# Quadrilaterals

MA 341 – Topics in Geometry
Lecture 23

# Theorems

1. A convex quadrilateral is cyclic if and only if opposite angles are supplementary.
   (Circumcircle, maltitudes, anticenter)
2. A convex quadrilateral is tangential if and only if opposite sides sum to the same measure.
   (Incircle, incenter)

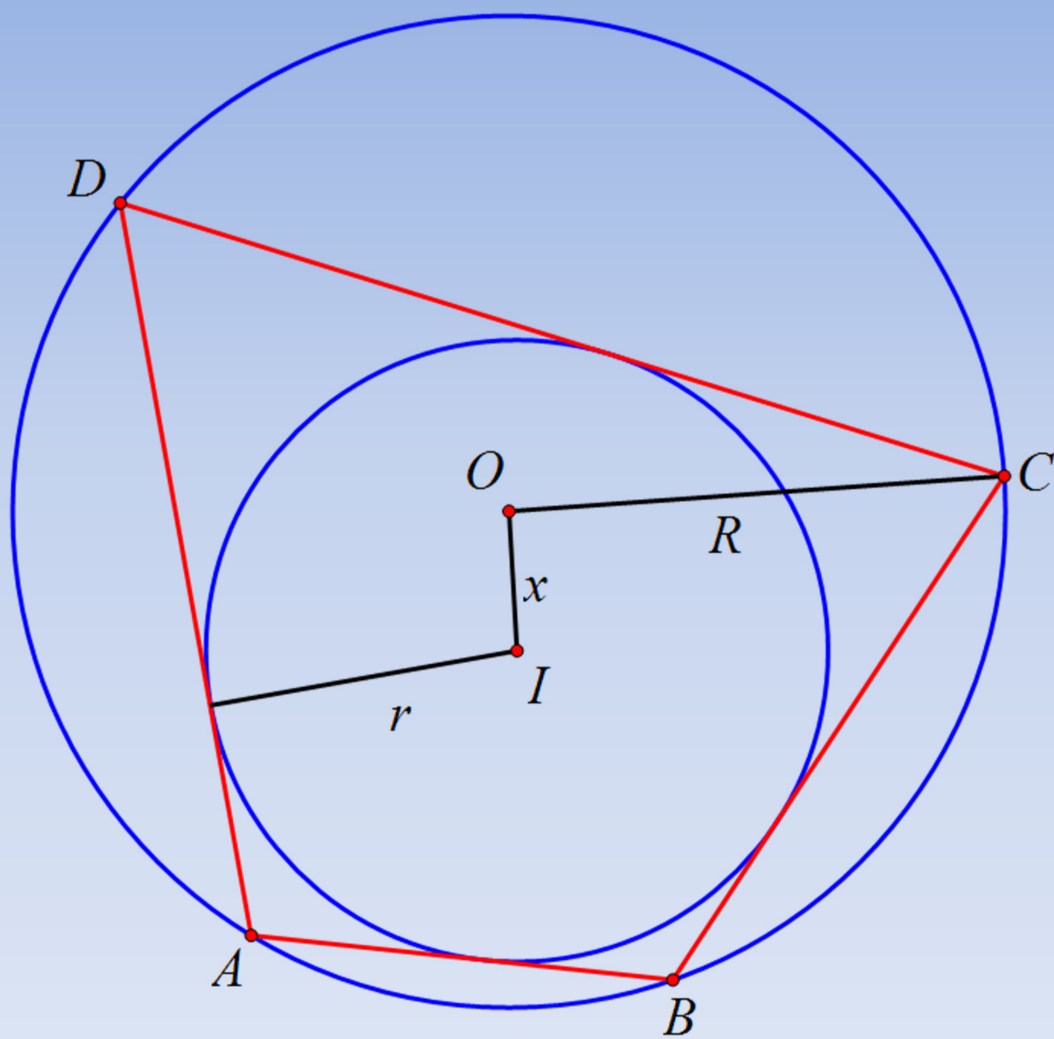
# Bicentric Quadrilaterals

A convex quadrilateral is bicentric if it is both cyclic and tangential.

A bicentric quadrilateral has both a circumcircle and an incircle.

A convex quadrilateral is bicentric if and only if $a + c = b + d$ and $A + C = 180 = B + D$

# Bicentric Quadrilaterals



# Bicentric Quadrilaterals

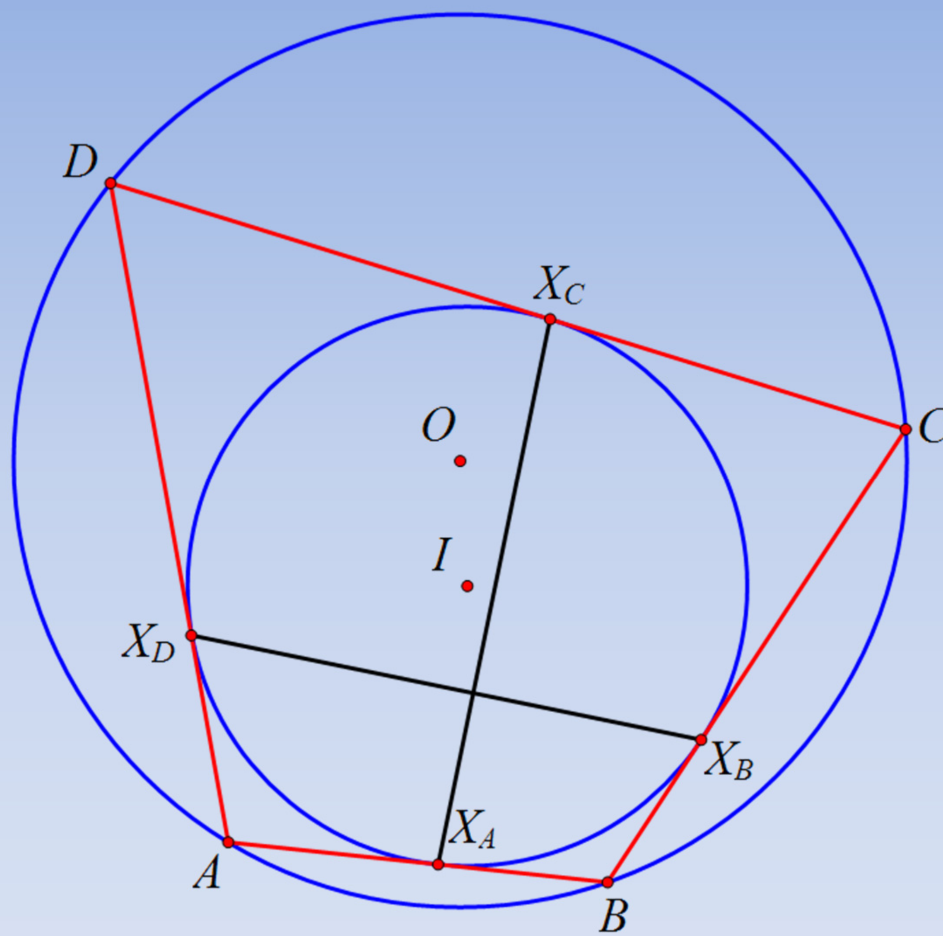
$X_A, X_B, X_C, X_D$ points of contact of incircle and quadrilateral

Bicentric iff

$X_AX_C \perp X_BX_D$ or

$$\frac{AX_A}{X_AB} = \frac{DX_C}{X_CC} \text{ or}$$

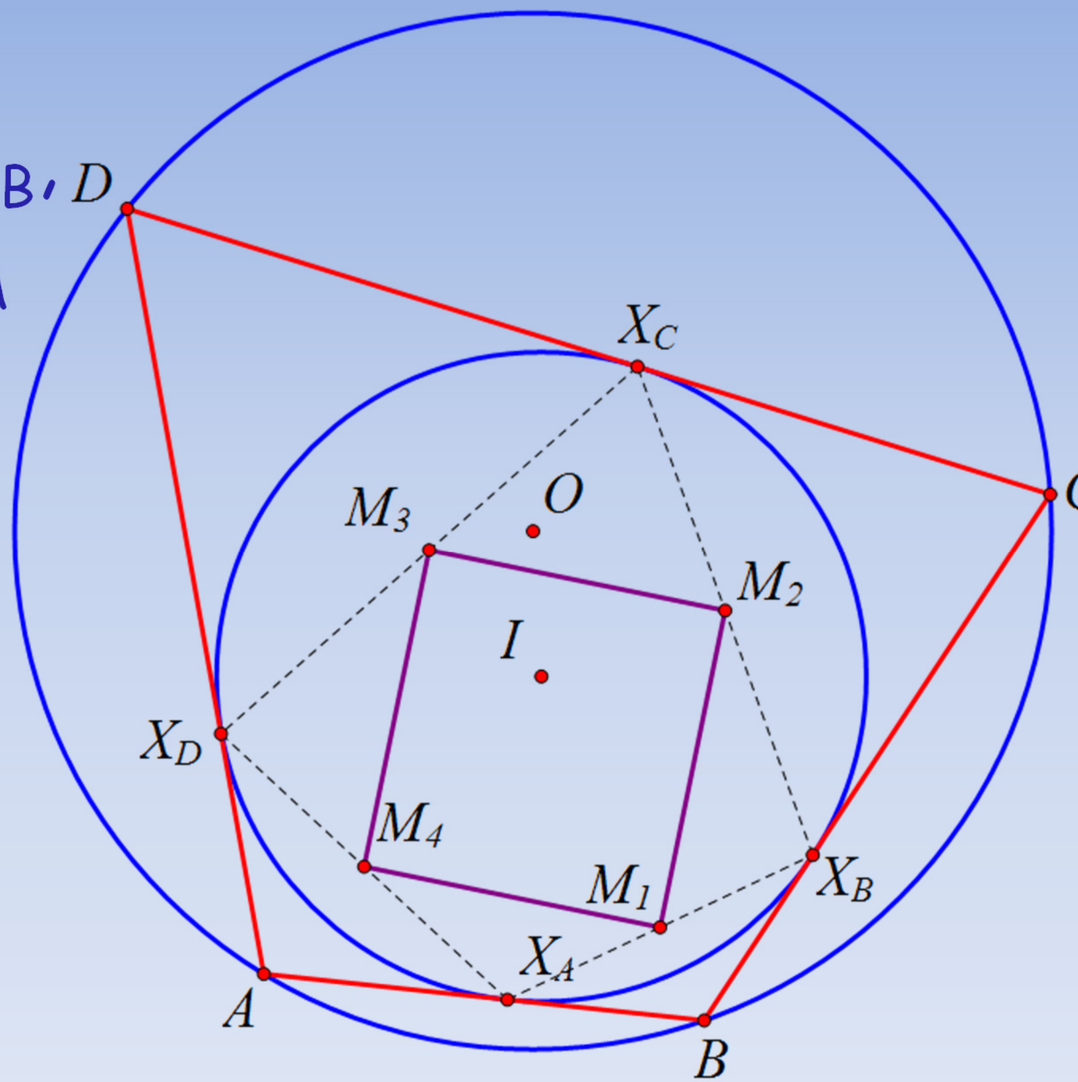
$$\frac{AC}{BC} = \frac{AX_A + CX_C}{BX_B + DX_D}$$



# Bicentric Quadrilaterals

$M_1$, $M_2$, $M_3$, $M_4$ midpoints of $X_AX_B$, $X_BX_C$, $X_CX_D$, $X_DX_A$

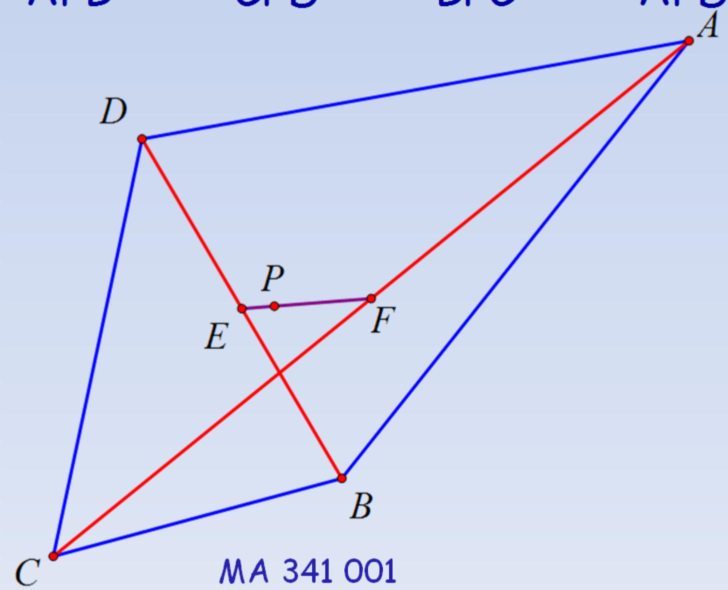
Bicentric iff $M_1M_2M_3M_4$ is a rectangle.



# Newton Line

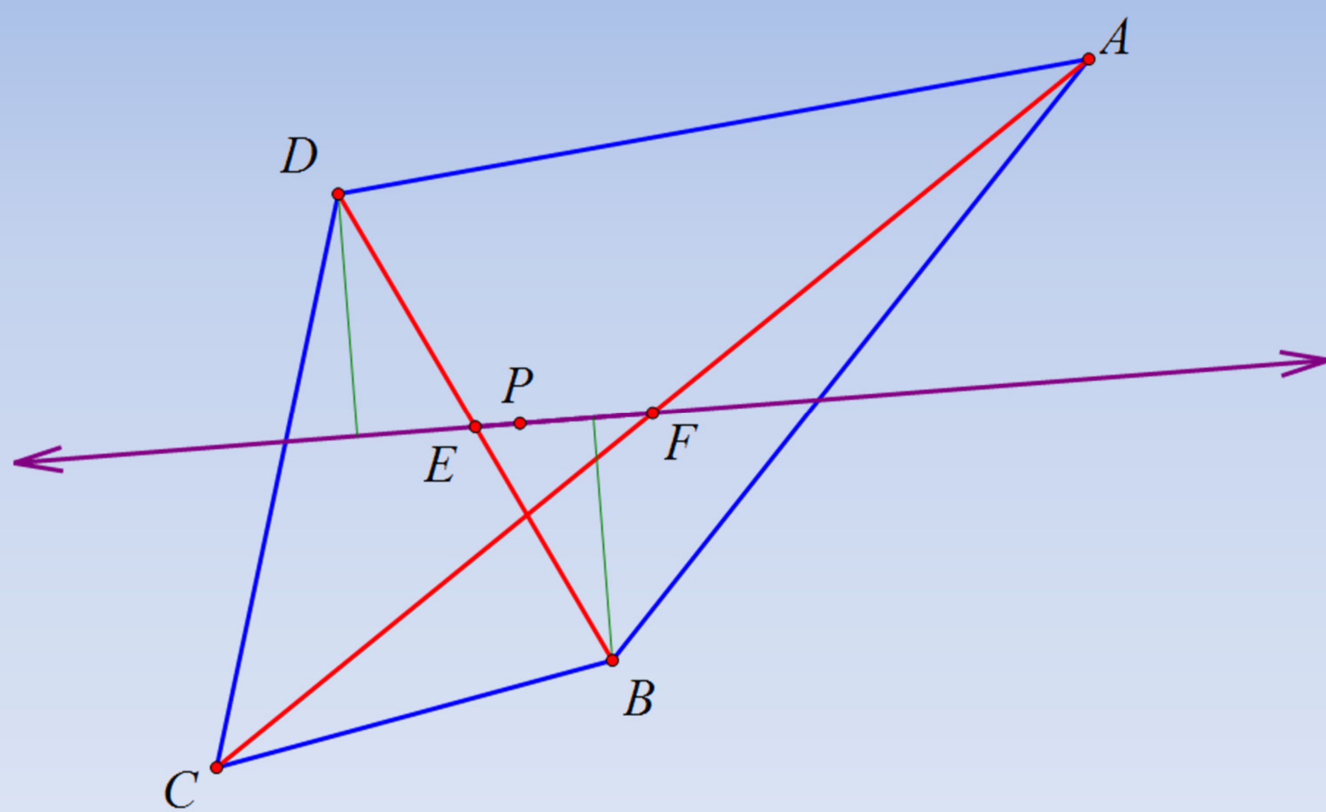
Theorem: (Léon Anne) Let ABCD be a quadrilateral that is not a parallelogram. P lies on the line joining the midpoints of the diagonals iff

$$K_{APB} + K_{CPD} = K_{BPC} + K_{APD}$$



# Proof

## Drop perpendicular from B & D to EF.



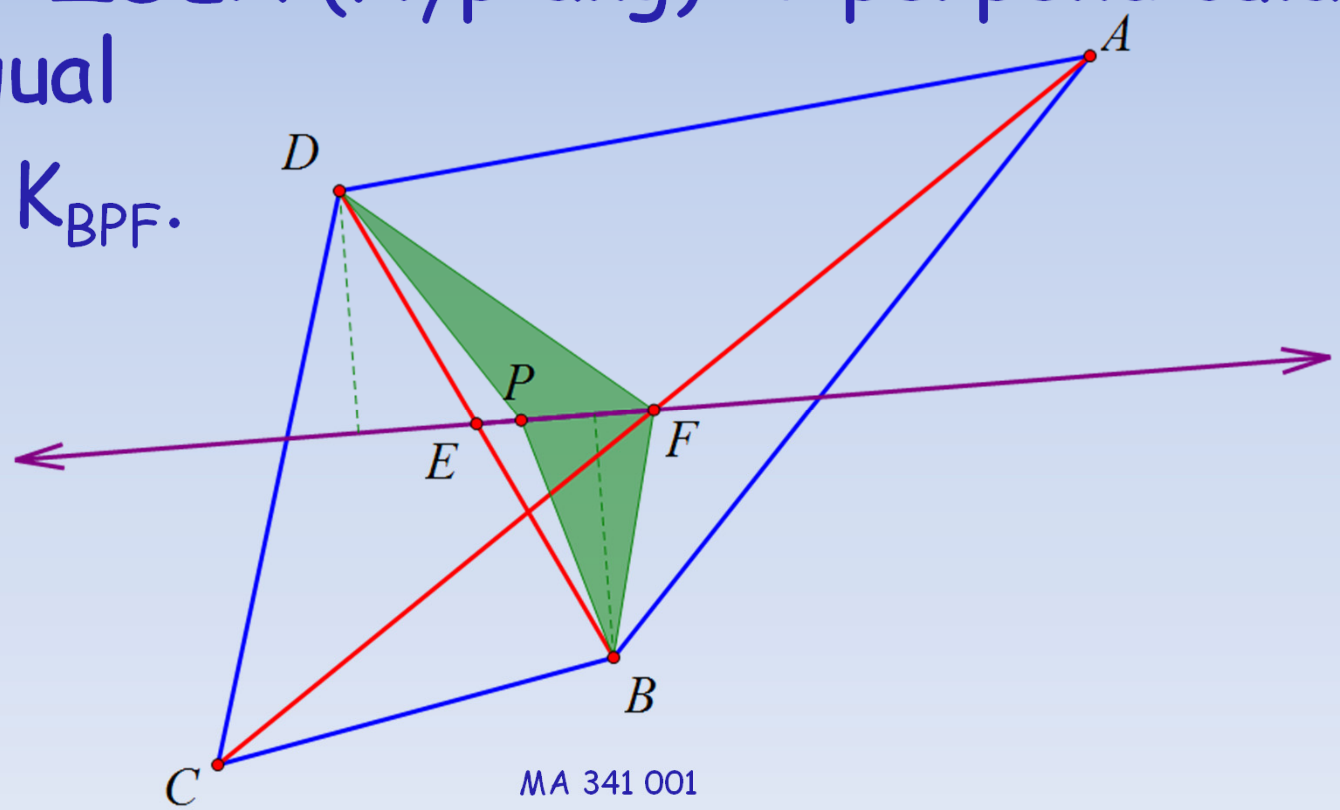
# Proof

E midpoint of BD,

$\angle EPB = \angle DEX$ (vertical angles),

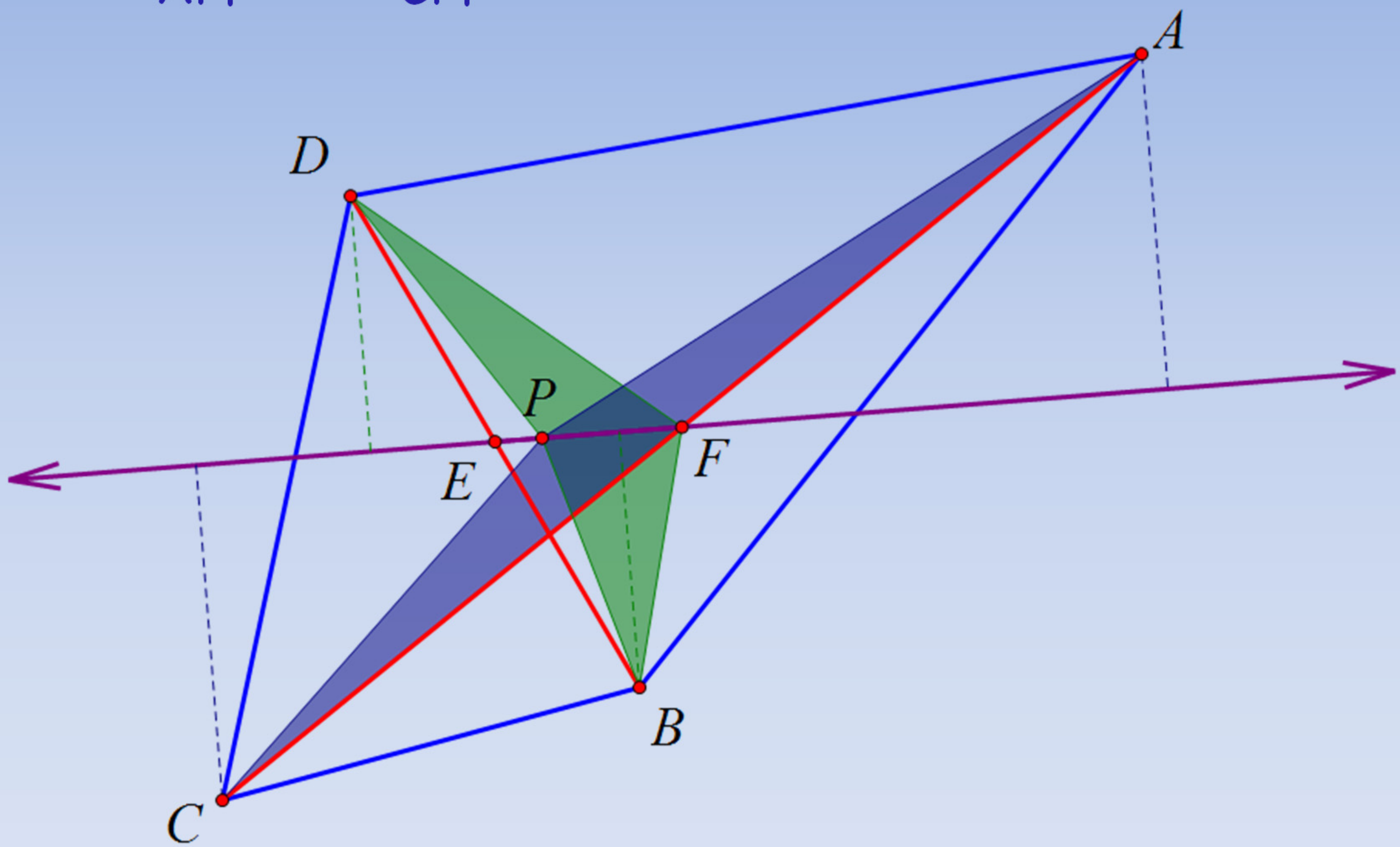
$\Delta EPB = \Delta DEX$ (Hyp-ang) $\Rightarrow$ perpendiculars are equal

$K_{DPF} = K_{BPF}$.



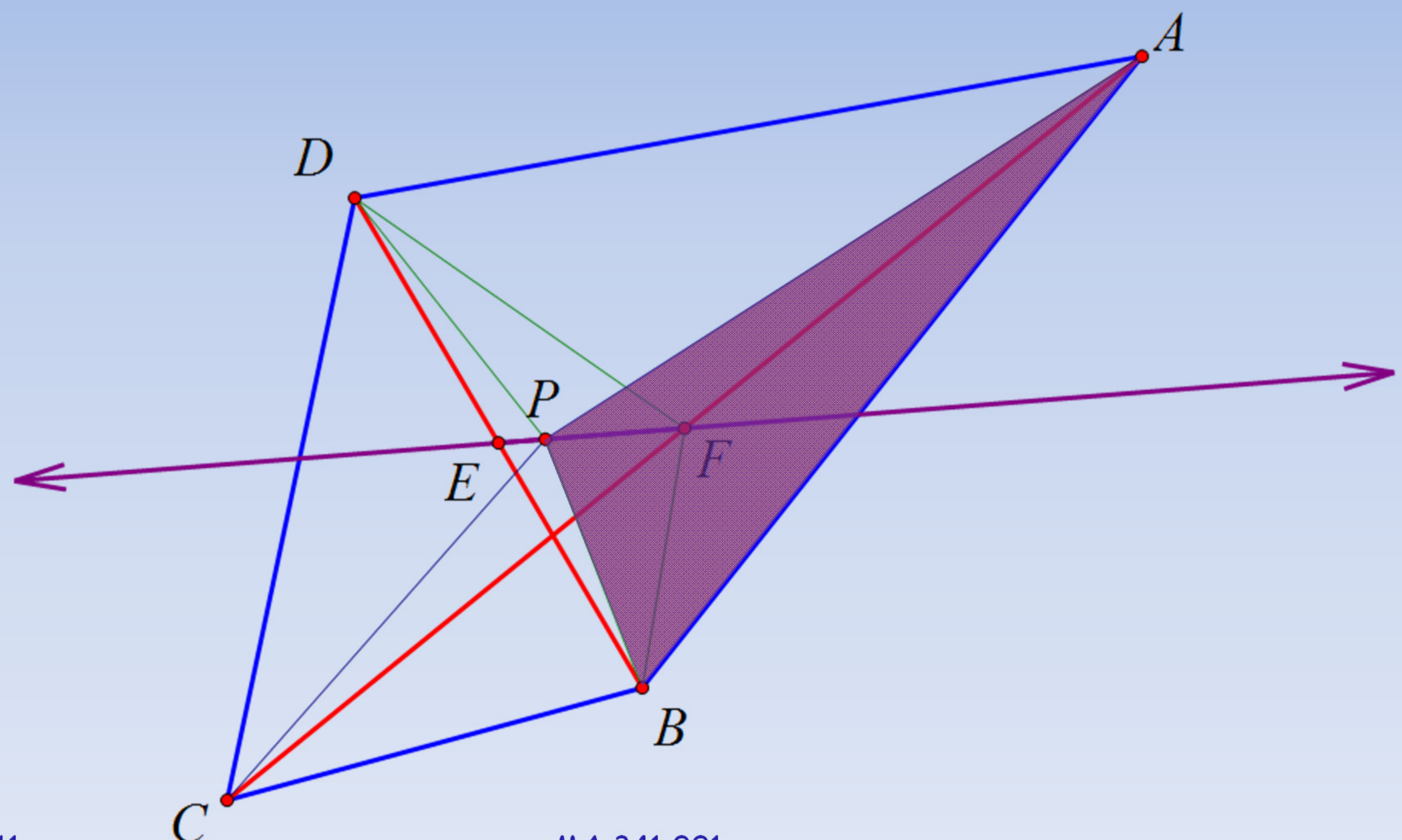
# Proof

Likewise $K_{APF} = K_{CPF}$.



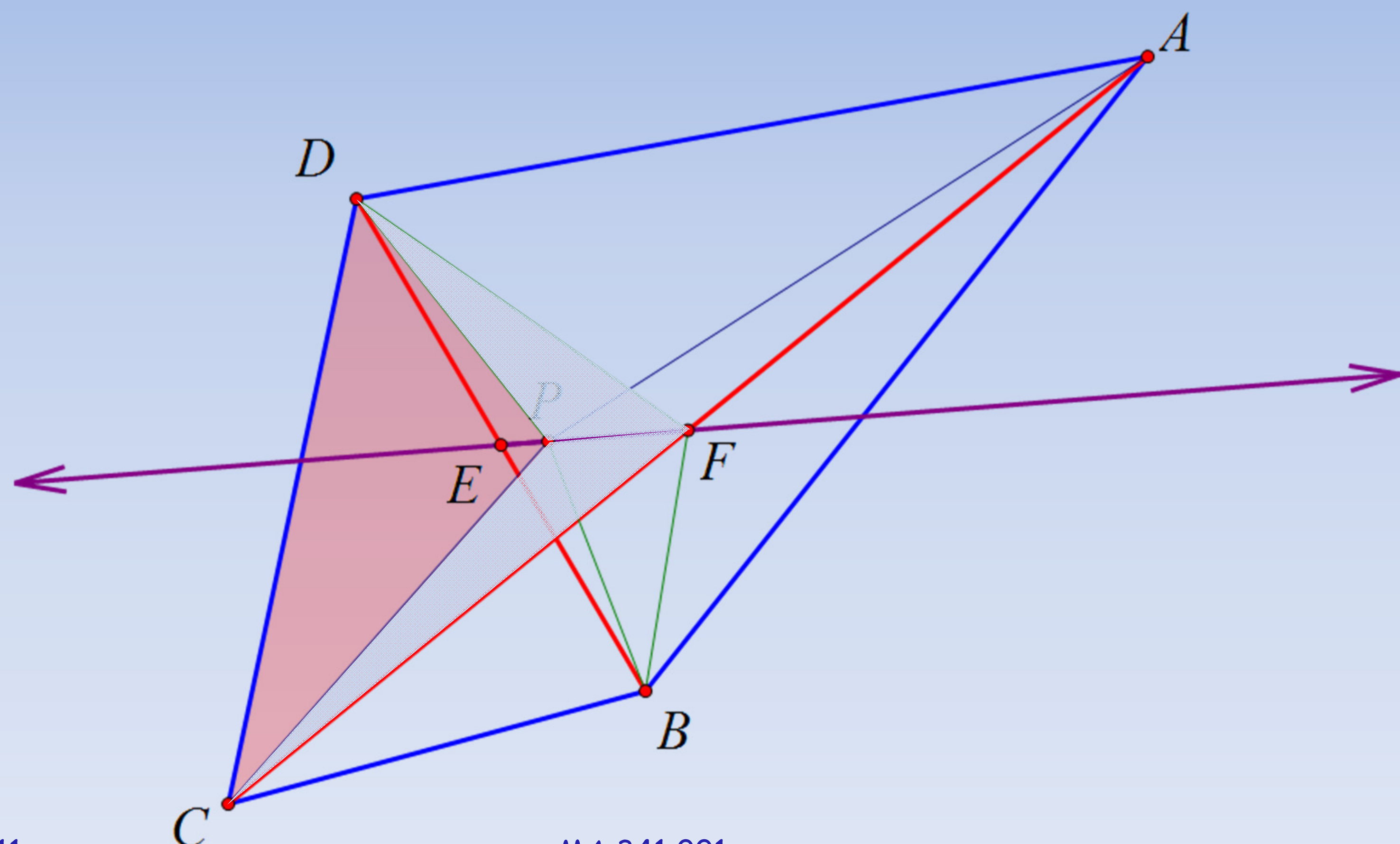
# Proof

$K_{APB} = K_{AFB} + K_{APF} + K_{BPF}$



# Proof

$K_{DPC} = K_{DFC} - K_{CPF} - K_{DPF}$



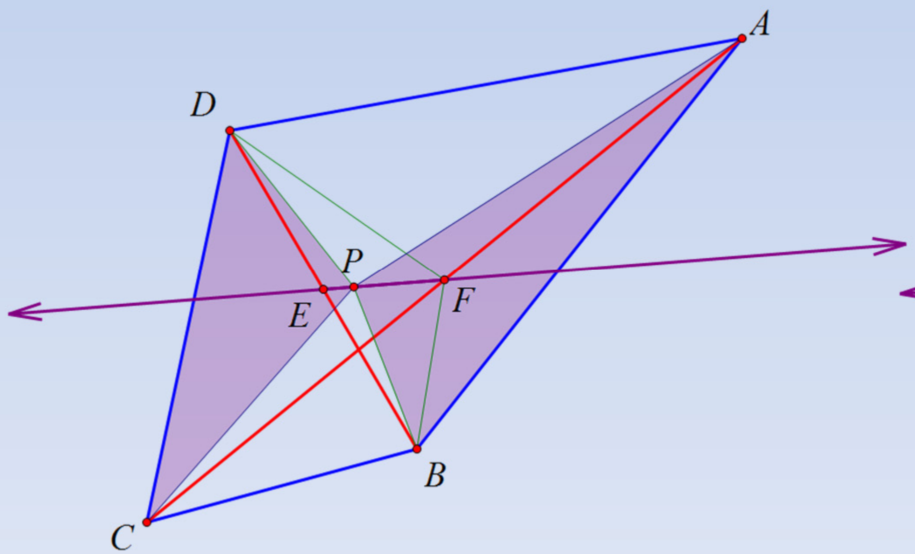
# Proof

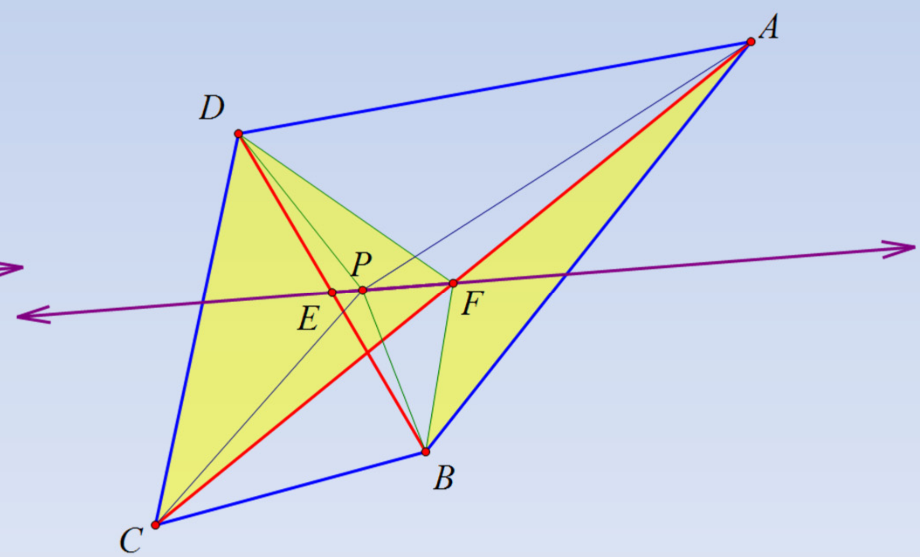
$K_{APB} = K_{AFB} + K_{APF} + K_{BPF}$

$K_{DPC} = K_{DFC} - K_{CPF} - K_{DPF}$

So

$K_{APB} + K_{DPC} = K_{AFB} + K_{DFC}$



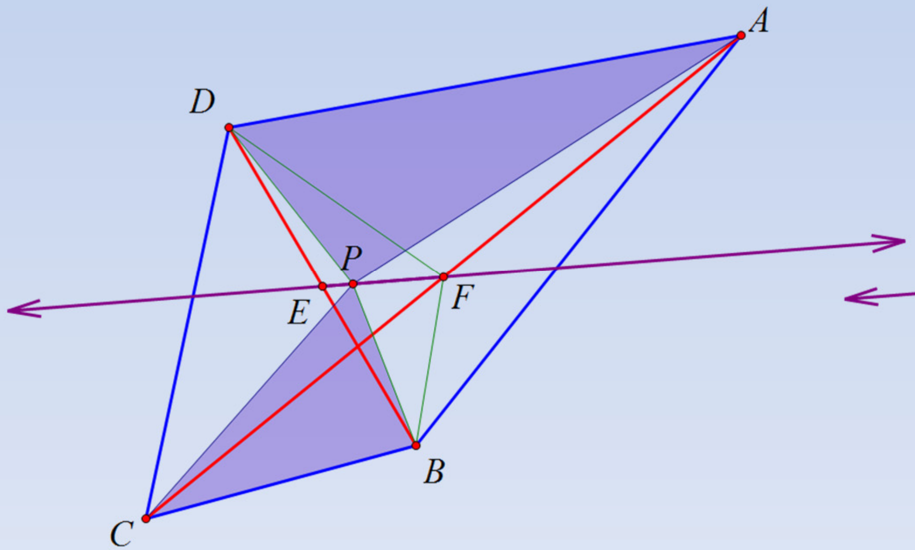
# Proof

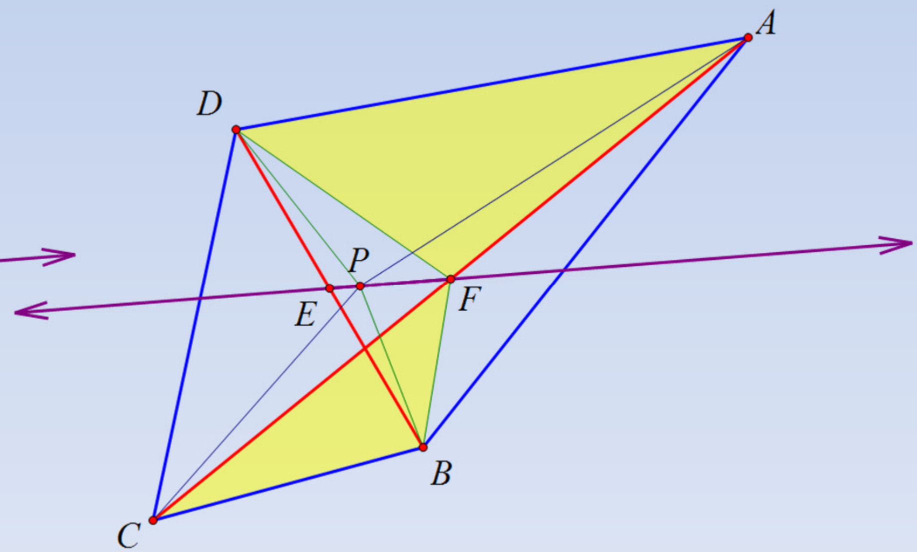
$K_{APD} = K_{AFD} + K_{DPF} + K_{APF}$

$K_{BPC} = K_{BFC} - K_{BPF} - K_{CPF}$

So

$K_{APD} + K_{BPC} = K_{AFD} + K_{CFB}$



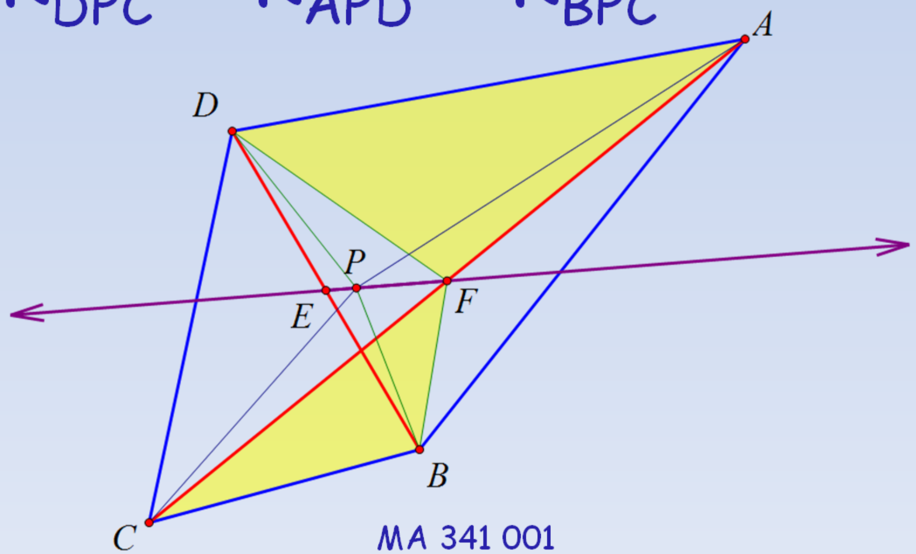
# Proof

$K_{APB} + K_{DPC} = K_{AFB} + K_{DFC}$

$K_{APD} + K_{BPC} = K_{AFD} + K_{CFB}$

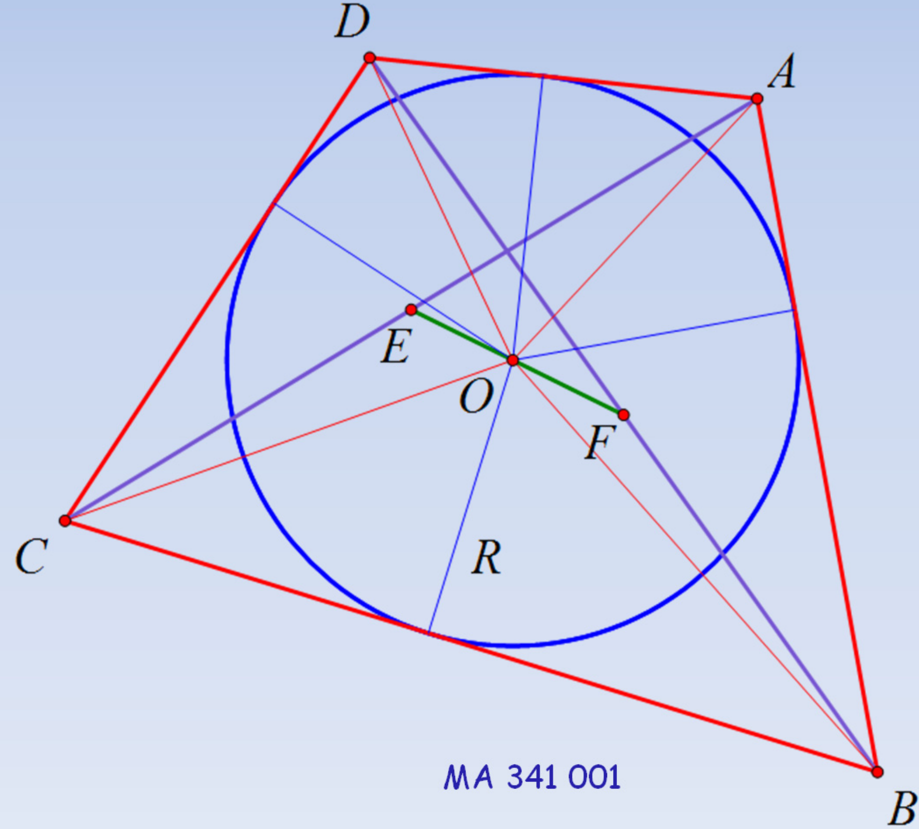
Also $K_{AFB} = K_{CFB}$ (equal bases, same height)
and $K_{DFC} = K_{AFD}$ (equal bases, same height)

So $K_{APB} + K_{DPC} = K_{APD} + K_{BPC}$



# Newton Line

Theorem: The center of the circle inscribed into a quadrilateral lies on the line joining the midpoints of the diagonals.



# Proof

The distance from O to each side is R.

The area of each triangle then is

$K_{AOB} = \frac{1}{2} R|AB|$ $K_{BOC} = \frac{1}{2} R|BC|$

$K_{COD} = \frac{1}{2} R|CD|$ $K_{DOA} = \frac{1}{2} R|AD|$

Since AB + CD = BC + AD, multiplying both sides by $\frac{1}{2}R$, we get that

$$K_{AOB} + K_{COD} = K_{BOC} + K_{DOA}$$

Thus, O lies on the Newton Line.

# Area

A bicentric quadrilateral is a cyclic quadrilateral, so Brahmagupta's Formula applies:

$$K = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

Since it is a tangential quadrilateral we know that a + c = s = b + d. Thus:

s – a = c, s – b = d, s – c = a and s – d = b or

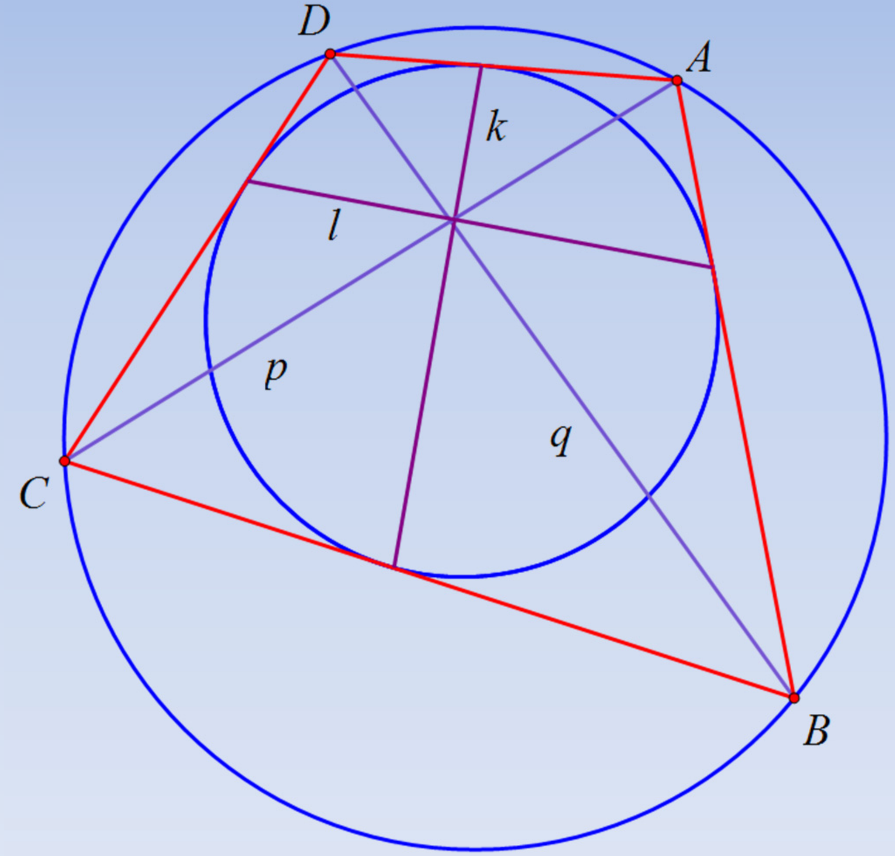
$$K = \sqrt{abcd}$$

# Area

k,l = tangency chords

p,q = diagonals

$$K = \frac{klpq}{k^2 + l^2}$$

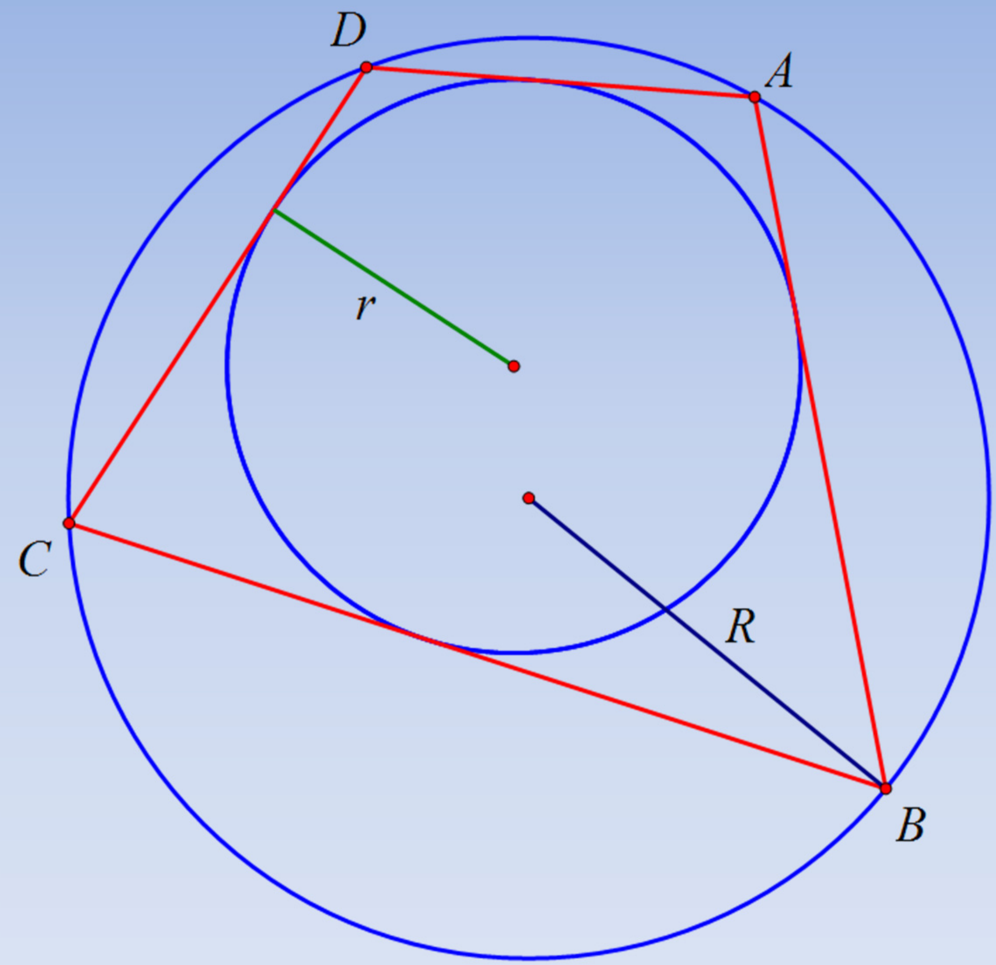


# Area

r = inradius

R = circumradius

$$4r^2 \leq K \leq 2R^2$$



# Inradius & Circumradius

Since it is a tangential quadrilateral

$$r = \frac{K}{a+c}$$

Since it is bicentric, we get

$$r = \frac{\sqrt{abcd}}{a+c}$$

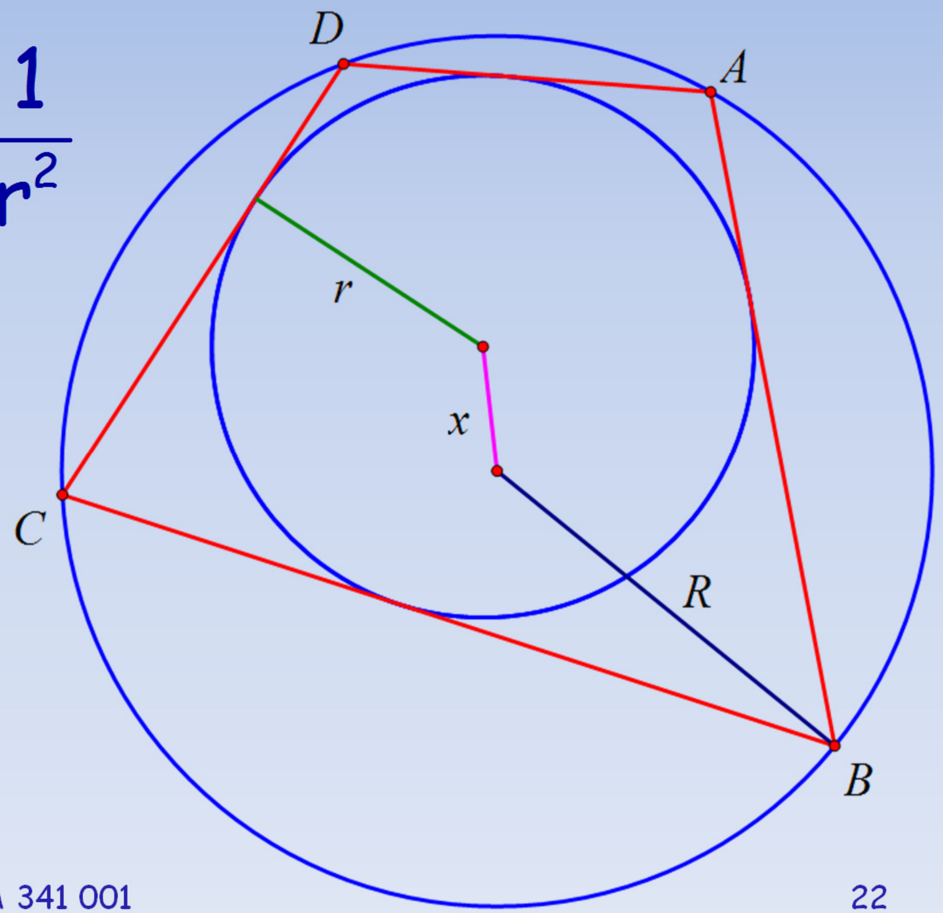
We gain little for the circumradius

$$R = \frac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{abcd}}$$

# Fuss' Theorem

Let x denote the distance from the incenter, I, and circumcenter, O. Then

$$\frac{1}{(R-x)^2}+\frac{1}{(R+x)^2}=\frac{1}{r^2}$$



# NOTE

Let d denote the distance from the incenter, I, and circumcenter, O, in a triangle, then Euler's formula says

$$\frac{1}{R-d}+\frac{1}{R+d}=\frac{1}{r}$$

or

$$2Rr=R^2-d^2$$

# Proof

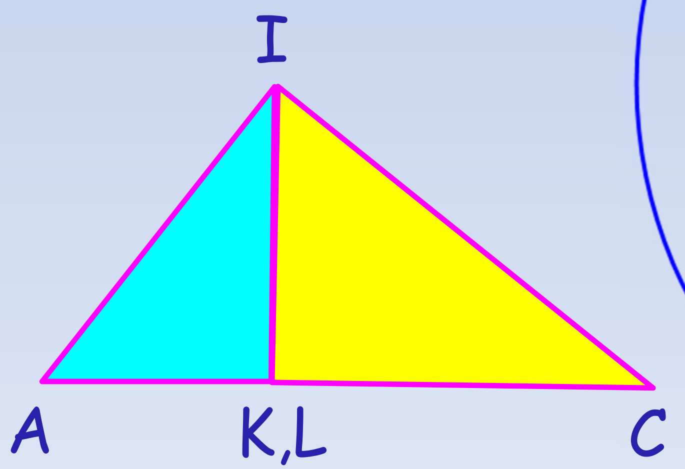
Let K,L be points of tangency of incircle

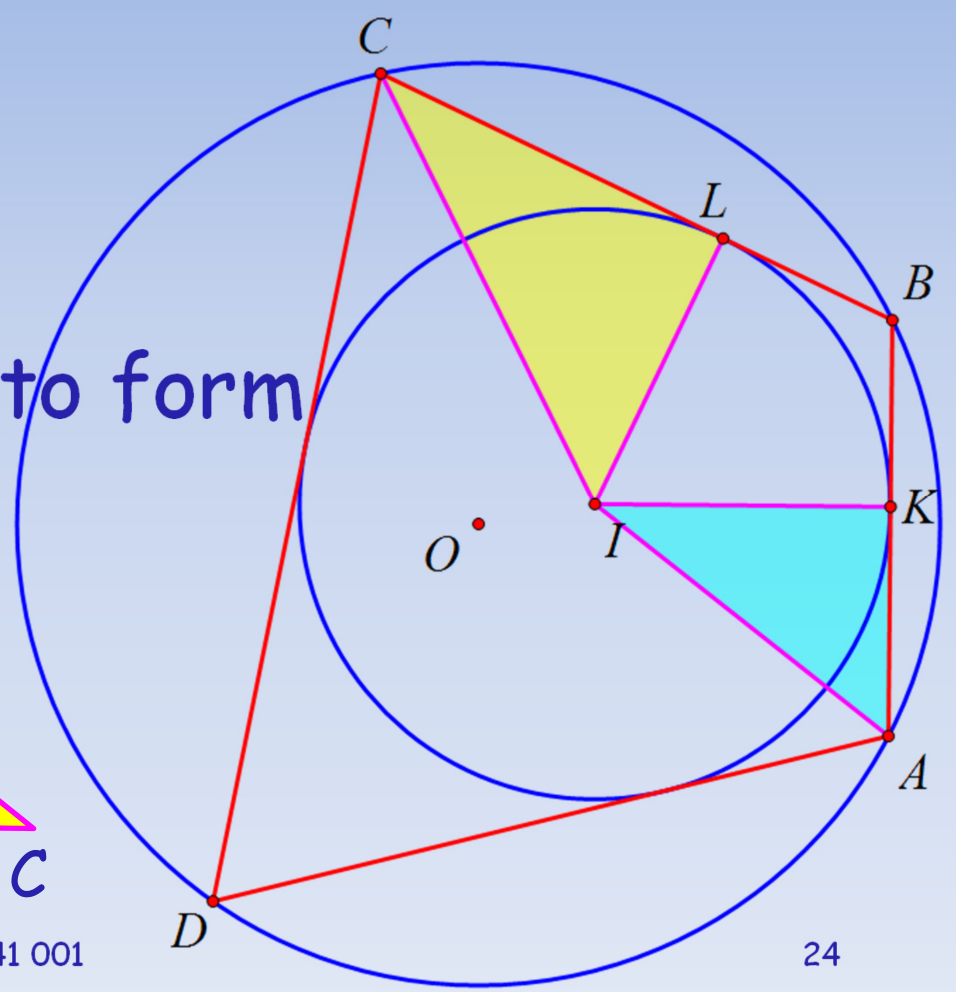
A + C = 180

IL = r = IK and

∠ILC = ∠IKA = 90

Join ΔILC and ΔIKA to form ΔCIA



# Proof

What do we know about $\Delta CIA$?

$A + C = 180$

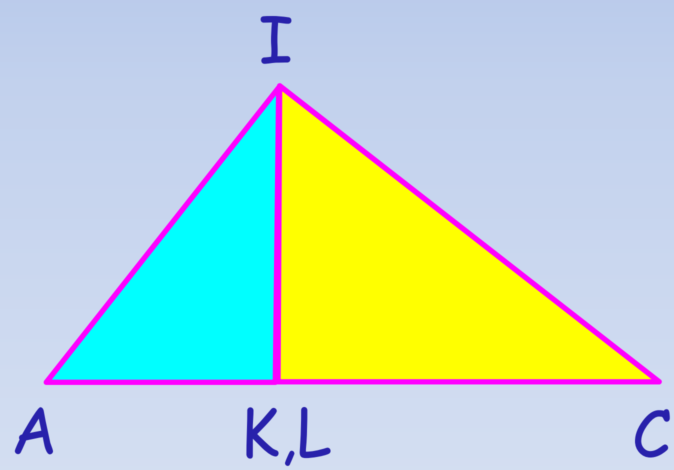
$\angle LCI = \frac{1}{2}C$ and $\angle IAK = \frac{1}{2}A$

$\angle LCI + \angle IAK = 90$

$\angle CIA = 90$

$\Delta CIA$ a right triangle

Hypotenuse = AK+CL



# Proof

Area = $\frac{1}{2}r(AK+CL) = \frac{1}{2}\, AI\ CI$ or

$$r(AK+CL) = AI\ CI$$

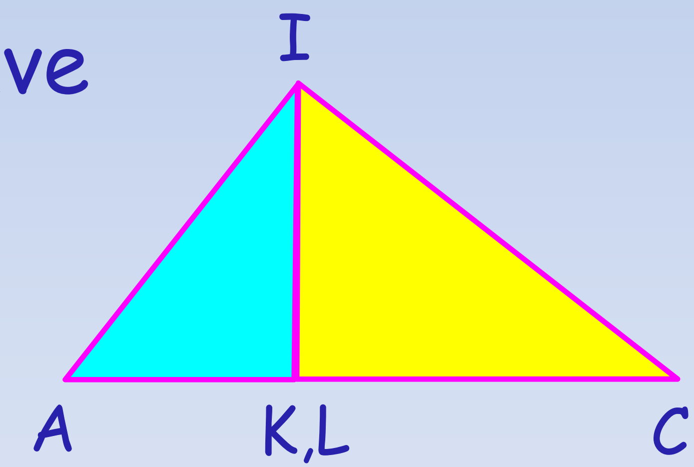
Pythagorean Theorem gives

$$(AK+CL)^2 = AI^2 + CI^2$$

From first equation we have

$r^2(AI^2 + CI^2) = AI^2\ CI^2$

$$\frac{1}{r^2} = \frac{1}{AI^2} + \frac{1}{CI^2}$$



# Proof

Extend AI and CI to meet circumcircle at F and E.

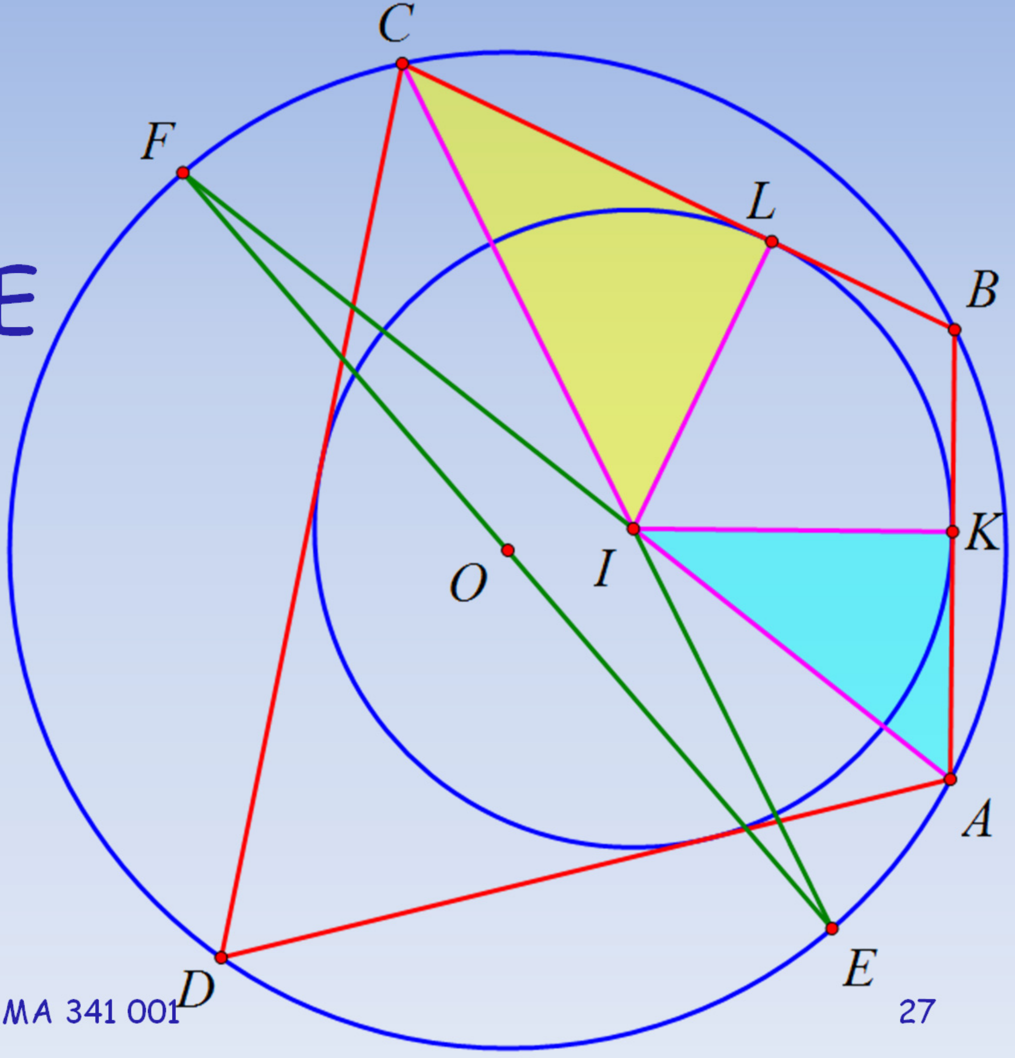
$\angle DOF + \angle DOE =$

$2\angle DAF + 2\angle DCE$

$= \angle BAD + \angle BCD$

$= 180$

EF is a diameter!



# Proof

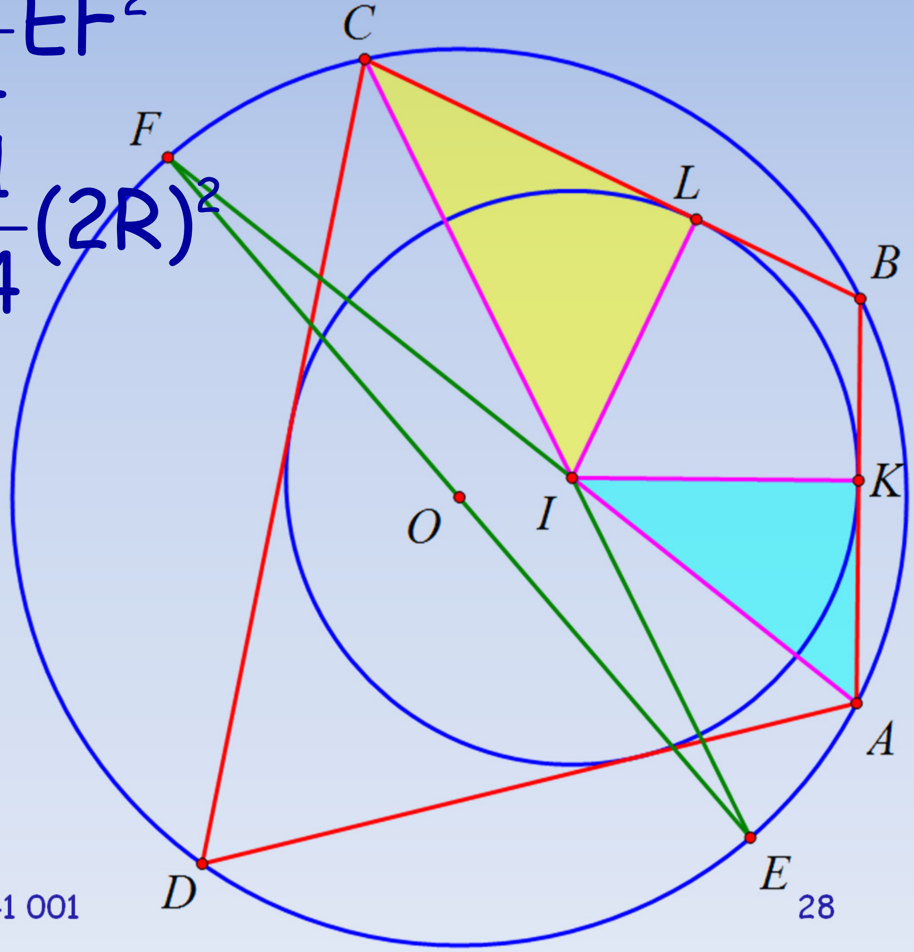
X = IO = median in ΔIFE.

$$IO^2 = \frac{1}{2}(IE^2 + IF^2) - \frac{1}{4}EF^2$$

$$IO^2 = \frac{1}{2}(IE^2 + IF^2) - \frac{1}{4}(2R)^2$$

$$IE^2 + IF^2 = 2IO^2 + 2R^2$$

$$= 2(x^2 + R^2)$$



# Proof

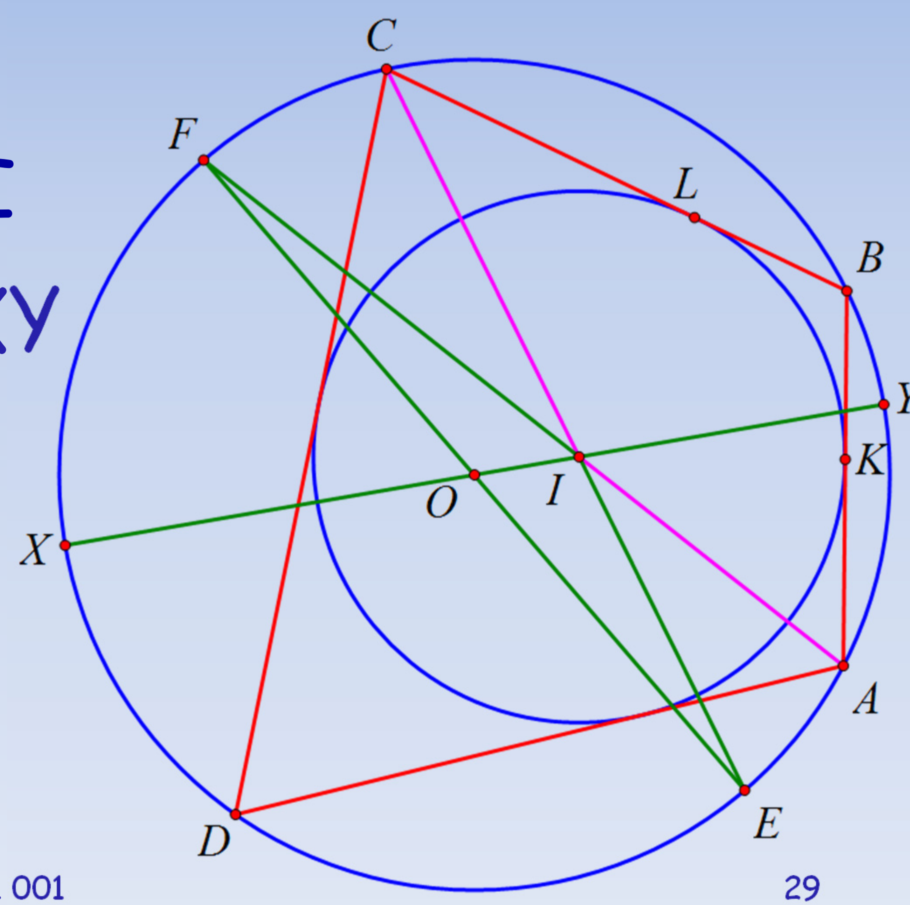
From chords CE and AF, we have

$$\frac{CI}{FI} = \frac{AI}{EI}$$

$$AI \bullet FI = CI \bullet EI$$

From chords CE and XY

$$\begin{aligned} CI \bullet EI &= XI \bullet YI \\ &= (R + x)(R - x) \\ &= R^2 - x^2 \end{aligned}$$



# Proof

$$AI \bullet FI = R^2 - x^2 \qquad AI \bullet FI = CI \bullet EI$$

$$\frac{1}{AI^2} + \frac{1}{CI^2} = \frac{FI^2}{CI^2 \bullet EI^2} + \frac{EI^2}{CI^2 \bullet EI^2}$$

$$= \frac{EI^2 + FI^2}{(R^2 - x^2)^2}$$

$$= \frac{2(R^2 + x^2)}{(R^2 - x^2)^2}$$

# Proof

$$\frac{1}{r^2} = \frac{2(R^2 + x^2)}{(R^2 - x^2)^2}$$

$$= \frac{((R+x)^2 + (R-x)^2)}{(R^2 - x^2)^2}$$

$$\frac{1}{r^2} = \frac{1}{(R+x)^2} + \frac{1}{(R-x)^2}$$

$$2r^2(R^2 + x^2) = (R^2 - x^2)^2$$

$$x = \sqrt{R^2 + r^2 - r\sqrt{4R^2 + r^2}}$$

# Other results

In a bicentric quadrilateral the circumcenter, the incenter and the intersection of the diagonals are collinear.

Given two concentric circles with radii R and r and distance x between their centers satisfying the condition in Fuss' theorem, there exists a convex quadrilateral inscribed in one of them and tangent to the other.

# Other results

Poncelet's Closure Theorem:

If two circles, one within the other, are the incircle and the circumcircle of a bicentric quadrilateral, then every point on the circumcircle is the vertex of a bicentric quadrilateral having the same incircle and circumcircle.