

Triangles II

Circumradius, Area, Medians,
Stewart's Theorem

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Question

Where do the perpendicular bisectors of the sides intersect the circumcircle?

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Claim

- 1) AY bisects $\angle BAC$
- 2) AZ bisects exterior angle

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Claim
 Let $Y = HD \cap \text{circle}$

$\angle BAY = \frac{1}{2} \angle BHY$
 $\angle BAC = \frac{1}{2} \angle BHC$
 $\angle BHY = \angle CHY = \frac{1}{2} \angle BHC$
 $2\angle BAY = \angle BHY$
 $= \frac{1}{2} \angle BHC$
 $= \angle BAC$
 $\angle BAY = \frac{1}{2} \angle BAC$

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Claim
 Let $Z = HD \cap \text{circle}$

$\angle ZAY = 90 = \angle BAY + \angle BAZ$
 $\angle BA\Omega + \angle BAC = 180$
 $\angle CA\Omega + \angle ZA\Omega = 90$
 $\angle BAY = \angle CA\Omega$
 $\angle BAY + \angle BAZ = \angle CA\Omega + \angle ZA\Omega$
 $\angle BAZ = \angle ZA\Omega$

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Question

Where do the perpendicular bisectors of the sides intersect the circumcircle?
 At one end is point of intersection of angle bisector with circumcircle
 The other end is point of intersection of exterior angle bisector with circumcircle.

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Extended Law of Sines

Theorem: Given $\triangle ABC$ with circumradius R , let a , b , and c denote the lengths of the sides opposite angles $\angle A$, $\angle B$, and $\angle C$, respectively. Then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

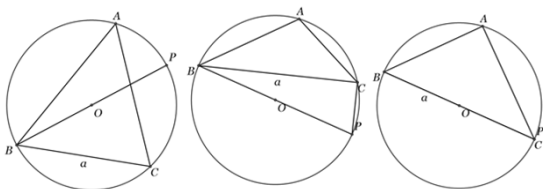
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Proof

Three cases:



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Proof

Case I: $\angle A < 90^\circ$

$BP = \text{diameter}$

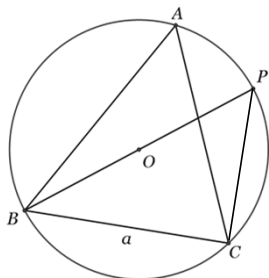
$\Rightarrow \triangle BCP$ right triangle

$BP = 2R$

$\Rightarrow \sin P = a/2R$

$\angle A = \angle P$

$\Rightarrow 2R = a/\sin A$



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Proof

Case II: $\angle A > 90^\circ$
 BP = diameter
 $\Rightarrow \triangle BCP$ right triangle
 BP = 2R
 $\Rightarrow \sin P = a/2R$
 $\angle A = \angle P$
 $\Rightarrow 2R = a/\sin A$

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Proof

Case III: $\angle A = 90^\circ$
 BP = a = diameter
 BP = 2R
 $2R = a = a/\sin A$

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Circumradius and Area

Theorem: Let R be the circumradius and K be the area of $\triangle ABC$ and let a, b, and c denote the lengths of the sides as usual. Then $4KR=abc$

$$K = \frac{abc}{4R}$$

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Proof

$$K = \frac{1}{2} ab \sin C$$

$$2K = ab \sin C$$

$$c/\sin C = 2R$$

$$\sin C = c/2R$$

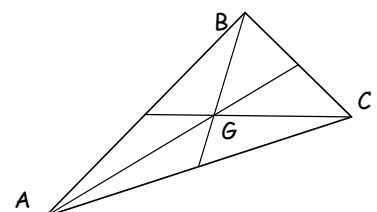
$$2K = abc/2R$$

$$4KR = abc$$

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Centroid

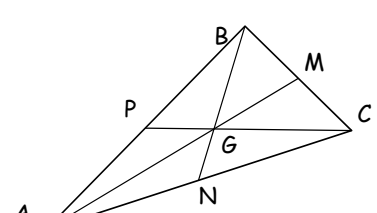
Theorem: The medians of a triangle meet in a single point, called the centroid, G .



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Centroid

We have shown this by Ceva's Theorem, so this is not new. However, there is something more.



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Centroid

$AG = 2 GM$
 $BG = 2 GN$
 $CG = 2 GP$

} The median is 2/3 the way from the vertex to the side.

How can we show this?

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Centroid

MN is the midsegment of $\triangle ABC$ so $MN \parallel AB$ and $MN = \frac{1}{2} AB$.

By Alternate Interior Angles
 $\angle ABG = \angle MNG$ and $\angle BAG = \angle NMG \Rightarrow \triangle ABG \sim \triangle MNG$

So

$$\frac{AG}{GM} = \frac{AB}{MN}$$

$$AG = 2GM$$

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Stewart's Theorem (1746)

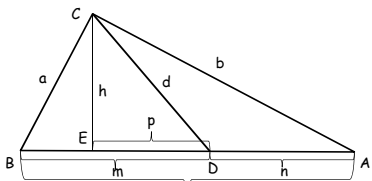
With the measurements given in the triangle below, the following relationship holds:

$$a^2n + b^2m = c(d^2 + mn)$$

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Stewart's Theorem (1746)

$CE \perp AB$ so we will apply the Pythagorean Theorem several times



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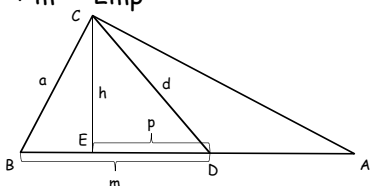
Stewart's Theorem (1746)

In $\triangle CEB$ $a^2 = h^2 + (m - p)^2$

In $\triangle CED$ $d^2 = h^2 + p^2$

$a^2 = d^2 - p^2 + (m - p)^2$

$a^2 = d^2 + m^2 - 2mp$



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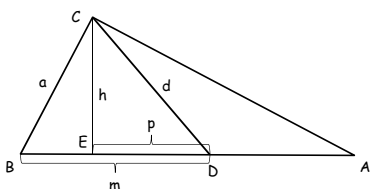
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Stewart's Theorem (1746)

In $\triangle CEA$ $b^2 = h^2 + (n + p)^2$

$b^2 = d^2 - p^2 + (n + p)^2$

$b^2 = d^2 + n^2 + 2np$



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Stewart's Theorem (1746)

$$a^2n = d^2n + m^2n - 2mnp$$

$$b^2m = d^2m + n^2m + 2mnp$$

$$a^2n + b^2m = d^2n + m^2n + d^2m + n^2m$$

$$= d^2(n + m) + mn(m + n)$$

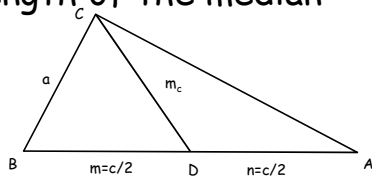
$$a^2n + b^2m = c(d^2 + mn)$$

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The length of the median



$$a^2n + b^2m = c(m_c^2 + mn)$$

$$\frac{a^2c}{2} + \frac{b^2c}{2} = c \left(m_c^2 + \frac{c^2}{4} \right)$$

$$m_c^2 = \frac{a^2}{2} + \frac{b^2}{2} - \frac{c^2}{4}$$

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The length of the medians

$$2m_a^2 = b^2 + c^2 - \frac{1}{2}a^2$$

$$2m_b^2 = a^2 + c^2 - \frac{1}{2}b^2$$

$$2m_c^2 = a^2 + b^2 - \frac{1}{2}c^2$$

For a 3-4-5 triangle this gives us that the medians measure:

$$m_a = \frac{\sqrt{73}}{2}; \quad m_b = \sqrt{13}; \quad m_c = \frac{5}{2}$$

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Example

Find x x=3

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Theorem 4

For any triangle, the sum of the lengths of the medians is less than the perimeter of the triangle.

N in AF so that $NF=AF$
 $ACNB$ is a parallelogram
 $BN=AC$
 In $\triangle ABN$, $AN < AB+BN$
 $2AF < AB + AC$
 $2m_a < b + c$

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Theorem 4

Similarly
 $2m_b < a + c$ and $2m_c < a + b$
 $2(m_a+m_b+m_c) < 2a+2b+2c$

$m_a + m_b + m_c < a + b + c$

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Theorem 5

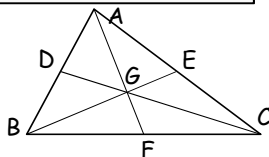
For any triangle, the sum of the lengths of the medians is greater than three-fourths the perimeter of the triangle.

$$BG + CG > BC$$

$$\frac{2}{3}m_c + \frac{2}{3}m_b > a$$

and

$$\frac{2}{3}m_a + \frac{2}{3}m_b > c \quad \frac{2}{3}m_a + \frac{2}{3}m_c > b$$



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Theorem 5

$$\frac{2}{3}m_b + \frac{2}{3}m_c + \frac{2}{3}m_a + \frac{2}{3}m_c + \frac{2}{3}m_a + \frac{2}{3}m_b > a+b+c$$

$$\frac{4}{3}(m_a + m_b + m_c) > a+b+c$$

$$m_a + m_b + m_c > \frac{3}{4}(a+b+c)$$

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Result

$$\frac{3}{4}(a+b+c) < m_a + m_b + m_c < a+b+c$$

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Theorem 6

The sum of the squares of the medians of a triangle equals three-fourths the sum of the squares of the sides of the triangle.

$$2m_a^2 = b^2 + c^2 - \frac{1}{2}a^2$$

$$2m_b^2 = a^2 + c^2 - \frac{1}{2}b^2$$

$$2m_c^2 = a^2 + b^2 - \frac{1}{2}c^2$$

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Theorem 6

$$2(m_a^2 + m_b^2 + m_c^2) = 2(a^2 + b^2 + c^2) - \frac{1}{2}(a^2 + b^2 + c^2)$$

$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$$

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Theorem 7

The sum of the squares of the lengths of the segments joining the centroid with the vertices is one-third the sum of the squares of the lengths of the sides.

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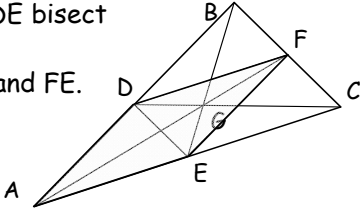
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Theorem 8

A median and the midline it intersects bisect each other.

Show AF and DE bisect each other.
 Construct DF and FE.
 $DF \parallel AE$
 $AD \parallel FE$

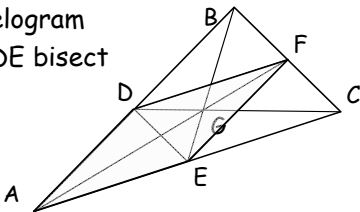


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Theorem 8

A median and the midline it intersects bisect each other.

ADFE a parallelogram
 Thus, AF and DE bisect each other.

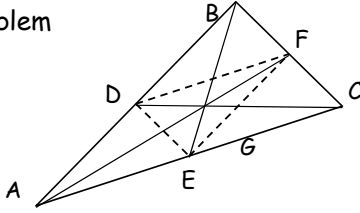


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Theorem 9

A triangle and its medial triangle have the same centroid.

This is HW Problem 2B.1.



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