varphi^{-1}(1)|m}R \wedge B^{|\varphi^{-1}(2)|m}R \wedge \dots \\
\Big\downarrow \\
F_i\, THH(R)[mn] \,/\, F_{i-1} THH(R)[mn]
\end{array}$$

Proof. We first describe the multiplication map on THH(R) , following [2]. The simplicial infinite loopspace THH(R) is given so that infinite loopspace in degree r has a spectrum , which is a realization of the smashproduct of r+1 copies of the spectrum R. That is, we can approximate the spectrum by

$$\Omega^{m(r+1)}(B^mR^{\wedge(r+1)}) \ .$$

The simplicial infinite loopspace $THH(R)^n$ is then in degree r approximated by

$$\{ \Omega^{m(r+1)} (B^mR^{\wedge(r+1)})\}^n .$$

The structure map μ is defined degreewise, and in degree r it can be approximated by

$$\{ \Omega^{m(r+1)} ((B^mR)^{\wedge(r+1)}) \}^n \to \Omega^{mn(r+1)}((B^mR)^{\wedge n(r+1)}) \to \Omega^{mn(r+1)}((B^{mn}R)^{\wedge r+1}) .$$

The spaces involved all carry a Σ_n-action. Using the trivializations we fixed, we can arrange that the maps extend to maps of bundles over skeletons of $B\Sigma_n$.

The map $S^1_+ \wedge B^mR \to THH(R)[m]$ is degreewise given, up to homotopy, by the inclusion

$$[r] \mapsto (B^mR \vee B^mR \vee \dots B^mR)_+ \to \Omega^{mr}((B^mR)^{\wedge(r+1)}) .$$

Here the component number s in the wedge is included by the adjoint of the map $S^{mr}(B^mR) \to (B^mR)^{\wedge(r+1)}$ which includes B^mR as factor number s in the smash product. The map from $T^n_+ \wedge (B^mR)^{\wedge n}$ to THH(R) is the n^{th} power of this map. To obtain the map of filtration quotients, we only have to compose these maps. First we have to identify the subspace of $\mathcal{T}_i / \mathcal{T}_{i-1}$ corresponding to the simplex σ . This will be a subspace of a smash product of wedges :

$$(B^mR \vee \dots B^mR)^{\wedge n}$$

The subspace will be given, by picking one component of each of the factors. Choose component number one in the first $\varphi^{-1}(1)$ factors, component number two in the next $\varphi^{-1}(2)$ factors, and so on. A direct computation of the composite restricted to this component then proves the lemma.

Corollary 2.8. Let $a \in H_*(B_m R)$. The image of $(\sigma_1 \otimes a^{\otimes n})$ represents

$$(1 \otimes a \otimes a \otimes \ldots \otimes a)$$

in $E^2(THH(R))$. The tensorproduct contains n+1 factors.

Proof. Apply 2.7. The group H is the trivial group, so the composite in 2.7 is just the inclusion of $S^j \wedge (B^mR)^{\wedge n}$ in $F_i THH(R)[mn] / F_{i-1} THH(R)[mn]$.

Now we can compute the twisted Dyer-Lashof operations on the spectrum homology of THH(R) . Let $x \in \lim_m H_{i+m}(B^mR) = \mathcal{R}$. Then the image of

$\sigma_1 \otimes x \in H_{i+m+1}(S^1_+ \wedge B^mR)$ in the homology of THH(R) represents the class $1 \otimes x \in \mathcal{R} \otimes \mathcal{R} = \lim H_{i+m+1} (F_1 \, THH(R)[m] / F_0 \, THH(R)[m])$. According to lemma 2.7, we have a commutative diagram

$$\begin{tikzcd}
E\Sigma_{n+} \overset{\Sigma_n}{\wedge} (S^1_+ \wedge B^mR)^{\wedge n} \arrow[r, "id\wedge(\varphi_1)^{\wedge n}"] \arrow[d] & E\Sigma_{n+} \overset{\Sigma_n}{\wedge} (THH(R)[m])^{\wedge n} \arrow[dd] \\
S^1 \wedge E\Sigma_{n+} \overset{\Sigma_n}{\wedge} (B^mR)^{\wedge n} \arrow[d] & \\
S^1_+ \wedge B^{mn}R \arrow[r] & THH(R)[mn]
\end{tikzcd}$$

In particular, we have

Lemma 2.9 . $\tilde{Q}^i (\lambda (\sigma_1 \otimes x)) = \lambda (\sigma_1 \otimes Q^i (x))$

The second problem which we have to solve, concerns the differentials in the spectral sequence. We want to compute

$$d_{p-1} (1 \otimes x \otimes \ldots \otimes x)$$

for certain $x \in \mathcal{R}$. The interesting case is when the tensor product contains p^i+1 factors .

The argument will be slightly different according to whether i = 1 or i > 1. In both cases, the idea of the proof is to compare THH(R) to the space

$$E\Sigma_{p^i+} \overset{\Sigma_{p^i}}{\wedge} (S^1)^{p^i}_+ \wedge (B^mR)^{\wedge p^i} .$$

As a preliminary, we consider the spectral sequence associated to this space. Actually, we make an additional simplification. Consider a space X which is a suspension. We can form the simplicial space

$$E\mathbb{Z}/p_+ \overset{\mathbb{Z}/p}{\wedge} \left((S^1)^{\wedge p} \wedge X^{\wedge p} \right) .$$

Since $S^1_+ \wedge X \to S^1 \wedge X$ is a split surjection up to homotopy, the Σ_p-equivariant map

$$(S^1_+)^{\wedge p} \wedge X^{\wedge p} \to S^p \wedge X^{\wedge p}$$

is a surjection, up to Σ_p-equivariant homotopy. In particular, the class

$$e_0 \otimes (\sigma_1 \otimes x)^{\otimes p} \in H_*(E\mathbb{Z}/p_+ \overset{\mathbb{Z}/p}{\wedge} (S^1_+ \wedge X)^{\wedge p})$$

is the image of the class

$$e_0 \otimes (\sigma(x))^{\otimes p} \in H_*(E\mathbb{Z}/p_+ \overset{\mathbb{Z}/p}{\wedge} (S^1 \wedge X)^{\wedge p}) .$$

under a homotopy section. We are led to consider the space

$$E\mathbb{Z}/p_+ \overset{\mathbb{Z}/p}{\wedge} \left(S^p \wedge X^{\wedge p} \right) .$$

with the filtration induced by the simplicial structure of $S^p = S^1 \wedge \ldots \wedge S^1$.

Inside this filtration, we have a shorter filtration, consisting of the three spaces

$$* \quad \subset \quad E\mathbb{Z}/p_+ \overset{\mathbb{Z}/p}{\wedge} \left(S^1 \wedge X^{\wedge p} \right) \subset E\mathbb{Z}/p_+ \overset{\mathbb{Z}/p}{\wedge} \left(S^p \wedge X^{\wedge p} \right)$$

Claim 2.12. The quotient of the two nontrivial spaces is homotopy equivalent to

$$S^2 \wedge (S^{p-2}_+ \wedge X^{\wedge p})$$

The boundary map is induced by the equivariant inclusion

$$S^{p-2} \subset S^\infty = E\mathbb{Z}/p$$

In particular, the boundary map on homology is dertermined by the remark after 2.1

To see this, first check that the cofibre of the map

$$S^{p-2}_+ \to S^0$$

which collapses the entire (p-2)-sphere to the basepoint, has cofibre equal to S^{p-1} . Suspending this cofibration , and forming the smashproduct with $E\mathbb{Z}/p$ we obtain a new cofibration

$$E\mathbb{Z}/p_+ \wedge S^1 \overset{\mathbb{Z}/p}{\wedge} X^{\wedge p} \to E\mathbb{Z}/p_+ \wedge S^p \overset{\mathbb{Z}/p}{\wedge} X^{\wedge p} \to E\mathbb{Z}/p_+ \wedge (S^2 \overset{\mathbb{Z}/p}{\wedge} S^{p-2}_+) \wedge X^{\wedge p}$$

The claim follows from this and the observation, that as $\mathbb{Z}/p$ acts freely on S^{p-2}, the following projection is a $\mathbb{Z}/p$-equivariant homotopy equivalence.

$$S^{p-2} \times E\mathbb{Z}/p \to S^{p-2} .$$

We can compute the long exact sequence in homology, belonging to the filtration determined by 2.12. In the notation of 2.1, the differential

$$\delta : H_*(S^2 \wedge S^{p-2}_+ \overset{\mathbb{Z}/p}{\wedge} X^{\wedge p}; \mathbb{Z}/p) \to H_{*-1}(S^1 \wedge E\mathbb{Z}/p_+ \overset{\mathbb{Z}/p}{\wedge} X^{\wedge p} ; \mathbb{Z}/p)$$

is given by the formula

$$\delta (\sigma_2 \otimes e_i \otimes x^{\otimes p}) = \sigma_1 \otimes e_i \otimes x^{\otimes p} .$$

In particular, the boundary of the class $\sigma^2 \otimes e_{p-2} \otimes x^{\otimes p}$ is $\sigma^1 \otimes e_{p-2} \otimes x^{\otimes p}$.

Applying the homotopy section, and noting that the simplicial filtration is a refinement of the short filtration, this shows that in the spectral sequence belonging to the simplicial structure of

$$E\mathbb{Z}/p_+ \overset{\mathbb{Z}/p}{\wedge} (S^1)^p_+ \wedge X^{\wedge p}$$

there is a nontrivial differential ,which maps the class in E^2 projecting to $\sigma_2 \otimes e_{p-2} \otimes x^p$ into a class projecting to $\sigma_1 \otimes e_{p-2} \otimes x^p$.
The two classes will be given by the two classes

$$\sigma_p \otimes x^{\otimes p} \in H_*(S^p \wedge X^{\wedge p})$$

$$\sigma_1 \otimes e_{p-2} \otimes x^{\otimes p} \in H_* (S^1 \wedge E\mathbb{Z}/p_+ \overset{\mathbb{Z}/p}{\wedge} X^{\wedge p})$$

Now, let X be an approximation of B^mR. The map

$$E\mathbb{Z}/p_+ \overset{\mathbb{Z}/p}{\wedge} \left((S^1)^p_+ \wedge X^{\wedge p} \right) \to E\mathbb{Z}/p_+ \overset{\mathbb{Z}/p}{\wedge} THH(R)^p \to THH(R)$$

preserves simplicial filtration, so the classes above will map to two classes in E^2(THH(R)) which are related through a differential d_p .

Applying lemma 2.7, we obtain

Lemma 2.13. Let $x \in H_n(R; \mathbb{Z}/p)$. Then we have the relation

$$d_{p-1}(1 \otimes \underbrace{x \otimes \ldots x}_{p}) = 1 \otimes Q^{np+p-n-1}(x) .$$

Now consider the general case. The symmetric group Σ_{p^i} has a p-Sylow subgroup $S_i(p) \subset \Sigma_{p^i}$. This Sylow subgroup is abstractly isomorphic to an iterated wreath product:

$$S_i(p) \cong \mathbb{Z}/p \wr \mathbb{Z}/p \wr \ldots \wr \mathbb{Z}/p .$$

This group acts on S^{p^i} by permutation of coordinates. The union of all fixed point sets of all nontrivial subgroups of $S_i(p)$ is a union F_{p^i} of (p^i-p+1)-dimensional spheres . The quotient can, using the case i = 1 treated above, be described as a smash product

$$(S^2 \wedge S^{p-2}_+)^{\wedge p^{i-1}}$$

where the action of $S_i(p)$ is induced from the action of $\mathbb{Z}/p$ on $S^2 \wedge S^{p-2}_+$.

The quotient

$$ES_i(p_+) \overset{S_i(p)}{\wedge} S^{p^i} \wedge X^{\wedge p^i} \Big/ ES_i(p_+) \overset{S_i(p)}{\wedge} F_{p^i} \wedge X^{\wedge p^i}$$

is homeomorpic to

$$ES_i(\ p\ )_+ \overset{S_i(\ p\ )}{\wedge} (\ S^2 \wedge S_+^{p-2}\ )^{\wedge p^{i-1}}$$

where the action on $(\ S^2 \wedge S_+^{p-2}\ )^{\wedge p^{i-1}}$ is induced from the action of $\mathbb{Z}/p$ on $S^2 \wedge S_+^{p-2}$. The cofibration giving rise to this quotient, corresponds to the short filtration in the case $i = 1$.

Alternatively, we can describe this quotient as the iterated power construction

$$E\mathbb{Z}/p_+ \overset{\mathbb{Z}/p}{\wedge} (\ E\mathbb{Z}/p_+ \overset{\mathbb{Z}/p}{\wedge} \ldots (\ S^2 \wedge S_+^{p-2}\ )^{\wedge p}\ )^{\wedge p}\ ) \ldots)^{\wedge p}\ .$$

The next highest quotient in the filtration of

$$ES_i(\ p\ )_+ \overset{S_i(\ p\ )}{\wedge} S^{p^i} \wedge X^{p^i}$$

induced from the fixed point sets in S^{p^i} is the space

$$(\ S^1 \wedge\ E\mathbb{Z}/p_+ \overset{\mathbb{Z}/p}{\wedge} X^{\wedge p}\ ) \wedge (\ (\ S^2 \wedge (\ S_+^{p-2} \overset{\mathbb{Z}/p}{\wedge} X^p\ )\ )^{\wedge(p-1)}\ ) \ldots \qquad .$$

We consider the somewhat more general situation, where we have a cofibration

$$X \to Y \to Z\ .$$

Then the p-power construction on Y has a filtration through the spaces

$$C_i = \{\ (e,\ y_1\ ,\ y_2\ ,\ \ldots\ y_p\ ) \in E\mathbb{Z}/p_+ \overset{\mathbb{Z}/p}{\wedge} Y^p\ ;\ \text{at most } i \text{ of } y_1\ ,\ y_2\ ,\ \ldots \text{ not in } X\ \}$$

Lemma 2.14. In this situation, if $z \in H_*(\ Z\ ;\ \mathbb{Z}/p\ )$, $x = \delta z \in H_*(\ X\ ;\ \mathbb{Z}/p\ )$, then

$$d_2\ (\ e_0 \otimes z^p\ ) = x \otimes z^{p-1}\ .$$

Here we have made the identifications

$$E^2_{p,*}\ (\ C\ ) = H_*(\ E\mathbb{Z}/p_+ \overset{\mathbb{Z}/p}{\wedge} Z^p\ )$$

$$E^2_{p-1,*}\ (\ C\ ) = H_*(\ X \wedge Z^{p-1}\ )\ .$$

Proof. Pick a chain $\bar{z}$ in $C_*(\ Y\ )$ which represents z after projection to $C_*(\ Z\ )$. Then, $\delta\ (\ e_0(\ \bar{z}\ )\ ) = (\ \delta\ \bar{z}\ ) \otimes \bar{z}^{\ p-1}$ represents the boundary of $e_0 \otimes z^p$ in $H(\ C_{p-1}\ )$. The claim follows, after reduction to homology.

W apply 2.14 to the filtration given by the inclusion

$$ES_i\ (\ p\ )_+ \overset{S_i(\ p\ )}{\wedge} F_{p^i} \wedge X^{\wedge p^i} \quad \subset \quad ES_i\ (\ p\ )_+ \overset{S_i(\ p\ )}{\wedge} S^{p^i} \wedge X^{\wedge p^i}$$

Inductively in i, each such cofibration arises from the previous one as the inclusion $C_{p-1} \subset C_p$. By repeated application of 2.14, the boundary of

$$e_0 \otimes (\ e_0 \ldots (\ \sigma_2 \otimes e_{p-2} \otimes x^{\otimes p}\ )^{\otimes p} \quad \ldots)^{\otimes p}$$

is given by

$$\big(\delta\,(\,\sigma_2 \otimes\; e_0 \otimes x^{\otimes p}\,)\big) \otimes (\,e_0 \otimes (\,\sigma_2 \otimes x^{\otimes p}\,)\,)^{\otimes p-1} \otimes \dots \quad .$$

By an application of 2.12 we finally obtain that the boundary equals

$$(\,\sigma_1 \otimes\; e_0 \otimes x^{\otimes p}\,) \otimes (\,e_0 \otimes (\,\sigma_2 \otimes x^{\otimes p}\,)\,)^{\otimes p-1} \otimes \dots$$

The rest of the argument is similar to the case $i = 1$. We obtain

Lemma 2.15. Let $x \in H_n(\,R\,;\,\mathbb{Z}/p\,)$. Let

$$\gamma_i(x) = 1 \otimes x \otimes x \dots \otimes x = \quad \in \mathcal{R}^{p^i+1}$$

Then the differential in the spectral sequence converging to spectrum homology of topological Hochschild homology is given by

d_{p-1} ?

$$d_p(\gamma_j(\,x\,)\,) = (\,1 \otimes Q^{np+p-n-1}(x))\,(\,\gamma_1(x))^{p-1}\,(\,\gamma_2(x))^{p-1} \dots (\gamma_{i-1}(x))^{p-1} \quad .$$

In particular, $d_i(\gamma_j(x)) = 0$ for $i \leq p-1$. d_{p-1} ??

We now specialize 2.9 and 2.15 to the cases $R = \mathbb{Z}/p$ and $R = \mathbb{Z}$. These applications depend on the determination of the relevant Dyer-Lashof operations for these rings. We collect these computations in the next section.

§3. In this paragraph we prove the technical lemmas 1.2 to 1.5.

We use the computations in §2 specialized to the case $\mathbb{Z}/p$. These computations relate differentials and extension problems in the spectral sequence converging to spectrum homology of $THH(\mathbb{Z}/p)$ to questions about the map

$$\mu : E\mathbb{Z}/p \;\times\; K(\mathbb{Z}/p,\,n\,)^{\wedge\, p} \;\to\; K(\mathbb{Z}/p, np\,)$$

classifying the cup product.

Recall from §1 the classes τ_i and ξ_i. Let n be large enough, so that these are defined in the homology of $K(\mathbb{Z}/p,\,n)$.

Lemma 3.1 . If n is large enough (in dependence of i), then

$$\mu_*\,(\,e_{p-2} \otimes \tau_i \otimes \dots \otimes \tau_i\,) \;=\; (\text{ unit }) \,\xi_{i+1} + (\text{ decomposable }) \quad ; \quad p \text{ odd}$$

$$\mu_*\,(\,e_{p-1} \otimes \tau_i \otimes \dots \otimes \tau_i\,) \;=\; (\text{ unit })\, \tau_{i+1} + (\text{ decomposable }) \quad ; \quad p \text{ odd}$$

$$\mu_*\,(\,e_1 \otimes \xi_i \otimes \xi_i\,) \;=\; \xi_{i+1} + (\text{ decomposable }) \quad ; \quad p = 2$$

Proof: For p odd, let $Q_0 = \beta$, the Bockstein, P^i the Steenrod powers, $R_1 = P^1$ and inductively

$$R_i = P^{p^i} R_{i-1} - R_{i-1} P^{p^i} \qquad 2p^i - 2$$

$$Q_i = P^{p^{i-1}} Q_{i-1} - Q_{i-1} P^{p^{i-1}} \qquad \text{dim } 2p^i - 1$$

Then Q_i is a primitive cohomology operation, dual to τ_i , and R_i is a primitive cohomology operation, dual to ξ_i .

For p = 2, let M_0 be the Bockstein, and let inductively

$$M_i = Sq^{2^i} M_{i-1} + M_{i-1} Sq^{2^i} \quad . \; 2^{i+1}-1, \text{ dual to } \xi_{i+1}, \; ??$$

We have to prove that (?) Q_{i+1} , R_{i+1} , and M_{i+1} evaluate nontrivially on the classes ($e_{p-2} \otimes \tau_i \otimes \ldots \otimes \tau_i$) , ($e_{p-1} \otimes \tau_i \otimes \ldots \otimes \tau_i$) and ($e_1 \otimes \xi_1 \otimes \xi_1$) respectively.

The case i = 0 is covered by the calculation of $P^1 = R_1$ in

$$H_* (E\mathbb{Z}/p_+ \overset{\mathbb{Z}/p}{\wedge} X^p ; \mathbb{Z}/p)$$

This calculation is implicit in [11]. For an explicit formula, see [7] theorem 9.4.

We claim that the general case reduces to the case i = 0 by induction. We prove the statements first in the case n = 1, and then use a product argument to obtain the case n > 1. For similar arguments, see [8].

We pass to cohomology. Recall that

$$H^* (K(\mathbb{Z}/2 , 1) ; \mathbb{Z}/2) = P(\iota)$$

$$H^* (K(\mathbb{Z}/p , 1) ; \mathbb{Z}/p) = \Lambda (\iota) \otimes P (\beta \iota)$$

where P( ) and Λ( ) denote an polynomial and an exterior algebra respectively.

We claim inductively in i that the following formulas are valid:

$$Q_i (e^0 \otimes \iota \otimes \ldots \otimes \iota) = (\text{unit}) \, e^{p-1} \otimes Q_{i-1}\iota \otimes \ldots Q_{i-1}\iota$$

$$R_i (e^0 \otimes \iota \otimes \ldots \otimes \iota) = (\text{unit}) \, e^{p-2} \otimes Q_{i-1}\iota \otimes \ldots Q_{i-1}\iota$$

$$M_i (e^0 \otimes \beta \iota \otimes \beta \iota) = e^1 \otimes M_{i-1}\iota \otimes M_{i-1}\iota$$

$$Q_i (e^0 \otimes \beta \iota \otimes \ldots \otimes \beta \iota) = 0$$

$$R_i (e^0 \otimes \beta \iota \otimes \ldots \otimes \beta \iota) = 0 \quad .$$

To prove these assertions, consider the projection

$$E\mathbb{Z}/p_+ \overset{\mathbb{Z}/p}{\wedge} (K(\mathbb{Z}/p , 1)_+)^{\wedge p} \to E\mathbb{Z}/p_+ \overset{\mathbb{Z}/p}{\wedge} (K(\mathbb{Z}/p , 1))^{\wedge p}$$

determined by a choice of basepoint. This map is injective on cohomology, so it suffices to prove our assertions for the source of the map.

In this space, we have a cup product decomposition

$$(e^{p-2} \otimes Q_{i-1}\iota \otimes \ \ldots Q_{i-1}\iota) = (e^{p-2} \otimes 1 \otimes \ldots \otimes 1) (e^{0} \otimes Q_{i-1}\iota \otimes \ \ldots Q_{i-1}\iota)$$

Using the Cartan formula for the primitive operation Q_{i-1} , and the relation that $P^{p^i}(x) = 0$ for deg(x) smaller than $2p^i$, since $i > 0$, so that $P^{p^i} (\iota) = 0$, we see:

$$Q_i (e^0 \otimes \iota \otimes \ldots \otimes \iota) = (P^{p^i} Q_{i-1} - Q_{i-1} P^{p^i}) (e^0 \otimes \iota \otimes \ldots \otimes \iota) =$$
$$= e^{p-\cancel{2}\,1} \otimes Q_{i-1}\iota \otimes \ \ldots Q_{i-1}\iota \qquad P^{p^k}(e^{p-1})$$

This proves the first assertion in the list, since the case $i = 0$ is known. The other assertions follows in the same way.

To get from our assertions about classifying spaces $K(\mathbb{Z}/p , 1)$ to the lemma, we again use the multiplicative structure. There is a map

$$f : (K (\mathbb{Z}/p , 1)_+)^{\wedge m} \wedge (K (\mathbb{Z}/p , 1)_+)^{\wedge n} \to K (\mathbb{Z}/p , m+2n)$$

classifying the cohomology class

$$(\iota \otimes \iota \otimes \ldots \otimes \iota) \otimes (\beta \iota \otimes \beta \iota \ldots \otimes \beta \iota) .$$

This map is injective on cohomology in small dimensions. To see this, recall that the dual of the Steenrod algebra is generated by classes defined in $K(\mathbb{Z}/p , 1)$ and $K(\mathbb{Z}/p , 2)$. Thus, we only have to prove that

$$Q_i (e^0 \otimes a \otimes a \otimes \ldots \otimes a) = (\text{unit})\, e^{p-1} \otimes Q_{i-1}a \otimes Q_{i-1}a \otimes \ldots \otimes Q_{i-1}a$$

$$P_i (e^0 \otimes a \otimes a \otimes \ldots \otimes a) = (\text{unit})\, e^{p-2} \otimes Q_{i-1}a \otimes Q_{i-1}a \otimes \ldots \otimes Q_{i-1}a$$

This follows from our formulas for $n = 1$ and the Cartan formula.

In case $p = 2$, we note that the map

$$K (\mathbb{Z}/2 , 1)_+)^{\wedge n} \to K (\mathbb{Z}/2 , n)$$

is injective on homology in small dimensions, and use the Cartan formula.

We can now prove the lemmas in §1.

Proof of 1.2. and 1.4.

According to lemma 2.9 we have

$$\begin{cases} [\lambda (\sigma_1 \otimes \xi_i)]^{\otimes 2} \quad Q^{2^i} (\lambda (\sigma_1 \otimes \xi_i)) = \lambda (\sigma_1 \otimes Q^{2^i} (\xi_i)) \\ [\lambda (\sigma_1 \otimes \tau_j)]^{\otimes p} = Q^{2p^{j+1}-2p^j}(\lambda (\sigma_1 \otimes \tau_j)) = \lambda (\sigma_1 \otimes Q^{2p^{j+1}-2p^j}(\tau_j)) . \end{cases}$$

The Lemma 3.1 says that $Q^{2^i} (\xi_i) = \mu_* (e_1 \otimes \xi_i \otimes \xi_i) = (\text{unit})\, \xi_{i+1} +$ decomposables, and similarily for τ_i . Since $\lambda (1 \otimes \xi_i)$ is a particular choice of a class representing λ_i , Lemmas 1.2 follows. Similarily for 1.4.

Proof of 1.3. This follows from lemma 2.15 , and the computation

$$Q^{np+p-n-2}(\tau_i) = \mu_*(e_{p-2} \otimes \tau_i \otimes \tau_i \otimes \ldots \otimes \tau_i) = \xi_i . \quad \xi_{i+1}$$

Proof of 1.5. This is again lemma 2.9, applied to the case

$$Q^4(\varphi(\sigma_1 \otimes \eta)) = \varphi(\sigma_i \otimes (Q^4(\eta)))$$

Since $Q^4(\eta)$ has dimension 6, it is decomposable. It follows that $\varphi(\sigma_i \otimes (Q^4(\eta)))$ is trivial.

For later reference, we also note

Lemma 3. 2. Let $\lambda : S^1_+ \wedge \mathbb{Z} \to THH(\mathbb{Z})$ be the map of spectra discussed in §2. Then, the image of the homology class $\sigma \otimes \xi_i$ unde λ represents

$$1 \otimes \xi_i \quad \epsilon \quad E^2(THH(\mathbb{Z})) .$$

In particular, under the homotopy equivalence of theorem 1.1, the fundamental class in cohomology of $K(\mathbb{Z}/p^i, 2p^i - 1)$ pulls back to a class evaluating nontrivially on $\sigma \otimes \xi_i$.

Proof. The first statement is a particular case of 2.7. The second statement follows from this and from the fact that the $1 \otimes \xi_i$ generates E^2 in this dimension (see the analysis of the spectral sequence in §1).

References

[1] Atiyah M. F. , Characters and cohomology of finite groups, Publ. IHES Vol 9 , Paris (1961).

[2] Bökstedt M. , Topological Hochschild homology, to appear.

[3] Bökstedt M. , The natural transformation from $K(\mathbb{Z})$ to $THH(\mathbb{Z})$, to appear.

[4] Cartan H. and Eilenberg S., " Homological algebra " , Princeton university press, Princeton (1956)

[5] Kassel C. La K-theorie stable , Bull. Soc. Math. France Vol 110, 381-416 (1982)

[6] Madsen I. Higher torsion in SG and BSG. Math. Z. 143, 55-80 (1975)

[7] May J. P. , A general algebraic approach to Steenrod operations, Lecture Notes in Math. 168, 153-231, Springer , Berlin (1970)

[8] Milgram R. J., " Unstable homotopy from the stable point of view " , Lecture Notes in Math. , Vol 368, Springer, Berlin-Heidekberg-New York (1974)

[9] Milnor J. The Steenrod algebra and its dual, Annals of Math. , Vol 67 150-171 (1958).

[10] Segal G., Categories and cohomology theories, Topology Vol 13 , 293-312 (1974).

[11] Steenrod N. E. and Epstein B. A. " Cohomology operations " , Princeton university press, Princeton (1962).

[12] Waldhausen F., Algebraic K-theory of topological spaces I, Proc. Symp. Pure Math., Vol 32, 35-60 AMS (1978).

[13] Waldhausen F., Algebraic K-theory of topological spaces II, Lecture notes in mathematics, Vol 763 , 356-394 , Springer, Berlin (1979).

[14] Waldhausen F., An outline of how manifolds relate to algebraic K-theory, to appear in proceedings of the Durham conference on homotopy theory (1985), edited by J. Jones.

[15] Woolfson R., Hyper-Γ-spaces and hyperspectra, Quart. J. Math, Vol 30 229-255 (1979).