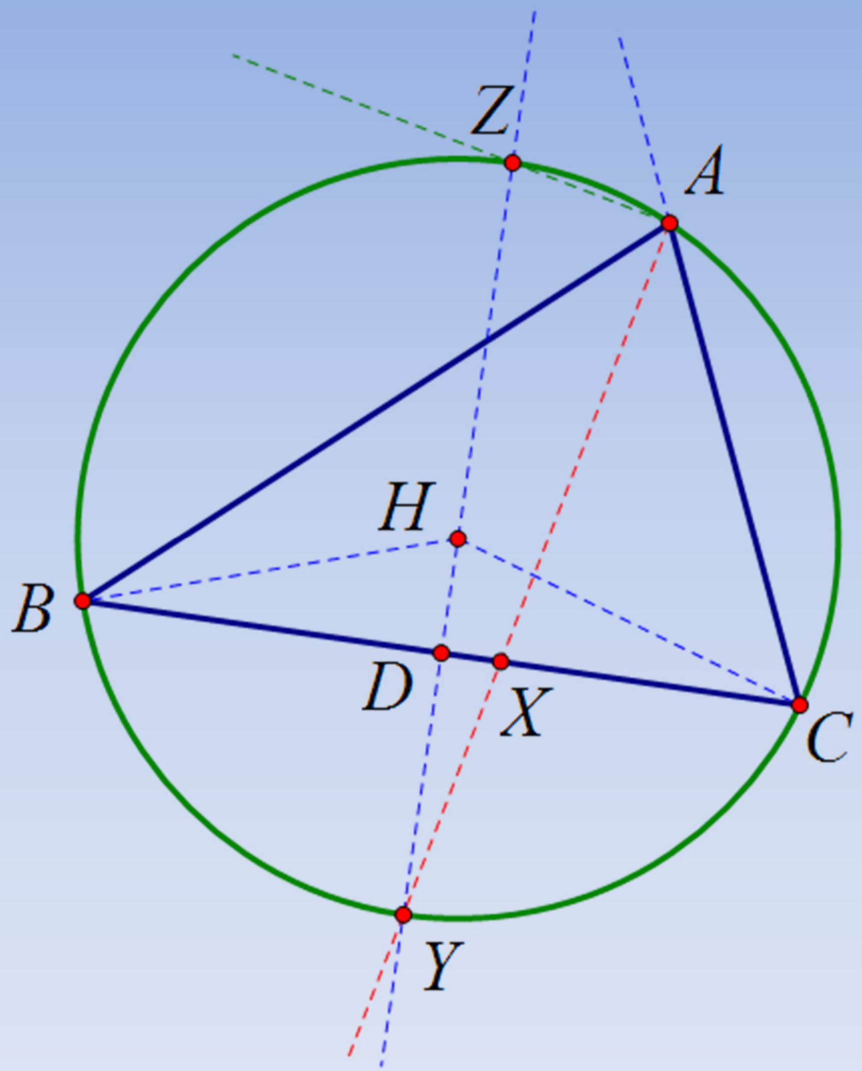
Triangles II

Circumradius, Area, Medians, Stewart's Theorem

Question

Where do the perpendicular bisectors of the sides intersect the circumcircle?

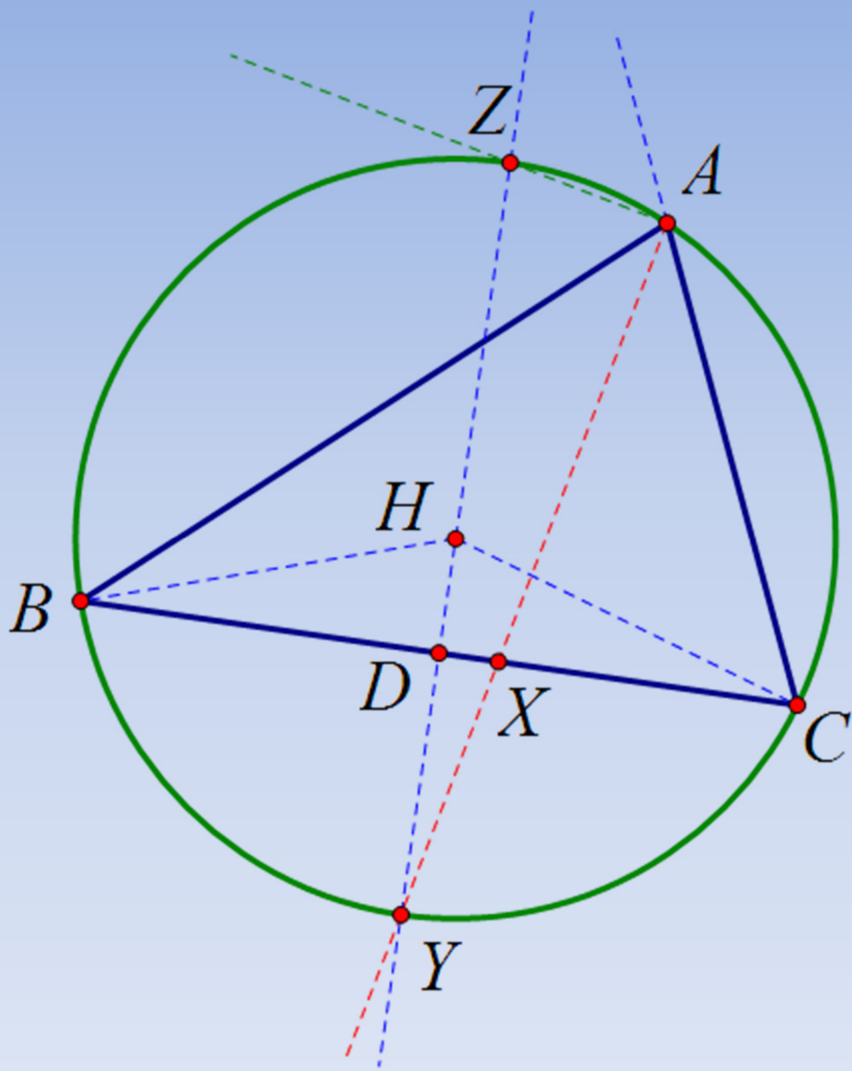
Claim



1) AY bisects $\angle BAC$

2) AZ bisects exterior angle

Claim



Let Y = HD∩ccircle

$\angle BAY = \frac{1}{2}\ \angle BHY$

$\angle BAC = \frac{1}{2}\ \angle BHC$

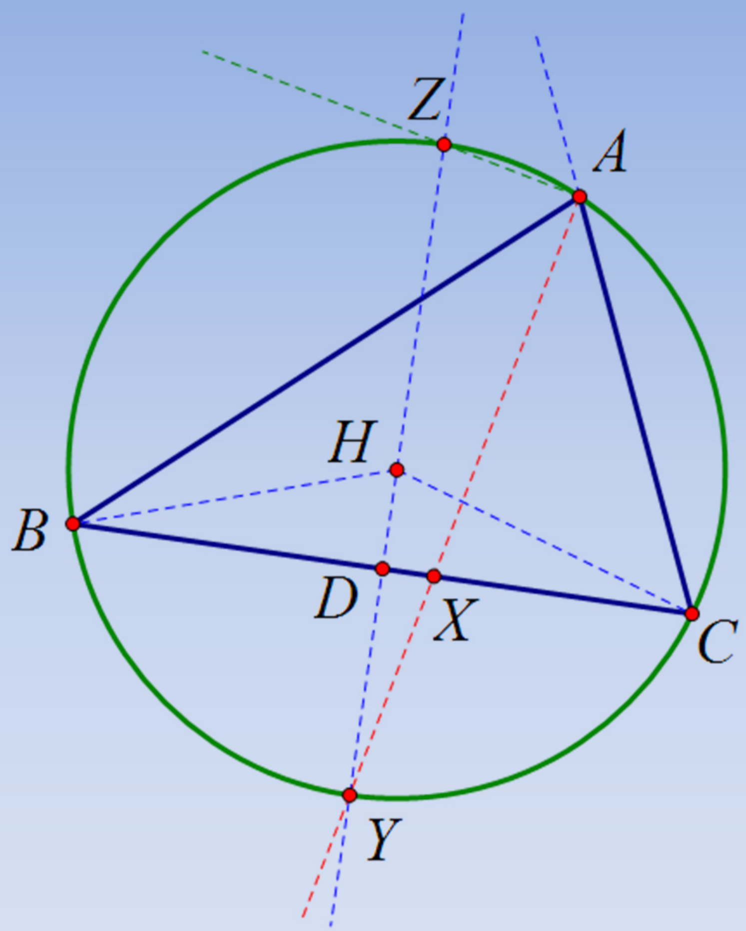
$\angle BHY = \angle CHY = \frac{1}{2} \angle BHC$

$$
\begin{aligned}
2\angle BAY &= \angle BHY \\
&= \tfrac{1}{2}\angle BHC \\
&= \angle BAC
\end{aligned}
$$

$\angle BAY = \frac{1}{2}\ \angle BAC$

Claim

Let Z = HD ∩ ccircle



$\angle ZAY = 90 = \angle BAY + \angle BAZ$

$\angle BA\Omega + \angle BAC = 180$

$\angle CAY + \angle ZA\Omega = 90$

$\angle BAY = \angle CAY$

$\angle BAY + \angle BAZ = \angle CAY + \angle ZA\Omega$

$\angle BAZ = \angle ZA\Omega$

Question

Where do the perpendicular bisectors of the sides intersect the circumcircle?

At one end is point of intersection of angle bisector with circumcircle

The other end is point of intersection of exterior angle bisector with circumcircle.

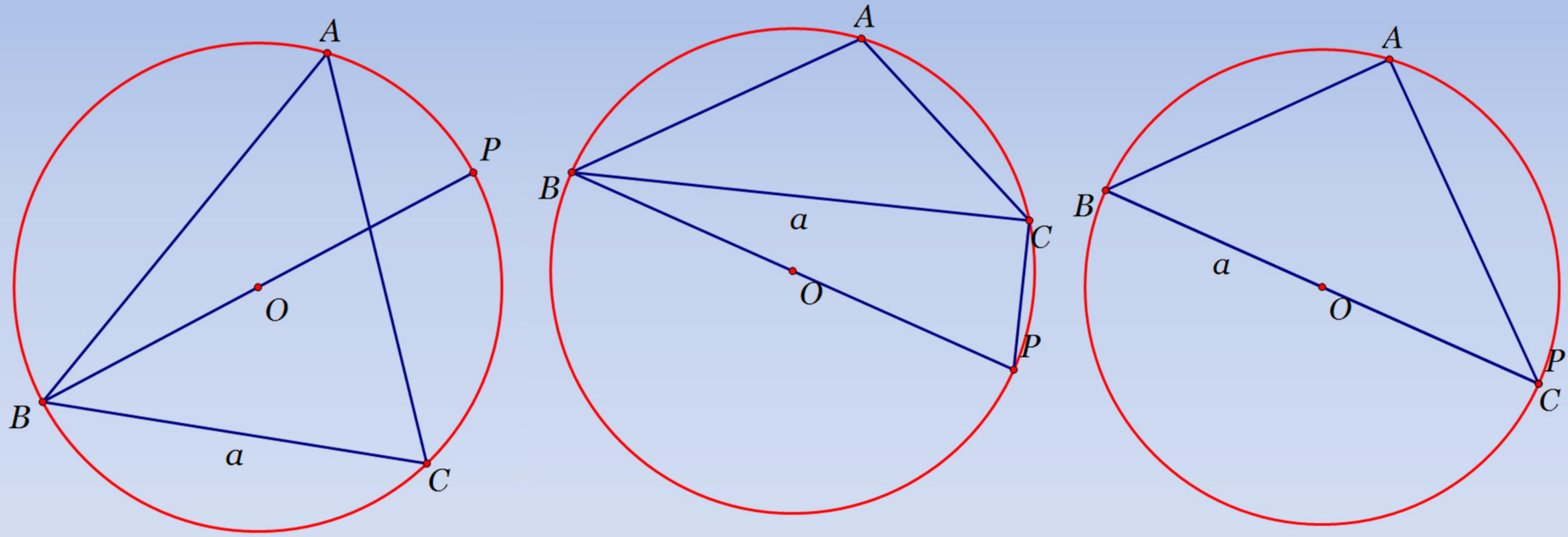
Extended Law of Sines

Theorem: Given ΔABC with circumradius R, let a, b, and c denote the lengths of the sides opposite angles $\angle A$, $\angle B$, and $\angle C$, respectively. Then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

Proof

Three cases:



Proof

Case I: $\angle A < 90°$

BP = diameter

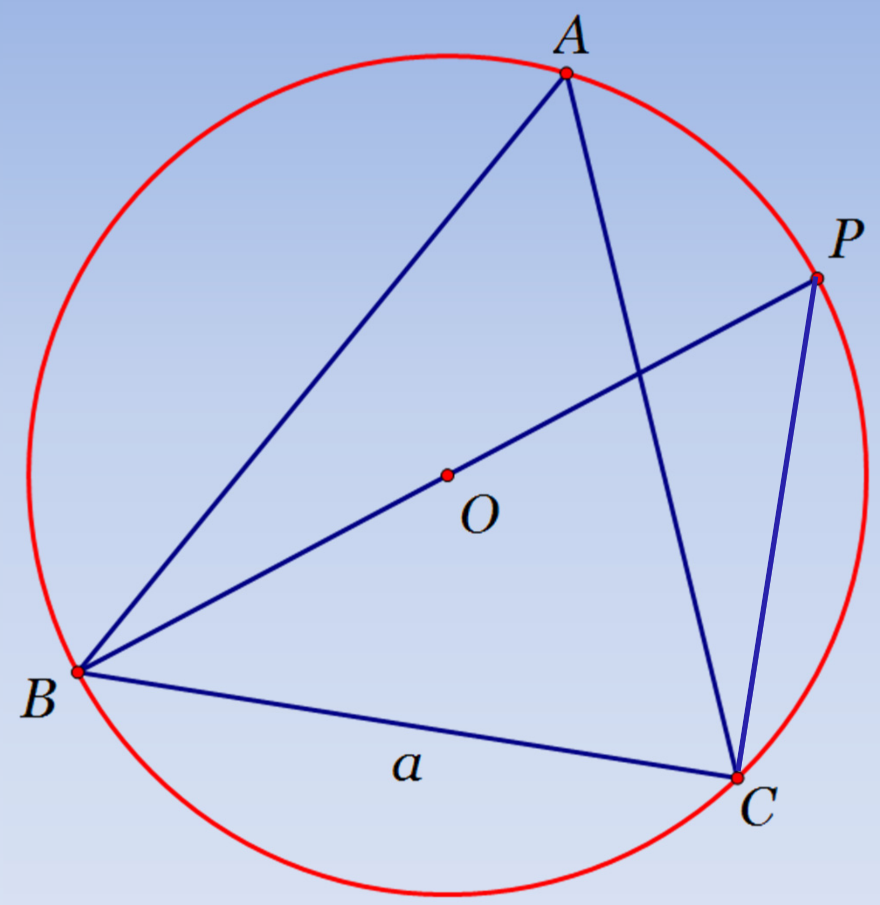
$\Rightarrow \Delta BCP$ right triangle

BP = 2R

$\Rightarrow \sin P = a/2R$

$\angle A = \angle P$

$\Rightarrow 2R = a/\sin A$



Proof

Case II: $\angle A > 90°$

BP = diameter

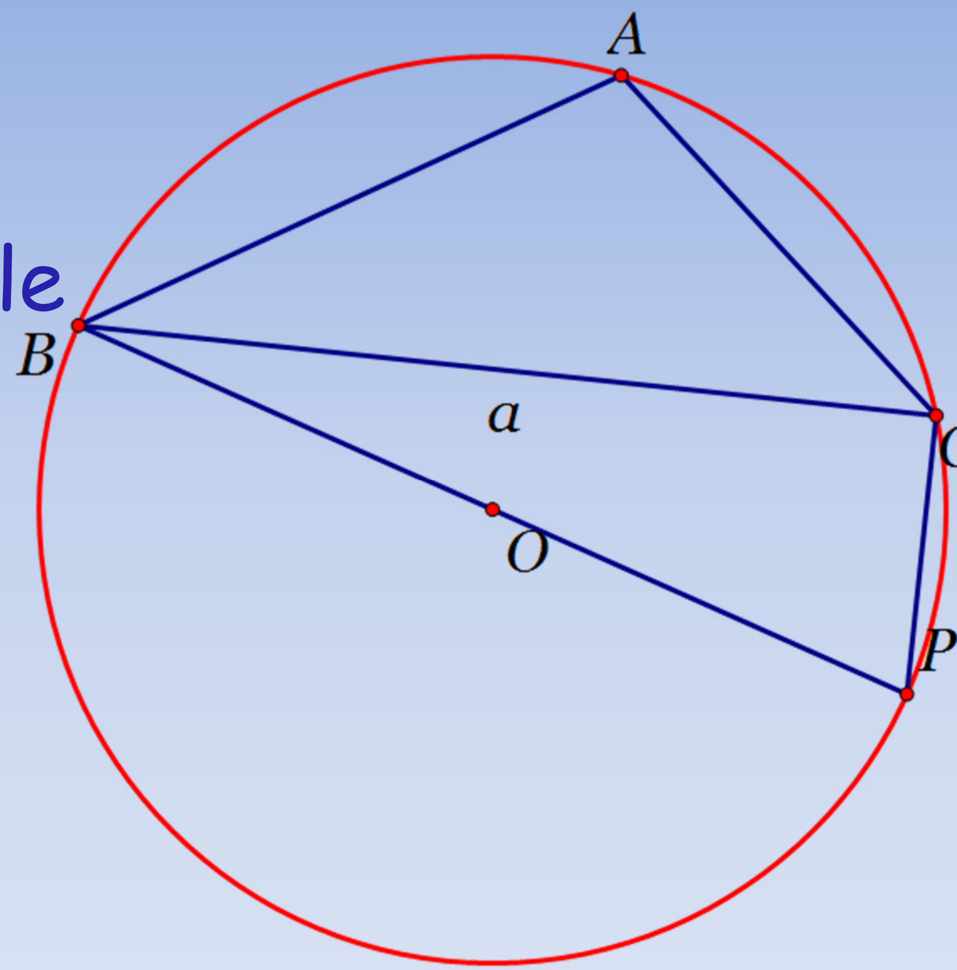
$\Rightarrow \Delta BCP$ right triangle

BP = 2R

$\Rightarrow \sin P = a/2R$

$\angle A = \angle P$

$\Rightarrow 2R = a/\sin A$



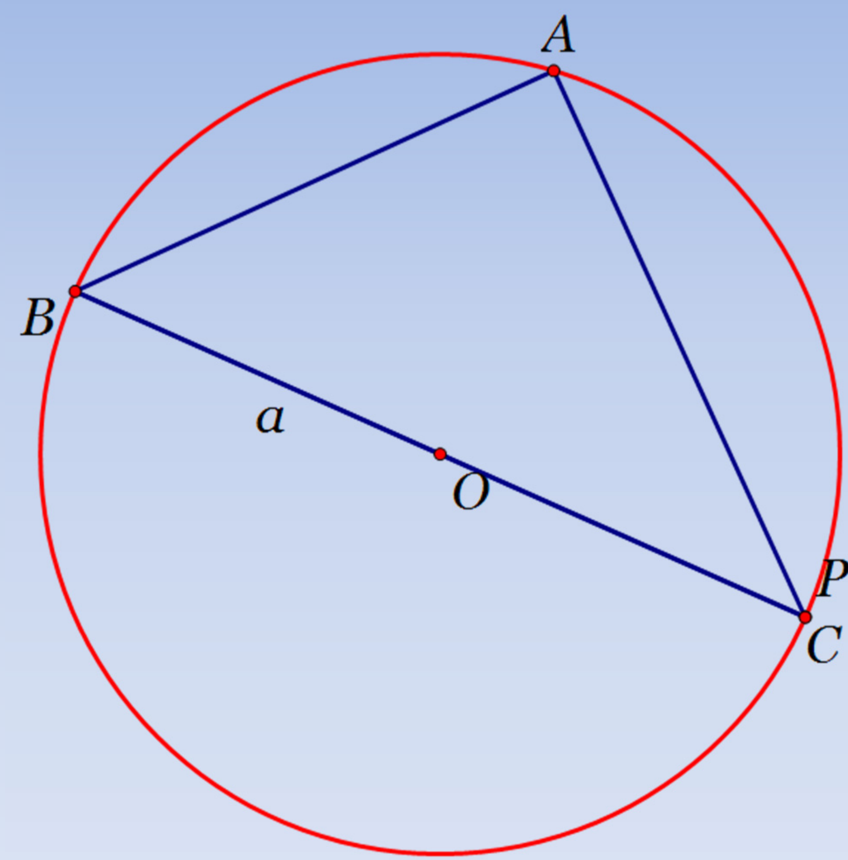
Proof

Case III: $\angle A = 90°$

BP = a = diameter

BP = 2R

2R = a = a/sin A



Circumradius and Area

Theorem: Let R be the circumradius and K be the area of ΔABC and let a, b, and c denote the lengths of the sides as usual. Then 4KR=abc

$$K = \frac{abc}{4R}$$

Proof

$$K = \frac{1}{2} ab \sin C$$

$$2K = ab \sin C$$

$$c/\sin C = 2R$$

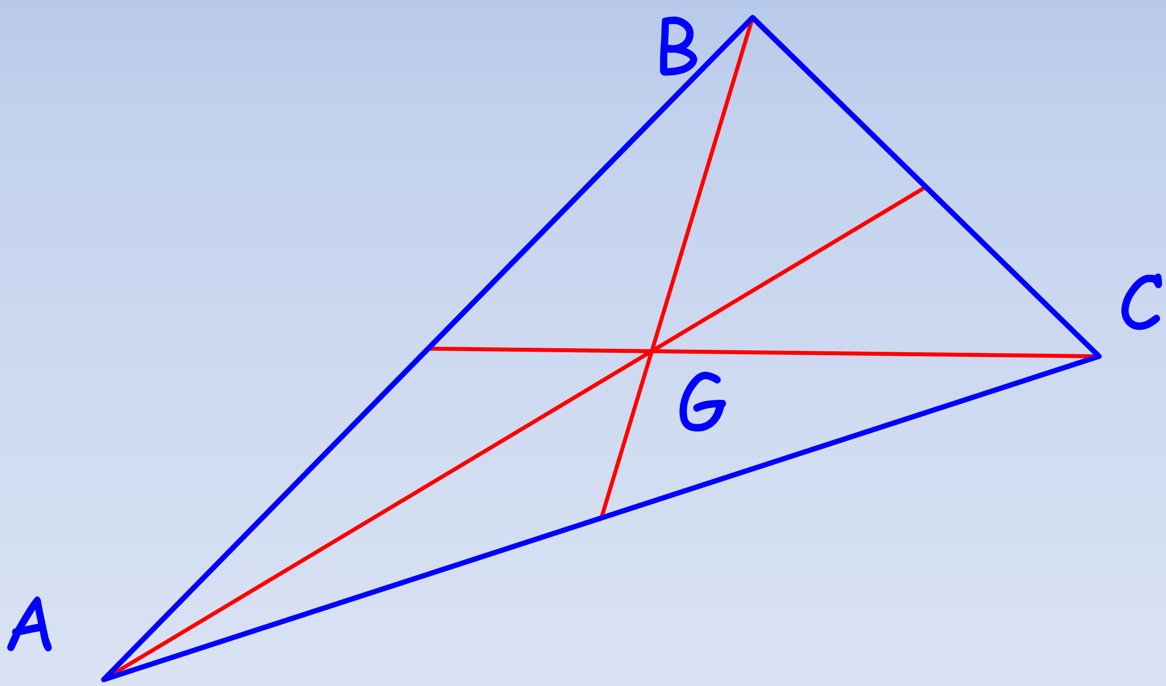
$$\sin C = c/2R$$

$$2K = abc/2R$$

$$4KR = abc$$

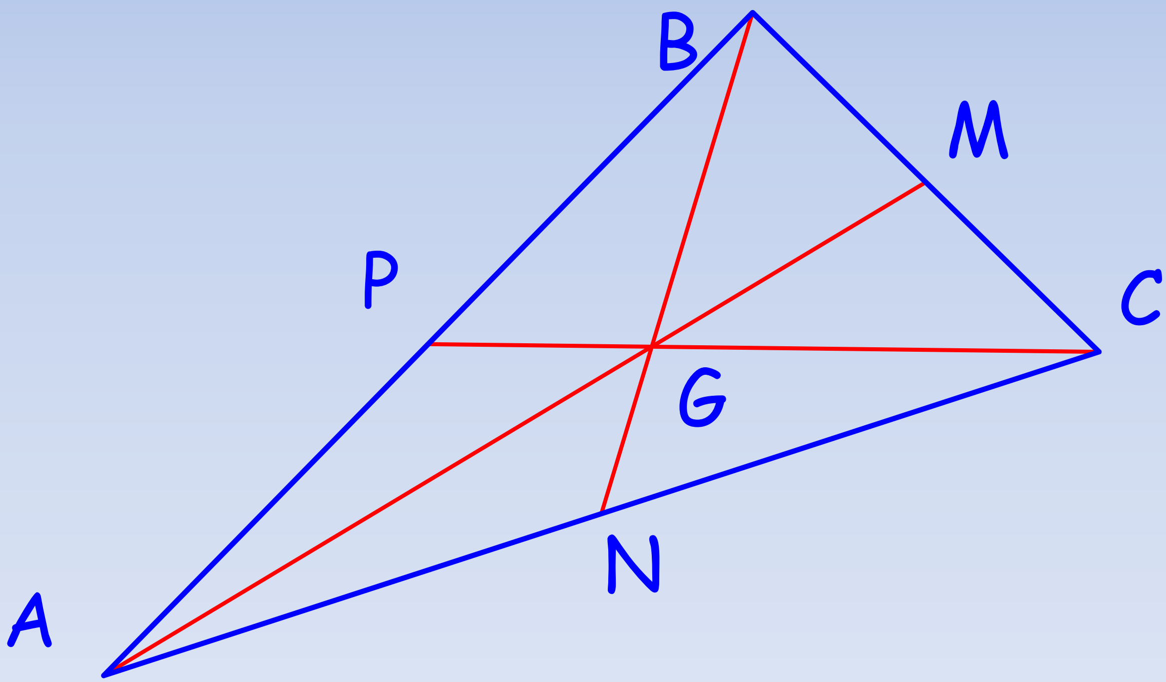
Centroid

Theorem: The medians of a triangle meet in a single point, called the centroid, G.



Centroid

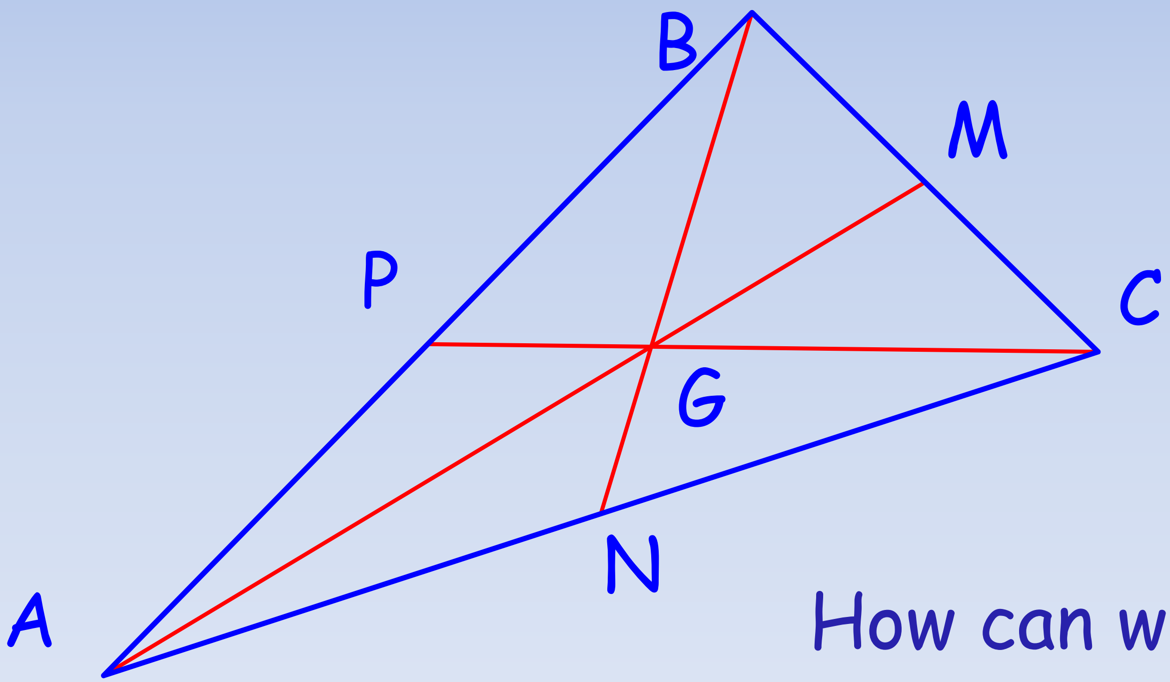
We have shown this by Ceva's Theorem, so this is not new. However, there is something more.



Centroid

AG = 2 GM
BG = 2 GN
CG = 2 GP

The median is 2/3 the way from the vertex to the side.



How can we show this?

Centroid

MN is the midsegment of ΔABC so $MN || AB$ and $MN = \frac{1}{2} AB$.

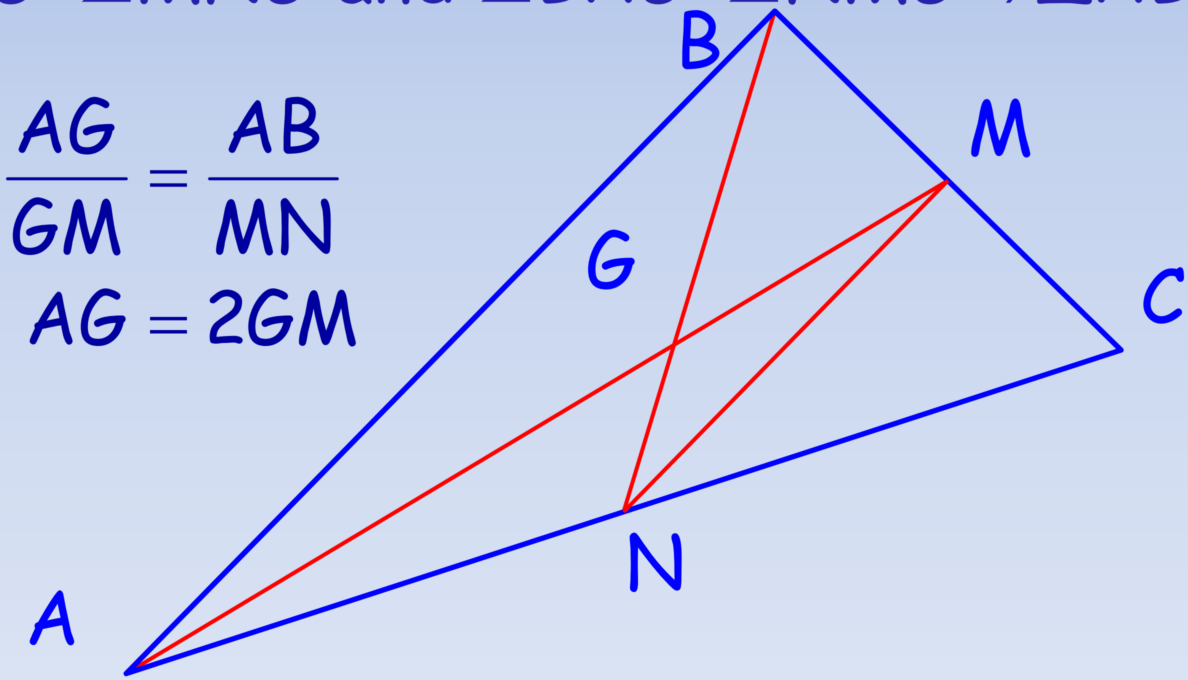
By Alternate Interior Angles

$\angle ABG = \angle MNG$ and $\angle BAG = \angle NMG \Rightarrow \Delta ABG \sim \Delta MNG$

So

$$\frac{AG}{GM} = \frac{AB}{MN}$$

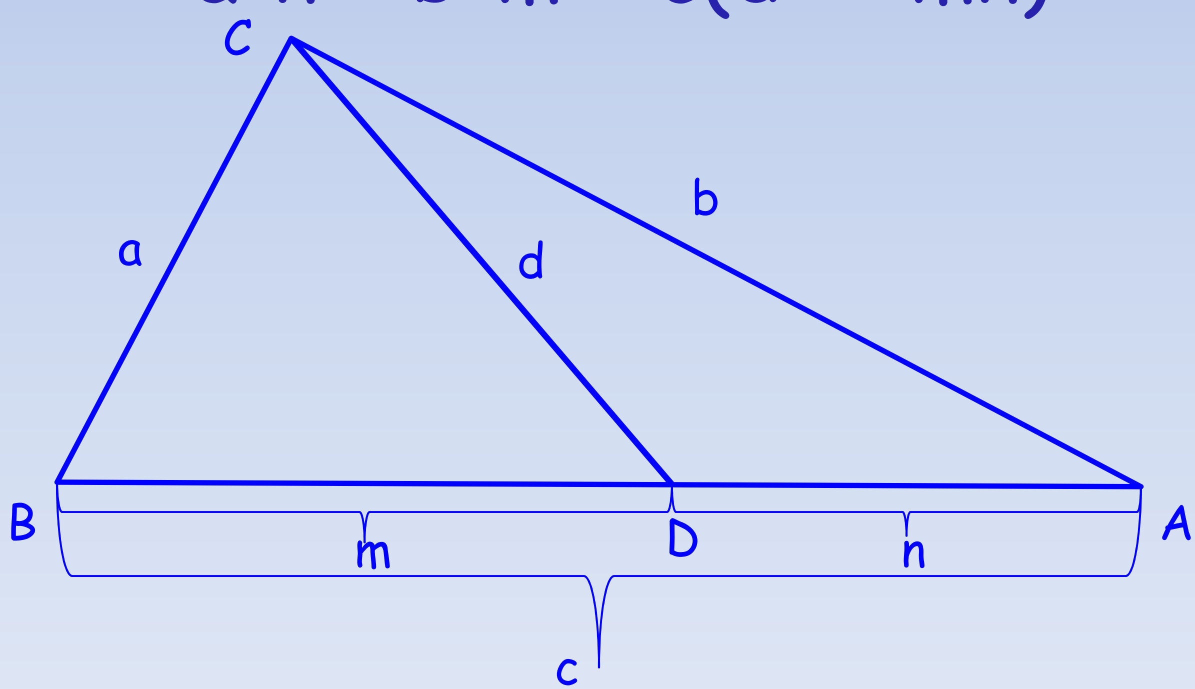
$$AG = 2GM$$



Stewart's Theorem (1746)

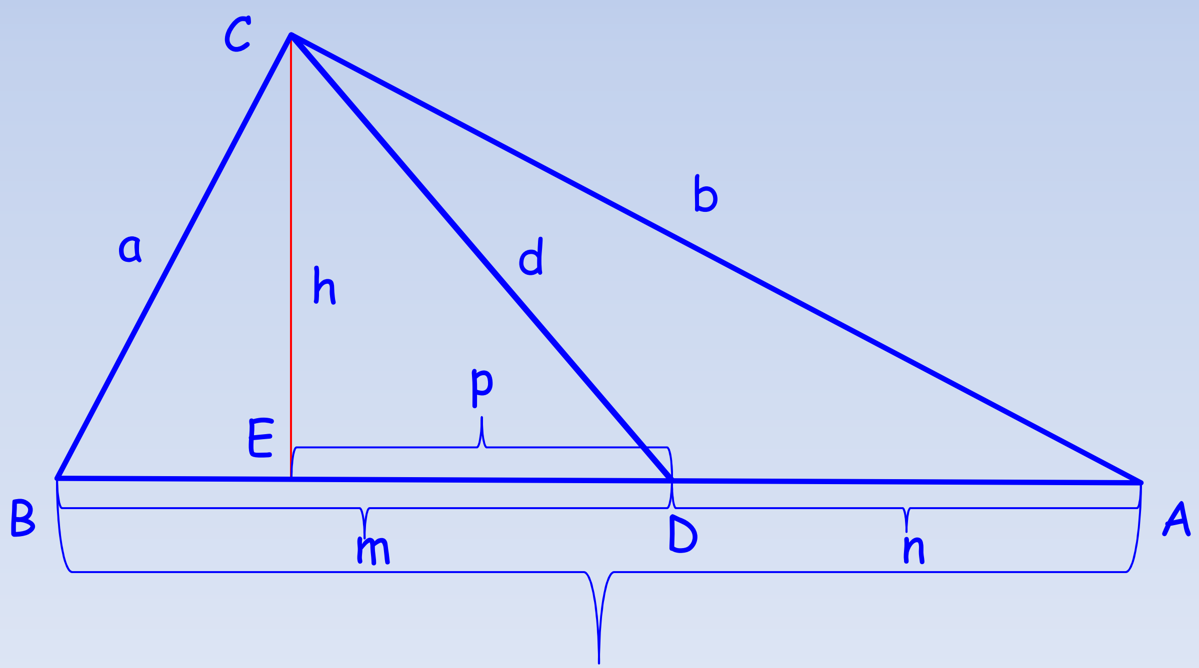
With the measurements given in the triangle below, the following relationship holds:

$$a^2n + b^2m = c(d^2 + mn)$$



Stewart's Theorem (1746)

CE⊥AB so we will apply the Pythagorean Theorem several times



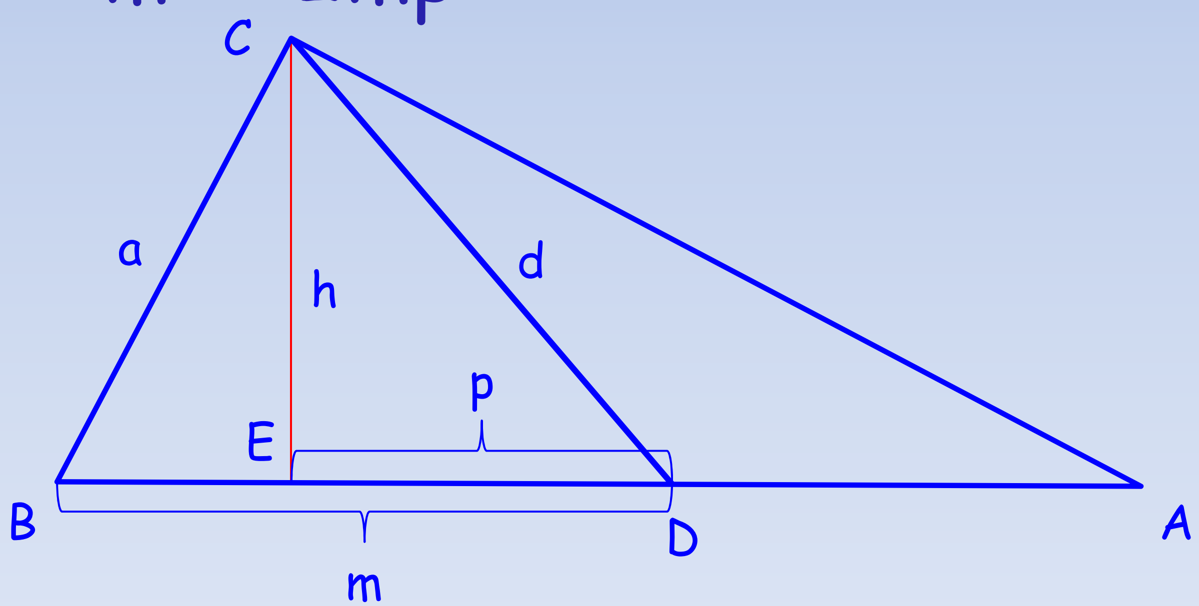
Stewart's Theorem (1746)

In ΔCEB $a^2 = h^2 + (m - p)^2$

In ΔCED $d^2 = h^2 + p^2$

$a^2 = d^2 - p^2 + (m - p)^2$

$a^2 = d^2 + m^2 - 2mp$

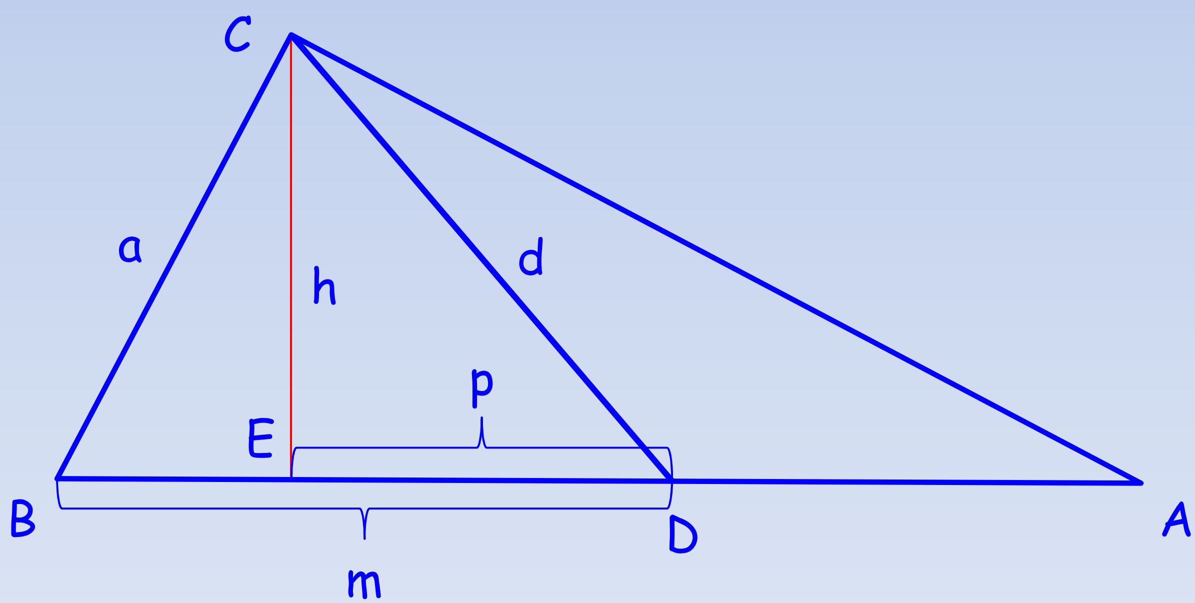


Stewart's Theorem (1746)

In ΔCEA $b^2 = h^2 + (n + p)^2$

$b^2 = d^2 - p^2 + (n + p)^2$

$b^2 = d^2 + n^2 + 2np$



Stewart's Theorem (1746)

$a^2n = d^2n + m^2n - 2mnp$

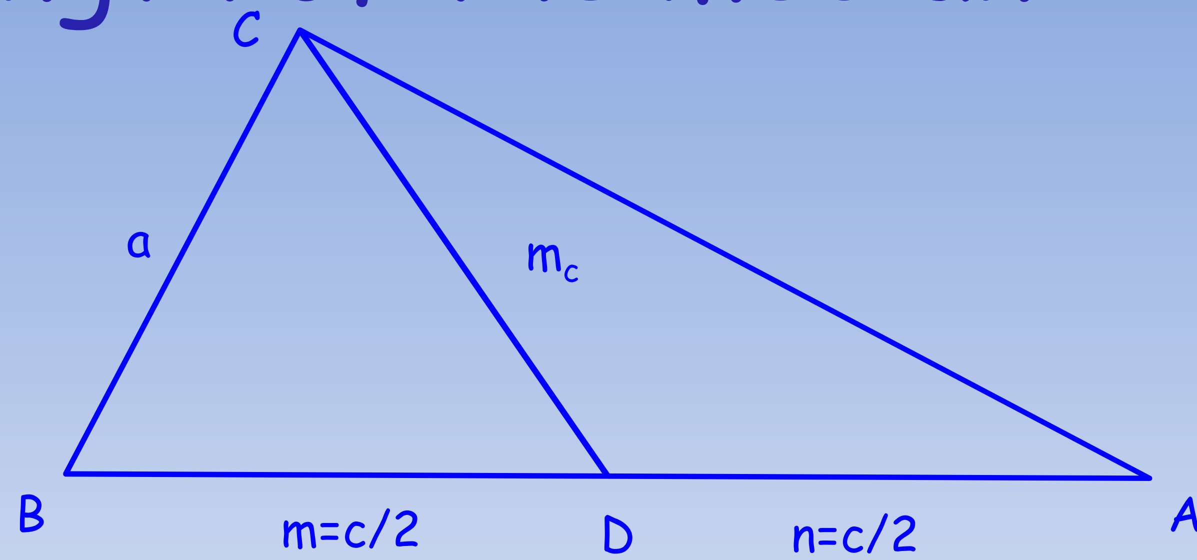
$b^2m = d^2m + n^2m + 2mnp$

$a^2n + b^2m = d^2n + m^2n + d^2m + n^2m$

$= d^2(n + m) + mn(m + n)$

$a^2n + b^2m = c(d^2 + mn)$

The length of the median



$$a^2 n + b^2 m = c(m_c^2 + mn)$$

$$\frac{a^2 c}{2} + \frac{b^2 c}{2} = c\left(m_c^2 + \frac{c^2}{4}\right)$$

$$m_c^2 = \frac{a^2}{2} + \frac{b^2}{2} - \frac{c^2}{4}$$

The length of the medians

$$2m_a^2 = b^2 + c^2 - \frac{1}{2}a^2$$

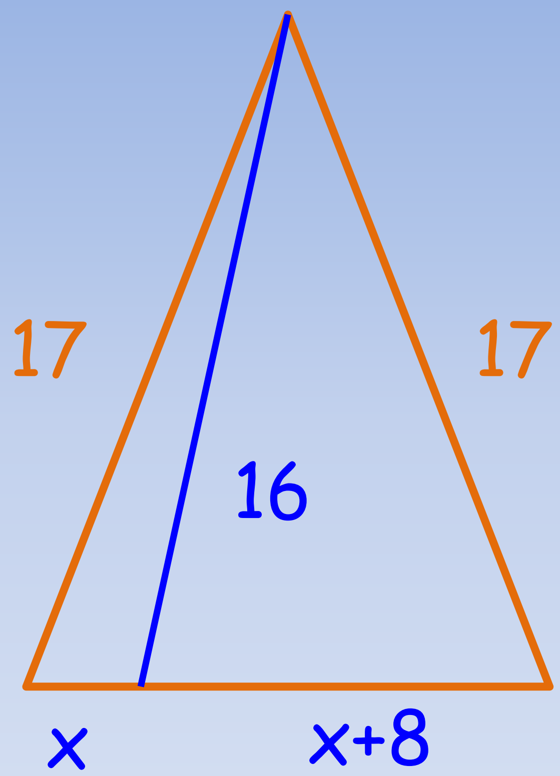
$$2m_b^2 = a^2 + c^2 - \frac{1}{2}b^2$$

$$2m_c^2 = a^2 + b^2 - \frac{1}{2}c^2$$

For a 3-4-5 triangle this gives us that the medians measure:

$$m_a = \frac{\sqrt{73}}{2}; \qquad m_b = \sqrt{13}; \qquad m_c = \frac{5}{2}$$

Example



Find x

x=3

Theorem 4

For any triangle, the sum of the lengths of the medians is less than the perimeter of the triangle.

N in AF so that NF=AF

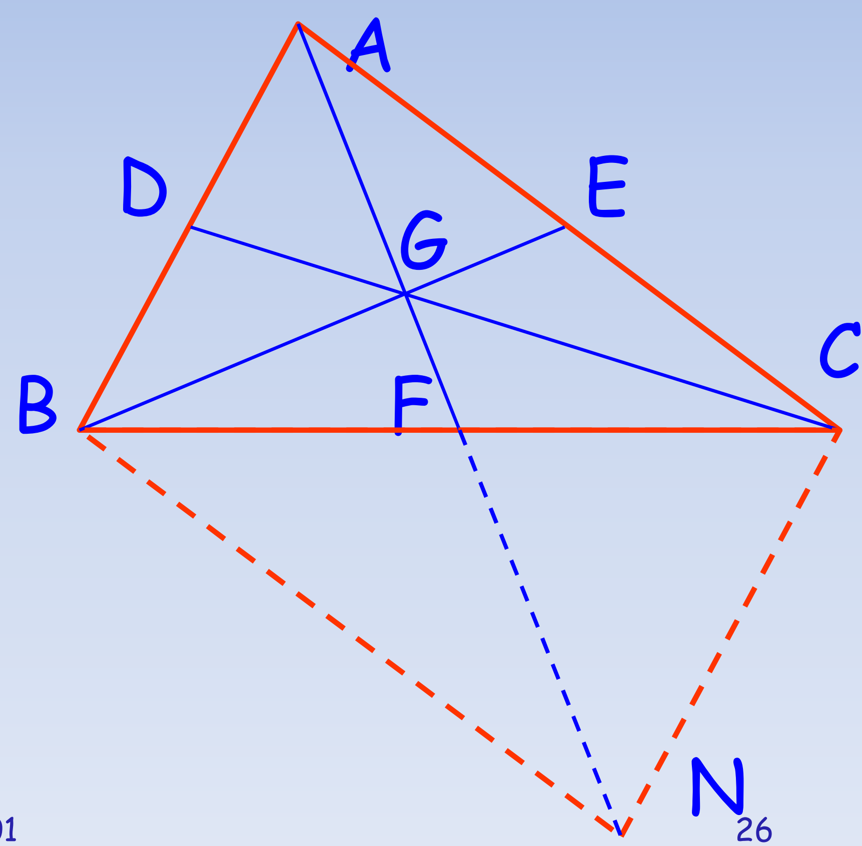
ACNB is a parallelogram

BN=AC

In ΔABN, AN < AB+BN

2AF < AB + AC

$2m_a < b + c$



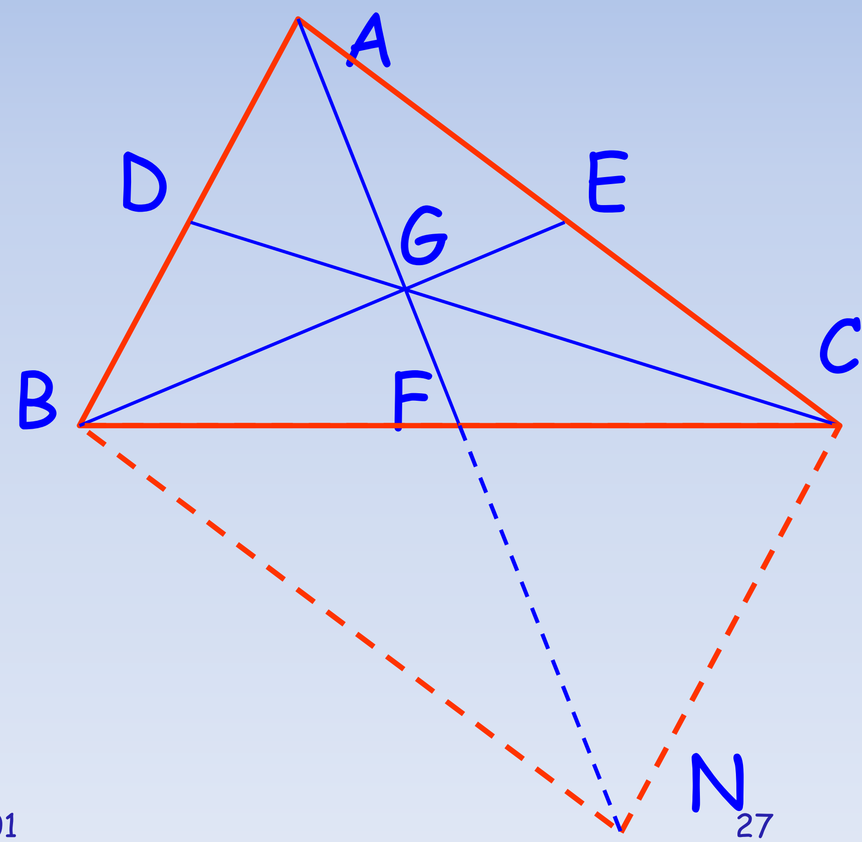
Theorem 4

Similarly

$2m_b < a + c$ and $2m_c < a + b$

$2(m_a+m_b+m_c) < 2a+2b+2c$

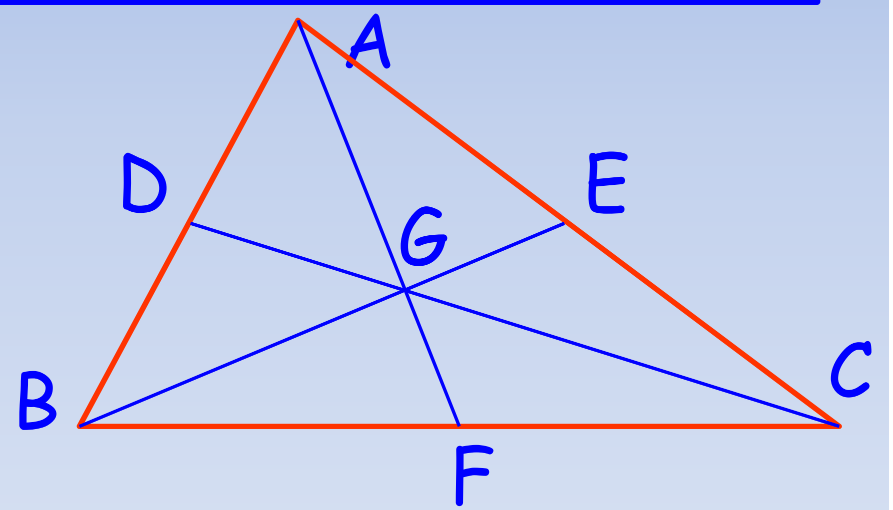
$m_a + m_b + m_c < a+ b+ c$



Theorem 5

For any triangle, the sum of the lengths of the medians is greater than three-fourths the perimeter of the triangle.

BG + CG > BC



$$\frac{2}{3}m_c + \frac{2}{3}m_b > a$$

and

$$\frac{2}{3}m_a + \frac{2}{3}m_b > c \qquad \frac{2}{3}m_a + \frac{2}{3}m_c > b$$

Theorem 5

$$\frac{2}{3}m_b + \frac{2}{3}m_c + \frac{2}{3}m_a + \frac{2}{3}m_c + \frac{2}{3}m_a + \frac{2}{3}m_b > a + b + c$$

$$\frac{4}{3}(m_a + m_b + m_c) > a + b + c$$

$$m_a + m_b + m_c > \frac{3}{4}(a + b + c)$$

Result

$$\frac{3}{4}(a+b+c) < m_a + m_b + m_c < a+b+c$$

Theorem 6

The sum of the squares of the medians of a triangle equals three-fourths the sum of the squares of the sides of the triangle.

$$2m_a^2 = b^2 + c^2 - \frac{1}{2}a^2$$

$$2m_b^2 = a^2 + c^2 - \frac{1}{2}b^2$$

$$2m_c^2 = a^2 + b^2 - \frac{1}{2}c^2$$

Theorem 6

$$2(m_a^2 + m_b^2 + m_c^2) = 2(a^2 + b^2 + c^2) - \frac{1}{2}(a^2 + b^2 + c^2)$$

$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$$

Theorem 7

The sum of the squares of the lengths of the segments joining the centroid with the vertices is one-third the sum of the squares of the lengths of the sides.

Theorem 8

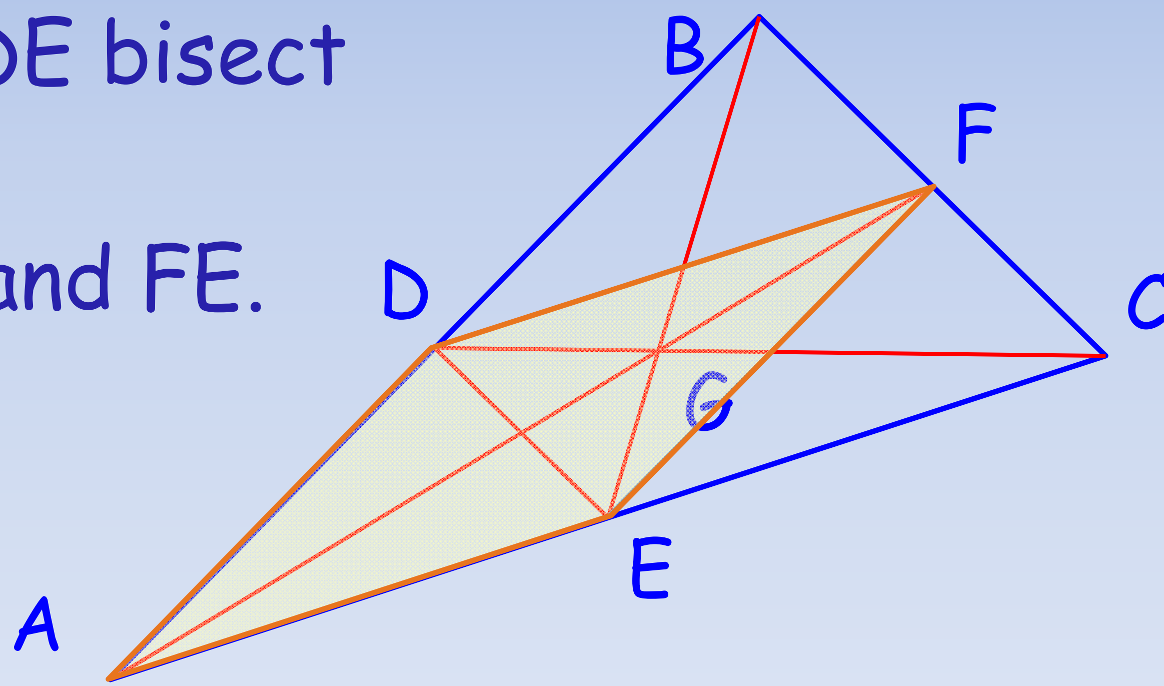
A median and the midline it intersects bisect each other.

Show AF and DE bisect each other.

Construct DF and FE.

DF||AE

AD||FG

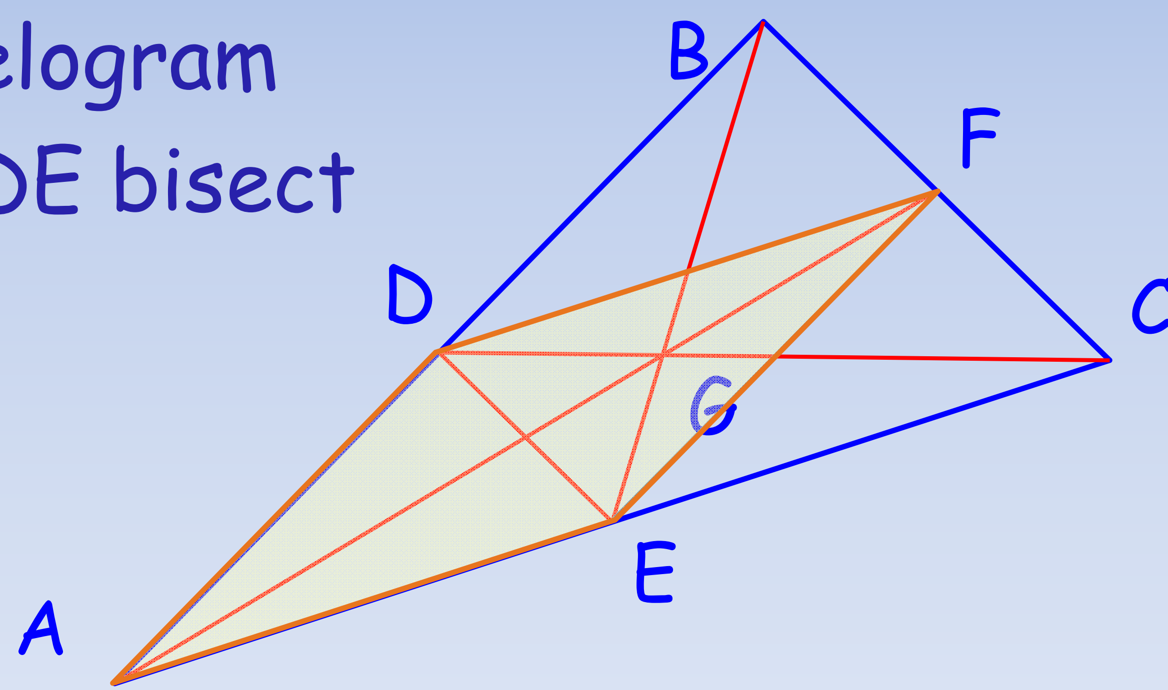


Theorem 8

A median and the midline it intersects bisect each other.

ADFE a parallelogram

Thus, AF and DE bisect each other.



Theorem 9

A triangle and its medial triangle have the same centroid.

This is HW Problem 2B.1.