THH of $\mathbb{Z}$ and $\mathcal{O}_K$ via Thom spectra

David Mehrle, Joe Stahl

R a ring spectrum, for example:

- any ordinary ring,
- $R = \mathbb{S}$,
- $R = \mathbb{S}_{\hat{p}}$,
- $R = \mathbb{S}_{\hat{p}}[\zeta_n]$ with $\gcd(n, p) = 1$

$\exists$ a topological group† $GL_1(R)$ with

$$\pi_0(GL_1(R)) = (\pi_0 R)^\times$$

$B\,GL_1(R)$ is a classifying space for invertible R-module spectra

For any space X and map $f\colon X \to B\,GL_1(R)$, one constructs an R-module spectrum Mf called the **Thom spectrum** of R.

Theorem (Blumberg–Cohen–Schlichtkrull): With reasonable commutativity assumptions,

$$\mathrm{THH}(Mf/R) \simeq Mf \otimes_{\mathbb{S}} BX.$$

This gives us a way to compute THH as a spectrum!

Theorem (Hopkins–Mahowald, Blumberg–Cohen–Schlichtkrull, Kitchloo):

$$\mathbb{F}_p \simeq M\big(\Omega^2 S^3 \xrightarrow{f_p} B\,GL_1(\mathbb{S}_{\hat{p}})\big)$$

$$\mathbb{Z}_p \simeq M\big(\underbrace{X \to \Omega^2 S^3}_{\text{universal cover}} \xrightarrow{f_p} B\,GL_1(\mathbb{S}_{\hat{p}})\big)$$

$$\mathbb{Z} \simeq M\big(X \to \prod_{p \text{ prime}} B\,GL_1(\mathbb{S}_{\hat{p}}) \xrightarrow{\sim} B\,GL_1(\mathbb{S})\big)$$

and for $(p, n) \neq (2, 1)$,

$$\mathbb{Z}/p^{n+1} \simeq M\big(\underbrace{X_{p^n} \to \Omega^2 S^3}_{p^n\text{-fold cover}} \xrightarrow{f_p} B\,GL_1(\mathbb{S}_{\hat{p}})\big)$$

Corollary:

$$\begin{aligned} \mathrm{THH}(\mathbb{F}_p) &\simeq \mathbb{F}_p \otimes_{\mathbb{S}} \Omega S^3 \\ \mathrm{THH}(\mathbb{Z}_p) &\simeq \mathbb{Z}_p \otimes_{\mathbb{S}} BX \\ \mathrm{THH}(\mathbb{Z}) &\simeq \mathbb{Z} \otimes_{\mathbb{S}} BX \end{aligned}$$

Corollary:

$$\begin{aligned} \pi_* \mathrm{THH}(\mathbb{F}_p) &\simeq H_*(\Omega S^3; \mathbb{F}_p) \\ \pi_* \mathrm{THH}(\mathbb{Z}) &\simeq H_*(BX; \mathbb{Z}) \end{aligned}$$

By the Serre spectral sequence, this gives

$$\pi_* \mathrm{THH}(\mathbb{F}_p) \cong \mathbb{F}_p[u]$$

$$\pi_n \mathrm{THH}(\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & (n = 0) \\ \mathbb{Z}/_m & (n = 2m-1, m > 0) \\ 0 & (\text{else}), \end{cases}$$

Our project: By taking an étale extension $\mathbb{S}_{\widehat{p}}[\zeta]$ of $\mathbb{S}_{\widehat{p}}$, we can find

$$\mathbb{F}_{p^n} \simeq M\big(\Omega^2 S^3 \xrightarrow{f_p} B\,GL_1(\mathbb{S}_{\widehat{p}}[\zeta])\big)$$

$$\mathbb{Z}_p[\zeta] \simeq M\big(\underbrace{X \to \Omega^2 S^3}_{\text{universal cover}} \xrightarrow{f_p} B\,GL_1(\mathbb{S}_{\widehat{p}}[\zeta])\big)$$

We can use these to compute $\mathrm{THH}(\mathbb{Z}_p[\zeta])$ and

$$\mathrm{THH}(\mathbb{F}_q) \simeq \mathbb{F}_q \otimes_{\mathbb{S}} \Omega S^3.$$

Further direction: realize $\mathcal{O}_K$ as a Thom spectrum, for K a p-adic/number field.

Problem: dealing with ramification