MA 320-001: Introductory Probability

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Combinations

Rule (Combinations)

If we start with n objects, each selection or combination, of r of these objects, with no reference to order, corresponds to r! permutations of size r from n objects. Thus the number of combinations of size r from a collection of size n is

$$C(n,r) = \binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!},\ 0 \leqslant r \leqslant n.$$

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- $\binom{n}{1} = \binom{n}{n-1} = n$,
- When $0 \leqslant n < r$, $\binom{n}{r} = 0$.

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$$\binom{20}{11} = \frac{20!}{11!(20-11)!} = 167960 \text{ ways .}$$

The Binomial Theorem

Rule

$$(x+y)^n = \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \binom{n}{2}x^2y^{n-2} + \cdots + \binom{n}{n-1}x^{n-1}y^1 + \binom{n}{n}x^ny^0$$

or

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{n-k}x^ky^{n-k}$$

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Replace $x = 2a$ and $y = -3b$.

Answer is $\binom{7}{5}x^5y^2 = 21(32a^5)(9b^2) = 6048a^5b^2$.

The Binomial Theorem: Applications

Corollary

For $n > 0$,

1. $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$, *and*
2. $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0$.

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Proof.

For (1), set $x = y = 1$.
For (2), set $x = 1$ and $y = -1$. □

The Multinomial Theorem

Theorem

For positive integers n, t, the coefficient of $x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}$ in the expansion of $(x_1 + x_2 + \cdots + x_t)^n$ is

$$\binom{n}{n_1, n_2, \cdots, n_t} = \frac{n!}{n_1! n_2! \cdots n_t!},$$

where each n_i is an integer with $0 \leqslant n_i \leqslant n$, $\forall 1 \leqslant i \leqslant t$, and $n_1 + n_2 + \cdots n_t = n$.

The Multinomial Theorem: Example

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2. What is the coefficient of xyz^5?
   Answer: $\binom{7}{1,1,5} = \frac{7!}{1!1!5!} = 42$.
3. What is the coefficient of x^3z^4.

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3. What is the coefficient of x^3z^4.
   Answer: $\binom{7}{3,0,4} = \frac{7!}{3!0!4!} = 35$.

Combinations with repetitions

Rule

The number of combinations of n objects taken r at a time, with repetition, is

$$C(n+r-1,r) = \binom{n+r-1}{r} = \frac{(n+r-1)!}{r!(n-1)!}$$

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Example

Seven high school freshmen stop at a restaurant where each of them has one of the following: a cheeseburger, a hot dog, a taco, or a fish sandwich. How many different purchases are possible (from the restaurant point of view)?

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c	cheeseburger
h	hot dog
t	taco
f	fish sandwich

1. c,c,h,h,t,t,f	x x | x x | x x | x
2. c,c,c,c,h,t,f	x x x x | x | x | x
3. c,c,c,c,c,c,f	x x x x x x ||| x
4. h,t,t,f,f,f,f	|x | x x | x x x x
5. t,t,t,t,t,f,f	| |x x x x x | x x
6. t,t,t,t,t,t,t	| |x x x x x x x |
7. f,f,f,f,f,f,f	| | | x x x x x x x

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4. h,t,t,f,f,f,f	|x | x x | x x x x
5. t,t,t,t,t,f,f	| |x x x x x | x x
6. t,t,t,t,t,t,t	| |x x x x x x x |
7. f,f,f,f,f,f,f	| | | x x x x x x x

The number of different purchases is $\binom{10}{7} = 120$.

Summary

Order is relevant	Repetitions are allowed	Type of result	Formula
Yes	No	Permutation	$P(n,r)$, $0 \leqslant r \leqslant n$
Yes	Yes	Arrangement	n^r, $n, r \geqslant 0$
No	No	Combination	$\binom{n}{r}$, $0 \leqslant r \leqslant n$
No	Yes	Combination with repetition	$\binom{n+r-1}{r}$, $n, r \geqslant 0$

More examples

Example

Prove that if n and k are positive integers with $n = 2k$, then $\frac{n!}{2^k}$ is an integer.

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Proof.

Consider n symbols, $x_1, x_1, x_2, x_2, \ldots, x_k, x_k$, then we have n objects with $n_1 = 2$ indistinguishable objects of a first type, $\ldots$, and $n_k = 2$ indistinguishable objects of an k^{th} type, where

$$n_1 + n_2 + \cdots + n_r = n, \text{ then there are,}$$

$$\frac{n!}{n_1!n_2!\cdots n_r!} = \frac{n!}{2!2!\cdots 2!} = \frac{n!}{2^k} \text{ (permutations with repetitions)}$$

(linear) arrangements of the given n objects, which is an integer. □

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$$\binom{n+r-1}{r} = \binom{4+10-1}{10} = \binom{13}{10} = 286.$$

2. Each vice president should receive at least \$100.

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2. Each vice president should receive at least \$100.

$$\binom{n+r-1}{r} = \binom{4+6-1}{6} = \binom{9}{6} = 84.$$

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$$\binom{n+r-1}{r} = \binom{4+6-1}{6} = \binom{9}{6} = 84.$$

3. Each vice president should receive at least \$100 and Mona, as executive vice president, gets at least \$500, then $\binom{4+2-1}{2} = \binom{5}{2} = 10$.