


Similarity

MA 341 - Topics in *Geometry*
Lecture 10



Similarity

- Definition
- AAA Theorem
- Proportionality
- Similarity and parallelism
- Similarity theorems
- Ceva's Theorem

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Definition

1. Definition: Two triangles are similar if their corresponding angles are equal.
2. Definition: Triangles are similar if they have the same shape, but can be different sizes.
3. Definition: Two geometric shapes are similar if there is a rigid motion of the plane that maps one onto the other.

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Definition

Working Definition: Two triangles are similar if their corresponding angles are equal.

$$\begin{aligned} \triangle ABC \sim \triangle DEF \\ \text{means that} \\ \angle A = \angle D, \angle B = \angle E, \angle C = \angle F \end{aligned}$$

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AA Similarity Theorem

Theorem: Given two triangles $\triangle ABC$ and $\triangle DEF$, suppose that $\angle A = \angle D$ and $\angle B = \angle E$. Then $\angle C = \angle F$, and so $\triangle ABC \sim \triangle DEF$.

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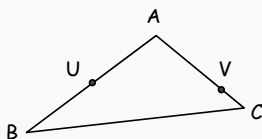
Proportionality Lemma

If U and V are points on sides AB and AC of triangle $\triangle ABC$, $UV \parallel BC$ if and only if

$$\frac{AU}{AV} = \frac{AB}{AC}$$

or

$$\frac{AU}{AB} = \frac{AV}{AC}$$



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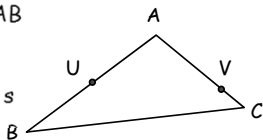
Proof

Let $r = \frac{UB}{AB}$ and $s = \frac{VC}{AC}$

$$AU = AB - UB = (1-r)AB$$

$$AV = (1-s)AC$$

$$\frac{AU}{AB} = 1-r \quad \frac{AV}{AC} = 1-s$$



So $AU/AB = AV/AC$ if and only if $r = s$.

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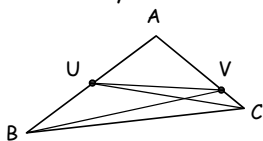
Proof

Now we need to show that $UV \parallel BC$ iff $r = s$.

Consider $\triangle ACB$ and $\triangle UCB$. They have same height. Thus:

$$\frac{K_{BUC}}{K_{ABC}} = \frac{UB}{AB} = r$$

$$K_{BUC} = rK_{ABC}$$



Similarly, $K_{BVC} = sK_{ABC}$.

Thus, $r = s$ iff $K_{BUC} = K_{BVC}$.

Thus, $UV \parallel BC$ iff $K_{BUC} = K_{BVC}$.

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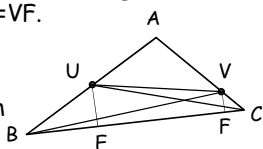
Proof

$\triangle BUC$ and $\triangle BVC$ have same base. Their areas are equal iff they have same height, or $UE = VF$. Thus, $UV \parallel BC$ iff $UE = VF$.

Note: $UE \parallel VF$.

Thus, if $UE = VF$ then $UVEF =$ parallelogram and $UV \parallel BC$.

If $UV \parallel BC$, then $UVEF$ is a parallelogram hence $UE = VF$.



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Proportionality Theorem

Theorem: If $\triangle ABC \sim \triangle DEF$, then the lengths of the corresponding sides are proportional.

Proof: We need to find k so that:

$$AB = k DE$$

$$AC = k DF$$

$$BC = k EF$$

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Proportionality Theorem

We want to show that $DE/AB=DF/AC$.

If $DE=AB$, then $\triangle ABC \cong \triangle DEF$ by ASA. In this case, $DF \cong AC$ and $DF/AC=1$.

So assume $DE < AB$.

Choose $U \in AB$ so that $AU=DE$ and construct UV parallel to BC with $V \in AC$.

$UV \parallel BC \Rightarrow \angle AUV = \angle B = \angle E$ and $\angle AVU = \angle C = \angle F$

$AU=DE \Rightarrow \triangle AUV \cong \triangle DEF$ by AAS $\Rightarrow AV=DF$.

$UV \parallel BC \Rightarrow AU/AB=AV/AC$, but $AU=DE$ & $AV=DF$

So $DE/AB=DF/AC$.

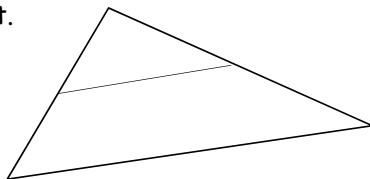
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Midline of a Triangle

A line joining two midpoints of a triangle is called a midline of the triangle. A segment joining two midpoints is called a midsegment.



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Midline Theorem

Theorem: In $\triangle ABC$ if U and V are the midpoints of AB and AC , respectively, then $UV \parallel BC$ and $UV = \frac{1}{2}BC$.

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SS Similarity Theorem

Theorem: Suppose that the sides of $\triangle ABC$ are proportional to the corresponding sides of $\triangle DEF$. Then $\triangle ABC \sim \triangle DEF$.

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SAS Similarity Theorem

Theorem: Suppose that in $\triangle ABC$ and $\triangle DEF$ we have that $\angle A = \angle D$ and $DE/AB = DF/AC$. Then $\triangle ABC \sim \triangle DEF$.

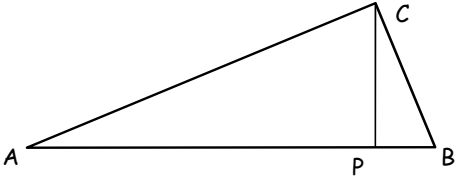
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Right Triangle Theorem

Theorem: Suppose $\triangle ABC$ is a right triangle with hypotenuse AB and CP is the altitude from C .
 $\triangle ACP \sim \triangle ABC \sim \triangle CBP$.

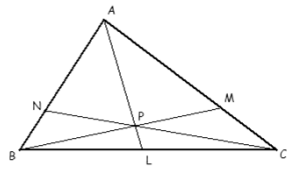


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Ceva's Theorem

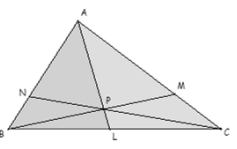
The three lines containing the vertices A , B , and C of $\triangle ABC$ and intersecting opposite sides at points L , M , and N , respectively, are concurrent if and only if

$$\frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = 1$$

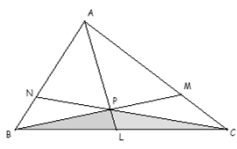


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Ceva's Theorem



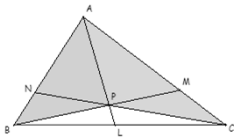
$$\frac{K(\triangle ABL)}{K(\triangle ACL)} = \frac{BL}{LC}$$



$$\frac{K(\triangle PBL)}{K(\triangle PCL)} = \frac{BL}{LC}$$

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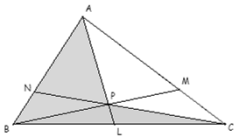
Ceva's Theorem



$$\frac{BL}{LC} = \frac{K(\triangle ABL) - K(\triangle PBL)}{K(\triangle ACL) - K(\triangle PCL)} = \frac{K(\triangle ABP)}{K(\triangle ACP)}$$

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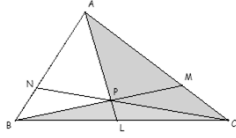
Ceva's Theorem



$$\frac{CM}{MA} = \frac{K(\triangle BMC) - K(\triangle PMC)}{K(\triangle BMA) - K(\triangle PMA)} = \frac{K(\triangle BCP)}{K(\triangle BAP)}$$

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Ceva's Theorem



$$\frac{AN}{NB} = \frac{K(\triangle ACN) - K(\triangle APN)}{K(\triangle BCN) - K(\triangle BPN)} = \frac{K(\triangle ACP)}{K(\triangle BCP)}$$

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Ceva's Theorem

$$\frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = \frac{K(\triangle ACP) \cdot K(\triangle ABP) \cdot K(\triangle BCP)}{K(\triangle BCP) \cdot K(\triangle ACP) \cdot K(\triangle ABP)} = 1$$

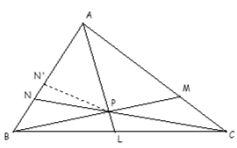
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Ceva's Theorem

Now assume that

$$\frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = 1$$

Let BM and AL intersect at P and construct CP intersecting AB at N', N' different from N.



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Ceva's Theorem

Then AL, BM, and CN' are concurrent and

$$\frac{AN'}{N'B} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = 1$$

From our hypothesis it follows that

$$\frac{AN'}{N'B} = \frac{AN}{NB}$$

So N and N' must coincide.

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