# Chapter 8

# Families of Functions

One very important area of real analysis is the study of the properties of real valued functions. We introduced the general properties of functions earlier in Chapter 6. In this chapter we want to look at specific functions and families of functions to see what properties really belong to the area of analysis separate from algebra or topology. We will take on the functions in a somewhat orderly manner.

## 8.1   Polynomials

The polynomials in the study of functions play a similar role to the integers in our study of numbers. They are our basic building blocks. Likewise, when we really start looking at the family of polynomials most of the questions we have are really algebraic in nature and not analytic, just like the study of integers.

In general a polynomial is an expression in which a finite number of constants and variables are combined using only addition, subtraction, multiplication, and non-negative whole number exponents (raising to a power).

A polynomial function is a function defined by evaluating a polynomial. For example, the function $f$ defined by $f(x) = x^3 - x$ is a polynomial function. Polynomial functions are an important class of *smooth* functions — smooth meaning that they are infinitely differentiable, *i.e.*, they have derivatives of all finite orders.

Because of their simple structure, polynomials are easy to evaluate, and are used extensively in numerical analysis for polynomial interpolation or to numerically integrate more complex functions. Computationally, they require only multiplication and addition. Any division is division of coefficients and subtraction is just the addition of the additive inverse.

Polynomials have a number of nice properties.

- A sum of polynomials is a polynomial.

- A product of polynomials is a polynomial.

- The derivative of a polynomial is a polynomial.

- The integral of a polynomial is a polynomial

Polynomials are used to approximate other functions, such as sine, cosine, and exponential. We will characterize those functions that can be approximated by polynomials later.

All polynomials have an expanded form, in which the distributive law has been used to remove all parentheses. (Some polynomials also have a factored form, in which parentheses appear.) In expanded form, a term of a polynomial is a part of the polynomial that is the product of a number (called the coefficient) and zero or more variables, where a variable that appears more than once is expressed as a power of that variable. Every polynomial in expanded form is a sum of terms, where subtraction is carried out by addition of terms with negative coefficients.

Every polynomial in one variable is equivalent to a polynomial with the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

This form is sometimes taken as the definition of a polynomial in one variable.

A polynomial equation is an equation in the form of a polynomial equal to zero or equal to another polynomial. The latter form is usually converted to the former by subtracting the second polynomial from both sides of the equation. When written as a polynomial equal to zero, the degree of the equation is the degree of the polynomial.

In elementary algebra, methods are given for solving all first degree and second degree polynomial equations in one unknown. The number of solutions may equal the degree, but it is necessary to consider multiplicity of solutions and complex numbers for this to be universally true, as described in the Fundamental Theorem of Algebra.

Determining the roots of polynomials, or "solving algebraic equations", is among the oldest problems in mathematics. Some polynomials, such as $f(x) = x^2 + 1$, do not have any real roots. If, however, the set of allowed candidates is expanded to the complex numbers, every (non-constant) polynomial has a root — in fact it has $n$ roots, counting multiplicity, if the degree of the polynomial is $n$. This is the statement of the Fundamental Theorem of Algebra.

### 8.1.1   Polynomial functions

For given numerical constants $a_0, \ldots, a_n$, $n > 0$, in some field (often $\mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$), with $a_n$ non-zero, then a polynomial (function) of degree $n$ is a function of the form

$$f(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + a_n x^n.$$

More concisely, a polynomial function can be written in sigma notation as

$$\sum_{k=0}^{n} a_k x^k.$$

The constants $a_0, \ldots, a_n$ are called the coefficients of the polynomial. $a_0$ is called the constant coefficient and $a_n$ is called the leading coefficient. When the leading coefficient is 1, the polynomial is said to be monic or normed.

Each summand $a_k x^k$ of the polynomial is called a term. A polynomial with one, two or three terms is called monomial, binomial or trinomial respectively.

## 8.1.2 Graphs

A polynomial function in one real variable can be represented by a graph.

- The graph of the zero polynomial $f(x) = 0$ is the x-axis.

- The graph of a degree 0 polynomial $f(x) = a_0$ , where $a_0 \neq 0$, is a horizontal line with $y$-intercept $a_0$.

- The graph of a degree 1 polynomial (or linear function) $f(x) = a_0 + a_1 x$ , where $a1 \neq 0$, is an line with $y$-intercept $a_0$ and slope $a_1$.

- The graph of a degree 2 or higher polynomial $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, where $a_n \neq 0$ and $n \geq 2$ is a continuous non-linear curve.

The best way to analyze the graph of a degree 2 or higher polynomial function is by its end behavior, the number of $x$-intercepts and the number of turning points.

### End behavior

There are four end behaviors which are direct results of whether $a_n$, the leading coefficient, is positive or negative and whether $n$, the degree of the polynomial, is even or odd.

1. If $a_n$ is positive and $n$ is even, the right end of the polynomial is in quadrant I while the left end is in quadrant II.

2. If $a_n$ is negative and $n$ is even, the right end is in quadrant IV while the left end is in quadrant III.

3. If $a_n$ is positive and $n$ is odd, the right end is in quadrant I while the left end is in quadrant III.

4. If $a_n$ is negative and $n$ is odd, the right end is in quadrant IV while the left end is in quadrant II.

### Number of $x$-intercepts

From the Fundamental Theorem of Algebra, a polynomial of degree $n$ has exactly $n$ complex roots, which may or may not be real. Therefore, the number of $x$-intercepts cannot exceed $n$. It also follows from the Fundamental Theorem of Algebra that the complex roots of a polynomial with real coefficients must exist in complex-conjugate pairs. This in particular implies that an even-degree polynomial with real coefficients may have no $x$-intercepts (because all its roots may be complex); an odd-degree polynomial with real coefficients, on the other hand, must have at least one $x$-intercept, since any pairing of roots into complex-conjugate pairs will necessarily leave at least one root unpaired for odd $n$. This (these) "unpaired" root(s) must therefore be real.

### Number of turning points

The number of turning points of an even-degree polynomial is any odd number less than the degree, while the number of turning points of an odd-degree polynomial is any even number less than the degree. For example, a degree 4 polynomial function can have 1 or 3 turning points whereas a degree 5 polynomial function can have 0, 2, or 4 turning points.

On your graphing calculator, graph the following examples of polynomials of low degree.

1. $f(x) = x^2 - x - 2 = (x + 1)(x - 2)$

2. $f(x) = \dfrac{x^3}{5} + \dfrac{4x^2}{5} - \dfrac{7x}{5} - 2 = \dfrac{1}{5}(x + 5)(x + 1)(x - 2)$

3. $f(x) = \frac{1}{14}(x + 4)(x + 1)(x - 1)(x - 3) + 0.5$

4. $f(x) = \frac{1}{20}(x + 4)(x + 2)(x + 1)(x - 1)(x - 3) + 2$

5. $f(x) = 13x^4 - 7x^3 + \frac{2}{3}x^2 - 5x + 3$

### Roots

A root or zero of a polynomial $p$ is a number $r$ so that $p(r) = 0$. The Fundamental Theorem of Algebra states that a polynomial of degree $n$ over the complex numbers has exactly $n$ complex roots (not necessarily distinct ones). Therefore a polynomial can be factored as

$$p(x) = a_n(x - r_1)(x - r_2) \cdots (x - r_n),$$

where each $r_i$ is a root of the polynomial $p$. For example, the polynomial $x^3 - x$ can be factored as $x(x-1)(x+1)$. An equation of the form $p(x) = 0$ is called a polynomial equation in $x$. The solutions to the equation are the roots of the polynomial. The real roots of a polynomial (without non-real coefficients) are the $x$-intercepts of the graph of $p(x)$ viewed as a polynomial function.

One important aspect of calculus is the project of analyzing complicated functions by means of approximating them with polynomials. The culmination of these efforts is Taylor's theorem, which roughly states that every differentiable function locally looks like a polynomial, and the Stone-Weierstrass theorem, which states that every continuous function defined on a compact interval of the real axis can be approximated on the whole interval as closely as desired by a polynomial. Polynomials are also frequently used to interpolate functions.

## 8.2 Rational Functions

A rational function is a function that is the quotient of two polynomial functions.

$$f(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomial functions, and $q(x)$ is not identically equal zero. Note that there is no condition that states $p(x)$ and $q(x)$ have no factors in common. In fact, you must realize that the two functions

$$f(x) = \frac{x^2 - 1}{x - 1} \quad \text{and} \quad g(x) = x + 1$$

are different as functions, though the are **algebraically** identical. Note that the domain of $f$ does not contain 1, while 1 is in the domain of $g$. Usually, we will want to consider the rational functions whose numerator and denominator have no factors in common, because it will give us essentially the same function just with a larger domain.

**Example 8.1** Let

$$f(x) = \frac{x^2 + x - 20}{x^2 - 3x - 18}.$$

To understand the behavior of this rational function it is useful to see its constituent polynomials in factored form. The polynomials in the numerator and the denominator of the above function would factor like this:

$$f(x) = \frac{(x + 5)(x - 4)}{(x + 3)(x - 6)}.$$

## 8.2.1 The Domain

By considering the constituent polynomials in factored form, the roots of the denominator are obvious. In the above example they are $x = -3$ and $x = 6$. That is, if $x$ takes on either of these two values, the denominator becomes equal to zero. Since we

can not divide by zero, the function is not defined for these two values of $x$. We say that the function is not defined at $x = -3$ and $x = 6$.

Other values for $x$ do not cause the function to become undefined, so, we say that the function is defined for all other values for $x$. In other words, all real numbers except $-3$ and 6 are allowed as inputs to this function. The domain for the function therefore is:

$$\{x \in \mathbb{R} \mid x \neq -3 \text{ or } 6\} = (-\infty, -3) \cup (-3, 6) \cup (6, \infty).$$

### 8.2.2   The $x$-intercepts

The $x$-intercepts for this function would be where the output, or $y$-value, equals zero. A rational function is a fraction, and a fraction is equal to zero when the numerator is equal to zero. For the above rational function this happens at the roots of this polynomial. The roots of the numerator polynomial are $x = -5$ and $x = 4$. That is, when $x$ takes on either of these two values the numerator becomes zero, and the output of the function, or $y$-value, also becomes zero.

Notice that the function must be defined at these values. In our other example $g(x) = \dfrac{x^2 - 1}{x - 1}$ even though the numerator is zero at $x = 1$, the function is not defined there. Thus, $x = 1$ cannot be an $x$-intercept.

### 8.2.3   The y-intercept

What about the $y$-intercept? Of course, this is where $x = 0$. As a matter of ease, it is often easier for students to use the unfactored form when computing the $y$-intercept. In the example above

$$f(0) = \frac{0^2 + 0 - 20}{x^2 - 3 \cdot 0 - 18} = \frac{-20}{-18} = \frac{10}{9}.$$

### 8.2.4   The Graph
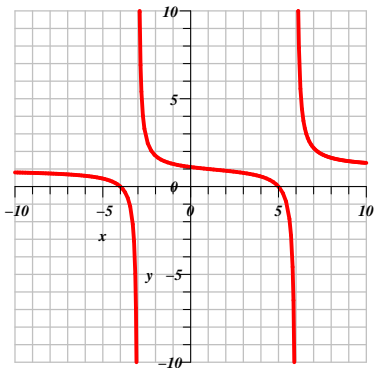


Consider a graph of this function in Figure 8.1. The graph of the function breaks at $x = -3$ and at $x = 6$. The function crosses the $x$-axis at $x = -5$ and $x = 4$. These are the same as the values which we calculated above for the $x$-intercepts. The function crosses the $y$-axis just a bit above 1, at about 1.1. This is the same location as the calculated $y$-intercept above.

### 8.2.5   End Behavior

The way that a function is shaped when the input approaches positive or negative infinity is called its *end*

Figure 8.1:

*behavior.* One might say that the end behavior of a function describes what the function looks like from far away.

In our graph notice how the graph is leveling off to a height of about $y = 1$ as $x$ gets large positively or negatively. That is, at the left and right edge of the graph the function starts to act like the function described $f(x) = 1$.

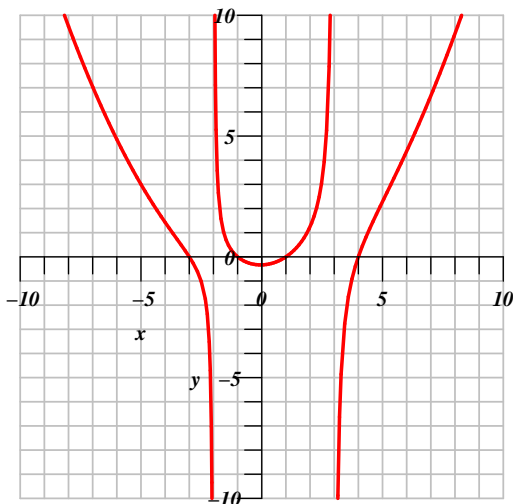What would you consider to be the end behavior of the following two functions?



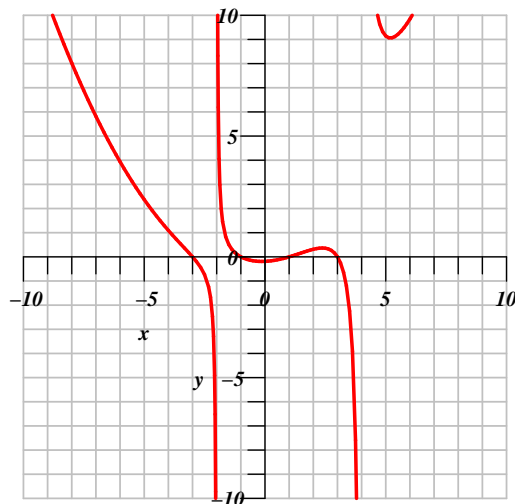Figure 8.2: $\dfrac{x^4 - 10x^2 + 9}{5x^2 - 10x - 40}$



Figure 8.3: $\dfrac{x^4 - 7x^3 + 11x^2 + 7x - 12}{5x^2 - 5x - 30}$

### 8.2.6   The Range

An examination of the graph above will indicate that all real numbers are available for output. The left most fork of the graph looks like it will rise not quite to $y = 1$, and the right most fork looks like it may not quite drop to $y = 1$; so, one might at first suggest that $y = 1$ is not in the range. However, if you examine the central portion of the graph, you can see that $y = 1$ is definitely a member of the range. So, all real numbers are present in the range.

## 8.3   Algebraic Functions

A function which can be constructed using only a finite number of elementary operations — addition, subtraction, multiplication, division, and exponentiation — together with the inverses of functions capable of being so constructed is called an **algebraic function**.

This class of functions contains polynomial functions and rational functions. It includes more though, since it includes the root functions, such as the square root

and the cube root. Examples of algebraic functions not previously considered would be functions like:

$$
\begin{aligned}
f(x) &= \sqrt{ax^3 + bx^2 + cx + d} \\
g(x) &= \sqrt[3]{x^4 - 8} \\
h(x) &= ax^{1/2} + bx^{1/3} + cx^{1/4}
\end{aligned}
$$

Functions involving expressions such as $\sqrt{x^2 + a^2}$, $\sqrt{x^2 - a^2}$, $\sqrt{a^2 - x^2}$, $\sqrt{ax + b}$, and $\sqrt{ax^2 + bx + c}$ are sometimes called *irrational functions*. These expressions often arise in calculus in antidifferentiation problems. Long lists of antiderivatives exist for them in multiple places. These, too, are algebraic functions.

Note that it is also possible to talk about an algebraic function that is implicitly defined by an algebraic equation in 2 variables. You might not be able to have an explicit representation of the function, but the values of the function, $y(x)$, are determined by the values of $x$. This is not essentially different from the above definition, but lends itself to generalizing to multiple variables.

## 8.4    Transcendental Functions

A function which is not algebraic is a ***transcendental function***. This includes the exponential, logarithmic, and trigonometric functions — which we will discuss later. It also includes many, many more functions called *special functions*. They are called this because they normally arise is special situations and are not as widely used as others. We will look at a few of these: the Lambert W function and glog. Many others exist that can be found in most Differential Equations and Numerical Analysis texts. A very important part of Real Analysis is finding ways to study and describe all functions, and this includes *special functions*.



Figure 8.4: $(-2)^x$ for carefully chosen values of $x \in [0, 2]$

### 8.4.1    Exponential Functions

We are used to functions where the variable is raised to a fixed power. The exponential function reverses the roles of the base and the exponent in power functions. To carefully define the exponential function we will need a number of things that we have not yet proven or derived. However, you can remember some of the properties from your earlier courses.

A function of the form $f(x) = a^x$ for $a \neq 1$ and $a > 0$ is called and *exponential function*. If we were to take $a = 1$ then this is just the constant function, which is
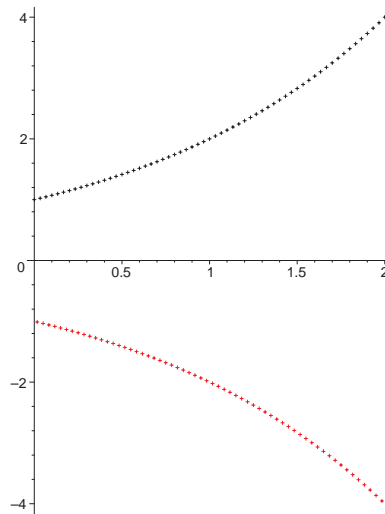
already contained in the set of polynomials. If we take $a < 0$, then the function is not defined for all real numbers. For example, if we consider the expression $(-2)^x$ and graph this over the interval from 0 to 2 we might get a graph like the following:

$e^x$ is always strictly positive for any $x$, so the domain of the function is the set of real numbers and the codomain is the set of positive real numbers. The exponential function obeys all of the normal laws of exponents i.e.

(a) $e^x e^y = e^{x+y}$.

(b) $e^{x+y} = e^x e^y$.

(c) $e^{-x} = \frac{1}{e^x}$.

(d) $e^{x-y} = \frac{e^x}{e^y}$.

In fact, we could start with the condition that $f(x + y) = f(x)f(y)$. Notice then that this makes $f(0) = [f(0)]^2$ so either $f(0) = 0$ or $f(0) = 1$. Since $f(x) = f(0+x) = f(0)f(x)$, if $f(0) = 0$, then we would have that $f(x) = 0$ for all $x$ and this is not that interesting. Therefore, $f(0) = 1$.

Note that this now implies that $f(-x) = 1/f(x)$, since

$$f(-x)f(x) = f(-x + x) = f(0) = 1.$$

It then follows that $f(x - y) = f(x)/f(y)$.

Now, let $f(1) = \alpha$. Then for any positive integer $n$, $f(n) = \alpha^n$ by induction, just like the homework. Then we can see that $f(-n) = 1/\alpha^n = \alpha^{-n}$. Let $r$ be a rational number and let $r = \frac{p}{q}$. Then

$$\alpha^p = f(p) = f(q \cdot \frac{p}{q}) = f(\frac{p}{q})^q.$$

From this it follows that $f(\frac{p}{q}) = \alpha^{p/q}$. Now, if we knew that $f$ were continuous, we could then show that $f(x) = \alpha^x$ for all real numbers $x$. This tells us that any continuous, real-valued function that satisfies $f(x + y) = f(x)f(y)$ must be an exponential function. What kind of function would satisfy $f(xy) = f(x) + f(y)$?

What can we say about $\alpha$? We know that if $\alpha = 1$, then $f(x) = 1$ and this would be a constant function. If we want this to be a real-valued function, then since we have shown that $f(\frac{1}{2}) = \sqrt{\alpha}$, we would want to keep $\alpha$ from being negative. If on the other hand we allowed complex values as output, then that restriction need not be applied. Also, if we were to have $\alpha = 0$, then $f(x)$ would be undefined for all $x < 0$. Thus, we will have the restrictions that $\alpha > 0$ and $\alpha \neq 1$.

Is this function a one-to-one function? What happens if $f(x) = f(y)$?

$$
\begin{aligned}
f(x) &= f(y) \\
\alpha^x &= \alpha^y \\
\alpha^{x-y} &= 1 \\
x - y &= 0 \\
x &= y
\end{aligned}
$$

Thus, the exponential function is one-to-one.

## 8.4.2   Logarithmic Functions

Since the exponential function is one-to-one, it then has an inverse function defined for $x > 0$ since the range of the exponential function is all positive reals.

The logarithm is the mathematical operation that is the inverse of exponentiation (raising a constant, the base, to a power). The logarithm of a number $x$ in base $b$ is the number $n$ such that $x = b^n$. Thus, we have that

$$\log_b(x) = n \Leftrightarrow b^n = x.$$

Logarithms reduce multiplication operations to addition, division to subtraction, exponentiation to multiplication, and roots to division. Therefore, logarithms are useful for making lengthy numerical operations easier to perform and, before the advent of electronic computers, they were widely used for this purpose in fields such as astronomy, engineering, navigation, and cartography. They have important mathematical properties and are still used in many ways.

The most widely used bases for logarithms are 10, 2 and the Euler constant $e \sim$ 2.718281828459045.... When *log* is written without a base, *i.e.* the $b$ is missing from $\log_b$, the intent can usually be determined from context:

- natural logarithm ($\log_e$) in mathematical analysis,
- common logarithm ($\log_{10}$) in engineering and when logarithm tables are used to simplify hand calculations,
- binary logarithm ($\log_2$) in information theory[1] and musical intervals

The notation $\ln(x)$ almost always means $\log_e(x)$, *i.e.*, the natural logarithm of $x$, but the implied base for $\log(x)$ varies by discipline:

- Mathematicians generally understand both $\ln(x)$ and $\log(x)$ to mean $\log_e(x)$ and write $\log_{10}(x)$ when the base-10 logarithm of $x$ is intended. Sometimes the term $\lg(x)$ is used for the base-2 logarithm of $x$.

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[1]Information theory is a discipline in applied mathematics involving the quantification of data with the goal of enabling as much data as possible to be reliably stored on a medium and/or communicated over a channel. The measure of data, known as information entropy, is usually expressed by the average number of bits needed for storage or communication. Applications of fundamental topics of information theory include ZIP files (lossless data compression), MP3s (lossy data compression), and DSL (channel coding). The field is at the crossroads of mathematics, statistics, computer science, physics and electrical engineering, and its impact has been crucial to success of the Voyager missions to deep space, the invention of the CD, the feasibility of mobile phones, the development of the Internet, the study of linguistics and of human perception, the understanding of black holes, and numerous other fields.

- Engineers, biologists, and some others write only $\ln(x)$ or $\log_e(x)$ when they mean the natural logarithm of $x$, and take $\log(x)$ to mean $\log_{10}(x)$ or, sometimes in the context of computing, $\log_2(x)$.

- On most calculators, the `LOG` button is $\log_{10}(x)$ and `LN` is $\log_e(x)$. However, in most commonly used computer programming languages, including C, C++, Java, Fortran, and BASIC, the `log` function returns the natural logarithm. The base-10 function, if it is available, is generally `log10`.

- Some people use $\text{Log}(x)$ (capital L) to mean $\log_{10}(x)$, and use $\log(x)$ with a lowercase l to mean $\log_e(x)$.

- The notation $\text{Log}(x)$ is also used by mathematicians to denote the principal branch of the (natural) logarithm function.

- Also frequently used is the notation $\text{blog}(x)$ instead of $log_b(x)$.

This chaos, historically, originates from the fact that the natural logarithm has nice mathematical properties (such as its derivative being $1/x$, and having a simple definition), while, back when logarithms were used to speed computations, the decimal logarithms were far better for that, so natural logarithms where unknown outside calculus classes while decimal logarithms were widely used in a variety of disciplines.

In computer science, the base 2 logarithm is sometimes written as $\lg(x)$ to avoid confusion. This usage was suggested by Edward Reingold and popularized by Donald Knuth. However, in Russian literature, the notation $\lg(x)$ is generally used for the base 10 logarithm, so even this usage is not without its perils.

While there are several useful identities, the most important for calculator use lets one find logarithms with bases other than those built into the calculator (usually $\log_e$ and $\log_{10}$). To find a logarithm with base $b$ using any other base $a$:

$$
\begin{aligned}
n = \log_b x \quad &\Leftrightarrow \quad b^n = x \\
\log_a(b^n) \;&=\; \log_a x \\
n \cdot \log_a b \;&=\; \log_a x \\
n \;&=\; \frac{\log_a x}{\log_a b} \\
\log_b x \;&=\; \frac{\log_a x}{\log_a b}
\end{aligned}
$$

Logarithms are useful in solving equations in which exponents are unknown. They have simple derivatives, so they are often used in the solution of integrals. The logarithm is one of three closely related functions. In the equation $b^n = x$, $b$ can be determined with radicals, $n$ with logarithms, and $x$ with exponentials.

Various quantities in science are expressed as logarithms of other quantities.

1. The negative of the base-10 logarithm is used in chemistry, where it expresses the concentration of hydronium ions (pH). The concentration of hydronium ions in neutral water is $10^{-7}$ at $25°C$, hence a pH of 7.

2. The bel (symbol B) is a unit of measure which is the base-10 logarithm of ratios, such as power levels and voltage levels. It is mostly used in telecommunication, electronics, and acoustics. It is used, in part, because the ear responds logarithmically to acoustic power. Invented by engineers of the Bell Telephone Laboratory to quantify the reduction in audio level over a 1 mile (1.6 km) length of standard telephone cable, it was originally called the transmission unit or TU, but was renamed in 1923 or 1924 in honor of the laboratory's founder and telecommunications pioneer Alexander Graham Bell. The bel was too large for everyday use, so the decibel (dB), equal to 0.1 bel (B), became more commonly used.

3. The neper is a similar unit which uses the natural logarithm of a ratio. It is not an international system unit but is accepted for use alongside that system. It is used to express ratios, such as gain to loss and voltage and current, and relative values. value of a ratio in nepers, $Np$, is given by

$$NP = \ln \frac{x_1}{x_2} = \ln x_1 - \ln x_2.$$

4. The Richter scale measures earthquake intensity on a base-10 logarithmic scale.

5. In spectrometry and optics, the absorbance unit used to measure optical density is equivalent to -1 B.

6. In astronomy, the apparent magnitude measures the brightness of stars logarithmically, since the eye also responds logarithmically to brightness.

The discovery of logarithms just before Newton's era had an impact in the scientific world which can be compared with the invention of the computer in the 20th century, because many calculations which were too laborious became feasible.

When the chronometer was invented in the 18th century, logarithms allowed all calculations needed for astronomical navigation to be reduced to just additions, speeding the process by one or two orders or magnitude. A table of logarithms with five decimals, plus logarithms of trigonometric functions, was enough for most astronomical navigation calculations, and those tables fit in a small book.

To compute powers or roots of a number, the common logarithm of that number was looked up and multiplied or divided by the radix. Interpolation could be used for still higher precision. Slide rules used logarithms to perform the same operations more rapidly, but with much less precision than using tables. Other tools for performing multiplications before the invention of the calculator include Napier's bones and mechanical calculators.

**History of Logarithms**

In the 17th century, Joost Bürgi, a Swiss clockmaker employed by the Duke of Hesse-Kassel, first discovered logarithms as a computational tool. He did not publish his

discovery, however, until 1620. The method of logarithms was first publicly propounded in 1614, in a book entitled *Mirifici Logarithmorum Canonis Descriptio*, by John Napier, Baron of Merchiston in Scotland, four years after the publication of his memorable discovery. This method contributed to the advance of science, and especially of astronomy, by making some difficult calculations possible. Prior to the advent of calculators and computers, it was used constantly in surveying, navigation, and other branches of practical mathematics. It supplanted the more involved *prosthaphaeresis*, which relied on trigonometric identities, as a quick method of computing products. Besides their usefulness in computation, logarithms also filled an important place in the higher theoretical mathematics.

At first, Napier called logarithms *artificial numbers* and antilogarithms *natural numbers*. Later, Napier formed the word logarithm to mean a number that indicates a ratio from the Greek $\lambda o \gamma o \sigma$ (logos) meaning proportion, and $\alpha \rho \iota \theta \mu o \sigma$ (arithmos) meaning number. Napier chose that because the difference of two logarithms determines the ratio of the numbers for which they stand, so that an arithmetic series of logarithms corresponds to a geometric series of numbers. The term antilogarithm was introduced in the late 17th century and, while never used extensively in mathematics, persisted in collections of tables until they fell out of favor with the advent of the electronic calculator.

Napier did not use a base as we now understand it, but his logarithms were, up to a scaling factor, effectively to base $1/e$. For interpolation purposes and ease of calculation, it is useful to make the ratio $r$ in the geometric series close to 1. Napier chose $r = 1 - 10^{-7} = 0.999999$, and Bürgi chose $r = 1 + 10^{-4} = 1.0001$. Napier's original logarithms did not have $\log 1 = 0$ but rather $\log 10^7 = 0$. Thus if $N$ is a number and $L$ is its logarithm as calculated by Napier, $N = 10^7 (1 - 10^{-7})L$. Since $(1 - 10^{-7})^{10^7}$ is approximately $1/e$, this makes $L/10^7$ approximately equal to $\log_{1/e} N/10^7$.

Prior to the advent of computers and calculators, using logarithms meant using tables of logarithms, which had to be created manually. Base-10 logarithms are useful in computations when electronic means are not available.

In 1617, Henry Briggs published the first installment of his own table of common logarithms, containing the logarithms of all integers below 1000 to eight decimal places. This he followed, in 1624, by his *Arithmetica Logarithmica*, containing the logarithms of all integers from 1 to 20,000 and from 90,000 to 100,000 to fourteen places of decimals, together with a learned introduction, in which the theory and use of logarithms are fully developed. The interval from 20,000 to 90,000 was filled up by Adriaan Vlacq, a Dutch computer; but in his table, which appeared in 1628, the logarithms were given to only ten places of decimals.

Vlacq's table was later found to contain 603 errors, but that was not regarded as a great number of errors "when it is considered that the table was the result of an original calculation, and that more than 2,100,000 printed figures are liable to error."[2]

---

[2]*Athenaeum*, 15 June 1872. See also the Monthly Notices of the Royal Astronomical Society for

An edition of Vlacq's work, containing many corrections, was issued at Leipzig in 1794 under the title *Thesaurus Logarithmorum Completus* by Jurij Vega.

Callet's seven-place table in 1795 gave the eight-place logarithms of the numbers between 100,000 and 108,000, in order to diminish the errors of interpolation, which were greatest in the early part of the table. The only important published extension of Vlacq's table was made by Mr. Sang 1871, whose table contained the seven-place logarithms of all numbers below 200,000.

Briggs and Vlacq also published original tables of the logarithms of the trigonometric functions.

Besides the tables mentioned above, a great collection, called *Tables du Cadastre*, was constructed under the direction of Gaspard de Prony, by an original computation, under the auspices of the French republican government of the 1700's. The work contained the logarithms of all numbers up to 100,000 to nineteen places, and of the numbers between 100,000 and 200,000 to twenty-four places. It exists only in manuscript, however, "in seventeen enormous folios," at the Observatory of Paris. It was begun in 1792 and was completed in two years.

## 8.5   Trigonometric Functions

The trigonometric functions have a long history in mathematics. The study of trigonometric functions dates back to Babylonian times, and a considerable amount of fundamental work was done by Persian and Greek mathematicians. In modern mathematics, they are functions of an angle; they are important when studying triangles and modeling periodic phenomena.

In Geometry we will commonly define them as ratios of two sides of a right triangle containing the angle, and later we equivalently define them as the lengths of various line segments from a unit circle. Mathematics has found many equivalent ways of defining them such as expressing them as infinite series or as solutions of certain differential equations, allowing their extension to positive and negative values and even to complex numbers.

In modern usage, there are six basic trigonometric functions. Especially in the case of the last four, these relations are often taken as the definitions of those functions, but one can define them equally well geometrically or by other means and then derive these relations. A few other functions were common historically (and appeared in the earliest tables), but are now seldom used, such as the versine $(1 - \cos\theta)$ and the exsecant $(\sec\theta - 1)$.

---

May 1872.

| Function | Abbreviation | Relationships |
|---|---|---|
| Sine | sin | $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$ |
| Cosine | cos | $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$ |
| Tangent | tan | $\tan\theta = \dfrac{1}{\cot\theta} = \dfrac{\sin\theta}{\cos\theta} = \cot\left(\frac{\pi}{2} - \theta\right)$ |
| Cotangent | cot | $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta} = \tan\left(\frac{\pi}{2} - \theta\right)$ |
| Secant | sec | $\sec\theta = \dfrac{1}{\cos\theta} = \csc\left(\frac{\pi}{2} - \theta\right)$ |
| Cosecant | csc | $\csc\theta = \dfrac{1}{\sin\theta} = \sec\left(\frac{\pi}{2} - \theta\right)$ |

**History**

The earliest use of sine appears in the *Sulba Sutras* written in ancient India from the 8th century BCE to the 6th century BCE. Trigonometric functions were later studied by Hipparchus of Nicaea (180-125 BCE), Ptolemy, Aryabhata (476550), Varahamihira, Brahmagupta, Muhammad ibn Musa al-Kwarizmi, Abu'l-Wafa, Omar Khayyam, Bhaskara II, Nasir al-Din Tusi, Ghiyath al-Kashi (14th century), Ulugh Beg (14th century), Regiomontanus (1464), Rheticus, and Rheticus' student Valentin Otho. Madhava (c. 1400) made early strides in the analysis of trigonometric functions in terms of infinite series. Leonhard Euler's *Introductio in analysin infinitorum* in 1748 was mostly responsible for establishing the analytic treatment of trigonometric functions in Europe, also defining them as infinite series and presenting *Euler's formula*, as well as the near-modern abbreviations sin., cos., tang., cot., sec., and cosec.

The use of trigonometric functions arises from the early connection between mathematics and astronomy. Early work with spherical triangles was as important as plane triangles.

The first work on trigonometric functions related to chords of a circle. Given a circle of fixed radius, 60 units were often used in early calculations, then the problem was to find the length of the chord subtended by a given angle. For a circle of unit radius the length of the chord subtended by the angle $x$ was $2\sin(x/2)$. The first known table of chords was produced by the Greek mathematician Hipparchus in about 140 BCE. Although these tables have not survived, it is claimed that twelve books of tables of chords were written by Hipparchus. This makes Hipparchus the founder of trigonometry in Europe.

The next Greek mathematician to produce a table of chords was Menelaus in about 100 AD. Menelaus worked in Rome producing six books of tables of chords which have been lost but his work on spherics has survived and is the earliest known

work on spherical trigonometry. Menelaus proved a property of plane triangles and the corresponding spherical triangle property known the *regula sex quantitatum.*

Ptolemy was the next author of a book of chords, showing the same Babylonian influence as Hipparchus, dividing the circle into 360 and the diameter into 120 parts. The suggestion here is that he was following earlier practice when the approximation 3 for $\pi$ was used. Ptolemy, together with the earlier writers, used a form of the relation $\sin^2 x + \cos^2 x = 1$, although of course they did not actually use sines and cosines but chords.

Similarly, in terms of chords rather than sine and cosine, Ptolemy knew the formulas

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Ptolemy calculated chords by first inscribing regular polygons of 3, 4, 5, 6 and 10 sides in a circle. This allowed him to calculate the chord subtended by angles of 36°, 72°, 60°, 90° and 120°. He then found a method of finding the cord subtended by half the arc of a known chord and this, together with interpolation allowed him to calculate chords with a good degree of accuracy. Using these methods Ptolemy found that $\sin 30'$ ($30' = $ half of 1°) which is the chord of 1° was, as a number in base 60, 0 31' 25". Converted to decimals this is 0.0087268 which is correct to 6 decimal places, the answer to 7 decimal places being 0.0087265.

The first actual appearance of the sine of an angle appears in the work of the Hindus. Aryabhata, in about 500, gave tables of half chords which now really are sine tables and used *jya* for our sin. This same table was reproduced in the work of Brahmagupta (in 628) and detailed method for constructing a table of sines for any angle were give by Bhaskara in 1150.

The Arabs worked with sines and cosines and by 980 Abu'l-Wafa knew that

$$\sin 2x = 2 \sin x \cos x$$

although it could have easily have been deduced from Ptolemy's formula

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

with $x = y$.

The Hindu word *jya* for the sine was adopted by the Arabs who called the sine *jiba*, a meaningless word with the same sound as *jya*. Now *jiba* became *jaib* in later Arab writings and this word does have a meaning, namely a 'fold'. When European authors translated the Arabic mathematical works into Latin they translated *jaib* into the word *sinus* meaning 'fold' in Latin. In particular Fibonacci's use of the term *sinus rectus arcus* soon encouraged the universal use of sine.

Chapters of Copernicus's book giving all the trigonometry relevant to astronomy was published in 1542 by Rheticus. Rheticus also produced substantial tables of sines

and cosines which were published after his death. In 1533 Regiomontanus's work *De triangulis omnimodis* was published. This contained work on planar and spherical trigonometry originally done much earlier in about 1464. The book is particularly strong on the sine and its inverse.

The term sine certainly was not accepted straight away as the standard notation by all authors. In times when mathematical notation was in itself a new idea many used their own notation. Edmund Gunter was the first to use the abbreviation sin in 1624 in a drawing. The first use of sin in a book was in 1634 by the French mathematician Hérigone while Cavalieri used *Si* and Oughtred *S*.

It is perhaps surprising that the second most important trigonometrical function during the period we have discussed was the versed sine, a function now hardly used at all. The versine is related to the sine by the formula

$$\operatorname{versin} x = 1 - \cos x.$$

It is just the sine turned (versed) through 90°.

The cosine follows a similar course of development in notation as the sine. Viète used the term *sinus residuae* for the cosine, Gunter (1620) suggested *co-sinus*. The notation *Si.2* was used by Cavalieri, *s co arc* by Oughtred and *S* by Wallis.

Viète knew formulas for $\sin nx$ in terms of $\sin x$ and $\cos x$. He gave explicitly the formulas (due to Pitiscus)

$$\begin{aligned}
\sin 3x &= 3\cos^2 x \sin x - \sin^3 x \\
\cos 3x &= \cos^3 x - 3\sin^2 x \cos x.
\end{aligned}$$

The tangent and cotangent came via a different route from the chord approach of the sine. These developed together and were not at first associated with angles. They became important for calculating heights from the length of the shadow that the object cast. The length of shadows was also of importance in the sundial. Thales used the lengths of shadows to calculate the heights of pyramids.

The first known tables of shadows were produced by the Arabs around 860 and used two measures translated into Latin as *umbra recta* and *umbra versa*. Viète used the terms *amsinus* and *prosinus*. The name tangent was first used by Thomas Fincke in 1583. The term *cotangens* was first used by Edmund Gunter in 1620.

Abbreviations for the tangent and cotangent followed a similar development to those of the sine and cosine. Cavalieri used *Ta* and *Ta.2*, Oughtred used *t arc* and *t co arc* while Wallis used *T* and *t*. The common abbreviation used today was used by Albert Girard in 1626, but tan was written over the angle. The abbreviation *cot* was first used by Jonas Moore in 1674.

The secant and cosecant were not used by the early astronomers or surveyors. These came into their own when navigators around the 15th Century started to prepare tables. Copernicus knew of the secant which he called the *hypotenusa*. Viète

knew the results

$$\frac{\csc x}{\sec x} = \cot x = \frac{1}{\tan x}$$
$$\frac{1}{\csc x} = \frac{\cos x}{\cot x} = \sin x.$$

The abbreviations used by various authors were similar to the trigonometric functions already discussed. Cavalieri used *Se* and *Se.2*, Oughtred used *se arc* and *sec co arc* while Wallis used *s* and $\sigma$. Albert Girard used *sec*, written above the angle as he did for the tangent.

The term *trigonometry* first appears as the title of a book *Trigonometria* by Pitiscus, published in 1595. Pitiscus also discovered the formulas for $\sin 2x$, $\sin 3x$, $\cos 2x$, and $\cos 3x$.

The 18th Century saw trigonometric functions of a complex variable being studied. Johann Bernoulli found the relation between $\sin^{-1} z$ and $\log z$ in 1702 while Cotes, in a work published in 1722 after his death, showed that

$$ix = \log(\cos x + i \sin x).$$

De Moivre published his famous theorem

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx$$

in 1722 while Euler, in 1748, gave the formula (equivalent to that of Cotes)

$$\exp(ix) = \cos x + i \sin x.$$

The hyperbolic trigonometric functions were introduced by Lambert.

Once we know something about similarity in triangles we should see some standard correspondence between the length of the sides of a triangle and the angles of the triangle. It is just these ratios that the trigonometric functions express, geometrically.

Just as we defined the exponential function above in terms of certain functional equations, we can define the trigonometric functions using functional equations based on properties like the sum and difference formulas. If we take these formulas and the Pythagorean identity as given then we can prove that only two real functions satisfy those conditions. Symbolically, we say that there exists exactly one pair of real functions $s$ and $c$ such that for all real numbers $x$ and $y$, the following equations hold:

$$\begin{aligned} s(x)^2 + c(x)^2 &= 1, \\ s(x+y) &= s(x)c(y) + c(x)s(y), \\ c(x+y) &= c(x)c(y) - s(x)s(y), \end{aligned}$$

with the added condition that

$$0 < xc(x) < s(x) < x \text{ for } 0 < x < 1.$$

Other derivations, starting from other functional equations, are also possible, and such derivations can be extended to the complex numbers.

**Properties of Trigonometric Functions**

The *Law of Sines* for an arbitrary triangle states

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Now, it is also known in the formulation:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

The common number $a/\sin A$ occurring in the theorem is the diameter of the circumcircle through the three points $A$, $B$ and $C$. The Law of Sines is useful for computing the lengths of the unknown sides in a triangle if two angles and one side are known. This is a common situation occurring in triangulation, a technique to determine unknown distances by measuring two angles and an accessible enclosed distance.

The *Law of Cosines* is merely an extension of the Pythagorean theorem

$$c^2 = a^2 + b^2 - 2ab\cos C,$$

or

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

The Law of Cosines is useful to determine the unknown data of a triangle if two sides and an angle are known. If the angle is not contained between the two sides, the triangle may not be unique.

There is also a *Law of Tangents*:

$$\frac{a+b}{a-b} = \frac{\tan\left[\frac{1}{2}(A+B)\right]}{\tan\left[\frac{1}{2}(A-B)\right]}.$$

**Geometric Realizations**

For the functions of the unit circle, note that each of the six functions of the angle correspond to certain segments associated with that angle in the unit circle.

If $\angle BOA = \theta$, then $OA = OB = 1$, $OD = EB = \cos\theta$ (the $x$-coordinate of the point $B$) and $OE = BD = \sin\theta$ (the $y$-coordinate of the point $B$). Now, $AC$ lies on the tangent to the circle at $A$ and is, hence, perpendicular to $OA$. Also, $\triangle BOD \sim \triangle COA$, so

$$AC = \frac{AC}{OA} = \frac{BD}{OD} = \frac{\sin\theta}{\cos\theta} = \tan\theta.$$
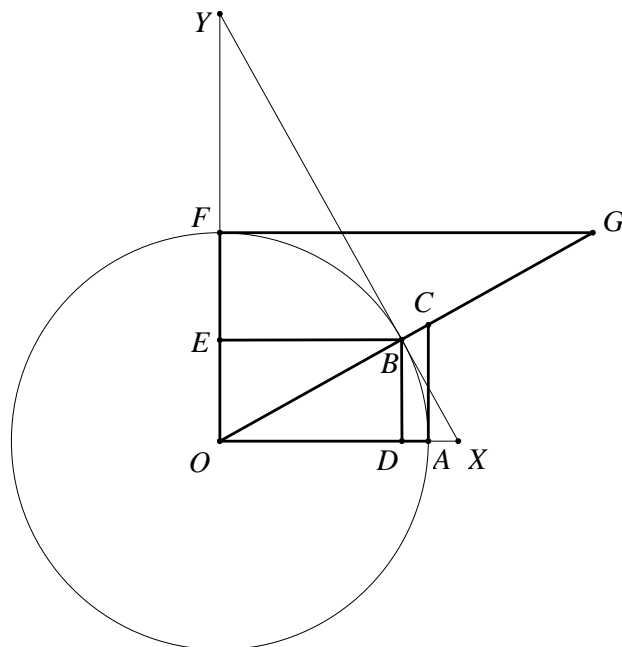
Figure 8.5: Unit Circle Realizations of trigonometric values

Also, from this similarity we have that

$$OC = \frac{OC}{OA} = \frac{OB}{OD} = \frac{1}{\cos \theta} = \sec \theta.$$

Likewise, $\triangle GFO \sim \triangle BEO$, and

$$FG = \frac{FG}{OF} = \frac{BE}{OE} = \frac{\cos \theta}{\sin \theta} = \cot \theta,$$

and

$$OG = \frac{OG}{OF} = \frac{OB}{EB} = \frac{1}{\sin \theta} = \csc \theta.$$

Thus, we have the following geometric realizations of the six functions:

| Function | Segment |
|----------|---------|
| $\sin \theta$ | $BD = OE$ |
| $\cos \theta$ | $OD = BE$ |
| $\tan \theta$ | $AC$ |
| $\cot \theta$ | $GF$ |
| $\sec \theta$ | $OC$ |
| $\csc \theta$ | $OG$ |

Note that if we look at the tangent to the circle at $B$, then we have points $X$ where the tangent intersects $OA$ and $Y$ where it intersects $OF$. The triangle $\triangle XBO$ is similar to $\triangle BDO$, so from this we get that $XB = \tan \theta$ and $OX = \sec \theta$. Likewise $\triangle OBY \sim \triangle BDO$ and this gives us that $BY = \cot \theta$ and $OY = \csc \theta$.

# 8.6 CORDIC Algorithm

We mentioned how we used to use log tables and trig tables to find the values of the exponential, logarithmic, and trigonometric functions before the advent of electronic calculators. It used to be someone job to calculate a lot of these formulas by hand — and they were called *computers*. Now, we have computers and calculators that are faster and more efficient. How do we know that they are more accurate? How do the electronic machines calculate that the sine of $22°$ is $\sin(22°) = 0.3746065934$ and why should we believe that it is correct? How do computers/calculators evaluate functions such as $e^x$, $\sqrt{x}$, $\sin(x)$ and $\arctan(x)$, to name just a few? How accurate are these calculations?

Years ago, if you used `BASICA` on an IBM computer to evaluate the natural log of 1.001, it would return $\ln(1.001) = 9.994461 \times 10^{-4}$ while the actual value is $9.995003331 \times 10^{-4}$. The computer was only getting the first 3 digits correct, while at that same time the TI-82 was returning all 10 digits correct. Clearly they are not using the same algorithm for computing the logarithm of a number.

Since on a computing device the operations of addition, subtraction, multiplication and division are relatively easy, while exponentiation is hard, it is best to try to replace the elementary functions by polynomials that will approximate the original function "well enough" on the given interval. That leaves plenty of questions: what polynomials? what is "well enough"?

The usual culprits that are used to approximate the elementary functions are

- Taylor series and Maclaurin series,
- Fourier series,
- Chebyshev series,
- Padé approximants, or
- Chebyshev-Padé approximants

Let's look at how we might compute the values of the tangent of a real number between $-1$ and $1$.

**Taylor Series** Let's look at the Taylor series for $\tan(x)$ centered at $x = 0$. From Calculus we know that

$$\tan(x) \approx \sum_{n=0}^{\infty} \frac{\frac{d^n}{dx^n}\left(\tan(x)|_{x=0}\right)}{n!} x^n$$

If we take $n = 10$ in this series we get the following polynomial to approximate $\tan(x)$:

$$P_9(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{7}{315}x^7 + \frac{62}{2835}x^9.$$

**Chebyshev Series** At the same time the Chebyshev series for $\tan(x)$ takes the following form. The Chebyshev polynomials, $T_n(x)$, are solutions to Chebyshev's

equation:

$$(1 - x^2)y'' - xy' + n^2 y = 0,$$

and satisfy the recurrence relation

$$T_0(x) = 1, \ T_1(x) = x, \ \dots T_{n+1}(x) = 2x \cdot T_n(x) - T_{n-1}x,$$

for $n > 0$. The Chebyshev polynomials are *orthogonal* on the interval $[-1, 1]$ with respect to the weight function $w(x) = 1/\sqrt{1 - x^2}$. This means that

$$\int_{-1}^{1} T_n(x) \cdot T_m(x) \cdot w(x) \, dx = 0, \ \text{if } m \neq n.$$

For such a family of orthogonal functions, we can write another function, say $f(x)$ in terms of our orthogonal family by

$$f(x) = \sum_{n=0}^{\infty} c_n T_n(x),$$

where

$$c_n = \frac{\int_{-1}^{1} f(x) \cdot T_n(x) \cdot w(x) \, dx'}{\int_{-1}^{1} (T_n(x))^2 \cdot w(x) \, dx}$$

This is how we get the following 9th order Chebyshev polynomial for $\tan(x)$.

$$T_9(x) = 1.0040x + 0.325610x^3 + 0.173344x^5 - 0.026645x^7 + 0.0846426x^9.$$

**Padé Approximant** If we are given a power series $f(x) = \sum c_n x^n$, then a Padé approximant is a rational function

$$[L/M] = \frac{a_0 + a_1 x + \cdots + a_L x^L}{1 + b_1 x + \cdots + b_M x^M}$$

whose Maclaurin series agrees with the given power series at least through order $L + M$. If it converges, it should converge faster than the original power series. For example if

$$
\begin{aligned}
f(x) &= 1 - \frac{1}{2}x + \frac{1}{3}x^2 + \dots \\
[1/0] &= 1 - \frac{1}{2}x \\
[0/1] &= \frac{1}{1 + \frac{1}{2}x} \\
[1/1] &= \frac{1 + \frac{1}{6}x}{1 + \frac{2}{3}x}
\end{aligned}
$$

The [5/5] Padé approximant for the tangent function is

$$P_{[5/5]}(x) = \frac{x - \frac{1}{9}x^3 + \frac{1}{945}x^5}{1 - \frac{4}{9}x^2 + \frac{1}{63}x^4}.$$

**Chebyshev-Padé Approximant** The $(m, n)$ Chebyshev-Padé approximation is the rational function $p(x)/q(x)$ with $\deg(p(x)) \le m$ and $\deg(q(x)) \le n$ such that the Chebyshev series expansion of $p(x)/q(x)$ has maximal initial agreement with the Chebyshev series expansion of $f$, usually through degree $m + n$. For the tangent function this is:

$$CP_{[5/5]}(x) = \frac{1.27604x - 0.141986x^3 + 0.00136920x^5}{1.27604 - 0.567336x^2 + 0.0203367x^4}.$$

So, we have the following four approximations to the tangent function

$$
\begin{aligned}
P_9(x) &= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{7}{315}x^7 + \frac{62}{2835}x^9 \\
T_9(x) &= 1.0040x + 0.325610x^3 + 0.173344x^5 - 0.026645x^7 + 0.0846426x^9 \\
P_{[5/5]}(x) &= \frac{x - \frac{1}{9}x^3 + \frac{1}{945}x^5}{1 - \frac{4}{9}x^2 + \frac{1}{63}x^4} \\
CP_{[5/5]}(x) &= \frac{1.27604x - 0.141986x^3 + 0.00136920x^5}{1.27604 - 0.567336x^2 + 0.0203367x^4}
\end{aligned}
$$

We can look at the error between these approximations and the actual values of the tangent function by graphing the absolute value of the error on the interval $[-1, 1]$. Let

$$
\begin{aligned}
ET(x) &= |P_9(x) - \tan(x)| \\
EC(x) &= |T_9(x) - \tan(x)| \\
EP(x) &= |P_{[5/5]}(x) - \tan(x)| \\
ECP(x) &= |CP_{5/5]}(x) - \tan(x)|
\end{aligned}
$$

The graph below is a graph of the absolute value of the error of each of these approximations on the interval $[-1, 1]$. The red curve is the error in the Chebyshev series approximation, the blue curve is error in the Padé approximant, the green curve is the error in the Taylor series approximation, and the magenta is the error in the Chebyshev-Padé approximant.

For $x = 0.25$ consider the following values:

$$
\begin{aligned}
P_9(0.25) &= 0.25534191905295 \\
T_9(0.25) &= 0.25535748371491 \\
P_{[5/5]}(0.25) &= 0.25534192122101 \\
CP_{[5/5]}(0.25) &= 0.25534196284698 \\
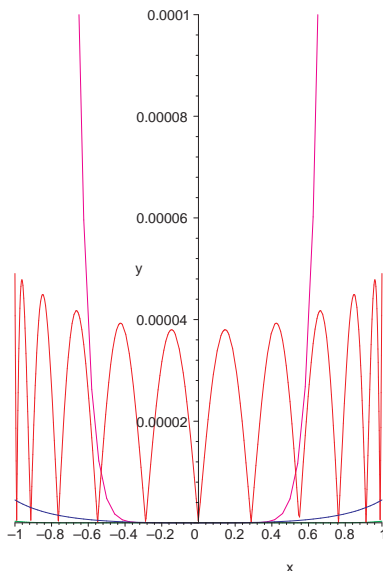\tan(0.25) &= 0.25534192122104
\end{aligned}
$$

Figure 8.6: Errors to approximations to tangent

Note that all are good to 4 decimal places and the Padé approximant is good to 13 decimal places.

## 8.6.1   CORDIC Algorithm

In 1959 James Volder developed the CORDIC algorithm in order to solve navigational problems with onboard computers in large military planes. He was allowed to use addition, subtraction, multiplication, division and a shift algorithm. With these he had to be able to compute angles and their functions. Later this was used to find the values of most of the elementary functions.

CORDIC stands for **CO**ordinate **R**otation **DI**gital **C**omputer. The algorithm uses formulas for functions of a sum as a basis for a sequence of recursive computations. This algorithm is used by Casio, TI, Sharp, and all current calculator manufacturers and by Intel and Motorola for their coprocessors.

The idea of the CORDIC algorithm was to evaluate the tangent of an angle or the arctangent of a real number. From there one can find the other functions.

Let's see how the CORDIC algorithm is used to compute $\tan(\theta)$. This is done in several steps. Take an angle $\theta$ and rotate a vector over this angle towards zero in a series of steps such that

1. the sum of all the steps taken equals $\theta$, and

2. accumulate the corresponding $x$ and $y$ increments at each step so that in the end $\frac{y}{x} = \tan\theta$.

To make the arithmetic simple and fast, you choose a set of angles $\phi(k) = \arctan(10^{-k})$. The TI 82 used 14 digit arithmetic, so on that calculator we would store the following 15 entry table:

| $k$ | $10^{-k}$ | $\arctan(10^{-k})$ | [13 digits] | index |
|---|---|---|---|---|
| 0 | 1. | .78539816339745 | .7853981633974 | 1 |
| 1 | .1 | .09966865249116 | .0996686524912 | 2 |
| 2 | .01 | .00999966668667 | .0099996666867 | 3 |
| 3 | .001 | .00099999966667 | .0009999996667 | 4 |
| 4 | .0001 | .00009999999967 | .0000999999997 | 5 |
| 5 | .00001 | .00001000000000 | .0000100000000 | 6 |
| 6 | .000001 | .00000100000000 | .0000010000000 | 7 |
| 7 | .0000001 | .00000010000000 | .0000001000000 | 8 |
| 8 | .00000001 | .00000001000000 | .0000000100000 | 9 |
| 9 | .000000001 | .00000000100000 | .0000000010000 | 10 |
| 10 | .0000000001 | .00000000010000 | .0000000001000 | 11 |
| 11 | .00000000001 | .00000000001000 | .0000000000100 | 12 |
| 12 | .000000000001 | .00000000000100 | .0000000000010 | 13 |
| 13 | .0000000000001 | .00000000000010 | .00000000000010 | 14 |
| 14 | .00000000000001 | .00000000000001 | | 15 |

The algorithm now is outlined as: Take $\theta$ to be between 0 and $\pi/2$ radians. To compute $Z = \tan(\theta)$

```
set K=0
set X=1
set Y=0
while (θ ≠ 0)
    while(θ < arctan(10⁻ᵏ))
    K = K + 1
    end while
θ=θ-arctan(10⁻ᵏ)
TEMP=X
X=X-10⁻ᵏ× Y
Y=Y+10⁻ᵏ× TEMP
end while
Z=Y/X
```

Why does this work? Well, the input angle, $\theta$, defines a vector from the origin whose length is fixed. The rotated coordinates $X'$ and $Y'$ as the vector rotates from

an initial position given by $(X, Y)$ are given by:

$$
\begin{aligned}
X' &= X\cos(\theta) + Y\sin(\theta) \\
Y' &= Y\cos(\theta) - X\sin(\theta) \\
\frac{X'}{\cos(\theta)} &= X + \tan(\theta) \times Y \\
\frac{Y'}{\cos(\theta)} &= Y - \tan(\theta) \times X
\end{aligned}
$$

Now, we broke $\theta$ into successively smaller parts and stored the arctangent, $\arctan(10^{-k})$, for each part.

Now, the sine and cosine function come from the identities:

$$
\begin{aligned}
\sin(\theta) &= \frac{\tan(\theta)}{\sqrt{1 + \tan^2(\theta)}} \\
\cos(\theta) &= \frac{1}{\sqrt{1 + \tan^2(\theta)}}
\end{aligned}
$$

**Inverse Trigonometric Functions**: To get the inverse tangent use the same constants as above. Enter with Y as the argument.

```
    set K=0
set X=1
set θ=0
while (Y≠ 0)
    while(Y < 10^{-k}× X)
    K = K + 1
    end while
θ=θ-arctan(10^{-k})
TEMP=X
X=X-10^{-k}× Y
Y=Y+10^{-k}× TEMP
end while
return
```

**Exponential Functions**: Use the constants: $\log(1 + 10^{-k})$, $k = 0, 1, 2 \ldots$. Then to find the exponential value $Y = \exp(X)$, use the following algorithm with $X$ as the argument.

```
    set K=0
set Y=1
while(K < 14)
Z=ln(1+10^{-k})
    while X ≥ Z
    X=X-Z
```
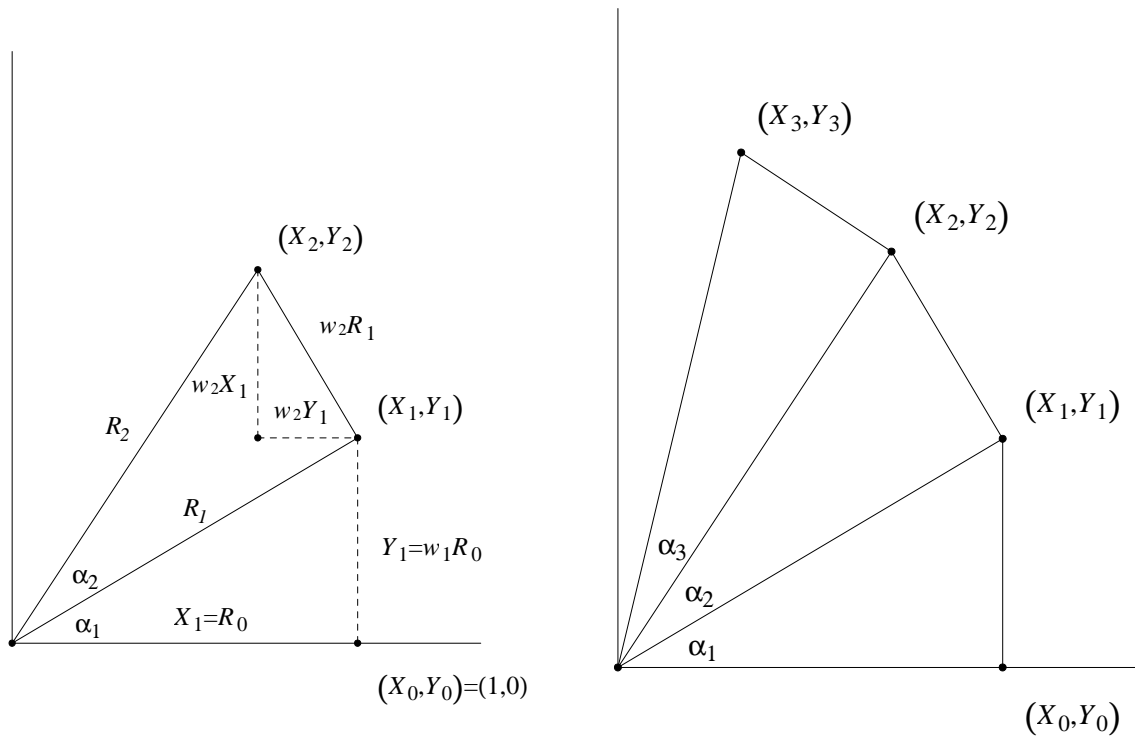
```
      Y=Y+10⁻ᵏ× Y
         end while
   K=K+1
   end while
   return Y
```

There are numerous references to the CORDIC algorithm on the Internet.

## 8.6.2   Alternative Method

There are many implementations of the CORDIC algorithm.



Suppose you want to find the trigonometric functions of a positive angle $\theta$ of less than $360°$. The algorithm finds acute angles $\alpha_1, \ldots, \alpha_n$ so that $\theta \approx \alpha_1 + \alpha_2 + \cdots + \alpha_n$, puts $w_i = tan\alpha_i$ and then performs the following calculations

$$
\begin{aligned}
X_0 &= 1, \quad Y_0 = 0, \\
X_i &= X_{i-1} - w_i Y_{i-1}, \\
Y_i &= Y_{i-1} + w_i X_{i-1},
\end{aligned}
$$

for $i = 1, 2, \ldots, n$. The situation for $n = 3$ is illustrated above. When this calculation is completed, the trigonometric functions of $\theta$ can be calculated using the usual trigonometric formulas applied to the coordinates $(X_n, Y_n)$. Note that $(X_n, Y_n)$ is probably not on the unit circle.

The computational power of the algorithm is enhanced by choosing $\alpha_1, \alpha_2, \ldots, \alpha_n$ so that $w_1, w_2, \ldots, w_n$ are simple one-significant digit numbers. Since most calculators use decimal digits, each $w_i$ will be $10^{-k}$ for some nonnegative integer $k$. For example if $X_{i-1} = 0.4214$ and $Y_{i-1} = 0.6582$ and $w_i = 0.0010$, then the computations necessary to compute $X_i$ and $Y_i$ involve multiplication by 0.0010 which can be considered a shift operation and addition or subtraction. If each $w_i$ is $10^{-k}$ for some $k$, then $\alpha_i$ must be $\arctan(10^{-k}$. (In a binary computer each $w_k$ is $2^{-k}$ and $\alpha_k$ is $\arctan(2^{-k})$ for some $k$.) The approximate values for the numbers $\arctan 10^{-k}$ for $k = 0, 1, 2, \ldots, m$ are stored in the calculator where $m$ is chosen with consideration to the number of digits of the calculator.

To choose the particular $\alpha_i$ for each given $\theta$, the following procedure is often used. Find the largest nonnegative integer $q_0$ so that $q_0 \arctan 1 < \theta$. If $q_0$ is greater than or equal to 1, then let $\alpha_1 = \alpha_2 = \cdots = \alpha_{q_0} = 45°(= \arctan 1)$. Now find the largest nonnegative integer $q_l$ so that

$$q_0 \arctan 1 + q_l \arctan 0.l \leq \theta.$$

If $q_l$ is greater than or equal to 1, then let all the $\alpha_i$ from $q_0 + 1$ to $q_0 + ql$ be $\arctan 0.l$ which is approximately $5.71°$. This process is continued until $\theta$ is approximately equal to $\sum_{j=0}^{k} q_j \arctan 10^{-j}$.

Let's apply this algorithm to compute $\tan 51°$. First we need the basic values of the $\alpha_i$'s.

| $k$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $\arctan 10^{-k}$ | 45° | 5.711° | 0.573° | 0.057° | 0.006° |

Now, $51° \approx 45° + 5.711° + 0(.573°) + 5(0.057°) + 0(0.006°)$, so we have that $q_0 = 1$, $q_1 = 1$, $q_2 = 0$, $q_3 = 5$ and $q_4 = 0$. Now, the computations are given in the following table following the rules $X_i = X_{i-1} - w_i Y_{i-1}$ and $Y_i = Y_{i-1} + w_i X_{i-1}$:

| $i$ | $\alpha_i$ | $\sum \alpha_i$ | $w_i$ | $X_i$ | $Y_i$ |
|---|---|---|---|---|---|
| 0 | | | | 1.0000 | 0.0000 |
| 1 | 45° | 45.000° | 1 | 1.0000 | 1.0000 |
| 2 | 5.711° | 50.711° | 0.1 | 0.9000 | 1.1000 |
| 3 | 0.057° | 50.768° | 0.001 | 0.8989 | 1.1009 |
| 4 | 0.057° | 50.825° | 0.001 | 0.8978 | 1.1018 |
| 5 | 0.057° | 50.882° | 0.001 | 0.8967 | 1.1027 |
| 6 | 0.057° | 50.939° | 0.001 | 0.8956 | 1.1036 |
| 7 | 0.057° | 50.996° | 0.001 | 0.8945 | 1.1045 |

Now we can compute $\tan 51° \approx Y_7/X_7 = 1.1045/0.8945 \approx 1.2348$.

For angles greater than $360°$ or less than $0°$, an appropriate (positive or negative) multiple of $2\pi$ or $360°$ is added to the angle to bring it into the range of $0°$ to

$360°$. Even though the algorithm has been described using degree measure, many calculators and computers also compute with radian measure and many use only radian measure internally. It turns out that many calculators do not accurately compute the trigonometric functions of large numbers, greater than $10^{10}$, because the value of $\pi$ used in the calculator to reduce the given number to between 0 and $2\pi$ is only given to 10 to 13 significant figures.

Turning now to inverse trigonometric functions, suppose one of $\arcsin w$, $\arccos w$, or $\arctan w$ is to be calculated, where there are restrictions on $w$ for the first two. For simplicity assume $w$ is greater than 0. In each case there are nonnegative numbers $x$ and $y$ so that $x^2 + y^2 = 1$ and so that either $y = w$, $x = w$, or $y/x = w$, respectively. In each case the measure of the angle $\theta$ whose tangent is $y/x$ needs to be found.

One version of the algorithm does the following. Let $(X_0, Y_0) = (x, y)$, and calculate $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ using the previous formulas where the $\alpha_i$ are chosen from the set $\{-\arctan 1, \arctan .1, -\arctan .01, \ldots\}$ so that $Y_i$ gets small but stays nonnegative. For example, if $(x, y) = (X_0, Y_0) = (.6, .8)$, $\alpha_l = -45°$, $(X_1, Y_1) = (1.4, 0.2)$, $\alpha_2 = -\arctan .1 = -5.7°$, $(X_2, Y_2) = (1.42, 0.06)$, $\alpha_3 = -\arctan 0.01$, *etc.* When the process is complete, $\theta = -(\alpha_1 + \ldots \alpha_n)$. Note that in this process nonnegative integers $q_i$ can be found so that $\theta \approx \sum_{j=0}^{m} q_j \arctan 10^{-j}$.

Another version of the algorithm starts out with $(X_0, Y_0) = (1, 0)$ and computes $(X_i, Y_i)$ with a predetermined set of angles, except for sign, where the sign of each angle is chosen by reference to the original coordinates $(x, y)$. Knowing the length of each $(X_i, Y_i)$ helps in the process.

For some other transcendental functions two different approaches are frequently used. First an algorithm for hyperbolic trigonometric functions can be generated in a similar manner to regular trigonometric functions. After hyperbolic trigonometric functions are computed, exponentials and then logarithms can be generated using the usual mathematical relationships.

$$
\begin{aligned}
e^w &= \cosh w + \sinh w \\
\frac{1}{2} \log w &= \tanh^{-1}\left(\frac{w-1}{w+1}\right)
\end{aligned}
$$