Logic, Proof, Axiom Systems

MA 341 – Topics in Geometry
Lecture 03

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \frac{22}{7} - \pi$$

$$111{,}111{,}111 \times 111{,}111{,}111 = 12{,}345{,}678{,}987{,}654{,}321$$

Rules of Reasoning

A tautology is a sentence which is true no matter what the truth value of its constituent parts.

Example:

$$\mathbf{P} \Rightarrow (\mathbf{P} \vee \mathbf{Q})$$

Logic Axiom 1

Every tautology is a rule of reasoning.

- $(P \Rightarrow Q) \Leftrightarrow (\sim Q \Rightarrow \sim P)$
  (The Contrapositive)
- $[P \wedge (P \Rightarrow Q)] \Rightarrow Q$
  (Modus ponens)
- $[(P \Rightarrow Q) \wedge (Q \Rightarrow R)] \Rightarrow (P \Rightarrow R)$
  (Law of Syllogism)
- $\sim(P \Rightarrow Q) \Leftrightarrow (P \wedge \sim Q)$ (basis for Proof by Contradiction)
- $[(P \Rightarrow R) \wedge (Q \Rightarrow R)] \Rightarrow [(P \vee Q) \Rightarrow R]$
  Proof by Cases

Logic Axiom 2

Let U denote a universal set. Each of the following is a rule of reasoning.

- $[\forall x, P(x) \Rightarrow Q(x)] \Rightarrow [\forall x, P(x) \Rightarrow \forall x, Q(x)]$
- $\forall x, P(x) \Leftrightarrow P(a)$ for any $a \in U$
- $\exists x, P(x) \Leftrightarrow [P(a)$ for some $a \in U]$

Logic Axiom 3

(Rule of Substitution) Suppose $P \Leftrightarrow Q$. Then P and Q may be substituted for one another in any sentence.

Logic Axiom 4

Every sentence of the type $\sim[\forall x, P(x)] \Leftrightarrow [\exists x$ so that $\sim P(x)]$ is true.

Logic Axiom 5

Every sentence of the type $\sim[\exists x$ so that $P(x)] \Leftrightarrow [\forall x, \sim P(x)]$ is true.

Mathematical System

Consists of

1. Set of undefined concepts,
2. Universal set
3. Set of relations
4. Set of operations
5. Set of logical axioms
6. Set of axioms
7. Set of theorems
8. Set of definitions
9. An underlying set theory

Proof

Suppose A_1, A_2,…,A_k are all the axioms and previously proved theorems of a mathematical system. A formal proof, or deduction, of a sentence P is a sequence of statements S_1, S_2,…,S_n, where

1. S_n is P and
2. One of the following holds
   a) S_i is one of A_1, A_2,…,A_k
   b) S_i follows from the previous statements by valid argument

Proving Conditionals

Suppose $A_1, A_2,\ldots,A_k$ are all the axioms and previously proved theorems of a mathematical system. A formal proof, or deduction, of a sentence P is a sequence of statements $S_1, S_2,\ldots,S_n$, where

1. S_n is P and
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Proof of $P \Rightarrow Q$

1. Assume P to be true.
2. Use P and all other theorems and axioms to deduce Q.
3. Once Q is deduced in this manner you have completed a proof of $P \Rightarrow Q$.

You have not shown that Q is true.

You have shown only that Q is true IF P is true

Rule of Conditional Proof
or
Deduction Theorem

Suppose that $A_1, A_2, \ldots, A_k$ are the axioms and previously proved theorems. To prove $P \Rightarrow Q$ is to show that

From $A_1, A_2, \ldots, A_k$ deduce $P \Rightarrow Q$ is valid.

To do this temporarily assume P to be an axiom and show that

From $A_1, A_2, \ldots, A_k$, P deduce Q is valid.

Proof by Contrapositive

We can prove $P \Rightarrow Q$ by proving $\sim Q \Rightarrow \sim P$.

Rule of conditional proof used to prove the contrapositive

Proving Biconditionals

Three techniques:

1. Prove $P \Rightarrow Q$ and $Q \Rightarrow P$.
2. Prove $P \Rightarrow Q$ and $\sim P \Rightarrow \sim Q$.
3. Prove an IFF string

Proving $\forall x\, P(x)$

1. Let x represent arbitrary element of universal set
2. Prove that P(x) is true
3. Since x was arbitrary we know that $\forall x\, P(x)$ is true.

The justification is Logical Axiom 2.

Proof by Cases

Proving a statement of the form:

$$(P \vee Q) \Rightarrow R$$

Use the tautology:

$$[P \Rightarrow R] \wedge [Q \Rightarrow R] \Leftrightarrow [(P \vee Q) \Rightarrow R]$$

Mathematical Induction

Let P(n) be a statement for any integer n.
Principle of Induction:

$$[P(1) \wedge (\forall k, P(k) \Rightarrow P(k+1))] \Rightarrow \forall n, P(n)$$

2 step process:

Basic Step: Prove P(1).

Inductive step: Prove $\forall k, P(k) \Rightarrow P(k+1)$

Proof by Contradiction

Contradiction: statement which is false no matter truth value of its constituent parts

$$R \wedge \sim R$$

Tautology: $[\sim P \Rightarrow (R \wedge \sim R)] \Rightarrow P$

Usually used to prove statement of type $P \Rightarrow Q$.
Deduction Theorem: Assume P and deduce $P \Rightarrow Q$
Contrapositive: Assume ~Q and deduce $\sim Q \Rightarrow \sim P$

Contradiction: Assume P AND ~Q and deduce $R \wedge \sim R$

Proving $\exists x\, P(x)$

Prove that there is at least one x so that P(x) is true. Here is one case where an example is good enough.

The other case is when you find a "counterexample" to show $\sim(\forall x\, P(x))$.

How to Create a Proof

Suggestions – NOT RULES

1. Translate to symbolic logic
2. Analogy – look at other proofs
3. Reverse engineering – you know where you want to end, figure out each step it would require to get there (analogy – trig identities)
4. Use definitions
5. Use **<u>previously</u>** proved theorems

Euclid's Five Axioms

Let the following be postulated

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any center and distance.
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles. (Euclid's Parallel Postulate)

Hilbert's Axioms

Incidence Axioms

I-1: For every point P and for every point Q not equal to P there exists a unique line that passes through P and Q.
I-2: For every line there exist at least two distinct points incident with it.
I-3: There exist three distinct points with the property that no line is incident with all three of them.

Hilbert's Axioms

Betweenness Axioms

B-1: If A*B*C, then A, B, and C are 3 distinct points all lying on the same line and C*B*A.

B-2: Given any two distinct points B and D, there exists points A, C, and E lying on BD such that A*B*D, B*C*D, and B*D*E.

B-3: If A, B, and C are three distinct point lying on the same line, then one and only one of the points is between the other two.

Hilbert's Axioms

Betweenness Axioms

B-4: (Plane Separation Axiom) For every line l and for any 3 points A, B, and C not lying on l:

i. If A and B are on the same side of l and B and C are on the same side of l, then A and C are on the same side of l.
ii. If A and B are on opposite sides of l and B and C are on opposite sides of l, then A and C are on the same side of l.

Hilbert's Axioms

Congruence Axioms

C-1: If A and B are distinct points and if A' is any point, then for each ray r emanating from A' there is a unique point B' on r such that $B' \neq A'$ and $AB \cong A'B'$.
C-2: If $AB \cong CD$ and $AB \cong EF$, then $CD \cong EF$. Also, every segment is congruent to itself.
C-3: If $A*B*C$ and $A'*B'*C'$ and $AB \cong A'B'$ and $BC \cong B'C'$, then $AC \cong A'C'$.
C-4: Given any $\angle BAC$ and given a ray $A'B'$ emanating from point A', $\exists!$ ray $A'C'$ on a given side of line $A'B'$ so that $\angle BAC \cong \angle B'A'C'$.

Hilbert's Axioms

Congruence Axioms

C-5: If $\angle A \cong \angle B$ and $\angle B \cong \angle C$ then $\angle A \cong \angle C$. Also, every angle is congruent to itself.

C-6: (Side-Angle-Side) If two sides and the included angle of one triangle are congruent respectively to two sides and the included angle of another triangle, then the two triangles are congruent.

Hilbert's Axioms

Continuity Axioms

Archimedes Axiom: If AB and CD are any segments, then there is a number n such that if segment CD is laid off n times on the ray AB emanating from A, then a point E is reached where n CD $\cong$ AE and B is between A and E.

Dedekind's Axiom: Suppose that the set of all points on a line l is the union $\Sigma_1 \cup \Sigma_2$ of two nonempty subsets such that no point of Σ_1 is between two points of Σ_2 and vice versa. Then there is a unique point, O, lying on l such that P_1*O*P_2 if and only if $P_1 \in \Sigma_1$ and $P_2 \in \Sigma_2$ and $O \neq P_1, P_2$.

Hilbert's Axioms

Continuity Axioms

Elementary Continuity Principle: If one endpoint of a segment is inside a circle and the other outside, then the segment intersects the circle.

Circular Continuity Principle: If a circle has one point inside and one point outside another circle, then the two circles intersect in two points.

Birkhoff's Axioms

B1. There exist nonempty subsets of the plane called lines, with the property that each two points belong to exactly one line.

B2. Corresponding to any two points A and B in the plane there exists a unique real number d(AB) = d(BA), the distance from A to B, which is 0 if and only if A = B.

Birkhoff's Axioms

B3. (Birkhoff Ruler Axiom) If k is a line and **R** denotes the set of real numbers, there exists a one-to-one correspondence (X->x) between the points X in k and the numbers x in **R** such that d(A,B) = |a-b| where A->a and B->b.

B4. For each line k there are exactly two nonempty convex sets R′ and R″ satisfying

a) $R' \cup k \cup R''$ is the entire plane,

b) $R' \cap R'' = \emptyset$, $R' \cap k = \emptyset$, and $R'' \cap k = \emptyset$

c) If X in R′ and Y in R″ then $XY \cap k \neq \emptyset$

Birkhoff's Axioms

B5. For each angle $\angle ABC$ there exists a unique real number x with $0 \le x \le 180$ which is the (degree) measure of the angle $x = \angle ABC°$.

B6. If ray $\overrightarrow{BD}$ lies in $\angle ABC$, then

$$\angle ABD° + \angle DBC° = \angle ABC°.$$

Birkhoff's Axioms

B7. If $\overrightarrow{AB}$ is a ray in the edge, k, of an open half plane H(k;P) then there exist a one-to-one correspondence between the open rays in H(k;P) emanating from A and the set of real numbers between 0 and 180 so that if $\overrightarrow{AX} \longrightarrow x$ then $\angle BAX° = x$.

Birkhoff's Axioms

B8. (SAS) If a correspondence of two triangles, or a triangle with itself, is such that two sides and the angle between them are respectively congruent to the corresponding two sides and the angle between them, the correspondence is a congruence of triangles.

SMSG Axioms

3 undefined terms: point, line, plane

22 axioms – a mixture of Birkhoff's and Hilbert's meant to make it easier to teach geometry in high school and college.