Congruency and Congruent Triangles

One of our
Basic Building Blocks

THEOREM

Theorem 2: Let l and m be distinct lines and let t be a transversal. The following are equivalent. (TFAE)

(1) l and m are parallel.

(2) Any two corresponding angles are congruent.

(3) Any two alternate interior angles are congruent.

(4) Any two alternate exterior angles are congruent.

(5) Any two same side interior angles are supplementary.

Proof

We will show that $1 \Rightarrow 3$ and $3 \Rightarrow 1$.

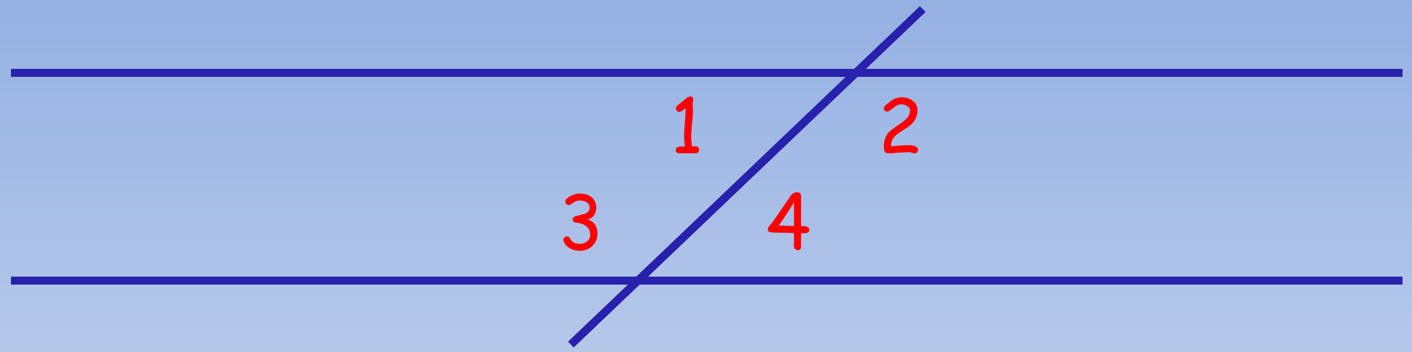
To complete this proof we could show either: $1 \Leftrightarrow 2$, $1 \Leftrightarrow 3$, $1 \Leftrightarrow 4$, $1 \Leftrightarrow 5$

OR: $1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4 \Leftrightarrow 5$

OR: $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 1$

OR: $1 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$

Proof: $1 \Rightarrow 3$



(Set up proof by contradiction).

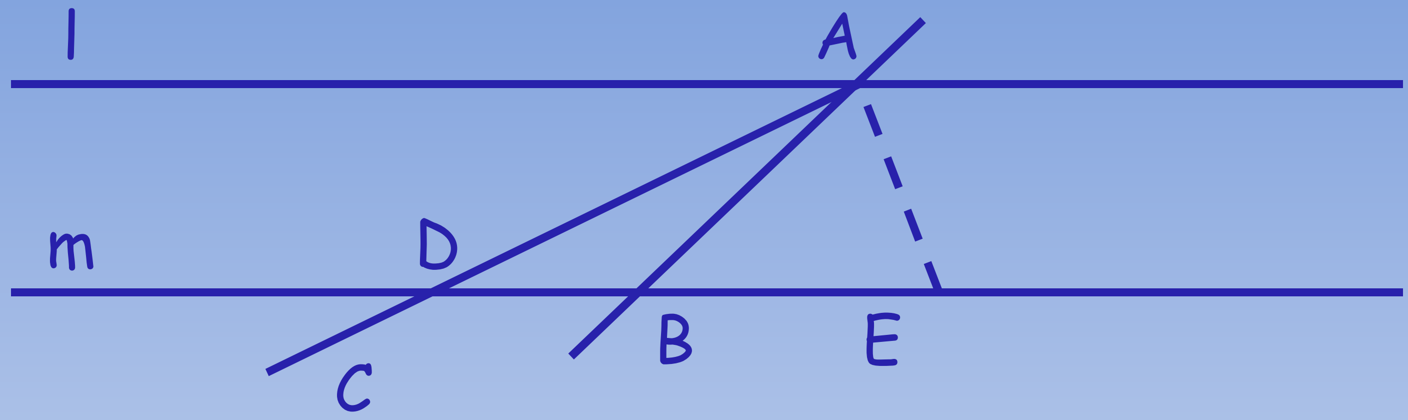
Assume l || m and $\angle 1 \neq \angle 4$.

We know

$$m\angle 1 + m\angle 2 = 180$$

and

$$m\angle 3 + m\angle 4 = 180.$$

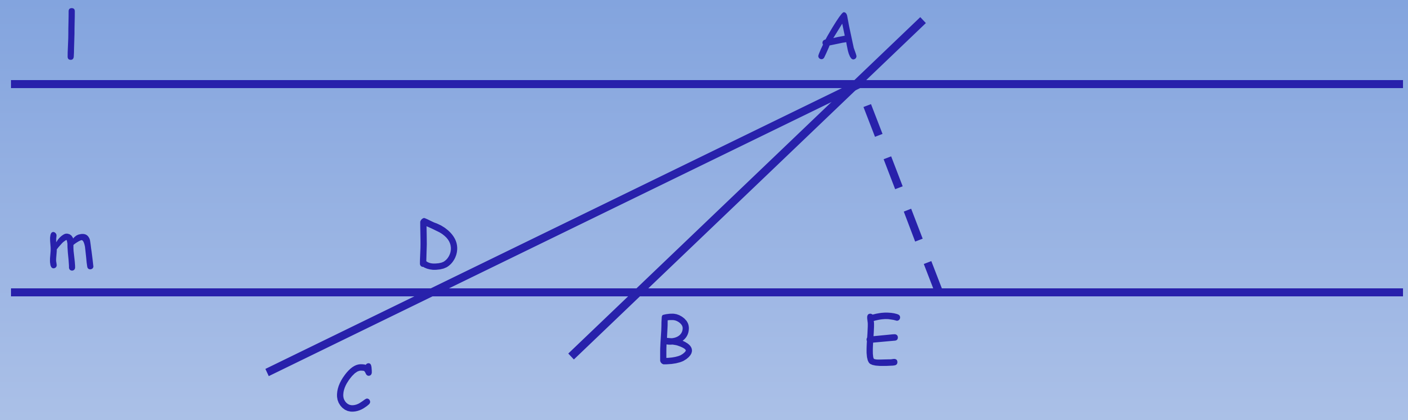


Since $\angle 1 \neq \angle 4$, we may assume $m\angle 1 > m\angle 4$. (Why?)

$\exists$ ray AC on opposite side of t from $\angle 4$ so that $m\angle CAB = m\angle ABE$.

Let $m \cap AC = D$.

$\exists$ E in m on opposite side of t from D so that AD = BE.



Then, AD = BE, $m\angle DAB = m\angle EBA$, and AB=AB.

Therefore by SAS $\Delta DAB \cong \Delta EBA$

$\Rightarrow m\angle DBA = m\angle 3 = m\angle BAE$

$\Rightarrow \angle DAB$ and $\angle BAE$ form linear pair

$\Rightarrow$ A, D, E collinear

$\Rightarrow$ A lies on m

$\Rightarrow$ l and m not parallel.

Thus we now have that

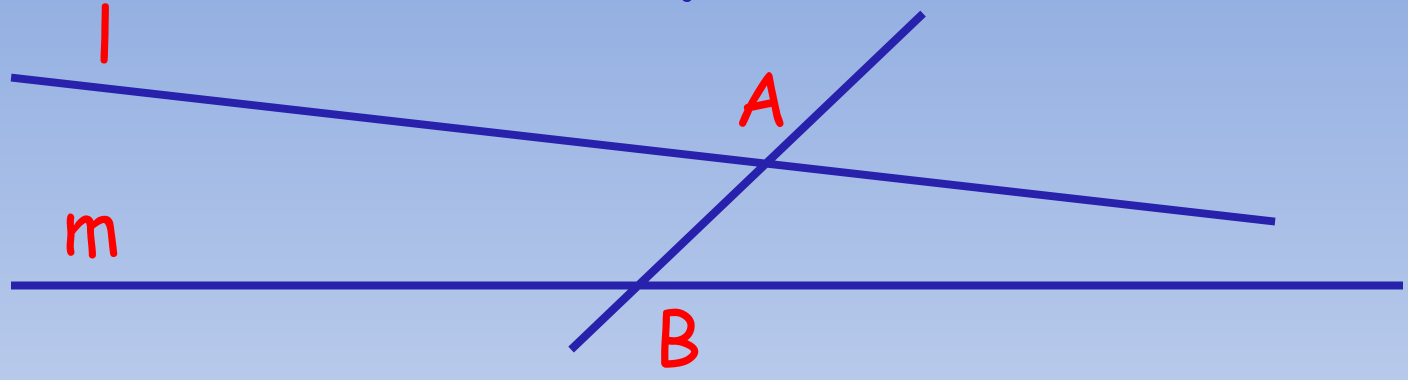
l and m not parallel

AND we are given that

l and m are parallel.

In other words we have $R \wedge \sim R$, a contradiction. Thus, $1 \wedge \sim 3$ leads to a contradiction so $1 \Rightarrow 3$

Proof: $3 \Rightarrow 1$

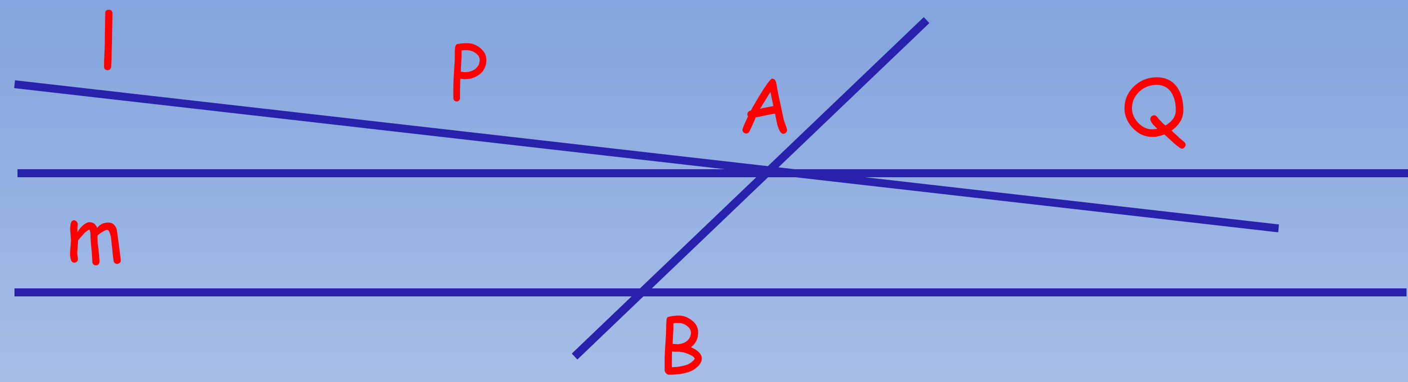


(Proof by Contrapositive).

[Assume ~1 and we need to deduce ~3.]

Assume l and m not parallel.

Let A and B be intersection of t with l and m.



P,Q on l so that P * A * Q

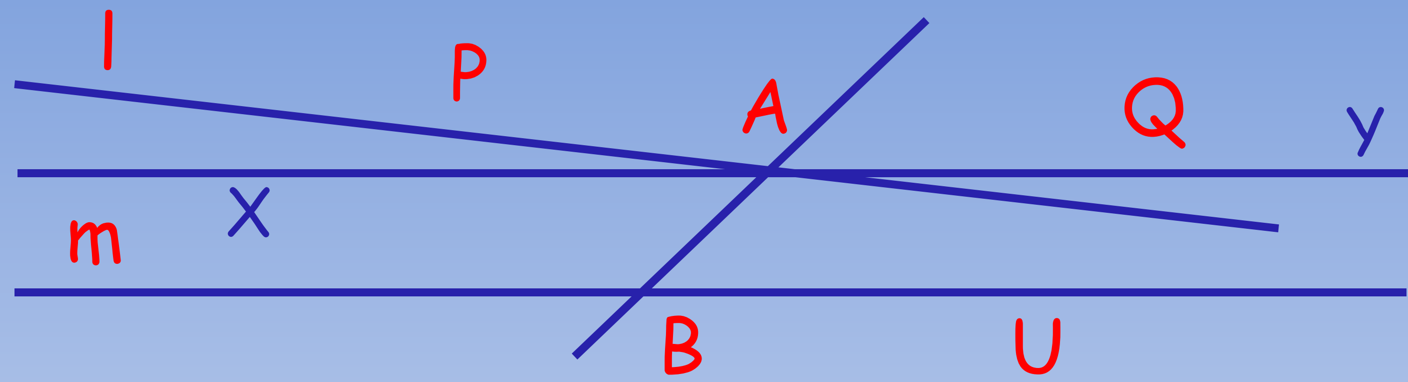
Let R be on m on same side of t as P

Let S be on m on same side of t as Q

$\exists$ line n through A parallel to m

Choose X,Y on n so that X * A * Y

$n \neq l$, so we may assume n is interior to $\angle PAB$.



Thus, $m\angle PAX > 0$.

By first part of proof,

$m\angle BAX = m\angle UBA = m\angle 4$

Thus,

$m\angle 1 = m\angle PAB = m\angle PAX + m\angle XAB >$
$m\angle UBA = m\angle 4$.

Thus, $m\angle 1 \neq m\angle 4$

Thus we now have that $\sim 1 \Rightarrow \sim 3$ which is logically equivalent to $3 \Rightarrow 1$.

Functional Definition

Two triangles $\Delta ABC \cong \Delta DEF$ are congruent if there is a one-to-one mapping of the plane to itself that sends ΔABC to ΔDEF and $f(A)=D$, $f(B)=E$, and $f(C)=F$.

In this case we say that A and D are corresponding points, as are B and E and C and F.

Functional Definition

Note that $\Delta ABC \cong \Delta DEF$ means:

$A \leftrightarrow D$ $B \leftrightarrow E$ $C \leftrightarrow F$

SO

$AB \leftrightarrow DE$ $AC \leftrightarrow DF$ $BC \leftrightarrow EF$

or ($AB \cong DE$, $AC \cong DF$, $BC \cong EF$)

AND

$\angle ABC \cong \angle DEF$, $\angle ACB \cong \angle DFE$, $\angle BAC \cong \angle EDF$

CPCTC

Corresponding
Parts of
Congruent
Triangles are
Congruent

What Else Does Congruence Imply?

If $\Delta ABC \cong \Delta DEF$ then we have ***six equalities***.

$AB = DE$

$AC = DF$

$BC = EF$

$m\angle ABC = m\angle DEF$

$m\angle BCA = m\angle EFD$

$m\angle CAB = m\angle FDE$

Reverse the Question

If we have ***six equalities***.

$AB = DE$

$AC = DF$

$BC = EF$

$m\angle ABC = m\angle DEF$

$m\angle BCA = m\angle EFD$

$m\angle CAB = m\angle FDE$

is $\Delta ABC \cong \Delta DEF$?

Fewer Equalities

Can we get by with fewer equalities?

What is the fewest number of equalities we can have and still guarantee congruence ?

Known Cases

Three sides (SSS)

Two sides and an angle (SAS)

Two angles and a side (ASA)

Two angles and a side (AAS)

Known Cases

Two sides and an angle (SSA) ?

Why not?

Three angles (AAA) ?

Why not?

Known Cases

Do two sides only ever determine a triangle ?

What about Hypotenuse-Leg ?

Verifiable?

How do we know that these cases that we have mentioned (SSS, SAS, AAS, ASA) are the basis for congruence?

Usual Approach

In most systems of axioms that we use for Geometry, we take the SAS result as an axiom and then prove the others from it and other propositions.

SSS Theorem

In ΔABC and ΔDEF if we have AB=DE, BC=EF and AC=DF, then $\Delta ABC \cong \Delta DEF$.

Proofs:

Isosceles Triangle Theorem

In ΔABC if $AB=AC$ then $\angle B = \angle C$.

Proofs: