Parallelograms Continued

MA 341 – Topics in Geometry
Lecture 08

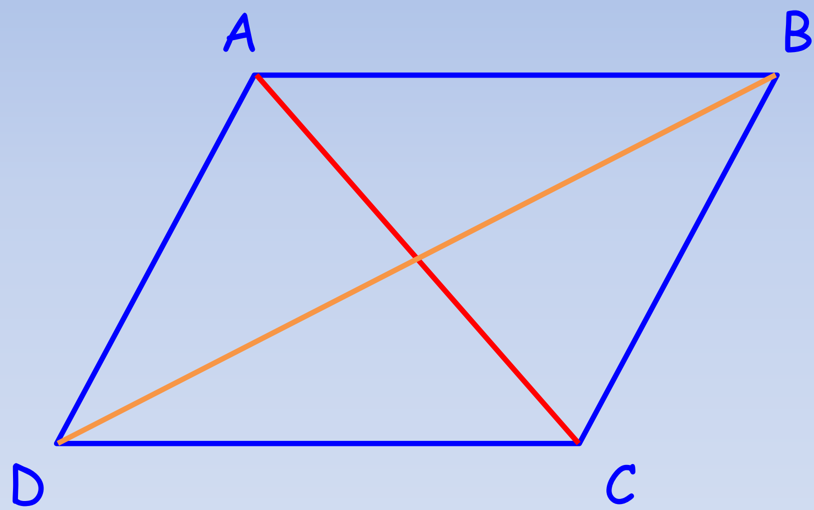
Parallelograms

Theorem: For every convex quadrilateral, ABCD, the following statements are equivalent.

a) ABCD is a parallelogram.
b) $\Delta ABC \cong \Delta CDA$ and $\Delta BAD \cong \Delta DCB$.
c) Pairs of opposite sides are congruent, i.e. $AB \cong CD$ and $AD \cong BC$.
d) Diagonals AC and BD bisect one another.
e) Pairs of opposite angles of ABCD are congruent, $\angle A \cong \angle C$ and $\angle B \cong \angle D$.

Parallelograms

- a) $\Rightarrow$ b) (DONE)



Parallelograms

b) $\Delta ABC \cong \Delta CDA$ and $\Delta BAD \cong \Delta DCB$.

$$\Rightarrow$$

c) Pairs of opposite sides are congruent, i.e. $AB \cong CD$ and $AD \cong BC$.

Parallelograms

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Parallelograms

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Parallelograms

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$$\Rightarrow$$

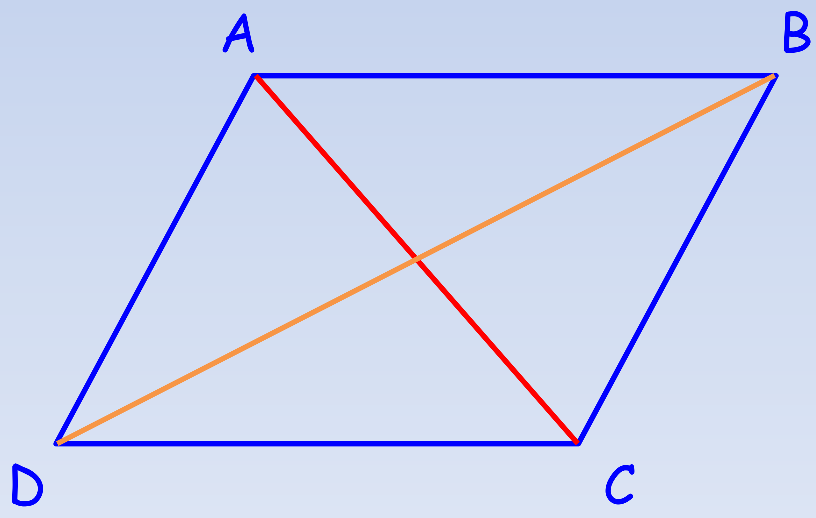
a) ABCD is a parallelogram.

Parallelograms

Theorem: ABCD is a parallelogram if and only if

$$AC^2 + BD^2 = AB^2 + BC^2 + CD^2 + DA^2.$$

The sum of the squares of the diagonals = sum of the squares of the sides.

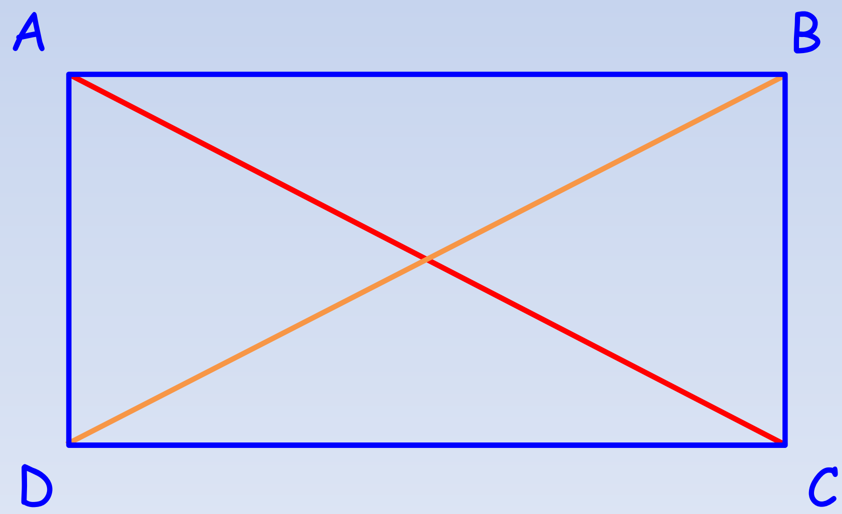


Comment

Why is this true for a rectangle?

$$AC^2 + BD^2 = AB^2 + BC^2 + CD^2 + DA^2.$$

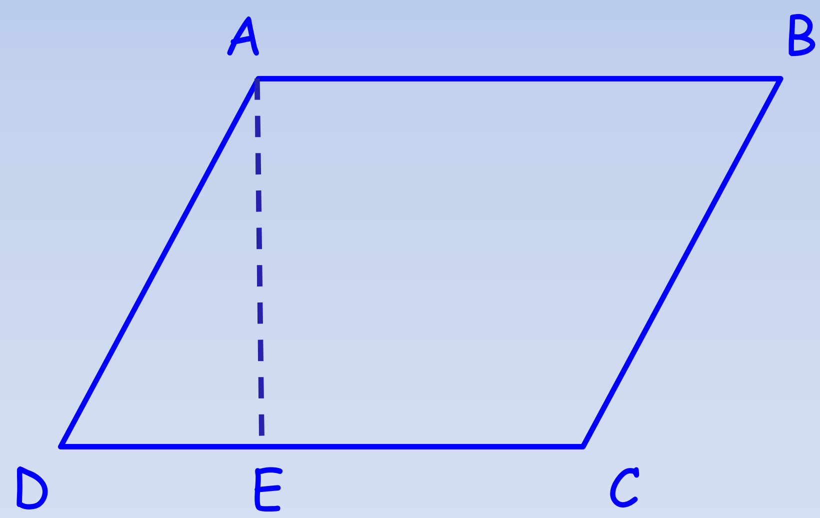
Would you expect it to be true for any parallelogram?



Proof

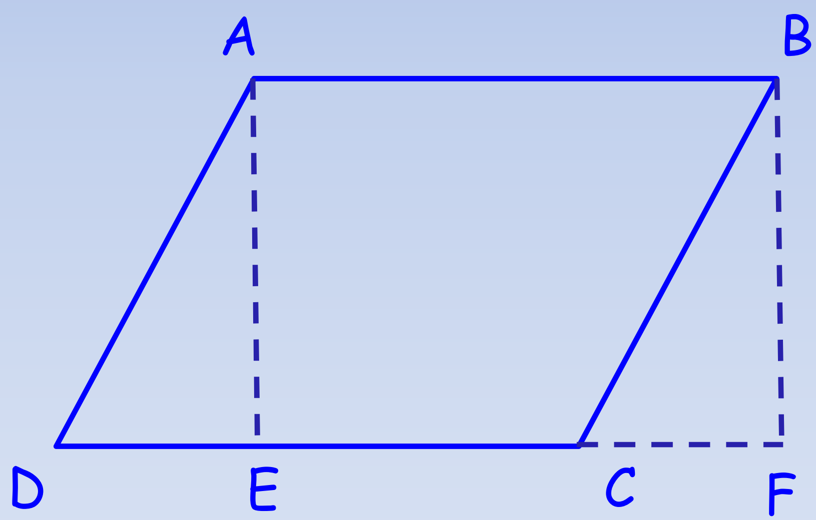
Assume ABCD is a parallelogram.

Drop a perpendicular from A to DC so that the foot of the perpendicular lies in the interior of DC.



Proof

Drop a perpendicular from B to DC so that the foot of the perpendicular lies in the exterior of DC.



Proof

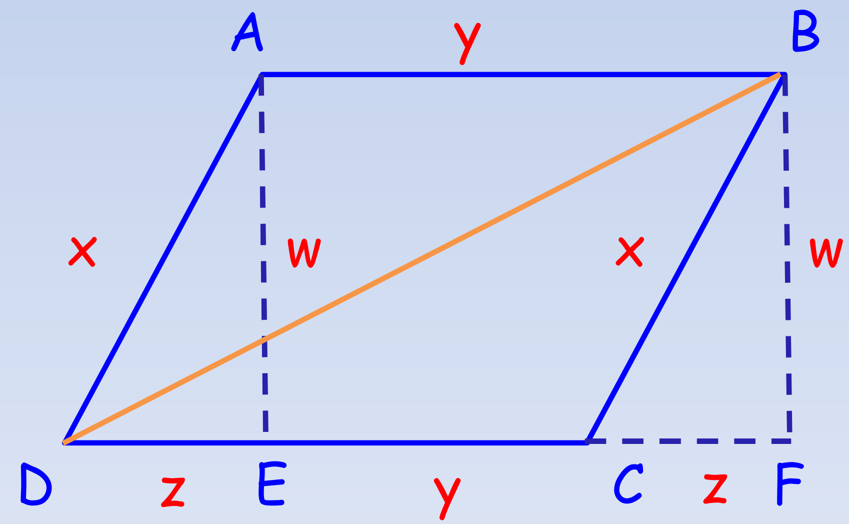
$\angle ADE \cong \angle BCF$, $AD \cong BC$, so $\Delta ADE \cong \angle BCF$.

Let $AD = BC = x$,

$AB = CD = y$,

$DE = CF = z$

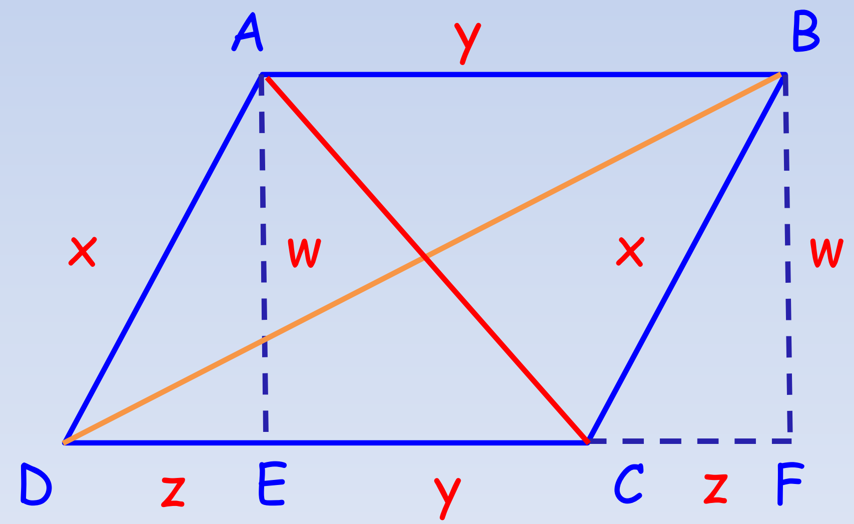
$AE = BF = w$



Proof

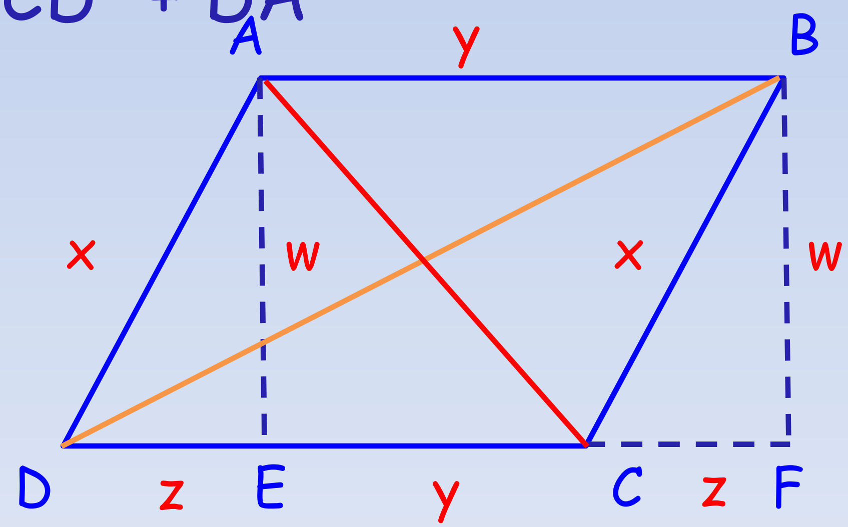
Pythagorean Theorem:

1) $w^2 = x^2 - z^2$
2) $BD^2 = (y + z)^2 + w^2$
3) $AC^2 = (y - z)^2 + w^2$



Proof

$$AC^2 + BD^2 = (y - z)^2 + w^2 + (y + z)^2 + w^2$$
$$= 2(y^2 + z^2 + w^2)$$
$$= 2(y^2 + z^2 + (x^2 - z^2))$$
$$= 2(x^2 + y^2)$$
$$= AB^2 + BC^2 + CD^2 + DA^2$$



Proof

($\Leftarrow$) We hold off on this until we have a neater way of proving it. We will not need it to prove anything else before then.

Area

- What type of units are used to describe area in the plane?

- What does this imply that area might be based on?

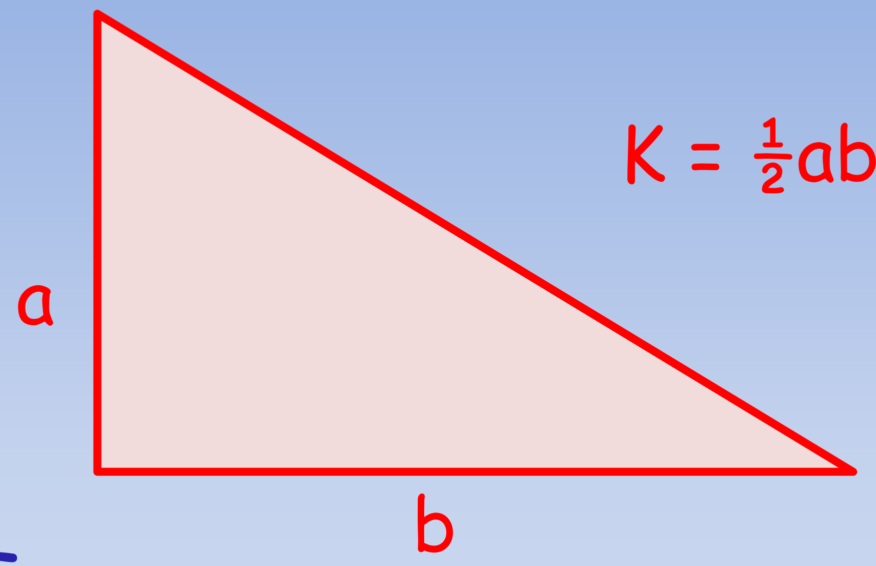
Choices

- Choice I:
  Define the area of a right triangle to be one-half the product of the two legs of the right triangle.

- Choice II:
  Define the area of a rectangle to be the product of the two adjacent sides of the rectangle.

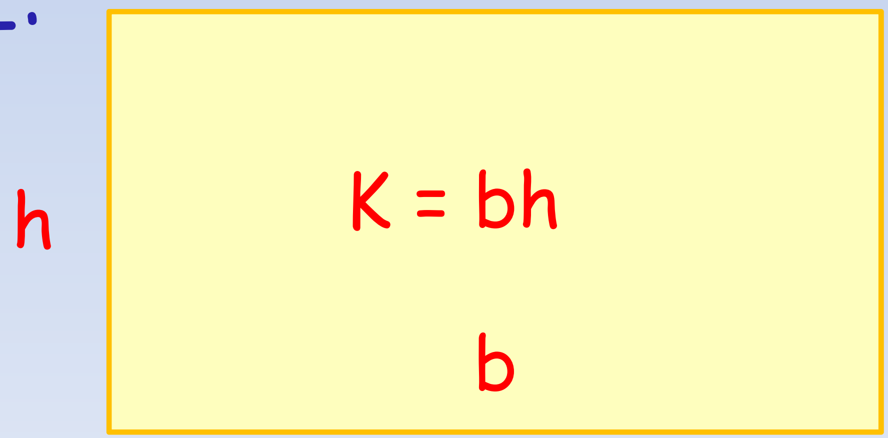
Choices

- Choice I:



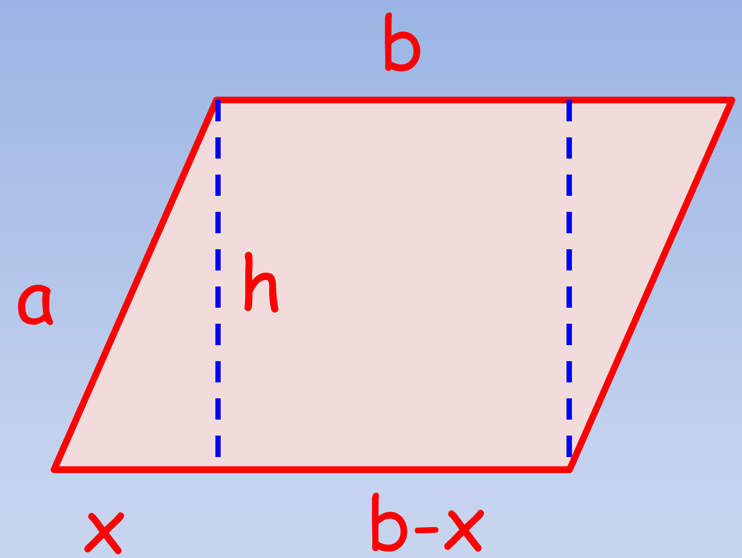
$K = \frac{1}{2}ab$

- Choice II:



Parallelogram

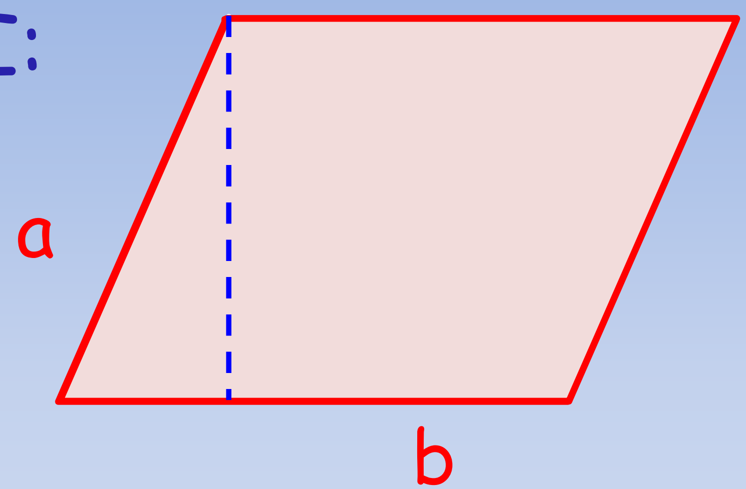
- Choice I:



$$K = \tfrac{1}{2}xh + h(b-x) + \tfrac{1}{2}xh$$
$$= xh + bh - xh$$
$$K = bh$$

Parallelogram

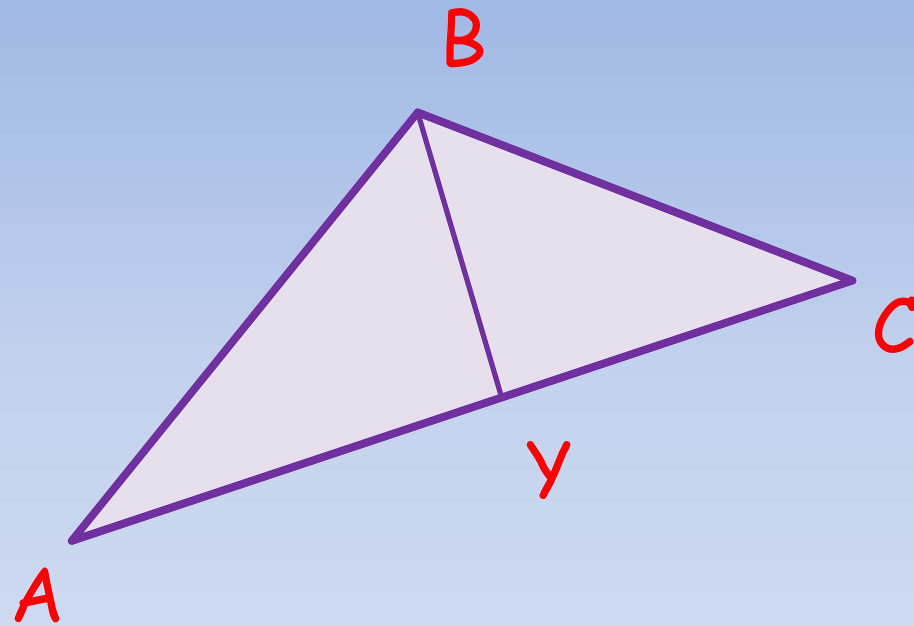
- Choice II:



$$K = bh$$

Arbitrary Triangle

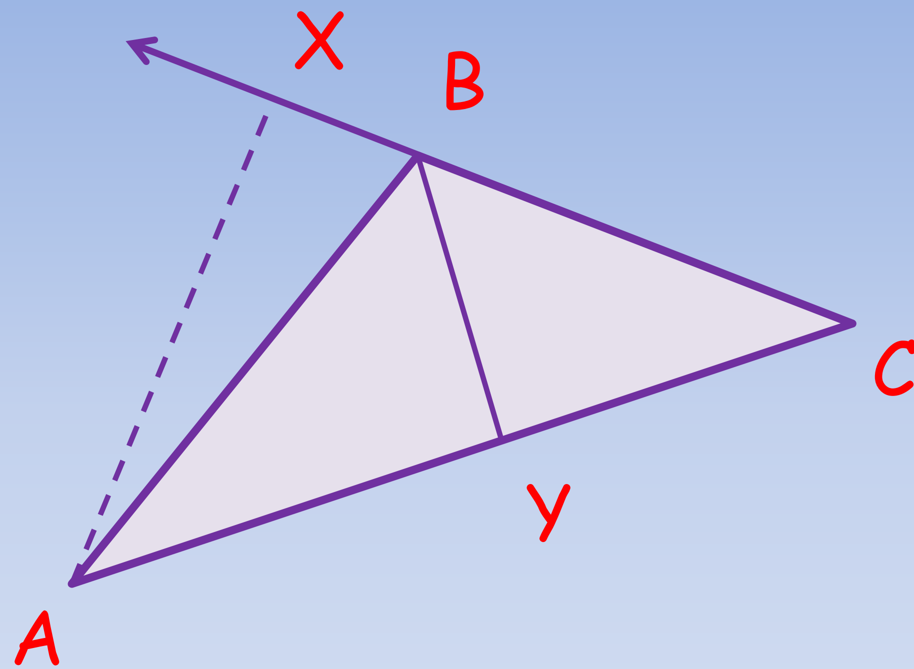
- Choice I:



$$K = \tfrac{1}{2}BY \cdot AY + \tfrac{1}{2}BY \cdot CY$$
$$K = \tfrac{1}{2}\, AC \cdot BY$$

Arbitrary Triangle

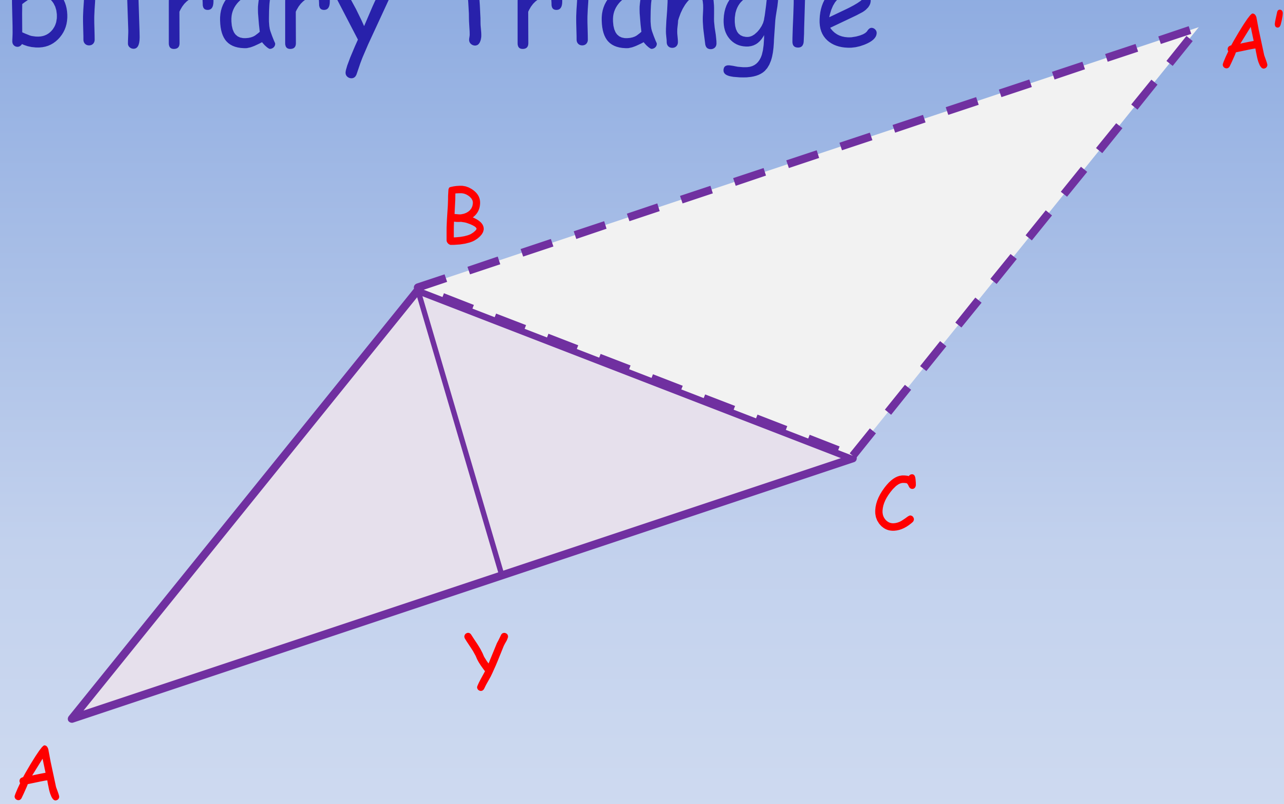
- Choice I:



Is $\frac{1}{2}$ AC•BY = $\frac{1}{2}$BC•AX ??!!?

Arbitrary Triangle

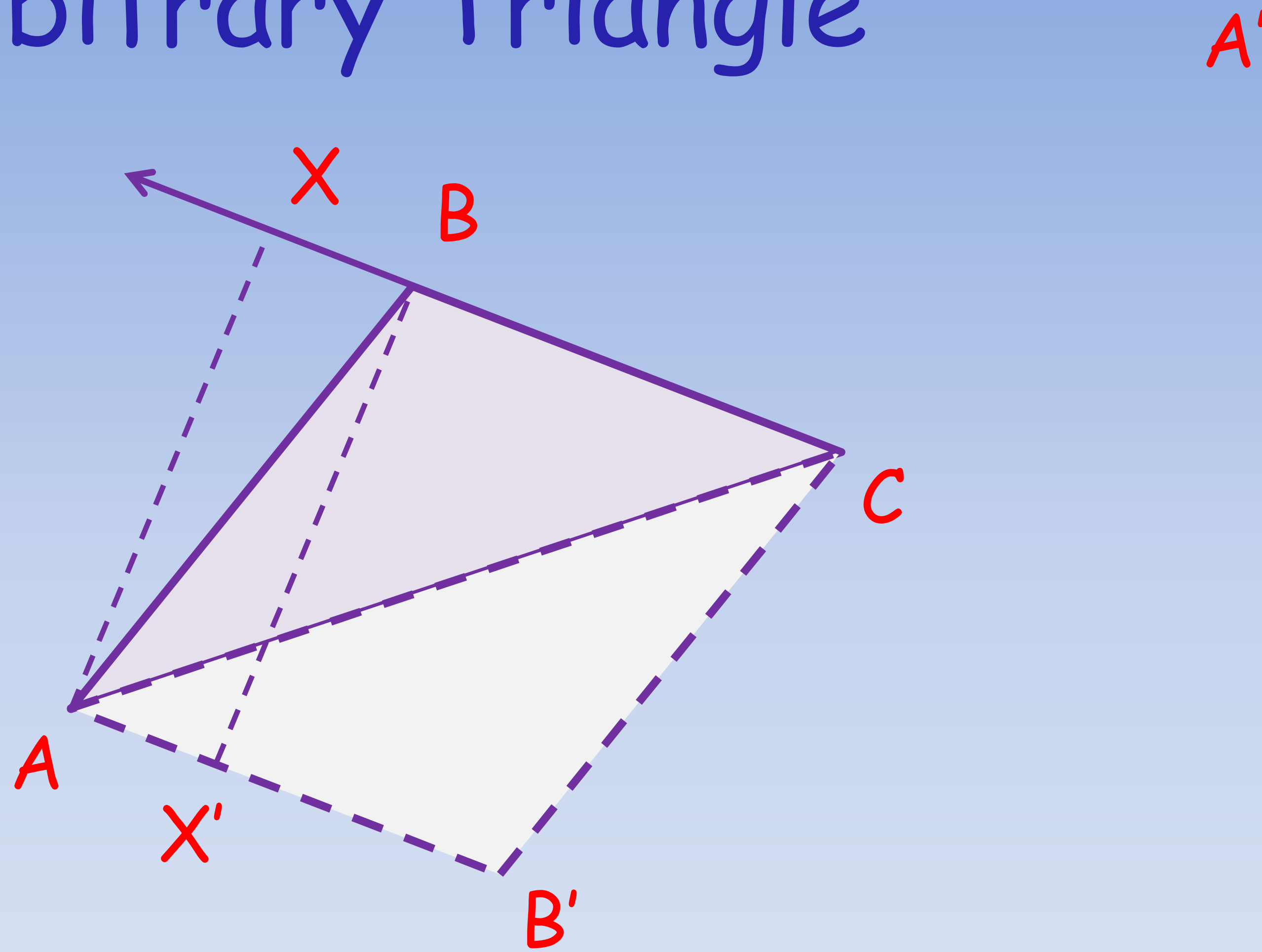
- Choice II:



$K = \frac{1}{2} AC \cdot BY$

Arbitrary Triangle

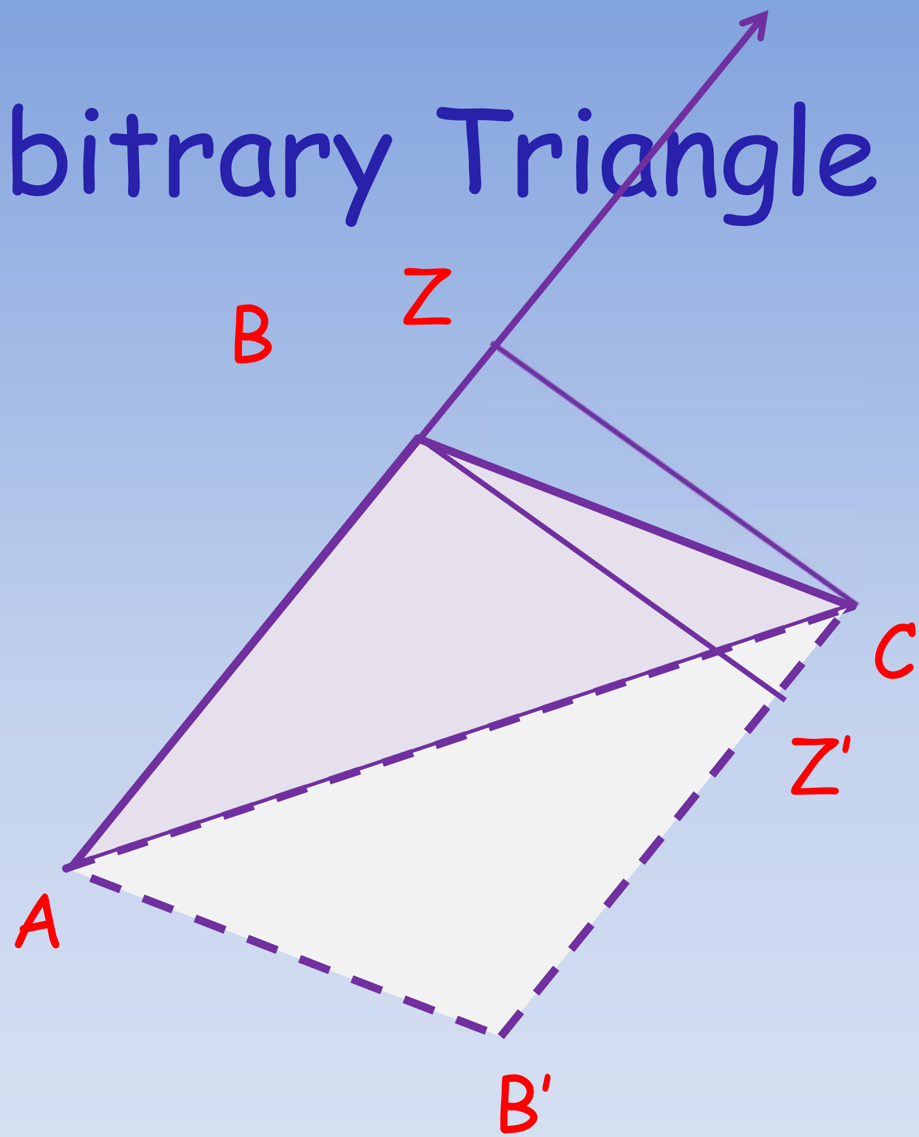
- Choice II:



K = ½ BC•BX′ = ½ BC•AX

Arbitrary Triangle

- Choice II:



$K = \frac{1}{2}\, AB \cdot BZ' = \frac{1}{2}\, AB \cdot CZ$

Comparison

- Don't all of those parallelograms have the same area?

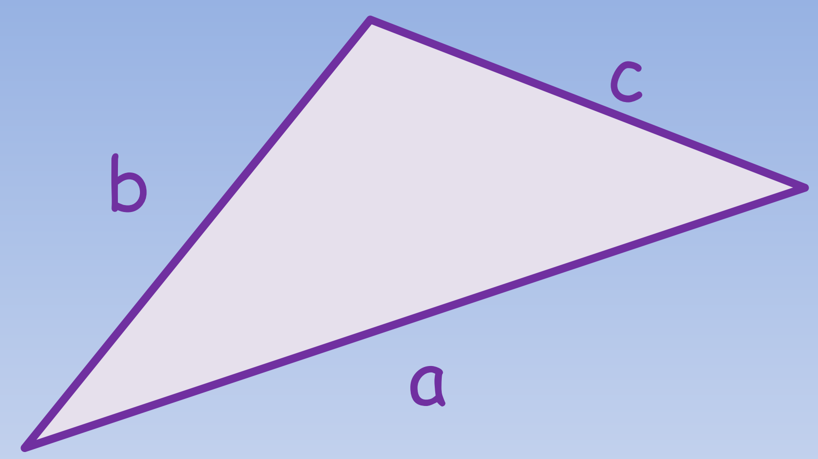
- We have then that:

$$K_{ABC} = \tfrac{1}{2}\, AX\, BC = \tfrac{1}{2}\, BY\, AC = \tfrac{1}{2}\, CZ\, AB$$

which finally shows it does not matter what you choose for the base of a triangle.

Heron's Formula

Let $s = \frac{1}{2}(a + b + c)$ be the semiperimeter.

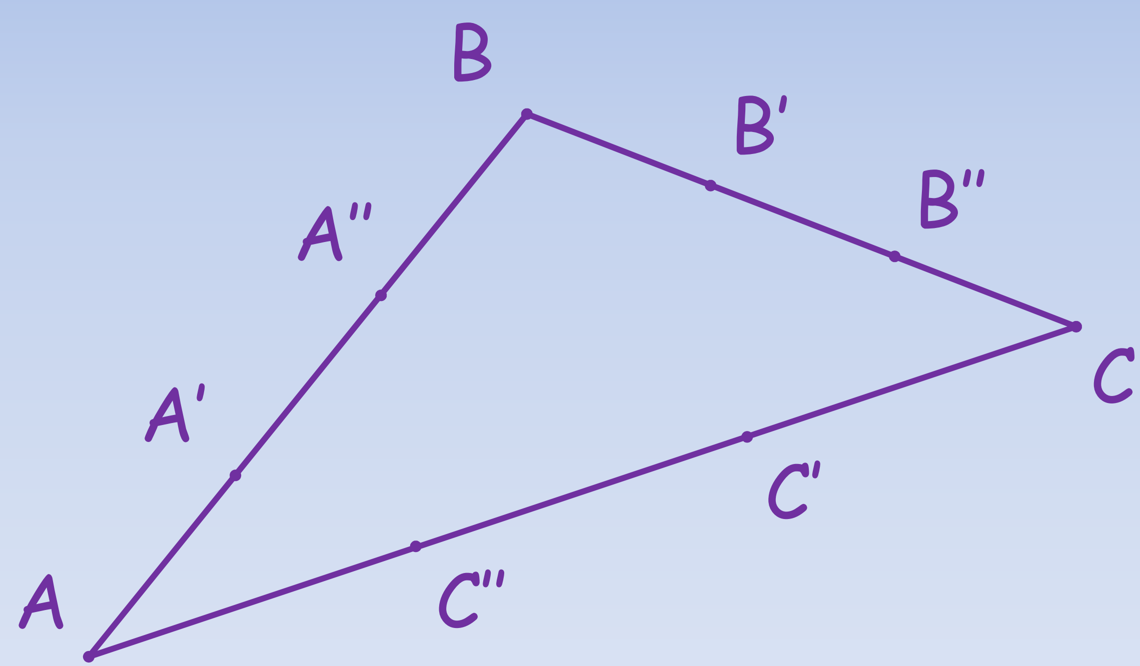


Theorem: (Heron of Alexandria)

$$K=\sqrt{s(s-a)(s-b)(s-c)}$$

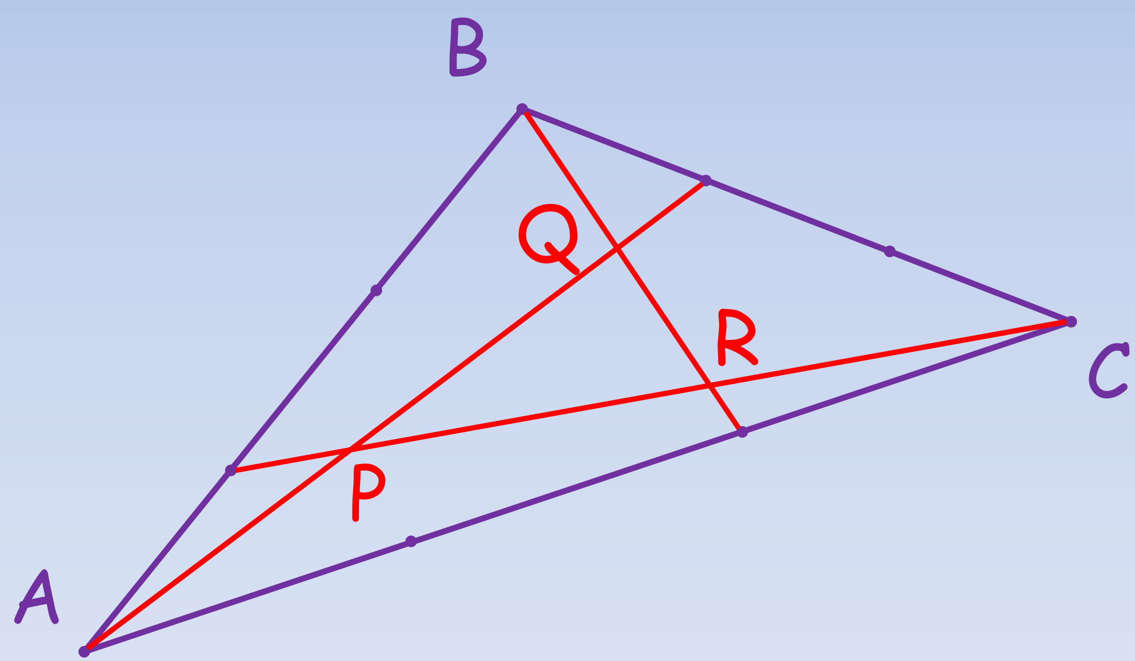
Subdividing a Triangle

- Start with a triangle ΔABC and trisect each side of the triangle.



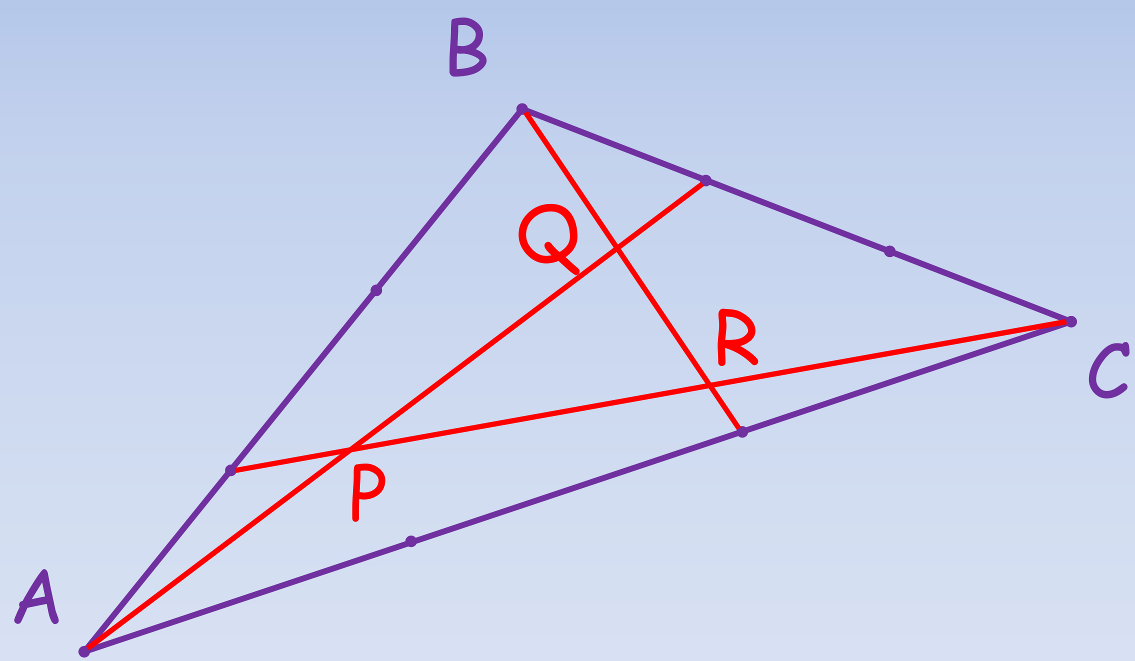
Subdividing a Triangle

- Join vertex to point 1/3 the way down the opposite side - counterclockwise.



Subdividing a Triangle

Theorem: $K_{PQR} = 1/7\ K_{ABC}$



Subdividing a Triangle

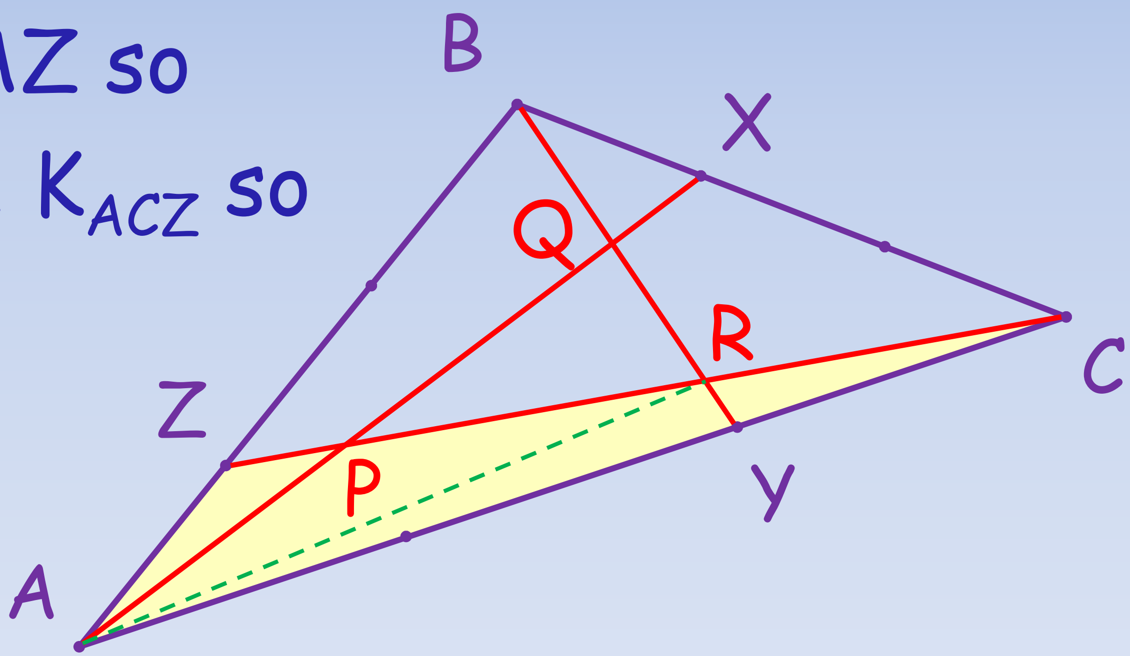
Let $K_{ABC} = 3$ and let $K_{CRY} = k$.

Claim: $K_{ARY} = 2k$

$BZ = 2AZ$ so

$K_{BCZ} = 2\ K_{ACZ}$ so

$K_{ACZ} = 1$



Subdividing a Triangle

$K_{ARZ} = 1 - K_{ARC} = 1 - 3k.$

Likewise $K_{BRZ} = 2\ K_{ARZ} = 2 - 6k$

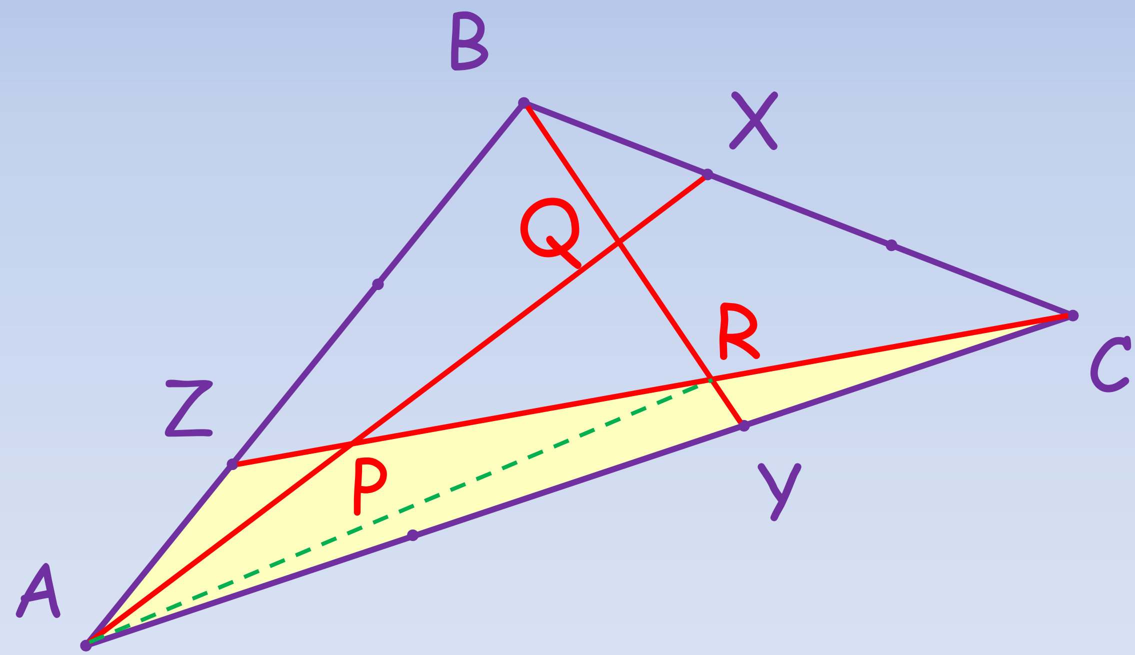
$BZ = 2AZ$ so

$K_{BCZ} = 2$ so

$K_{BCR} = 2 - K_{BRZ}$

$= 2 - (2 - 6k)$

$= 6k$



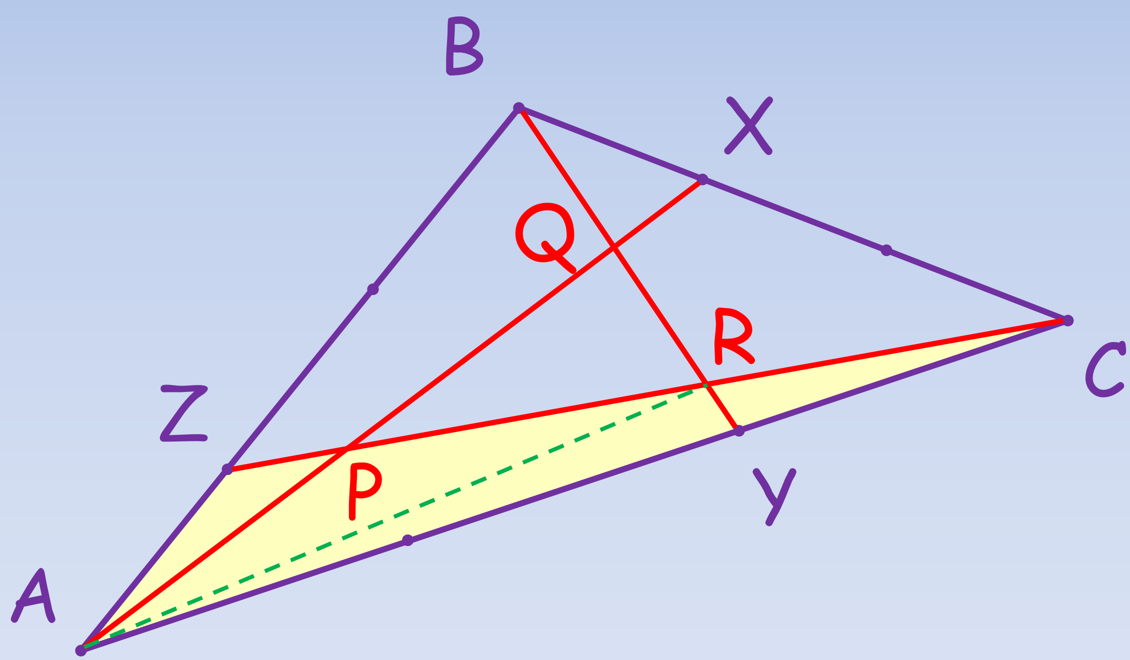
Subdividing a Triangle

Again $K_{BCY} = 1$

$K_{CRY} = K_{BCY} - K_{BCR} = 1 - 6k$

$k = 1 - 6k$

$K_{CRY} = k = 1/7$



Likewise:

$K_{BQX} = 1/7$

$K_{APZ} = 1/7$

Subdividing a Triangle

$K_{CRY} = 1/7,\ K_{BQX} = 1/7,\ K_{APZ} = 1/7$

$K_{BQPZ} = 1 - K_{BQX} - K_{APZ} = 5/7$

$K_{BRZ} = 2 - 6k = 8/7$

$K_{QPR} = 8/7 - 5/7 = 3/7$

$= K_{ABC}/7$