The natural transformation from K(Z) to THH(Z)

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In [2], [3] we have studied the topological Hochschild homology. In particular we have shown that there is a map

$$K(R) \longrightarrow THH(R)$$

of rings up to homotopy. In this paper, we will show that this map for $R = \mathbb{Z}$ is nontrivial on homotopy groups in positive degrees. As an application we will show that the map

$$K(\mathbb{Z}) \longrightarrow K^{et}(\mathbb{Z})$$

[4] is not a 2-primary equivalence, in contradiction to a conjecture in [5].

I want to thank I.Madsen for pointing out an error in an earlier version of this paper.

Let R be an FSP in the sense of [2]. This determines a ring up to homotopy [8].

R is a functor from the category of finite, pointed, simplicial spaces, with a product

$$\mu : R(X) \wedge R(Y) \to R(X \wedge Y)$$

The product is assumed to be associative, with a unit. It is also assumed that the limit system $\Omega^n R(S^n)$ stabilizes.

The corresponding ring is

$$\lim_n \Omega^n R(S^n)$$

Examples are the identity functor, and the functor which to a simplicial set associates the free abelian simplicial group generated by it. These examples correspond to the rings up to homotopy QS^0 resp. $\mathbb{Z}$.

For such an R , we defined the K-theory $K(R)$ and the topological Hochschild homology $THH(R)$. There are maps [2]:

$$(\text{homotopy units of } \lim \Omega^n F(S^n)) \to K(F) \to THH(F)$$

By naturality, we obtain a commutative diagram

$$\begin{array}{ccc} BG & \longrightarrow & B\,\mathbb{Z}/2 \\ \downarrow & & \downarrow \\ A(*) & \longrightarrow & K(\mathbb{Z}) \\ \downarrow & & \downarrow \\ QS^o = THH(QS^o) & \longrightarrow & THH(\mathbb{Z}). \end{array}$$

Theorem 1.1. The map

$$\pi_{2p-1}(K(\mathbb{Z})) \longrightarrow \pi_{2p-1}(THH(\mathbb{Z})) \approx \mathbb{Z}/p$$

is surjective.

The rest of this section is devoted to a proof of 1.1. In [11] it is shown that the boundary map

$$\pi_{2p-1}(K(\mathbb{Z})) \longrightarrow \pi_{2p-2}(K(\mathbb{Z}),A(*))$$

is surjective onto the first nontrivial homotopy group. In addition

$$\mathbb{Z}/p \approx \pi_{2p-2}(B\,\mathbb{Z}/2,\ BG) \longrightarrow \pi_{2p-2}(K(\mathbb{Z}),A(*))$$

is an isomorphism. It follows that 1.1 reduces to the statement

Claim 1.2. $\pi_{2p-2}(BSG) \longrightarrow \pi_{2p-2}(QS^o, THH(\mathbb{Z}))$ is surjective onto the image of

$$\partial_{THH} : \pi_{2p-1}(THH(\mathbb{Z})) \longrightarrow \pi_{2p-2}(QS^o, THH(\mathbb{Z})).$$

Recall from [3] that $THH(\mathbb{Z})$ is a product of Eilenberg-MacLane spectra, so that

$$\pi_i\, THH(\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i = 2j \\ \mathbb{Z}/j & i = 2j-1\ . \end{cases}$$

It is also known that if p is odd

$$\pi_i(QS^o)_p = \begin{cases} \mathbb{Z}/p & i = 0 \\ 0 & 0 < i < 2p-3 \\ \mathbb{Z}/p & i = 2p-3 \\ 0 & 2p-3 < i < 4p-3 \end{cases}$$

and

$$\pi_2(QS^o) \approx \mathbb{Z}/2\ .$$

The map $QS^0 \longrightarrow THH(\mathbb{Z})$ factors over the inclusion $\mathbb{Z} \longrightarrow THH(\mathbb{Z})$. In particular, it is trivial on homotopy groups in positive dimensions. It follows that

$$\pi_{2p-2}(QS^0, THH(\mathbb{Z})) = \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z}/2 & p = 2 \\ \mathbb{Z}/p & p \text{ odd}. \end{cases}$$

Let R be a commutative FSP. We have the following commutative diagram

1.3
$$\begin{array}{ccc} S^1 \times R^* & \longrightarrow & S^1 \times R \\ \downarrow & & \downarrow \\ \Lambda B R^* & \longrightarrow & THH(R) \\ & \searrow \quad \swarrow & \\ & BR^* & \end{array}$$

where $S^1 \times R^* \longrightarrow \Lambda BR^*$ is obtained by the inclusion of R^* in the free loop space on BR^* twisted by the circle action on ΛBR^*.

The map $R^* \longrightarrow R$ is the inclusion of the homotopy units in R, and the map

$$S^1 \times R \longrightarrow THH(R)$$

is given by combining the inclusion $R \longrightarrow THH(R)$ with the action of S^1 on $THH(R)$. For details, see [2].

The relevance to us is that the map $BR^* \longrightarrow THH(R)$ factors as

$$BR^* \longrightarrow K(R) \longrightarrow THH(R).$$

We can get a hold on this map by computing the map

$$S^1 \times R \longrightarrow THH(R)$$

and then applying diagram 1.3.

In [3] it is shown that the following map is nontrivial:

$$S^1 \wedge \underline{\underline{\mathbb{Z}}} \xrightarrow{f} S^1_+ \wedge \underline{\underline{\mathbb{Z}}} \xrightarrow{g} \underline{\underline{THH(\mathbb{Z})}} .$$

Here f is the map splitting up to homotopy the map given by a choice of basepoint. The map g corresponds to the map of spaces

$$S^1 \times \mathbb{Z} \longrightarrow THH(\mathbb{Z})$$

It is proved in [3], lemma 3.2, that the composition of the latter map with the map detecting $\pi_{2p-1}(THH(\mathbb{Z})) \approx \mathbb{Z}/p$,

$$THH(\underline{\underline{\mathbb{Z}}}) \longrightarrow S^{2p-1} \wedge \underline{\underline{\mathbb{Z}}}/p ,$$

represents the Steenrod power P^1.

Consider the diagram of spectra

$$\begin{array}{ccccc} S^1_+ \wedge \underline{\underline{QS^o}} & \longrightarrow & S^1 \wedge \underline{\mathbb{Z}} & \longrightarrow & S^1_+ \wedge (\underline{\mathbb{Z}} / \underline{\underline{GS^o}}) \\ \downarrow & & \downarrow & & \downarrow \\ \underline{\underline{QS^o}} = \underline{\underline{THH(QS^o)}} & \longrightarrow & \underline{\underline{THH(\mathbb{Z})}} & \longrightarrow & \underline{\underline{THH(\mathbb{Z})}} / \underline{\underline{THH(QS^o)}} \end{array} .$$

The map $THH(QS^o) \longrightarrow THH(\mathbb{Z})$ factors over the inclusion $\mathbb{Z} \longrightarrow THH(\mathbb{Z})$.

We obtain a new diagram at spectra.

$$\begin{array}{ccccc} S^1 \wedge \underline{\underline{QS^o}} & \longrightarrow & S^1 \wedge \underline{\mathbb{Z}} & \longrightarrow & S^1 \wedge (\underline{\mathbb{Z}} / \underline{\underline{GS^o}}) \\ \downarrow & & \downarrow & & \downarrow \\ \underline{\underline{\mathbb{Z}}} & \longrightarrow & \underline{THH(\mathbb{Z})} & \longrightarrow & S^{2p-1} \wedge \underline{\mathbb{Z}/p} \; ! \end{array}$$

The nontrivial p-torsion in $\pi_{2p-3}\underline{\underline{(QS^o)}}$ is detected by p^1, [10], VI. 5.2. Thus, the generator a of the p-torsion

$$\pi_{2p-2}(S^1 \wedge \underline{\underline{QS^o}})$$

is detected by P^1 in the sense that P^1 is nontrivial on the cofibre of a. It follows that $\pi_{2p-1}(S^1 \wedge \underline{\mathbb{Z}}/\underline{\underline{QS^o}})$ maps nontrivially into $\pi_{2p-1}(S^{2p-1} \wedge \underline{\underline{\mathbb{Z} / p}})$.

The corresponding map on the space level is a map of cofibres

$$\begin{array}{ccccc} S^1_+ \wedge QS^o & \longrightarrow & S^1_+ \wedge \mathbb{Z} & \longrightarrow & C_1 \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & THH(\mathbb{Z}) & \longrightarrow & C_2 \end{array} .$$

C_1 and C_2 are wedges indexed by $\mathbb{Z}$. Translation by 1 induces homeomorphisms of the cofibres permuting the wedge components. The wedge component corresponding to the 0-component of $S^1_+ \wedge \mathbb{Z}$ resp. $THH(\mathbb{Z})$ are mapped as the cofibres in the following diagram.

$$\begin{array}{ccccc} S^1 \times (QS^o)_o & \longrightarrow & S^1 & \longrightarrow & C \simeq S^1 \wedge S^1_+ \wedge (QS^o)_o \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & THH(\mathbb{Z})_o & \longrightarrow & THH(\mathbb{Z})_o \end{array}$$

Since the generator of the p-torsion of $\pi_{2p-1}(S^1_+ \wedge (\Sigma^1 (\underline{\underline{QS^o}})_o)$ can be desuspended to a generator of

$$\pi_{2p-1}(C) \approx \mathbb{Z} / p$$

this group maps isomorphically to the corresponding homotopy group of $THH(\mathbb{Z})$.

We can translate this statement into the corresponding statement about the 1-component.

Using 1.3 we conclude that the composite

$$S^1_+ \wedge S^1 \wedge SG \longrightarrow S^1 \wedge (\Lambda\, BSG) \longrightarrow THH(\mathbb{Z})_1$$

is nontrivial on π_{2p-1} . The image of

$$\pi_{2p-1}(S^1 \wedge BSG) \longrightarrow \pi_{2p-1}(S^1 \wedge \Lambda\, BSG)$$

agrees with the image of

$$\pi_{2p-1}(S^1 \wedge S^1_+ \wedge SG) \longrightarrow \pi_{2p-1}(S^1 \wedge \Lambda\, BSG) .$$

Claim 1.2 follows.

§ 2. The contradiction.

Let $K^{et}(R)$ denote the étale K-theory [4] of the ring R. There is a map

$$K(R) \longrightarrow K^{et}(R) .$$

Let p be a prime. It is conjectured that if $1/p \in R$, then the map above is close to being an isomorphism after completing at p.

For instance, in [5] it is conjectured that

$$K(\mathbb{Z}[\tfrac{1}{2}])^\wedge_2 \longrightarrow K^{et}(\mathbb{Z}[\tfrac{1}{2}])^\wedge_{(2)}$$

is a homotopy equivalence.

In this paragraph, I will show that this cannot be the case. From now on, all spaces will be completed at 2.

Recall that there is a fibre square of rings up to homotopy [1], [5]

$$\begin{array}{ccccc} K^{et}(\mathbb{Z}[\frac{1}{2}]) & \longrightarrow & K(\mathbb{R}) & = & \mathbb{Z} \times BO \\ \downarrow & & \downarrow & & \\ K(\mathbb{F}_3) & \longrightarrow & K(\mathbb{C}) & = & \mathbb{Z} \times BU \end{array} .$$

Assume that the map $K(\mathbb{Z}[\frac{1}{2}]) \longrightarrow K^{et}(\mathbb{Z}[\frac{1}{2}])$ is a homotopy equivalence. We want to derive a contradiction from this.

We can reconstruct the homotopy type of $K(\mathbb{Z})$ using the localization sequence

$$K(\mathbb{Z}/2) \longrightarrow K(\mathbb{Z}) \longrightarrow K(Z[\tfrac{1}{2}])$$

and Quillen's computation [7]

$$K(\mathbb{Z}/2) \xrightarrow{\ \sim\ } \mathbb{Z}$$

(everything is completed at 2!).

Consider the commutative diagram of rings up to homotopy

$$\begin{array}{ccc} QS^0 & \longrightarrow & K(\mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & THH(\mathbb{Z}) \end{array}$$

We claim that if $K(\mathbb{Z})$ as above is given by the assumption $K(\mathbb{Z}[\frac{1}{2}]) = K^{et}(\mathbb{Z}[\frac{1}{2}])$ we obtain a contradiction. This contradiction arises on comparing the infinite loop maps of the 0-component and the 1-components in the diagram.

The map $QS^0 \longrightarrow K(\mathbb{Z}) \longrightarrow K^{et}(\mathbb{Z})$ factors over the ring J. There are fibrations

$$\mathbb{Z} \times BSO \longrightarrow J \longrightarrow K^{et}(\mathbb{Z}[\tfrac{1}{2}])_i$$

where $i = 0,1$, and X_i denotes the i^{th} component at X. It follows that we have diagrams of infinite loop spaces whose rows are fibrations

$$\begin{array}{ccccc} J_i & \longrightarrow & K^{et}(\mathbb{Z}[\frac{1}{2}])_i & \longrightarrow & B(\mathbb{Z} \times BSO) \\ \uparrow & & \uparrow & & \uparrow \\ (QS^0)_i & \longrightarrow & K(\mathbb{Z})_i & \longrightarrow & X_i \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & THH(\mathbb{Z})_i & \longrightarrow & THH(\mathbb{Z})_i \end{array}$$

Under our assumption, we would have factorizations by infinite loop maps

$$K(\mathbb{Z})_i \xrightarrow{f_i} BBSO \xrightarrow{g_i} THH(\mathbb{Z})_i \ .$$

There is a fibration, see for instance [6], § 24.

$$U \longrightarrow B(\mathbb{Z} \times BSO) \longrightarrow BSpin$$

This is related to the description of $K^{et}(\mathbb{Z})$. The map $K(\mathbb{F}_3) \longrightarrow \mathbb{Z} \times BU$ occurring in the pullback computing $K^{et}(\mathbb{Z}[\frac{1}{2}])$ is the fibre of a map

$$\mathbb{Z} \times BU \longrightarrow BU$$

which on the 0-component is given by $\psi^3 - id$, and on the 1-component by ψ^3/id.

It follows that $K^{et}(\mathbb{Z}[\frac{1}{2}])_i$ is homotopy equivalent to the fibre of the map

$$BO \longrightarrow BU \xrightarrow{\psi^3 - 1} BU$$

and that under our assumption $K(\mathbb{Z})_i$ is the fibre of the map

$$BSO \xrightarrow{\psi^3 - 1} BSU \ .$$

In other words, we have commutative diagram

$$\begin{array}{ccccc} Spin & \longrightarrow & SU & \longrightarrow & BBSO \\ \downarrow & & \downarrow & & \| \\ J_i & \longrightarrow & K(\mathbb{Z})_i & \longrightarrow & BBSO \end{array}$$

In § 3 we prove the following

Lemma 2.2 a) $H^7(BBSO ; \mathbb{Z}/4) \approx \mathbb{Z}/4 \oplus \mathbb{Z}/2$

b) The map $f : H^7(BBSO ; \mathbb{Z}/4) \longrightarrow H^7(SU ; \mathbb{Z}/4)$ is trivial on 2-torsion

c) A generator g of $\mathbb{Z}/4 \subset H^7(BBSO ; \mathbb{Z}/4)$ maps nontrivially under f^*

d) The reduction of g to $H^7(BBSO ; \mathbb{Z}/2)$ has coproduct

$$\Delta g = g \otimes 1 + 1 \otimes g + a \otimes Sq^1 a + Sq^1 a \otimes a$$

where a and $Sq^1 a$ generate

$H^3(BBSO ; \mathbb{Z}/2) \cong \mathbb{Z}/2$ and
$H^4(BBSO ; \mathbb{Z}/2) \cong \mathbb{Z}/2$, respectively.

The maps

$$H^3(THH(\mathbb{Z})_i ; \mathbb{Z}/2) \longrightarrow H^3(BBSO ; \mathbb{Z}/2)$$

are injective by 1.1.

Recall from [2], theorem 1.1. that the unique element x of

$$H^7(THH(\mathbb{Z})_i ; \mathbb{Z}/2)$$

which is reduced from $H^7(,\mathbb{Z}/2)$ is primitive if $i = 0$, and has diagonal

$$\Delta x = x_7 \otimes 1 + 1 \otimes x_7 + x_3 \otimes Sq^1 x_3 + Sq^1 x_3 \otimes x_3$$

if $i = 1$. It follows that x_7 maps nontrivially if and only if $i = 1$. But then

$$H^7(THH(\mathbb{Z})_i ; \mathbb{Z}/4) \longrightarrow H^7(BBSO ; \mathbb{Z}/4)$$

hits an element of order 4 if and only if $i = 1$.

From Lemma 2.2 it follows that the composite

$$SU \longrightarrow K(\mathbb{Z}) \longrightarrow BBSO \longrightarrow THH(\mathbb{Z})_i$$

induces a nontrivial map on

$$H^7(- ; \mathbb{Z}/4)$$

if and only if $i = 1$. Since the two maps $SU \longrightarrow THH(\mathbb{Z})$ only differ by a translation, this gives the contradiction.

§ 3. Homology of BBSO

There is a fibration

$$SU \xrightarrow{f} BBSO \longrightarrow B\,Spin\ .$$

The purpose of this section is to show the following statement.

Lemma 1) The map $f : H^7(BBSO\ ;\ \mathbb{Z}/4) \longrightarrow H^7(SU\ ;\ \mathbb{Z}/4)$ is trivial on 2-torsion.

2) $H^7(BBSO\ ;\ \mathbb{Z}/4) \simeq \mathbb{Z}/4 \oplus \mathbb{Z}/2$.

A generator of $\mathbb{Z}/4$ maps nontrivially under f^*.

The main reference is [9]. In the notation of Stasheff

$$H^*(B\,Spin\ ;\ \mathbb{Z}/2) \approx \mathbb{Z}/2[w_i \mid i \neq 2^j + 1]$$
$$H^*(BBSO\ ;\ \mathbb{Z}/2) \approx E[c_i \mid j \geq 3].$$

Further, the image of w_i is e_j for $i \neq 2^j + 1$, and e_{2^j+1} maps to an indecomposable in $H^*(SU\ ;\ \mathbb{Z}/2)$.

$H^7(BBSO\ ;\ \mathbb{Z}/2)$ is generated by $e_3\,e_4$ and e_7. Since e_4 and e_7 are in the image of $H^*(B\,Spin)$, it follows that f^* is trivial on $H^7(BBSO\ ;\ \mathbb{Z}/2)$. We now have to examine the higher torsion of BBSO.

Sq^1 is given on $H^*(BBSO\ ;\ \mathbb{Z}/2)$ as follows (for $n \leq 8$)

n	0	1	2	3	4	5	6	7	8
H^nBBSO	1	0	0	$e_3 \longrightarrow$	e_4	e_5	$e_6 \longrightarrow$	e_7	e_8
								e_3e_4	$e_3e_5 \longrightarrow e_4e_j$.

There is a higher Bochstein of order 8 connecting e_3e_4 and e_8; e_5 is the reduction of a free generator of $H^5(BBSO\ ;\ \mathbb{Z})$.

$$H^*(Spin\ ;\ \mathbb{Z}/2) \approx \mathbb{Z}/2\ [t_3\ t_5\ t_7\ \ldots\ ]$$

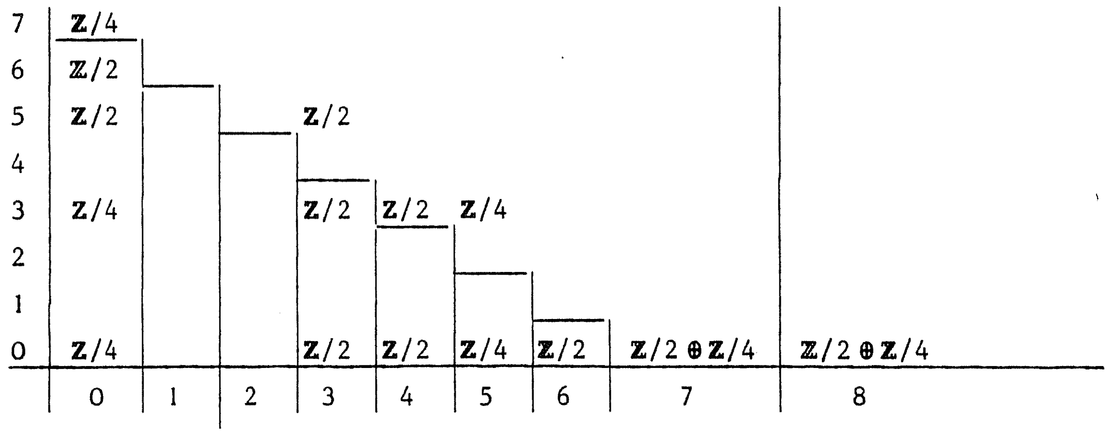
where $(t_{2p+1})^{2^i}$ suspends to $w_{(2p+1)2^i+1}$ in $H^*(B\,Spin\ ;\ \mathbb{Z}/2)$. Sq^1 is given on $H^*(Spin\ ;\ \mathbb{Z}/2)$ as follows:

n	0	1	2	3	4	5	6	7	8
H^nSpin	1	0	0	t_3	0	$t_5 \longrightarrow$	t_3^2	t_7	t_3t_5 .

The classes t_3 and t_7 are reductions of free generators of $H^*(Spin\ ;\ \mathbb{Z})$.

Now consider the spectral sequence associated to $\mathbb{Z}/4$ - cohomology of the fibration

$$\text{Spin} \longrightarrow SU \longrightarrow BBSO$$



$E^2 = H^*(BBSO, H^*(\text{Spin}, \mathbb{Z}/4))$

The spectral sequence converges to $H^*(SU ; \mathbb{Z}/4) \simeq \mathbb{Z}/4 \oplus H^*(SU ; \mathbb{Z}) \simeq \mathbb{Z}/4 \oplus E(a_3\ a_5\ a_7\ \ldots)$.

Lemma 2) follows from part 1 and the fact that a class in $H^7(BBSO ; H^0(\text{Spin}\ \mathbb{Z}/4)$ survives. There are not enough classes in the spectral sequence to hit all of it. $1 \Rightarrow 2$.

To prove part 1 of the lemma, notice that the unique element of $H^7(BBSO ; \mathbb{Z}/4)$ which is divisible by two is hit by a differential (for instance, since it is in the image of

$$H^7(BBSO ; \mathbb{Z}/2) \longrightarrow H^7(BBSO ; \mathbb{Z}/4)).$$

To finish the argument, we need to find a different element of order 2 in $H^7(BBSO ; \mathbb{Z}/4)$ which is hit by a differential.

There is an element x of order 2 in $H^7(B\,\text{Spin} ; \mathbb{Z}/4)$ which reduces to $w_7 \in H^7(B\,\text{Spin} ; \mathbb{Z}/2)$.
(Since $Sq^1\, w_6 = w_7$). The image of x in $H^7(BBSO ; \mathbb{Z}/4)$ is a class y which reduces to $e_7 \in H^7(BBSO ; \mathbb{Z}/2)$. Since the 2-divisible element reduces to 0, x is a different one.

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