Chapter 11

Uniform Continuity

We saw in the exercises that there are some functions that are badly discontinuous, such as the characteristic function of the rationals on the reals:

\[ f(x) = \begin{cases} 
1 & x \in \mathbb{Q} \\
0 & \text{otherwise}. 
\end{cases} \]

When we think of continuous functions, we tend to think of the usual functions from precalculus and calculus — polynomials, trigonometric functions, exponential functions, and so forth. These are continuous, yet somehow seem to be more than just meeting the definition of continuity.

By Theorem 10.1 we know that \( f : \mathbb{R} \to \mathbb{R} \) is continuous on a set \( S \subseteq \text{dom}(f) \) if and only if

for each \( a \in S \) and \( \epsilon > 0 \) there is a \( \delta > 0 \) so that if \( x \in \text{dom}(f) \) and \( |x - a| < \delta \) then \( |f(x) - f(a)| < \epsilon \).

From this definition we see that the choice of \( \delta \) depends both on the point \( a \in S \) and on the particular \( \epsilon > 0 \).

As an example, consider the function \( f(x) = 1/x^2 \) on the set \((0, +\infty)\). We know that \( f \) is continuous on this interval. Let \( a > 0 \) and \( \epsilon > 0 \). Now, we will need to show that \( |f(x) - f(a)| < \epsilon \) for \( |x - a| \) sufficiently small.

\[
|f(x) - f(a)| = \left| \frac{1}{x^2} - \frac{1}{a^2} \right| = \left| \frac{a^2 - x^2}{a^2x^2} \right| = \left( \frac{a - x}{a^2x^2} \right) \left( \frac{a + x}{a^2x^2} \right).
\]

If \( |x - a| < \frac{a}{2} \), then \( \frac{a}{2} < |x| < \frac{3a}{2} \) and \( |x + a| < \frac{5a}{2} \). Thus, if \( |x - a| < \frac{a}{2} \), then

\[
|f(x) - f(a)| < \left| \frac{a - x}{a^2x^2} \right| \cdot \frac{5a}{2} = \frac{10|x - a|}{a^3}.
\]

Thus if we let \( \delta = \min \{ \frac{a}{2}, \frac{a\epsilon}{10} \} \), then

\( |x - a| < \delta \) implies that \( |f(x) - f(a)| < \epsilon \).
Therefore, we have now shown that the conditions of Theorem 10.1 hold for \( f \) on \((0, +\infty)\). Note that \( \delta \) depends on both \( \epsilon \) and on \( a \). Even if we fix \( \epsilon \), \( \delta \) gets small when \( a \) is small. This shows that our choice of \( \delta \) depends on the value of \( a \) as well as \( \epsilon \), though this might seem to be because of sloppy estimates. However, we can see that the value of \( \delta \) must depend on \( a \) as well as \( \epsilon \), though this might seem to be because of sloppy estimates. How ever, we can see that it is very useful to know when the \( \delta \) in this condition can be chosen to depend only on \( \epsilon > 0 \) and the set \( S \), so that \( \delta \) does not depend on the particular point \( a \).

**Definition 11.1** Let \( f : \mathbb{R} \to \mathbb{R} \) be defined on \( S \subseteq \mathbb{R} \). Then \( f \) is uniformly continuous on \( S \) if

for each \( \epsilon > 0 \) there is a \( \delta > 0 \) so that if \( x, y \in S \) and \( |x - y| < \delta \) then \( |f(x) - f(y)| < \epsilon \).

We will say that \( f \) is uniformly continuous if it is uniformly continuous on \( \text{dom}(f) \).

Note that this says that if \( f \) is uniformly continuous on \( S \) then for any given \( \epsilon > 0 \) the choice of \( \delta > 0 \) works for the entire set \( S \).

Note that if a function is uniformly continuous on \( S \), then it is continuous for every point in \( S \). By its very definition it makes no sense to talk about a function being uniformly continuous at a point.

Now, we can show that the function \( f(x) = \frac{1}{x^2} \) is uniformly continuous on any set of the form \([a, +\infty)\). To do this we will have to find a \( \delta \) that works for a given \( \epsilon \) at every point in \([a, +\infty)\). We have

\[
f(x) - f(y) = \frac{(y - x)(y + x)}{x^2 y^2}.
\]

We want to see if we can prove that the term \( \frac{x + y}{x^2 y^2} \) is bounded by some number \( M \) on \([a, +\infty)\). Once we have done that we can take \( \delta = \frac{\epsilon M}{2} \). Now,

\[
\frac{x + y}{x^2 y^2} = \frac{1}{x^2 y} + \frac{1}{x y^2} \leq \frac{1}{a^3} + \frac{1}{a^3} = \frac{2}{a^3}.
\]

Thus, we will take

\[
\delta = \frac{\epsilon a^3}{2}.
\]

**Question:** How would we show that the function \( g(x) = x^2 \) is uniformly continuous on \([-5, 5]\)?

**Theorem 11.1** If \( f \) is continuous on a closed interval \([a, b]\), then \( f \) is uniformly continuous on \([a, b]\).
PROOF: Assume that $f$ is not uniformly continuous on $[a, b]$. Then there is an $\epsilon > 0$ such that for each $\delta > 0$ the implication

\[ |x - y| < \delta \text{ implies } |f(x) - f(y)| < \epsilon \]

fails. Therefore, for each $\delta > 0$ there exists at least a pair of points $x, y \in [a, b]$ such that $|x - y| < \delta$ but $|f(x) - f(y)| \geq \epsilon$.

Thus, for each $n \in \mathbb{N}$ there exist $x_n, y_n \in [a, b]$ so that $|x_n - y_n| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| > \epsilon$. By the Bolzano-Weierstrass Theorem (6.14) there exists a subsequence \( \{x_{n_k}\} \subset \{x_n\} \) that converges. Moreover, if $x_0 = \lim_{k \to \infty} x_{n_k}$, then $x_0 \in [a, b]$. Clearly we will also have to have that $x_0 = \lim_{k \to \infty} y_{n_k}$. Since $f$ is continuous at $x_0$ we have

\[ f(x_0) = \lim_{k \to \infty} f(x_{n_k}) = \lim_{k \to \infty} f(y_{n_k}), \]

so

\[ \lim_{k \to \infty} [f(x_{n_k}) - f(y_{n_k})] = 0. \]

Since $|f(x_{n_k}) - f(y_{n_k})| \geq \epsilon$ for all $k$, we have a contradiction. This leads us to conclude that $f$ is uniformly continuous on $[a, b]$.

Note that in view of this theorem the following functions are uniformly continuous on the indicated sets: $x^{45}$ on $[a, b]$, $\sqrt{x}$ on $[0, a]$, and $\cos(x)$ on $[a, b]$.

**Theorem 11.2** If $f$ is uniformly continuous on $A$ and $\{x_n\}$ is a Cauchy sequence in $A$, then $\{f(x_n)\}$ is a Cauchy sequence.

PROOF: Let $\{x_n\}$ be a Cauchy sequence in $A$ and let $\epsilon > 0$. Since $f$ is uniformly continuous on $A$, there is a $\delta > 0$ so that if $x, y \in A$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.

Since $\{x_n\}$ is a Cauchy sequence, there is an $N \in \mathbb{N}$ so that if $m, n > N$ then $|x_m - x_n| < \delta$. Thus, this implies that if $m, n > N$ then $|f(x_m) - f(x_n)| < \epsilon$, which proves that $\{f(x_n)\}$ is a Cauchy sequence.

As an example consider the function $f(x) = 1/x^2$ on $(0, 1)$. Let $x_n = 1/n$ for $n \in \mathbb{N}$. This clearly forms a Cauchy sequence in $(0, 1)$. However, the function takes the values $f(x_n) = n^2$ and the sequence $\{n^2\}$ is clearly not a Cauchy sequence. Thus, $f$ cannot be a uniformly continuous function on $(0, 1)$.

We define a function $\hat{f}$ to be an extension of $f$ if $\text{dom}(f) \subseteq \text{dom}(\hat{f})$ and $f(x) = \hat{f}(x)$ for all $x \in \text{dom}(f)$.

**Theorem 11.3** A real-valued function $f$ on $(a, b)$ is uniformly continuous on $(a, b)$ if and only if it can be extended to a continuous function $\hat{f}$ on $[a, b]$. 

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Proof: First, suppose that \( f \) can be extended to a continuous function \( \hat{f} \) on \([a, b]\). Then \( \hat{f} \) is uniformly continuous on \([a, b]\) by Theorem 11.1, so clearly \( f \) is uniformly continuous on \((a, b)\).

Now, suppose that \( f \) is uniformly continuous on \((a, b)\). We need to define \( f(a) \) and \( f(b) \) in such a way that the extension will be continuous. We will show how to deal with \( \hat{f}(a) \) and the other extension is handled similarly.

Let \( \{x_n\} \) be a sequence in \((a, b)\) that converges to \( a \). Since the sequence converges it must be a Cauchy sequence. Thus, \( \{f(x_n)\} \) is also a Cauchy sequence. Therefore, it converges. Let’s call this Condition A.

Let \( \{x_n\} \) and \( \{y_n\} \) be two sequences in \((a, b)\) that both converge to \( a \). Define a new sequence \( \{u_n\} \) by interleaving \( x_n \) and \( y_n \):

\[
\{u_n\}_{n=1}^\infty = \{x_1, y_1, x_2, y_2, x_3, y_3, \ldots \}
\]

It should be clear that \( \lim_{n \to \infty} u_n = a \). Thus, \( \lim_{n \to \infty} f(u_n) \) exists by Condition A. Since \( \{f(x_n)\} \) and \( \{f(y_n)\} \) are both subsequences of \( \{f(u_n)\} \) they must converge and converge to the same limit. Thus,

\[
\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} f(y_n).
\]

Let’s call this Condition B.

Thus, define \( \hat{f}(a) = \lim_{n \to \infty} f(s_n) \) for any sequence \( \{x_n\} \) in \((a, b)\) converging to \( a \). Condition A guarantees that this limit exists, and Condition B guarantees that this limit is well-defined and unique. This implies that \( \hat{f} \) is continuous at \( a \).

As an example consider the function \( f(x) = \sin(x)/x \) for \( x \neq 0 \). We can extend this function on \( \mathbb{R} \) by

\[
\hat{f}(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}
\]

The fact that \( \hat{f} \) is continuous at \( x = 0 \) implies that \( \hat{f} \) is uniformly continuous on \((a, 0)\) and \((0, b)\) for any \( a < 0 < b \). In fact, \( \hat{f} \) is uniformly continuous on \( \mathbb{R} \).

Theorem 11.4 Let \( f \) be continuous on an interval \( I \). Let \( I^\circ \) be the interval obtained by removing from \( I \) any endpoints that happen to be in \( I \). If \( f \) is differentiable on \( I^\circ \) and if \( f' \) is bounded on \( I^\circ \), then \( f \) is uniformly continuous on \( I \).

Proof: Let \( M \) be a bound for \( f' \) on \( I \) so that \( |f'(x)| \leq M \) for all \( x \in I^\circ \). Let \( \epsilon > 0 \) and let \( \delta = \frac{\epsilon}{M} \). Consider \( a, b \in I \) where \( a < b \) and \( |b - a| < \delta \). By the Mean Value Theorem there exists \( x \in (a, b) \) so that

\[
f'(x) = \frac{f(b) - f(a)}{b - a},
\]

so

\[
|f(b) - f(a)| = |f'(x)| \cdot |b - a| \leq M |b - a| < M \delta = \epsilon.
\]

Thus, \( f \) is uniformly continuous on \( I \).
Why is uniform continuity important? One of the reasons for studying uniform continuity is its application to the integrability of continuous functions on a closed interval, i.e. proving that a continuous function on a closed interval is integrable. To see how this might work with Riemann sums consider a continuous nonnegative real-values function \( f \) defined on \([0, 1] \). For \( n \in \mathbb{N} \) and \( k = 0, 1, 2, \ldots, n - 1 \), let

\[
M_{k,n} = \text{lub}\{f(x) \mid x \in \left[\frac{k}{n}, \frac{k+1}{n}\right]\}
\]

\[
m_{k,n} = \text{glb}\{f(x) \mid x \in \left[\frac{k}{n}, \frac{k+1}{n}\right]\}
\]

Then the sum of the areas of the rectangles in Figure 11.2 equals

\[
U_n = \frac{1}{n} \sum_{k=0}^{n-1} M_{k,n}
\]

and the sum of the areas of the rectangles in Figure 11.1 equals

\[
L_n = \frac{1}{n} \sum_{k=0}^{n-1} m_{k,n}.
\]

The function \( f \) is Riemann integrable if the numbers \( U_n \) and \( L_n \) are close together for large \( n \), in other words, if

\[
\lim_{n \to \infty} (U_n - L_n) = 0.
\]

In that case we define

\[
\int_0^1 f(x) \, dx = \lim_{n \to \infty} U_n = \lim_{n \to \infty} L_n.
\]

In order to prove that the above limit is 0, we actually need uniform continuity. Note that

\[
0 \leq U_n - L_n = \frac{1}{n} \sum_{k=0}^{n-1} (M_{k,n} - m_{k,n})
\]

for all \( n \). Let \( \epsilon > 0 \). By our previous theorem, \( f \) is uniformly continuous on \([0, 1]\), so there exists \( \delta > 0 \) so that

\[
x, y \in [0, 1] \text{ and } |x - y| < \delta \text{ imply } |f(x) - f(y)| < \epsilon.
\]
Now, choose an $N$ so that $\frac{1}{N} < \delta$. If $n > N$ then for $i = 0, 1, 2, \ldots, n - 1$ we know that there exist $x_i, y_i \in [\frac{1}{n}, \frac{i+1}{n}]$ satisfying $f(x_i) = m_{i,n}$ and $f(y_i) = M_{i,n}$. Since $|x_i - y_i| \leq \frac{1}{n} < \frac{1}{N} < \delta$, the above shows that $M_{i,n} - m_{i,n} = f(y_i) - f(x_i) < \epsilon$, so that

$$0 \leq U_n - L_n = \frac{1}{n} \sum_{i=0}^{n-1} (M_{i,n} - m_{i,n}) < \frac{1}{n} \sum_{i=0}^{n-1} \epsilon = \epsilon.$$ 

Which proves the limit as desired.

11.1 Limits of functions

If $f$ is continuous at $x = a$ we are tempted to write $\lim_{x \to a} f(x) = f(a)$ except that we have not defined how to find a limit of a function, only limits of sequences. We need to formalize the concept of a limit of a function at a point.

Since we will be interested in left-hand limits, right-hand limits, ordinary limits and limits at infinity, we will start with the following definition.

**Definition 11.2** Let $S \subseteq \mathbb{R}$, and let $a$ be a real number or the symbol $\infty$ or $-\infty$ that is the limit of some sequence in $S$, and let $L$ be a real number or the symbol $\infty$ or $-\infty$. We write

$$\lim_{x \to a^\delta} f(x) = L$$

if $f$ is a function defined on $S$ and fore every sequence $\{x_n\}$ in $S$ with limit $a$ we have $\lim_{n \to \infty} f(x_n) = L$.

This is a slightly different definition than that upon which we will eventually finalize. It has the advantage that we can continue to use the power of sequences, about which we know a lot.

Note that from our definition a function $f$ is continuous at $a \in \text{dom}(f) = S$ if and only if $\lim_{x \to a^\delta} f(x) = f(a)$. Also, note that the limits, when they exist, are unique. From this we will generate the usual definitions.

**Definition 11.3**

a) For $a \in \mathbb{R}$ and $f: \mathbb{R} \to \mathbb{R}$ we write $\lim_{x \to a} f(x) = L$ provided $\lim_{x \to a^\delta} f(x) = L$ for some set $S = J \setminus \{a\}$ where $J$ is an open interval containing $a$. $\lim_{x \to a} f(x)$ is called the two-sided limit of $f$ at $a$. Note that $f$ does not have to be defined at $a$ and, even if $f$ is defined at $a$, the value $f(a)$ does not have to be equal to the limit. In fact, $f(a) = \lim_{x \to a} f(x)$ if and only if $f$ is defined on an open interval containing $a$ and $f$ is continuous at $a$.

b) For $a \in \mathbb{R}$ and $f: \mathbb{R} \to \mathbb{R}$ we write $\lim_{x \to a^+} f(x) = L$ provided $\lim_{x \to a^+} f(x) = L$ for some open interval $S = (a, b)$. $\lim_{x \to a^+} f(x)$ is the right hand limit of $f$ at $a$. Again, $f$ does not have to be defined at $a$. 

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Let $f: \mathbb{R} \to \mathbb{R}$ be a function defined on the set $S = (c, a)$. $\lim_{x \to a^-} f(x)$ is the left-hand limit of $f$ at $a$.

Theorem 11.5 Let $f_1$ and $f_2$ be functions for which the limits $\lim_{x \to a^-} f_1(x) = L_1$ and $\lim_{x \to a^-} f_2(x) = L_2$ exist and are finite. Then

i) $\lim_{x \to a^-} (f_1 + f_2)(x)$ exists and equals $L_1 + L_2$;

ii) $\lim_{x \to a^-} (f_1 f_2)(x)$ exists and equals $L_1L_2$;

iii) $\lim_{x \to a^-} (f_1/f_2)(x)$ exists and equals $L_1/L_2$ provided $L_2 \neq 0$ and $f_2(x) \neq 0$ for $x \in S$.

Proof: The hypotheses imply that both $f_1$ and $f_2$ are defined on $S$ and that $a$ is the limit of some sequence in $S$. It is clear that the functions $f_1 + f_2$, $f_1 f_2$ and $f_1/f_2$ are defined on $S$, the latter if $f_2(x) \neq 0$ for $x \in S$.

Let $\{x_n\}$ be a sequence in $S$ with limit $a$. By our hypotheses we have $L_1 = \lim_{n \to \infty} f_1(x_n)$ and $L_2 = \lim_{n \to \infty} f_2(x_n)$. By our theorems on convergent sequences we have that

$$\lim_{n \to \infty} (f_1 + f_2)(x_n) = \lim_{n \to \infty} f_1(x_n) + \lim_{n \to \infty} f_2(x_n) = L_1 + L_2,$$

and

$$\lim_{n \to \infty} (f_1 f_2)(x_n) = \left[ \lim_{n \to \infty} f_1(x_n) \right] \cdot \left[ \lim_{n \to \infty} f_2(x_n) \right] = L_1L_2.$$

Thus, condition (b) in the definition holds for $f_1 + f_2$ and $f_1 f_2$, so that (i) and (ii) hold. Part (iii) holds by a similar argument.

Theorem 11.6 Let $f$ be a function for which the limit $L = \lim_{x \to a^-} f(x)$ exists and is finite. If $g$ is a function defined on the set $\{ f(x) \mid x \in S \} \cup \{ L \}$ that is continuous at $L$, then $\lim_{x \to a^-} g \circ f(x)$ exists and equals $g(L)$.

Example 11.1 Why does $g$ have to be continuous at $x = L$? Consider the following example. Let

$$f(x) = 1 + x \sin \frac{\pi}{x}, \ x \neq 0 \quad \text{and} \quad g(x) = \begin{cases} 4 & x \neq 1 \\ -4 & x = 1 \end{cases}$$

Now, note that

$$\lim_{x \to 0} f(x) = 1 \quad \lim_{x \to 1} g(x) = 4$$

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but what about \( \lim_{x \to 0} g(f(x)) \)? Let \( x_n = \frac{2}{n} \) for \( n \in \mathbb{N} \), then

\[
f(x_n) = 1 + \frac{2}{n} \sin \left( \frac{n\pi}{2} \right) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 1 \pm \frac{2}{n} \neq 1 & \text{if } n \text{ is odd} \end{cases}
\]

Thus,

\[
g(f(x_n)) = \begin{cases} -4 & \text{if } n \text{ is even} \\ 4 & \text{if } n \text{ is odd} \end{cases}
\]

Now, \( \lim_{n \to \infty} x_n = 0 \) so \( \{x_n\} \) converges, but \( \lim_{x \to 0} g(f(x)) \) cannot exist.

**Theorem 11.7** Let \( f \) be a function defined on \( S \subseteq \mathbb{R} \), let \( a \in \mathbb{R} \) be a real number that is the limit of some sequence in \( S \), and let \( L \) be a real number. Then \( \lim_{x \to a} f(x) = L \) if and only if for each \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that if \( x \in S \) and \( |x - a| < \delta \) then \( |f(x) - L| < \epsilon \).

**Corollary 11.1** Let \( f \) be a function defined on \( J \setminus \{a\} \) for some open interval \( J \) containing \( a \), and let \( L \) be a real number. Then \( \lim_{x \to a} f(x) = L \) if and only if for each \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that if \( 0 < |x - a| < \delta \) then \( |f(x) - L| < \epsilon \).

**Corollary 11.2** Let \( f \) be a function defined on some open interval \( (a, b) \), and let \( L \) be a real number. Then \( \lim_{x \to a^+} f(x) = L \) if and only if for each \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that if \( a < x < a + \delta \) then \( |f(x) - L| < \epsilon \).

**Theorem 11.8** Let \( f \) be a function defined on \( J \setminus \{a\} \) for some open interval \( J \) containing \( a \). Then \( \lim_{x \to a} f(x) \) exists if and only if the limits \( \lim_{x \to a^+} f(x) \) and \( \lim_{x \to a^-} f(x) \) both exist and are equal, in which case all three limits are equal.