# Logic and Proof

On my first day of school my parents dropped me off at the wrong nursery. There I was...surrounded by trees and bushes!

# Requirements for Proof

1. Mutual understanding of the words and symbols used
2. Acceptance of certain statements called axioms without justification
3. Agreement on how and when one statement follows logically from another, that is, agreement on certain rules of reasoning.

# Sets

A set is a collection of objects satisfying some condition.

It is possible to have no objects in a set. This set is called the **empty set** and is denoted by { } or Ø

# Set Notation

Roster method: {a,b,c,…z}

Set builder notation:

$$A = \{ x \mid P(x) \text{ is satisfied } \}$$

Set elements:

- $a \in A$
- a is in $A$
- a is a member of $A$
- a is an element of $A$

# Set Notation

Set elements:

$a \notin A$ means a is not a member of A

$A \subset B$ means that A is a subset of B which means every element of A is also an element of B

⟺ if $a \in A$ then $a \in B$

Note: $\emptyset \subset A$ for every set A

# Set Equality

A = B means that every element of A is also and element of B and vice versa.

$A = B \Longleftrightarrow A \subset B$ and $B \subset A$

# Set Intersection

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

# Set Union

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

# Set Complement

$$A^C = A' = CA = \{x \mid x \notin A\}$$

Note that

$$A \cap A^C = \emptyset$$

$$A \cup A^C = U$$

# Mathematical Statements

A declarative sentence which is true or false, but not both, is called a statement.

Examples: 1 + 1 = 3

2 - 1 = 1

The grass is blue.

# Set Variables

A **variable** is a symbol or an icon that can be used to represent various elements of the universal set.

Examples: He is a Wildcat.

This is true if "he=Adolf Rupp". This is not true if "he=Rick Pitino".

# Logical Connectives & Truth Tables

If P and Q are statements, then the statement P and Q is called the **conjunction** of P and Q.

Notation: $P \wedge Q$

Examples:

# Truth Values

| P | Q | $P \wedge Q$ |
|---|---|---|
| True | True | True |
| True | False | False |
| False | True | False |
| False | False | False |

# Logical Connectives & Truth Tables

If P and Q are statements, then the statement P or Q is called the **disjunction** of P and Q.

Notation: $P \vee Q$

Note: mathematicians use the inclusive "or"

Examples:

# Truth Values

| $P$ | $Q$ | $P \vee Q$ |
|---|---|---|
| True | True | True |
| True | False | True |
| False | True | True |
| False | False | False |

# Negation

If P is a statement, then the statement not P is called the **negation** of P.

Notation: ~P (sometimes –P or ¬P)

We assume that either P or ~P is true, but not both and not neither

# Negation

$\sim(P \land Q)$ is equivalent to $\sim P \lor \sim Q$.

$\sim(P \lor Q)$ is equivalent to $\sim P \land \sim Q$

$\sim(\sim P)$ is equivalent to $P$

# The Conditional

If P and Q are statements, then the statement if P then Q is called the **conditional statement**

Notation: $P \Rightarrow Q$

Examples: If it is cold, it is snowing.

If it is blue, it is UK.

# Truth Values

| $P$ | $Q$ | $P \Rightarrow Q$ |
|---|---|---|
| True | True | True |
| True | False | False |
| False | True | True |
| False | False | True |

See additional notes in Blackboard for further discussion.

# Truth Values

P = "The animal is a tiger."

Q = "The animal is a mammal."

When **must** the statement be false?

| P | Q | $P \Rightarrow Q$ |
|---|---|---|
| is a tiger | is a mammal | |
| is a tiger | is not a mammal | |
| is not a tiger | is mammal | |
| is not a tiger | is not a mammal | |

# Truth Values

Statements of the form "If P then Q" do not mean the same thing in math as in ordinary English.

**In ordinary English**,

"If P then Q" may suggest **order** of occurrence.

"If we go outside, the neighbors will see us" implies that the neighbors will see us **after** we go outside.

If P then Q" can suggest **causation**. This example has the connotation that the neighbors will see us **because** we went outside.

# The Conditional

$$P \Rightarrow Q$$

If P then Q $\leftarrow$ PREFERED

Q if P

P only if Q

Q provided P

Q when P

# Related Statements

$$P \Rightarrow Q$$

Conditional: If P, then Q ($P \Rightarrow Q$)

Converse: If Q, then P ($Q \Rightarrow P$)

Inverse: If not P, then not Q ($\sim P \Rightarrow \sim Q$)

Contrapositive: If not Q, then not P

$$(\sim Q \Rightarrow \sim P)$$

(Conditional logically equiv to Contrapositive)

(Converse logically equiv to Inverse )

# Related Statements

If there is smoke, then there is fire.

Converse:

If there is fire, then there is smoke.

Inverse:

If there is no smoke, then there is no fire.

Contrapositive:

If there is no fire, then there is no smoke.

# The Biconditional

If P and Q are statements, then the statement P if and only if Q is called the biconditional **statement**

Notation: $P \Leftrightarrow Q$

Meaning: $P \Leftrightarrow Q$ means $P \Rightarrow Q$ and $Q \Rightarrow P$.

# Truth Values

| $P$ | $Q$ | $P \Leftrightarrow Q$ |
|---|---|---|
| True | True | True |
| True | False | False |
| False | True | False |
| False | False | True |

# Compound Statements

If x is perpendicular to y and y is perpendicular to z, then x is parallel to z.

Let

**P**: x is perpendicular to y

**Q**: y is perpendicular to z

**R**: x is parallel to z.

$$P \wedge Q \Rightarrow R$$

| P | Q | R | $P \wedge Q \Rightarrow R$ |
|---|---|---|---|
| True | True | True | True |
| True | True | False | False |
| True | False | True | True |
| True | False | False | True |
| False | True | True | True |
| False | True | False | True |
| False | False | True | True |
| False | False | False | True |

# Universal Quantifier

If P is a statement that depends on a variable x, then the universal quantifier is

For every x, P(x)

**for every = for each = for all**

$$\forall x, P(x)$$

# Existential Quantifier

If P is a statement that depends on a variable x, then the existential quantifier is

There exists an x such that P(x)

**There is at least one = for at least one = some**

$$\exists x, P(x)$$

# Translate the following

| | |
|---|---|
| $\forall x\ \forall y,\ x+y = 0$ | For every x and for every y, x+y = 0. |
| $\forall x\ \exists y,\ x+y=0$ | For every x there exists a y so that x+y = 0. |
| $\exists x\ \forall y,\ x+y=0$ | There exists an x so that for every y, x+y = 0. |
| $\exists x\ \exists y,\ x+y=0$ | There exists an x and there exists a y so that x+y = 0. |

# Translate the following

For every x, if x is even, then there exists a y such that x = 2y.

$$\forall x\ (x \text{ is even} \Rightarrow \exists\ y,\ x = 2y)$$

# Rules of Reasoning

A tautology is a sentence which is true no matter what the truth value of its constituent parts.

Example:

$$\mathbf{P} \Rightarrow (\mathbf{P} \vee \mathbf{Q})$$

# Logic Axiom 1

Every tautology is a rule of reasoning.

- $(P \Rightarrow Q) \Leftrightarrow (\sim Q \Rightarrow \sim P)$ (The Contrapositive)
- $[P \wedge (P \Rightarrow Q)] \Rightarrow Q$ (Modus ponens)
- $[(P \Rightarrow Q) \wedge (Q \Rightarrow R)] \Rightarrow (P \Rightarrow R)$ (Law of Syllogism)
- $\sim(P \Rightarrow Q) \Leftrightarrow (P \wedge \sim Q)$ (basis for Proof by Contradiction)
- $[(P \Rightarrow R) \wedge (Q \Rightarrow R)] \Rightarrow [(P \vee Q) \Rightarrow R]$ Proof by Cases

# Logic Axiom 2

Let U denote a universal set. Each of the following is a rule of reasoning.

- $[\forall x, P(x) \Rightarrow Q(x)] \Rightarrow [\forall x, P(x) \Rightarrow \forall x, Q(x)]$
- $\forall x, P(x) \Leftrightarrow P(a)$ for any $a \in U$
- $\exists x, P(x) \Leftrightarrow [P(a)$ for some $a \in U]$

# Logic Axiom 3

(Rule of Substitution) Suppose $P \Leftrightarrow Q$. Then P and Q may be substituted for one another in any sentence.

# Logic Axiom 4

Every sentence of the type $\sim[\forall x, P(x)] \Leftrightarrow [\exists x$ so that $\sim P(x)]$ is true.

# Logic Axiom 5

Every sentence of the type $\sim[\exists x$ so that $P(x)] \Leftrightarrow [\forall x, \sim P(x)]$ is true.