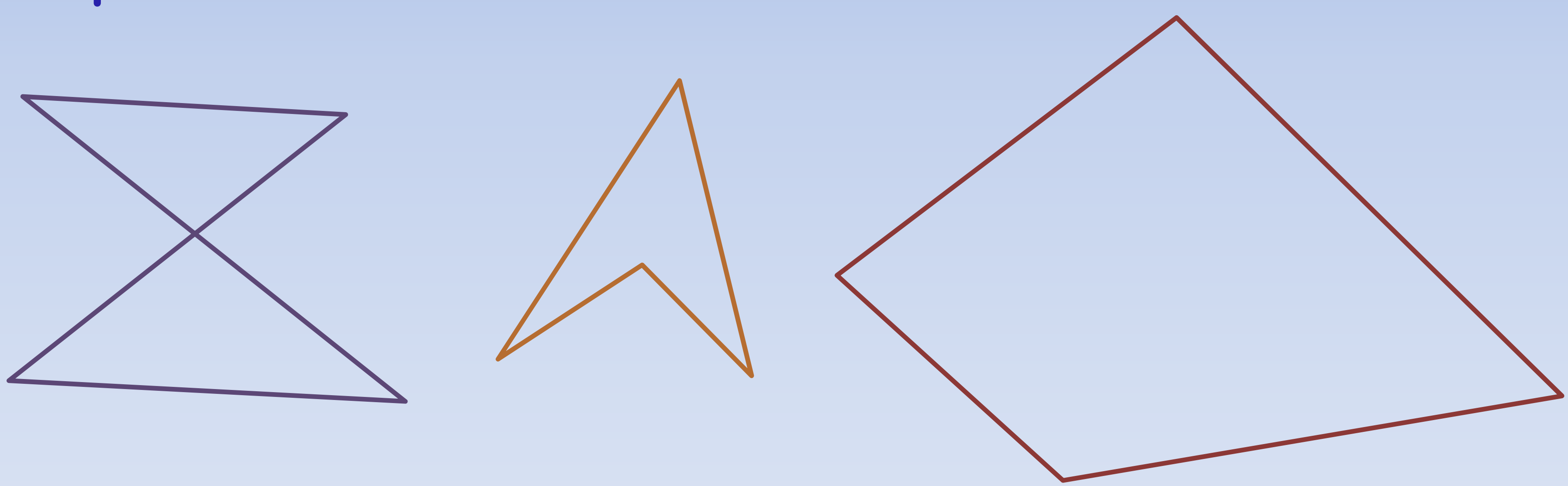
Parallelograms

MA 341 – Topics in Geometry
Lecture 05

Definitions

A quadrilateral is a polygon with 4 distinct sides and four vertices. Is there a more precise definition?



P_1 P_2 P_3

Definitions

Quadrilateral

Quadrangle

Tetragon

These will all mean the same object: 4 line segments that intersect only at the four endpoints (vertices).

Important Types of Quadrilaterals

Rectangles

Squares

Parallelograms

Rhombus

Trapezoid (trapezium)

Kites

The definitions are important.

Rectangles

A quadrilateral is a rectangle if …

it contains 4 right angles.

What is the difference between the definition of an object and properties that an object has?

Squares

A quadrilateral is a <u>square</u> if …

it is a rectangle with all sides congruent.

it is a rectangle with adjacent sides congruent.

Are these the same thing?

Parallelograms

A quadrilateral is a <u>parallelogram</u> if …

opposite sides are parallel.

Is a rectangle a parallelogram?

Rhombi or Rhombuses

A quadrilateral is a rhombus if …

all four sides have the same length.

Is a square a rhombus?

Trapezoid

A quadrilateral is a <u>trapezoid</u> if …

contains one pair of parallel sides.

Is a parallelogram a trapezoid?

Kites

A quadrilateral is a kite if …

the four sides can be grouped into two pairs of equal-length sides that are next to each other.

Kites

Theorems

Theorem: Opposite sides of a parallelogram are congruent.

Proof

Is the converse to this statement true?

Theorems

Theorem: In quadrilateral □ABCD if AB≅CD and AD≅BC, then ABCD is a parallelogram.

Proof: Homework Problem #1

Theorems

Theorem: A quadrilateral is a parallelogram if and only if its diagonals bisect one another.

Proof:
(1) If ABCD is a parallelogram then AC and BD bisect each other.

Theorems

Proof:

(2) In ABCD if AC and BD bisect each other then ABCD is a parallelogram.

Properties of Parallelograms

- Opposite sides of a parallelogram are equal in length.
- Opposite angles of a parallelogram are equal in measure.
- The area of a parallelogram is the product of the base the height.
- Opposite sides of a parallelogram will never intersect.
- The area is twice the area of a triangle created by one of its diagonals.
- The area equals to the magnitude of the vector cross product of two adjacent sides.

Properties of Parallelograms

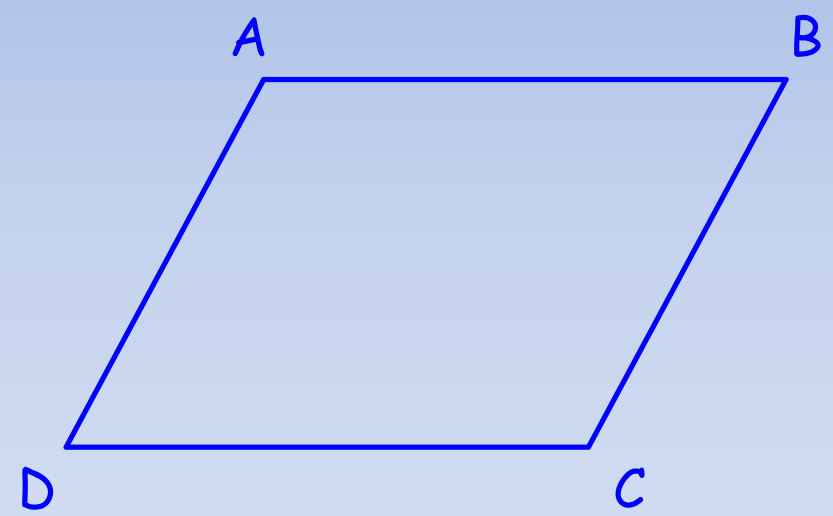
- The diagonals of a parallelogram bisect each other.
- Any non-degenerate affine transformation takes a parallelogram to another parallelogram.
- A parallelogram has rotational symmetry of order 2 (through 180°). If it also has two lines of reflectional symmetry then it must be a rhombus or a rectangle.
- The perimeter of a parallelogram is 2(a + b) where a and b are the lengths of adjacent sides.

Properties of Parallelograms

- Consecutive angles of a parallelogram are supplementary.

Interesting Facts - Parallelograms

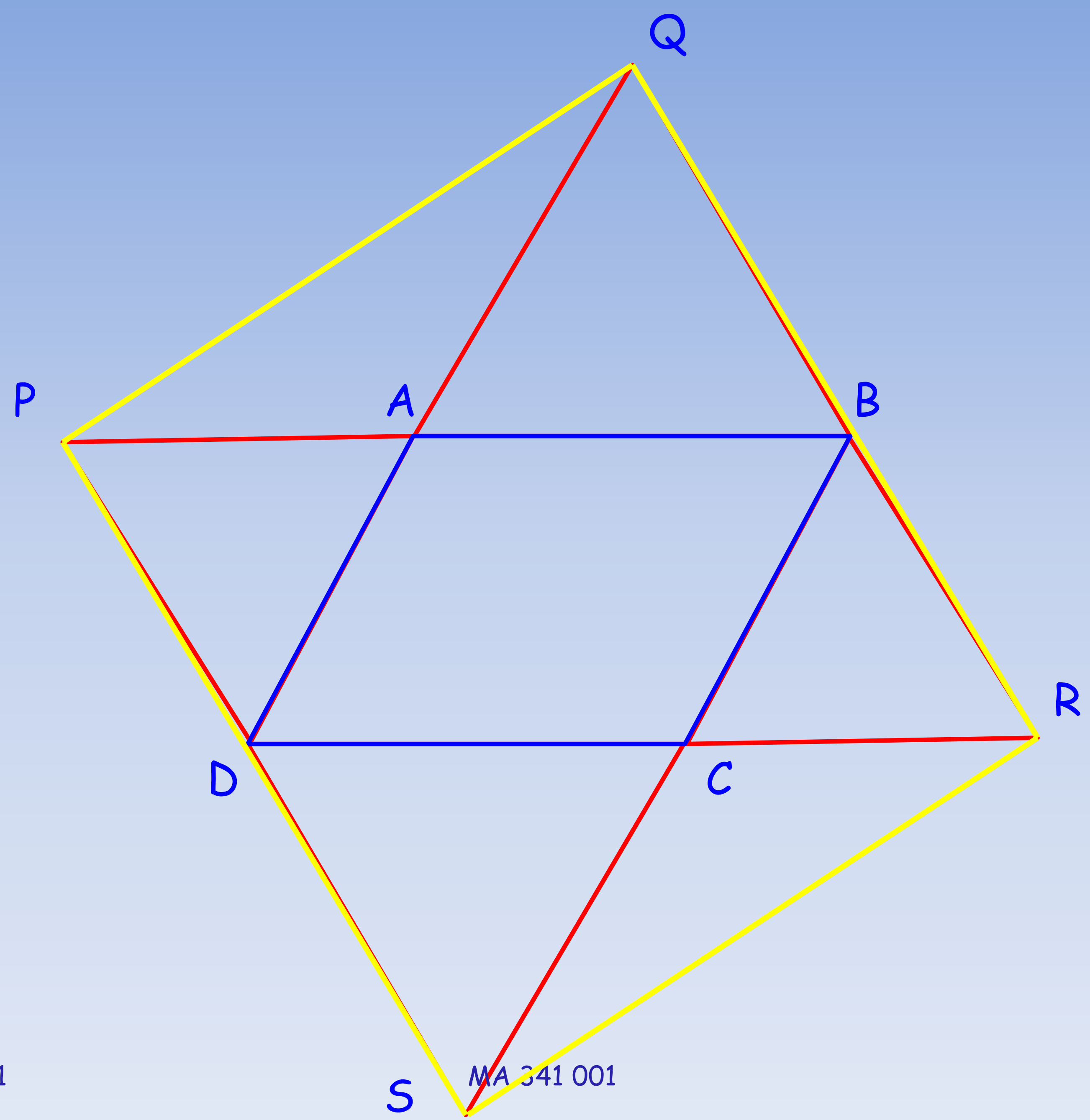
- Start with a parallelogram, ABCD.



Interesting Facts - Parallelograms

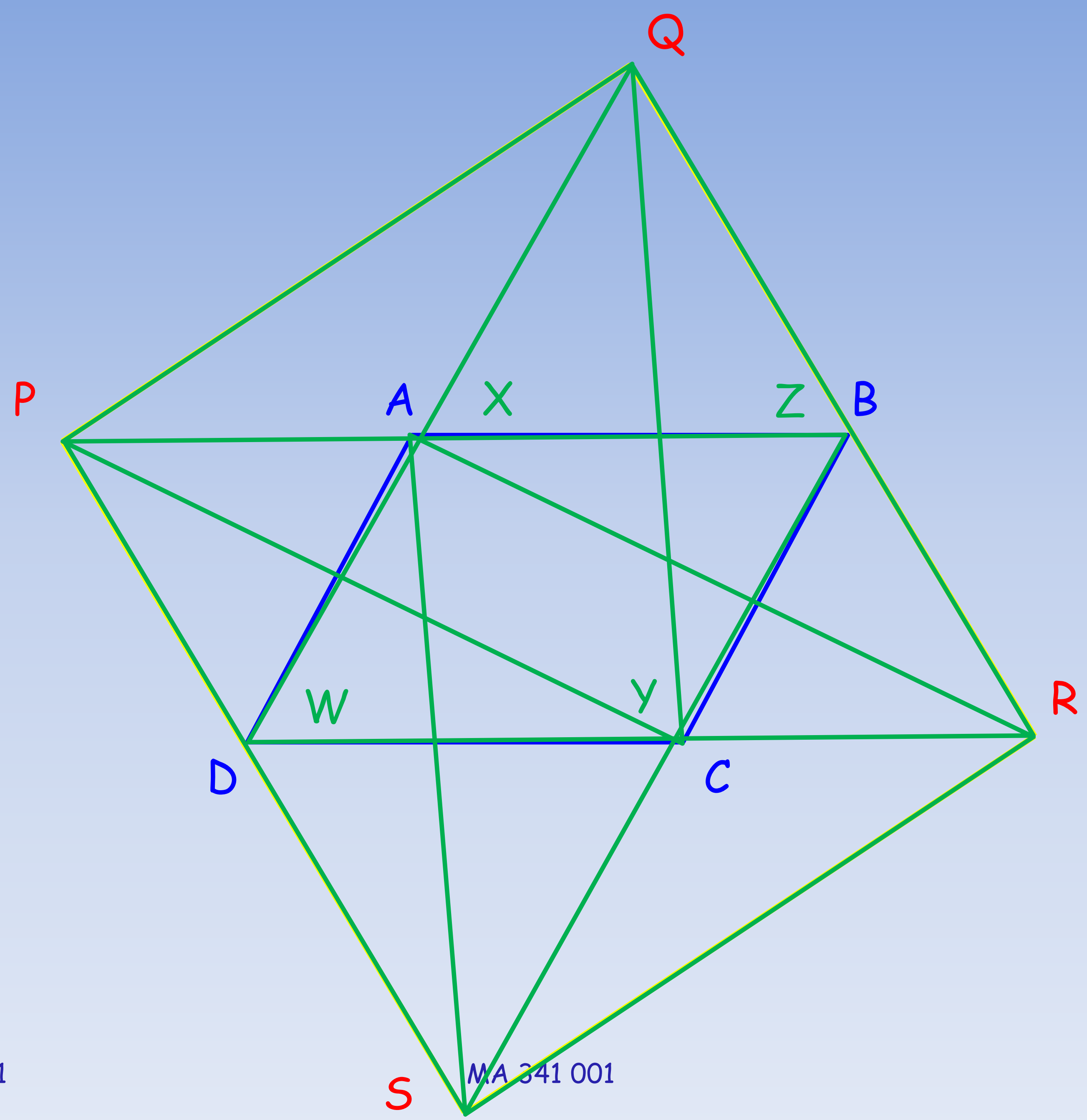
- Construct an equilateral triangle on the outside of each side of the parallelogram.

- The vertices, PQRS, form a parallelogram.


Q
P
A
B
R
D
C
S

Interesting Facts - Parallelograms

- Construct an equilateral triangle on the inside of each side of the parallelogram PQRS.


Q
P
A
X
Z
B
W
Y
R
D
C
S

Interesting Facts - Parallelograms

- XZYW is a parallelogram.

- Is XZYW = ABCD or is it coincidence?

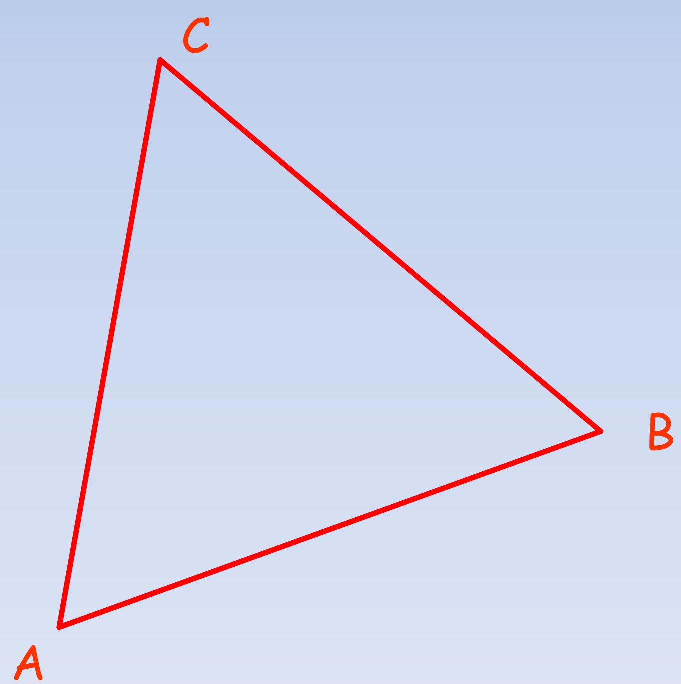
Interesting Facts - Parallelograms

- We can write P in terms of A and D.
- In fact, we can write the third vertex of an equilateral triangle in terms of the other two vertices – using complex numbers!

Aside-Equilateral Triangles

In the plane of ΔABC we can choose a Cartesian coordinate system so that we can identify points in the plane with complex numbers – (a,b) <-> a + bi.

Let T denote the rotation through 120° in the positive (counterclockwise) direction. This means that $T^3 = 1$, or, since $T \neq 1$,



Aside-Equilateral Triangles

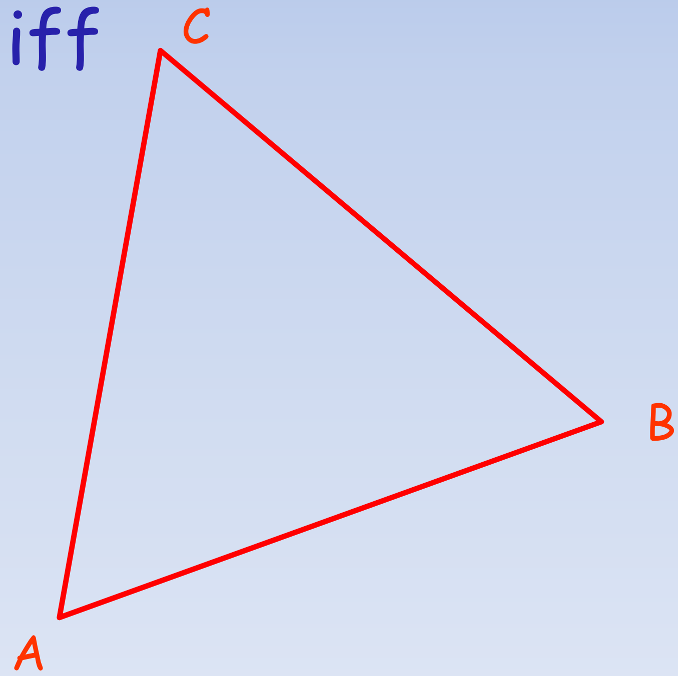
$T^3 - 1 = 0$

$(T - 1)(T^2 + T + 1) = 0$ and since $T \neq 1$,

$T^2 + T + 1 = 0$

Theorem: ΔABC is equilateral iff

$$A + TB + T^2C = 0.$$



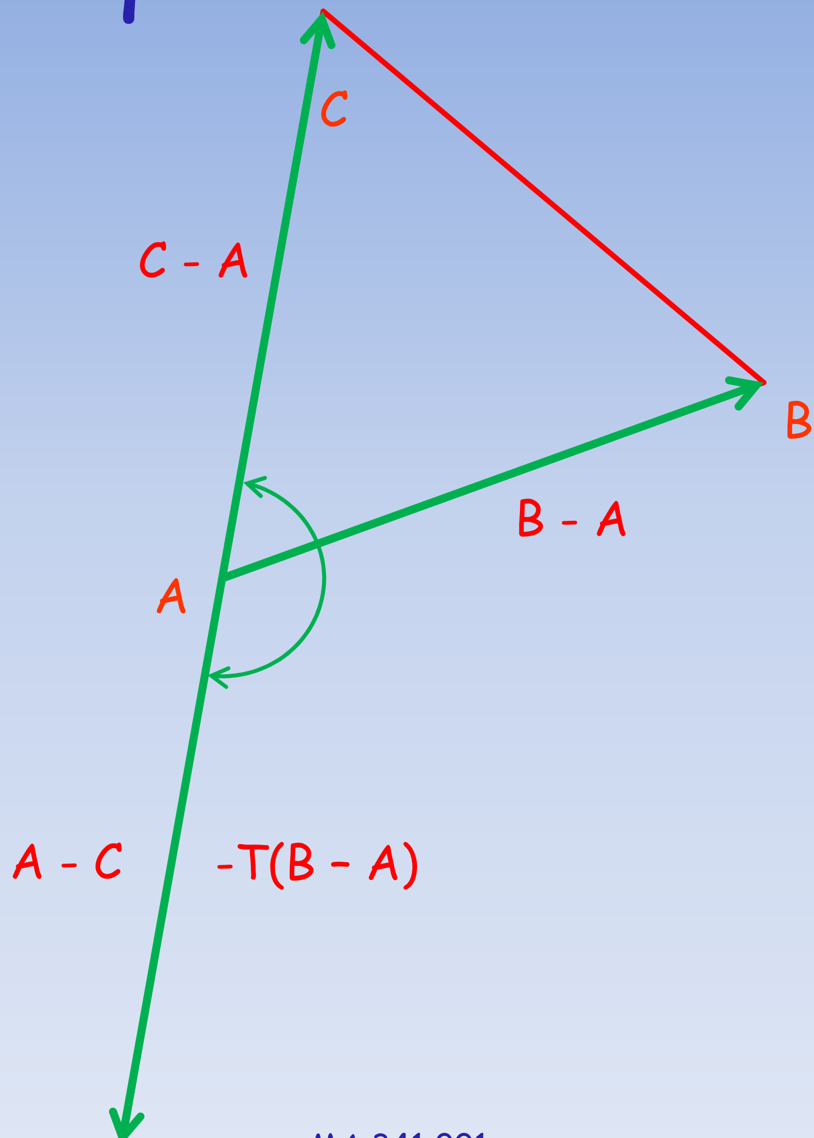
Aside-Equilateral Triangles

Proof:

ΔABC is equilateral iff each side could be obtained from another side by a rotation through 60^o around their common vertex.

A rotation through 120^o in the opposite direction, brings one side in the direction exactly opposite to the other making their sum equal to 0.

Aside-Equilateral Triangles

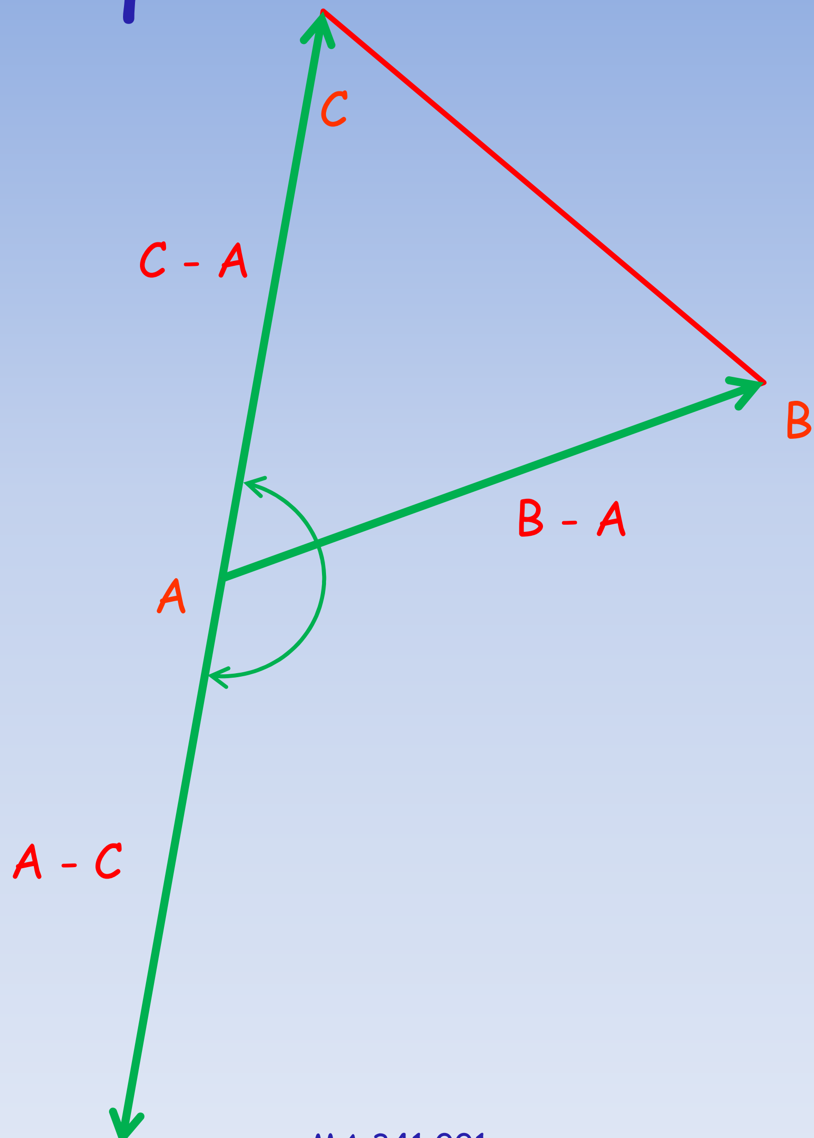


Aside-Equilateral Triangles

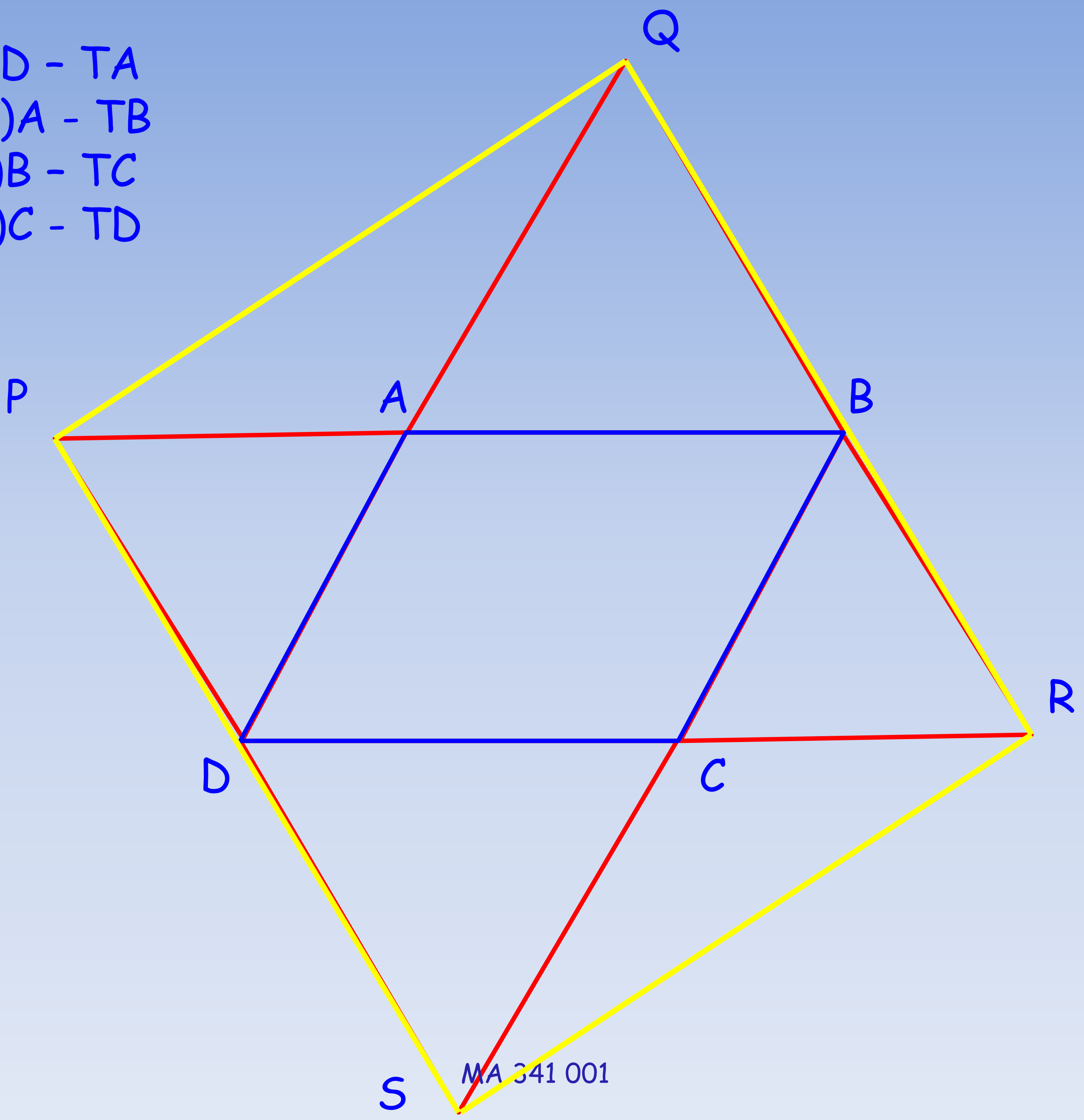
$T(B - A) + C - A = 0$

$C + TB - (1+T)A = 0$

$C = (1+T)A - TB$

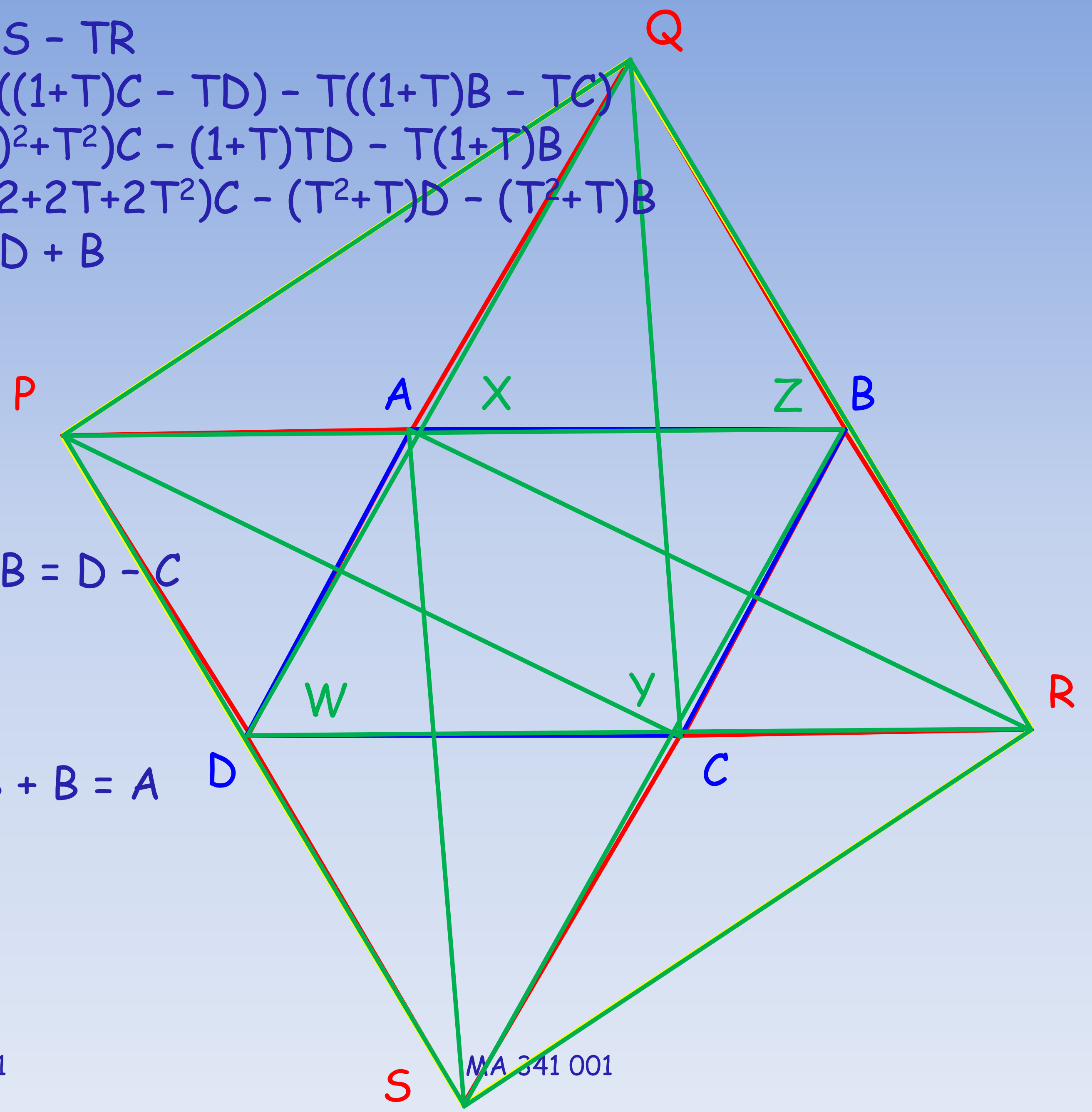


$P = (1+T)D - TA$
$Q = (1+T)A - TB$
$R = (1+T)B - TC$
$S = (1+T)C - TD$



$T^2 + T + 1 = 0$

$$\begin{aligned} X &= (1+T)S - TR \\ &= (1+T)((1+T)C - TD) - T((1+T)B - TC) \\ &= ((1+T)^2+T^2)C - (1+T)TD - T(1+T)B \\ &= (-1 + 2+2T+2T^2)C - (T^2+T)D - (T^2+T)B \\ &= -C + D + B \end{aligned}$$



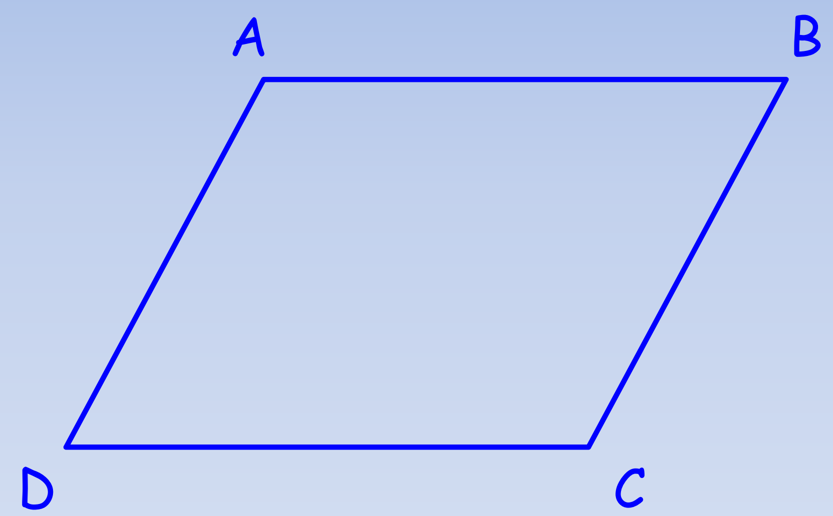
BUT $A - B = D - C$

So

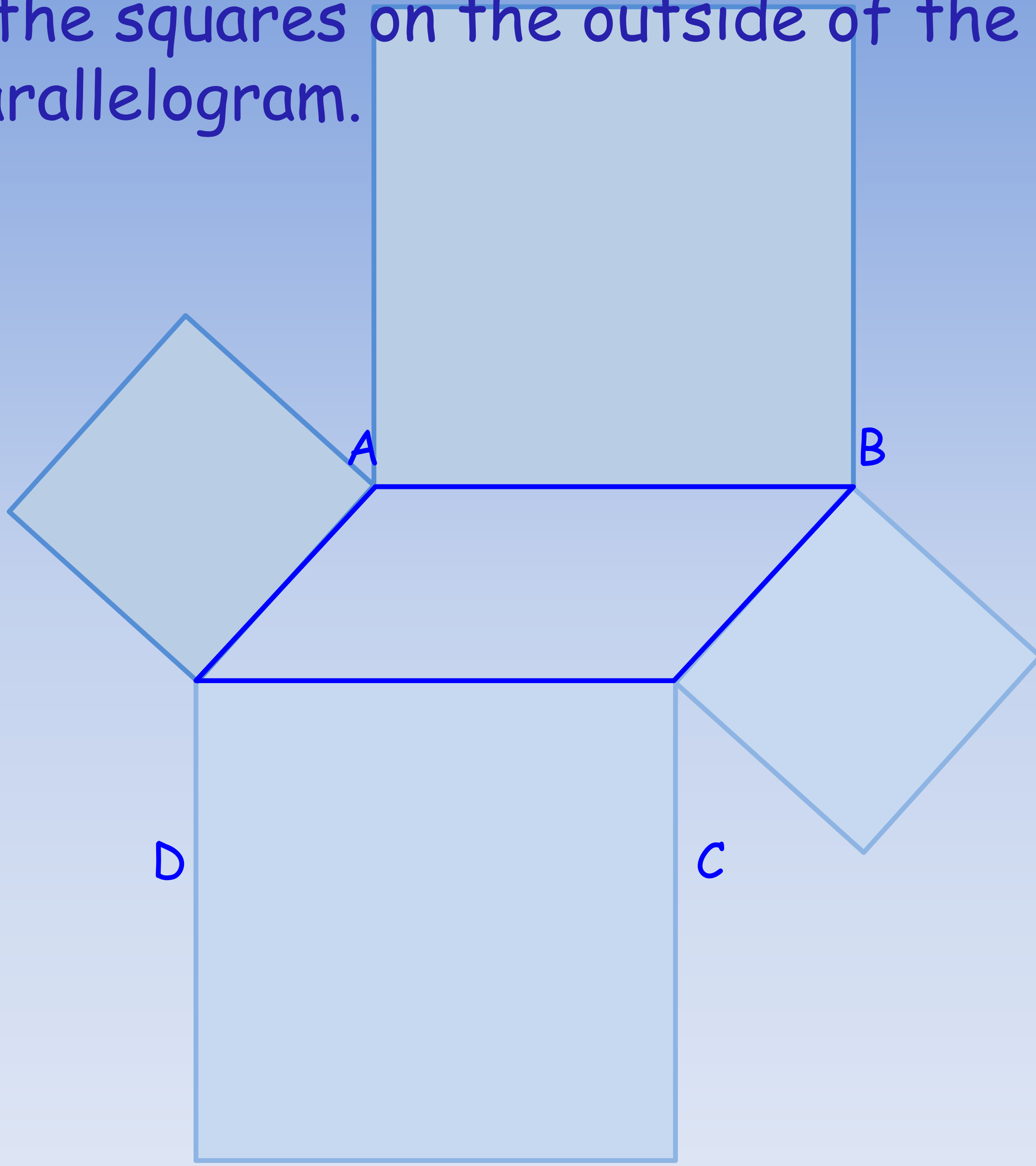
$X = A - B + B = A$

Interesting Facts - Parallelograms

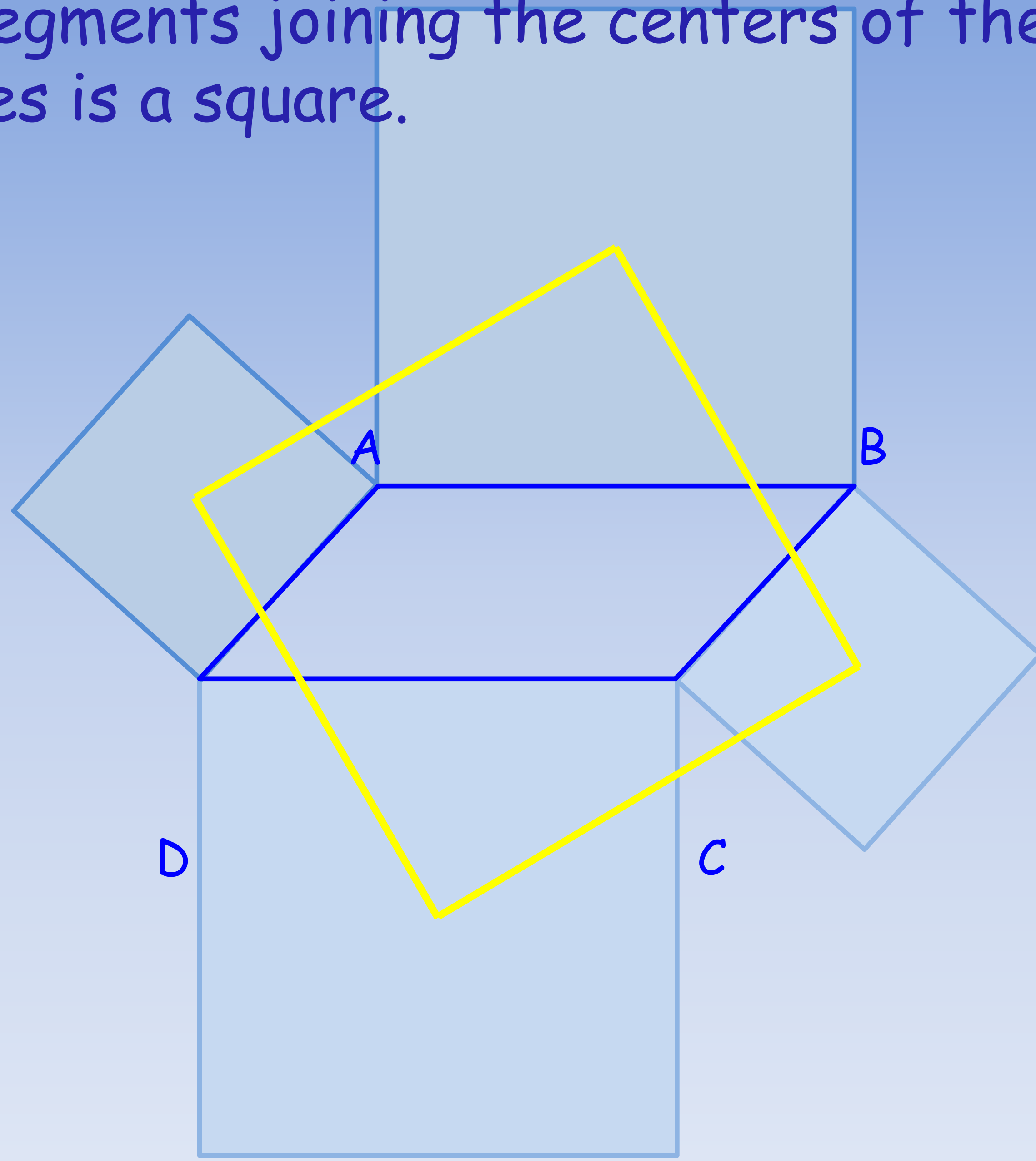
- Start with a parallelogram, ABCD.



- Draw the squares on the outside of the sides of the parallelogram.



- The segments joining the centers of these squares is a square.



Varignon's Theorem

- Let ABCD be a convex quadrilateral and X, Y, Z, and W the midpoints of the sides. XYZW is a parallelogram.