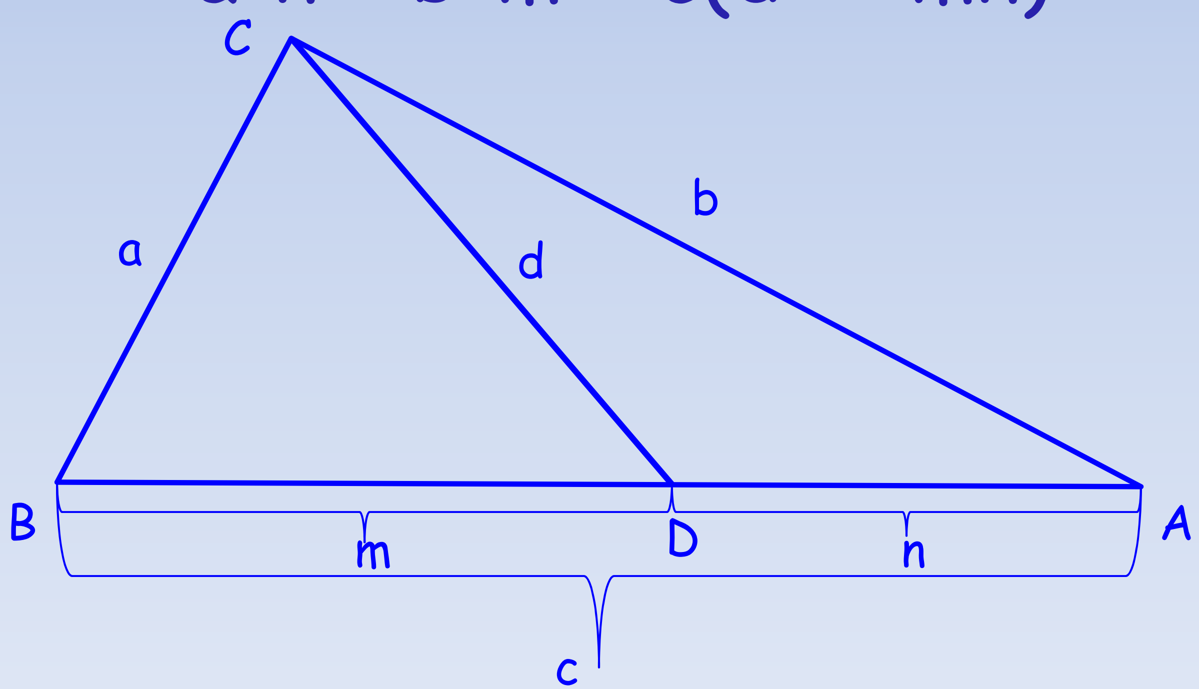
Triangles III

Stewart's Theorem, Orthocenter, Euler Line

Stewart's Theorem (1746)

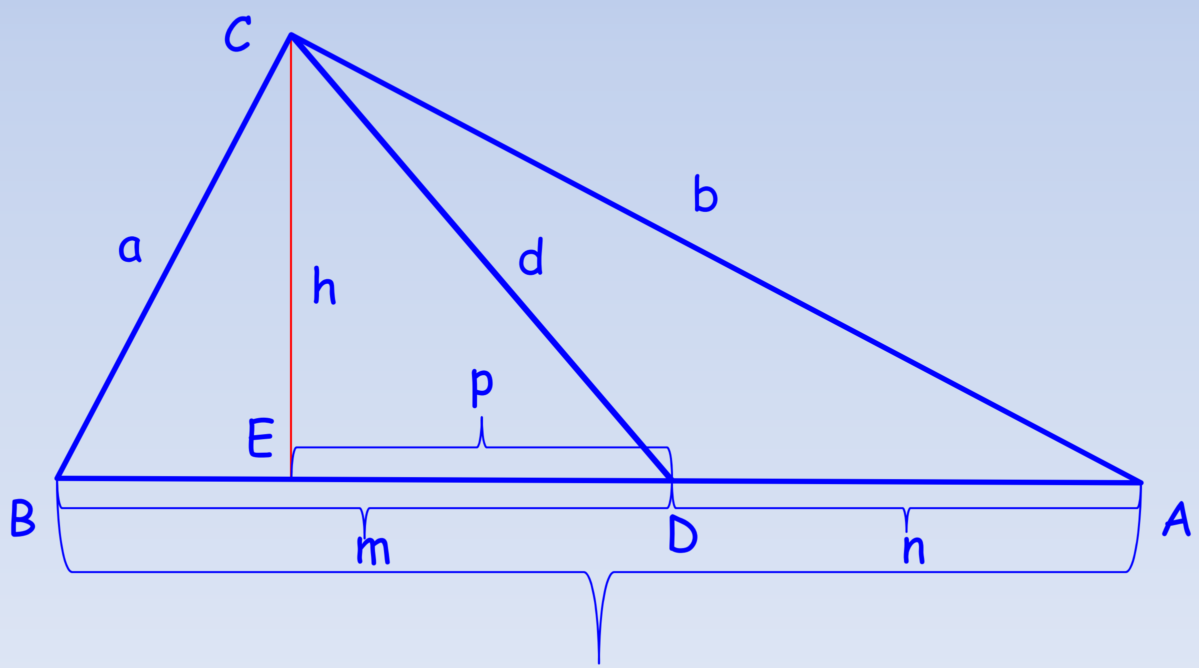
With the measurements given in the triangle below, the following relationship holds:

$$a^2n + b^2m = c(d^2 + mn)$$



Stewart's Theorem (1746)

CE⊥AB so we will apply the Pythagorean Theorem several times



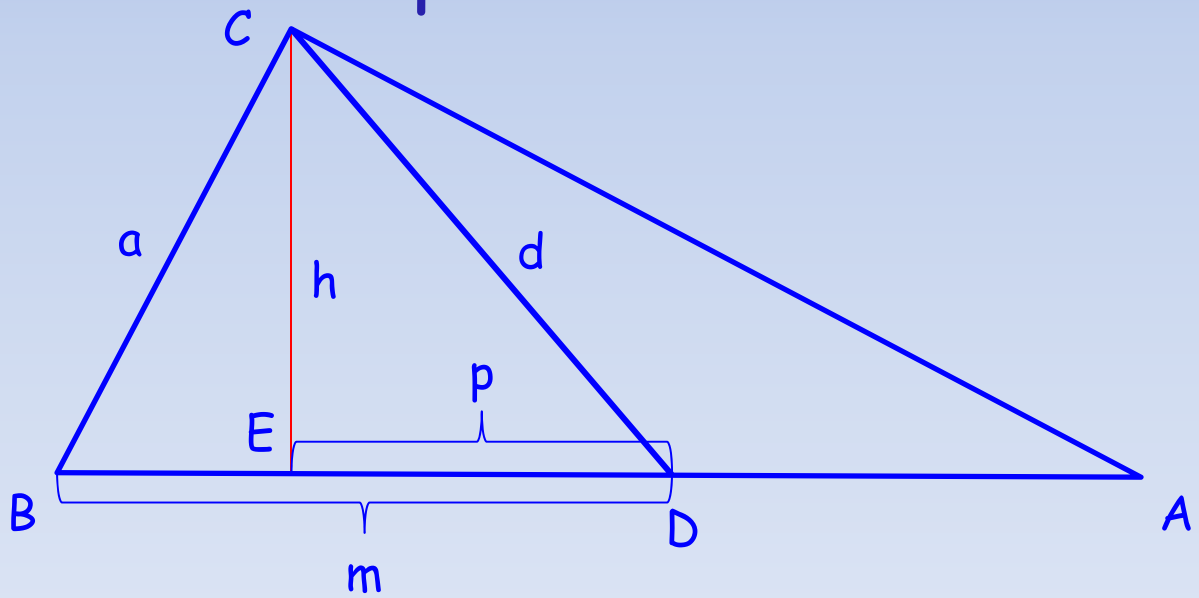
Stewart's Theorem (1746)

In ΔCEB $a^2 = h^2 + (m - p)^2$

In ΔCED $d^2 = h^2 + p^2$

$a^2 = d^2 - p^2 + (m - p)^2$

$a^2 = d^2 + m^2 - 2mp$

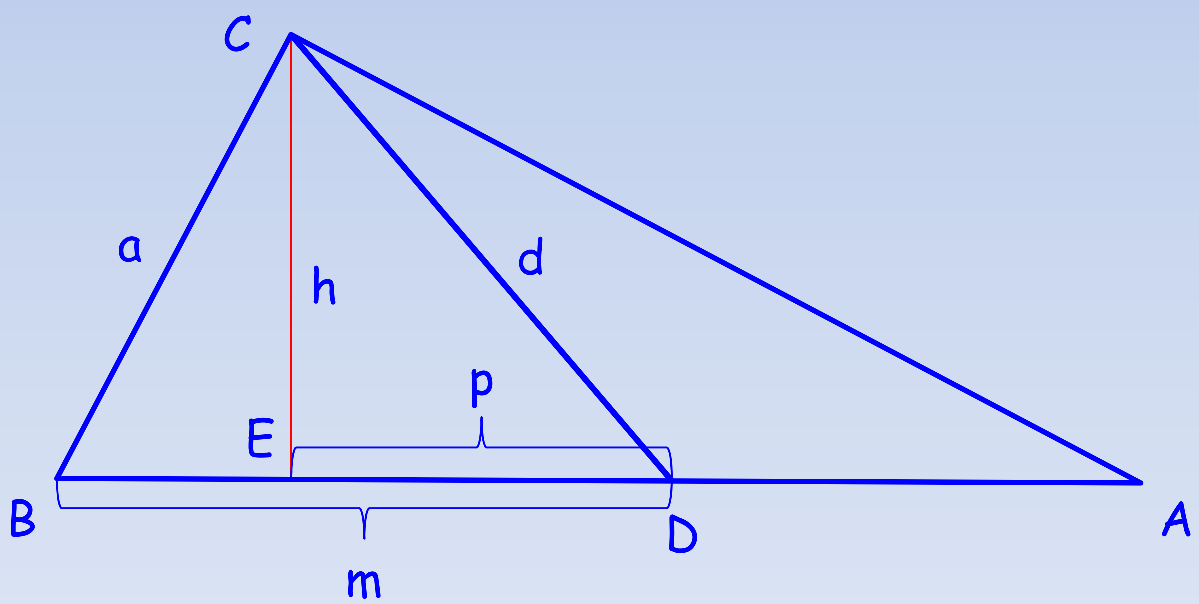


Stewart's Theorem (1746)

In ΔCEA $b^2 = h^2 + (n + p)^2$

$b^2 = d^2 - p^2 + (n + p)^2$

$b^2 = d^2 + n^2 + 2np$



Stewart's Theorem (1746)

$a^2n = d^2n + m^2n - 2mnp$

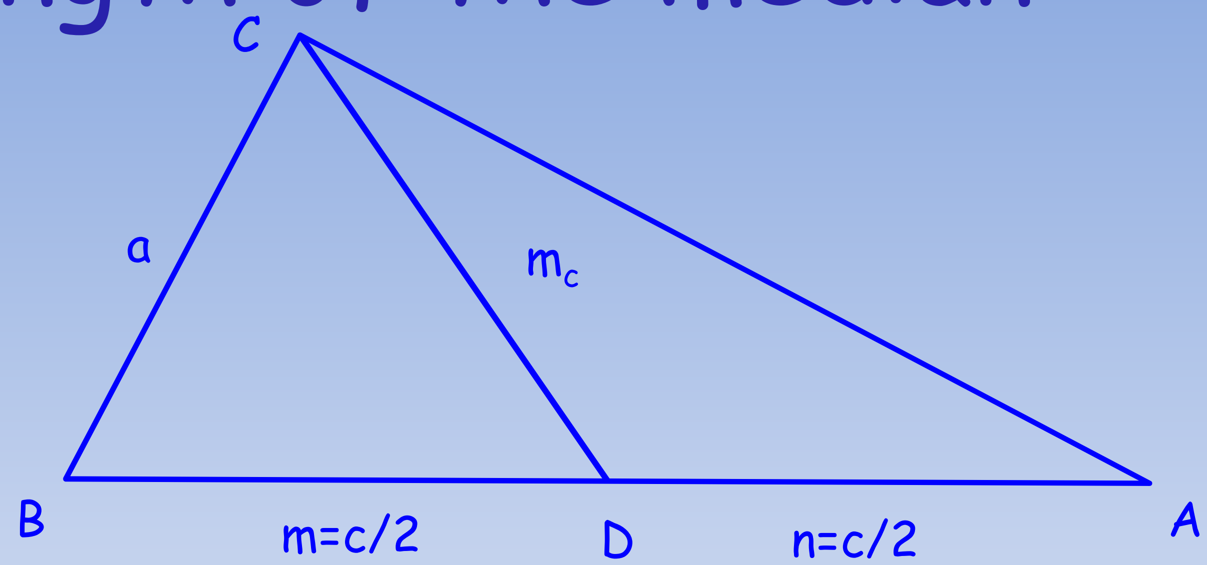
$b^2m = d^2m + n^2m + 2mnp$

$a^2n + b^2m = d^2n + m^2n + d^2m + n^2m$

$= d^2(n + m) + mn(m + n)$

$a^2n + b^2m = c(d^2 + mn)$

The length of the median



$$a^2n + b^2m = c(m_c^2 + mn)$$

$$\frac{a^2c}{2} + \frac{b^2c}{2} = c\left(m_c^2 + \frac{c^2}{4}\right)$$

$$m_c^2 = \frac{a^2}{2} + \frac{b^2}{2} - \frac{c^2}{4}$$

The length of the medians

$$2m_a^2 = b^2 + c^2 - \frac{1}{2}a^2$$

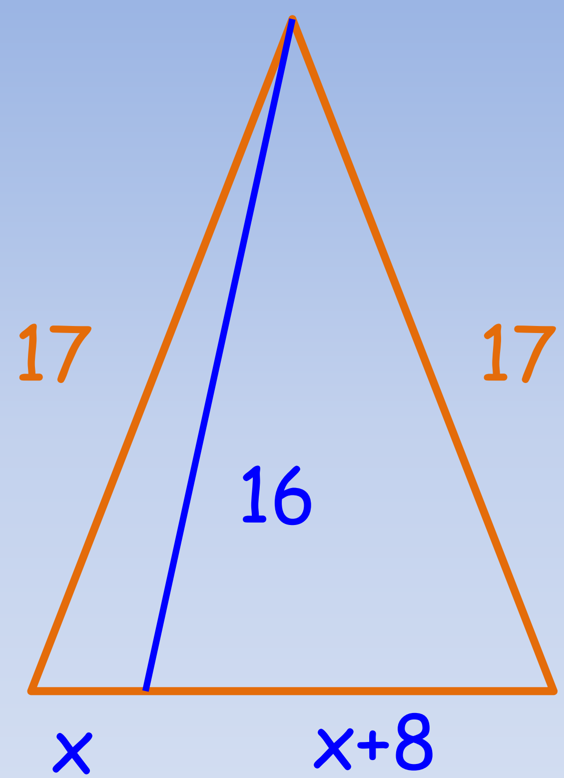
$$2m_b^2 = a^2 + c^2 - \frac{1}{2}b^2$$

$$2m_c^2 = a^2 + b^2 - \frac{1}{2}c^2$$

For a 3-4-5 triangle this gives us that the medians measure:

$$m_a = \frac{\sqrt{73}}{2}; \qquad m_b = \sqrt{13}; \qquad m_c = \frac{5}{2}$$

Example



Find x

x=3

Theorem 4

For any triangle, the sum of the lengths of the medians is less than the perimeter of the triangle.

N in AF so that NF=AF

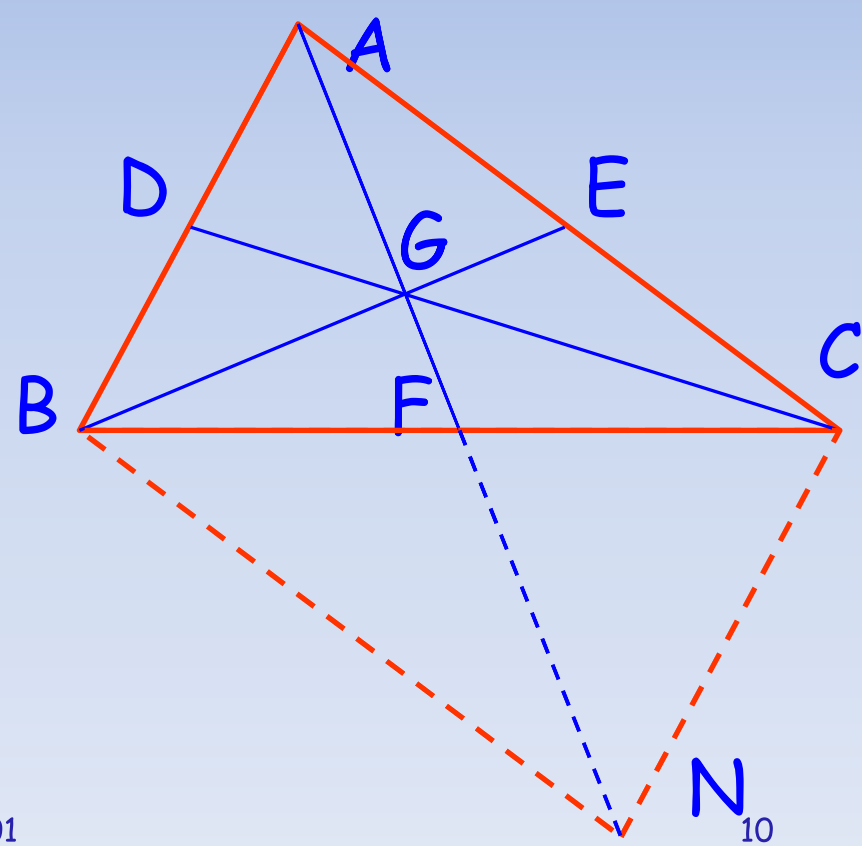
ACNB is a parallelogram

BN=AC

In ΔABN, AN < AB+BN

2AF < AB + AC

$2m_a < b + c$



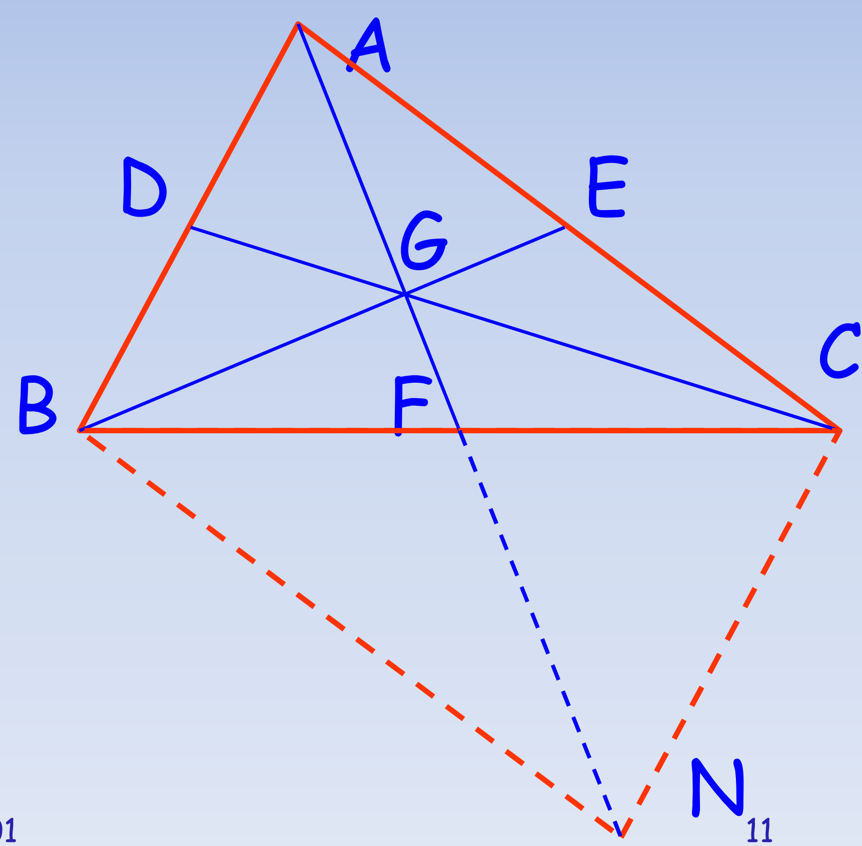
Theorem 4

Similarly

$2m_b < a + c$ and $2m_c < a + b$

$2(m_a+m_b+m_c) < 2a+2b+2c$

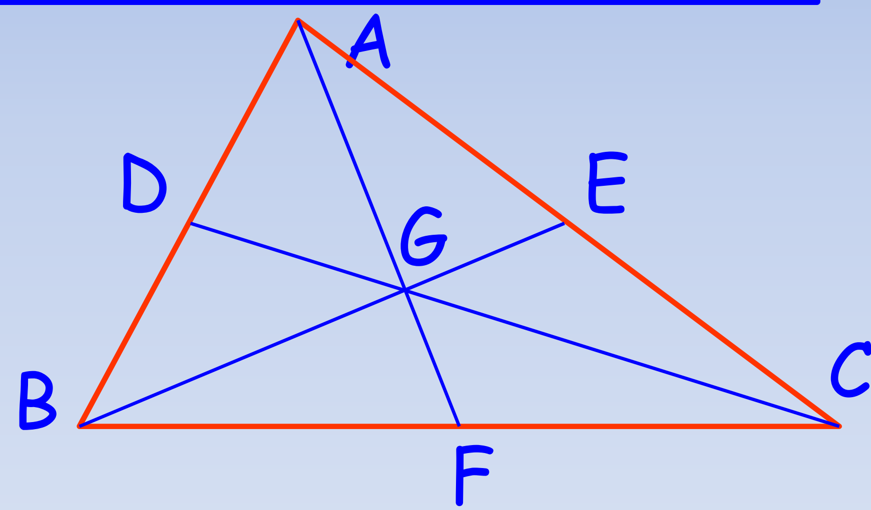
$m_a + m_b + m_c < a+ b+ c$



Theorem 5

For any triangle, the sum of the lengths of the medians is greater than three-fourths the perimeter of the triangle.

BG + CG > BC



$$\frac{2}{3}m_c + \frac{2}{3}m_b > a$$

and

$$\frac{2}{3}m_a + \frac{2}{3}m_b > c \qquad \frac{2}{3}m_a + \frac{2}{3}m_c > b$$

Theorem 5

$$\frac{2}{3}m_b + \frac{2}{3}m_c + \frac{2}{3}m_a + \frac{2}{3}m_c + \frac{2}{3}m_a + \frac{2}{3}m_b > a + b + c$$

$$\frac{4}{3}(m_a + m_b + m_c) > a + b + c$$

$$m_a + m_b + m_c > \frac{3}{4}(a + b + c)$$

Result

$$\frac{3}{4}(a+b+c) < m_a + m_b + m_c < a+b+c$$

Theorem 6

The sum of the squares of the medians of a triangle equals three-fourths the sum of the squares of the sides of the triangle.

$$2m_a^2 = b^2 + c^2 - \frac{1}{2}a^2$$

$$2m_b^2 = a^2 + c^2 - \frac{1}{2}b^2$$

$$2m_c^2 = a^2 + b^2 - \frac{1}{2}c^2$$

Theorem 6

$$2(m_a^2 + m_b^2 + m_c^2) = 2(a^2 + b^2 + c^2) - \frac{1}{2}(a^2 + b^2 + c^2)$$

$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$$

Theorem 7

The sum of the squares of the lengths of the segments joining the centroid with the vertices is one-third the sum of the squares of the lengths of the sides.

Theorem 8

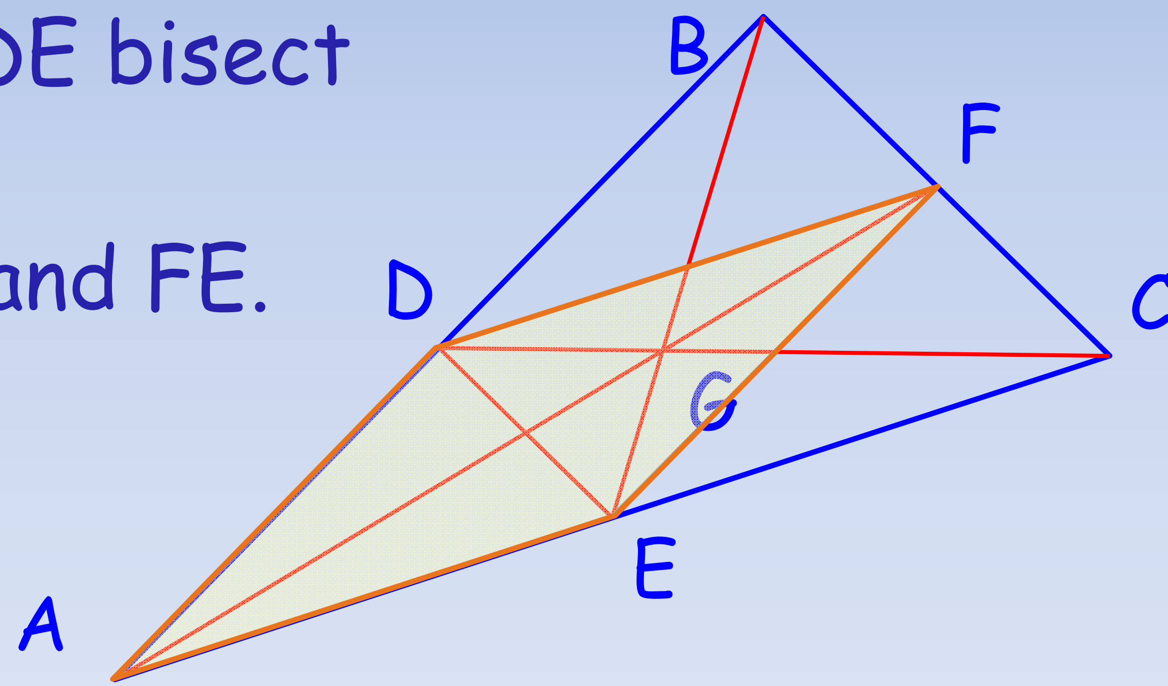
A median and the midline it intersects bisect each other.

Show AF and DE bisect each other.

Construct DF and FE.

DF||AE

AD||FG

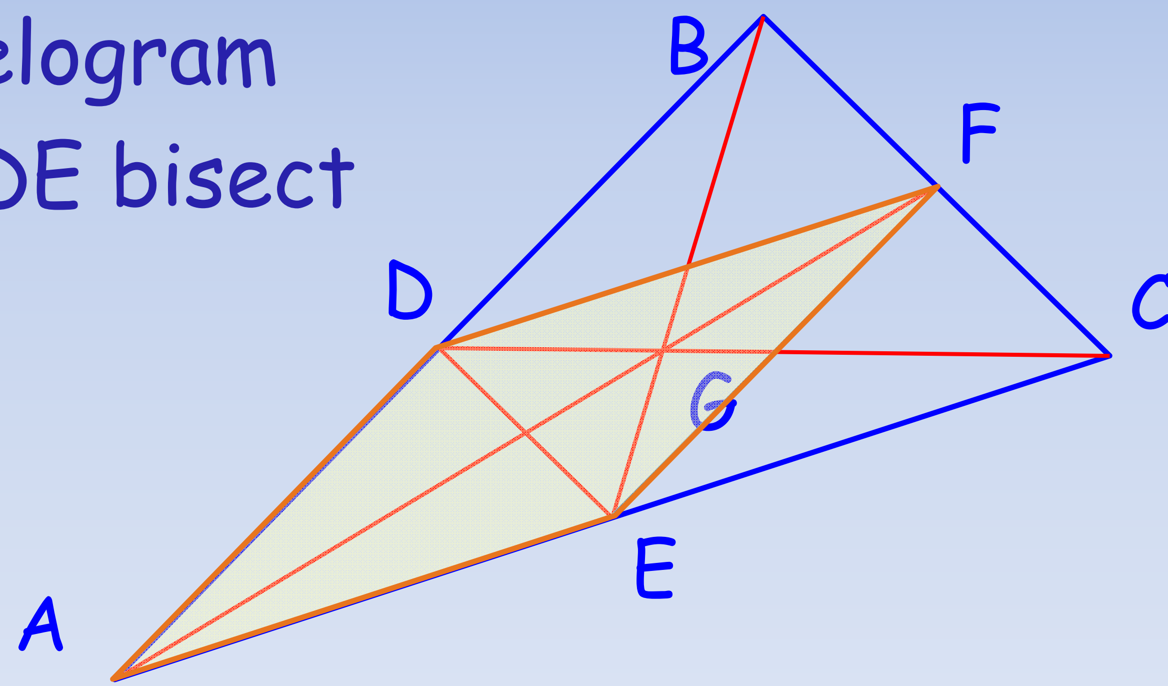


Theorem 8

A median and the midline it intersects bisect each other.

ADFE a parallelogram

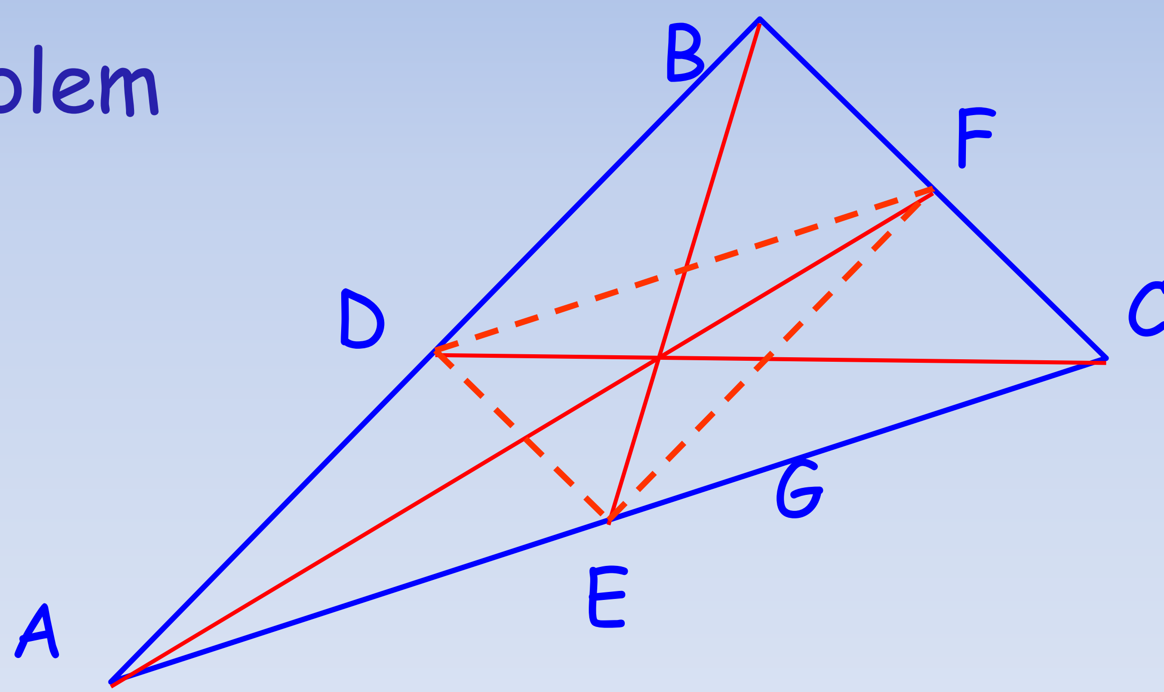
Thus, AF and DE bisect each other.



Theorem 9

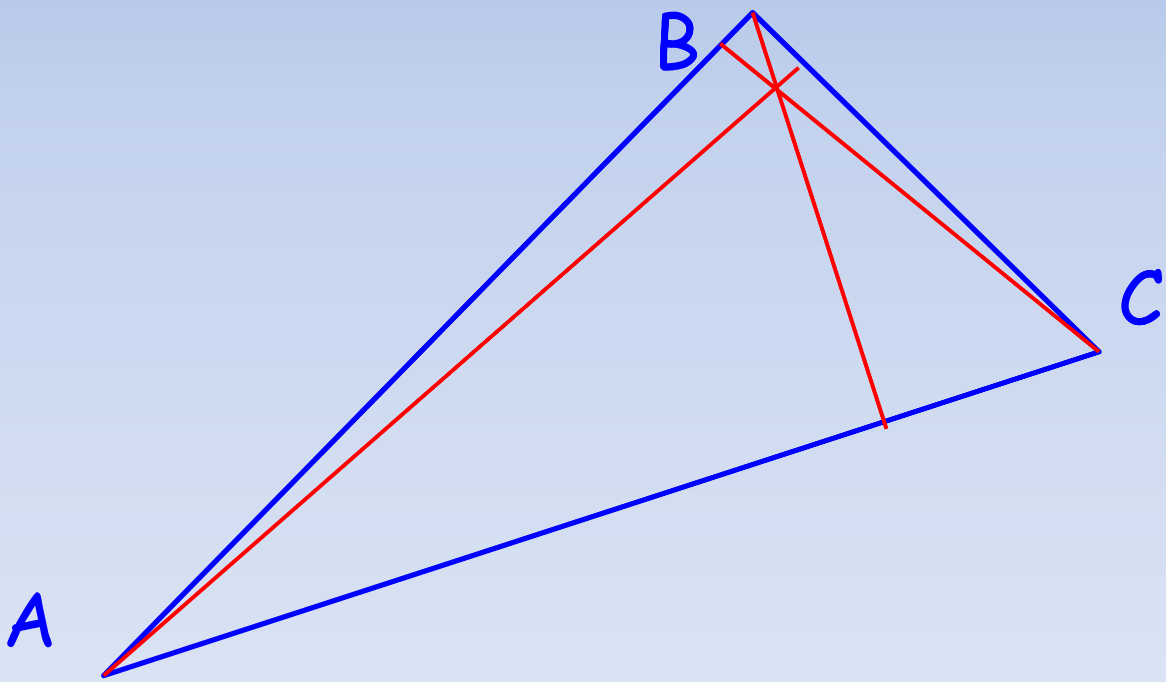
A triangle and its medial triangle have the same centroid.

This is HW Problem 2B.1.



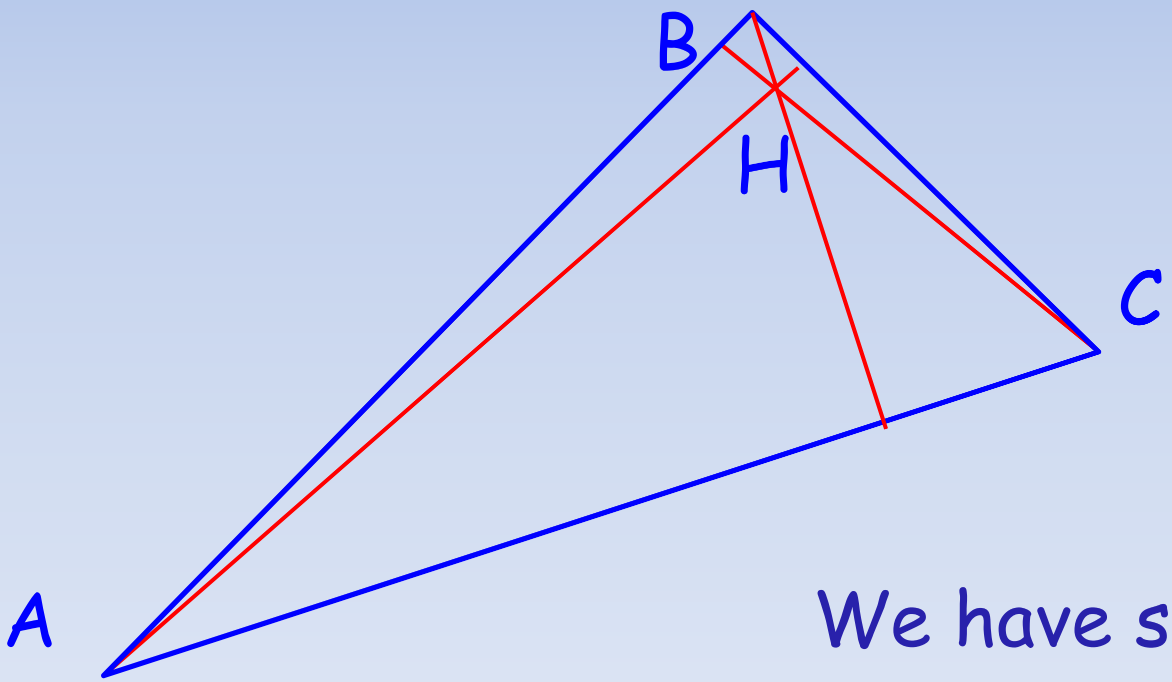
Orthocenter

Definition: In ΔABC the foot of a vertex to the side opposite that vertex is called an altitude of the triangle.



Orthocenter

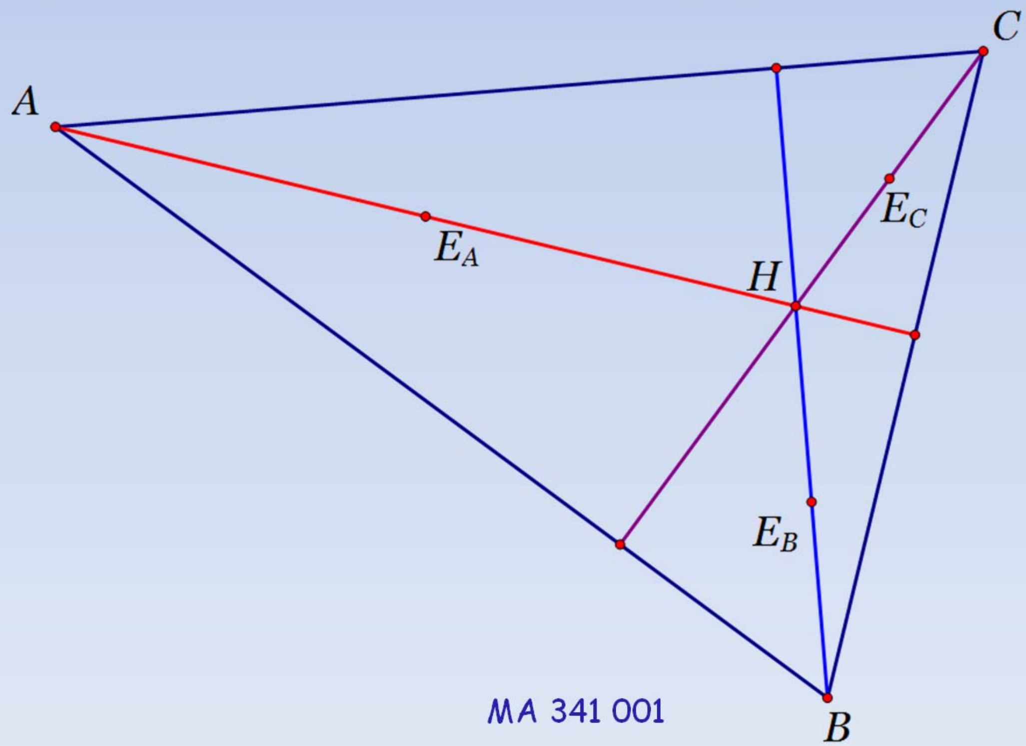
Theorem: The altitudes of a triangle meet in a single point, called the orthocenter, H.



We have shown this.

Euler Points

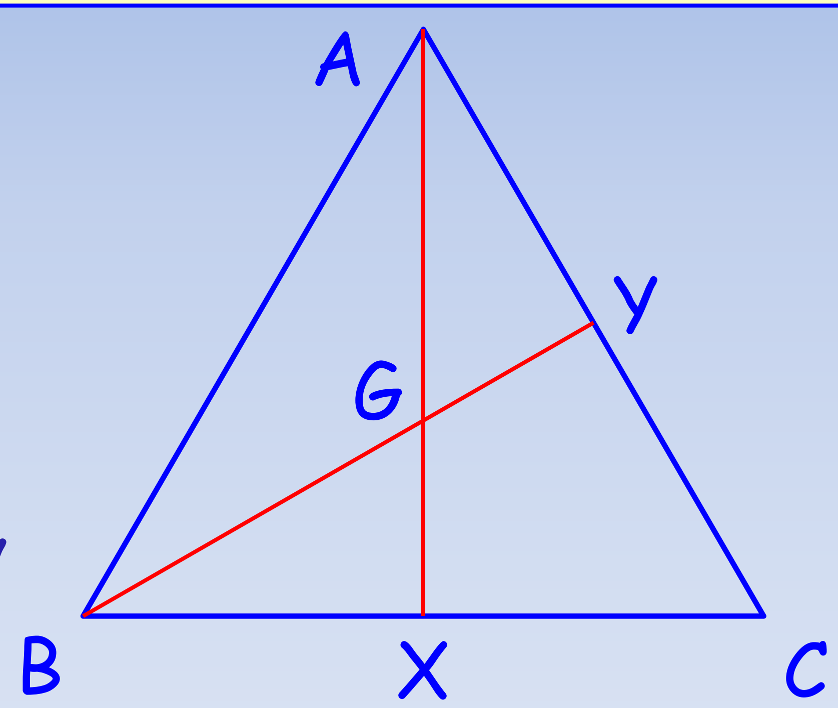
- The midpoint of the segment from the vertex to the orthocenter is called the Euler point of ΔABC opposite the side.



Circumcenter & Centroid

If the circumcenter and the centroid coincide, the triangle must be equilateral.

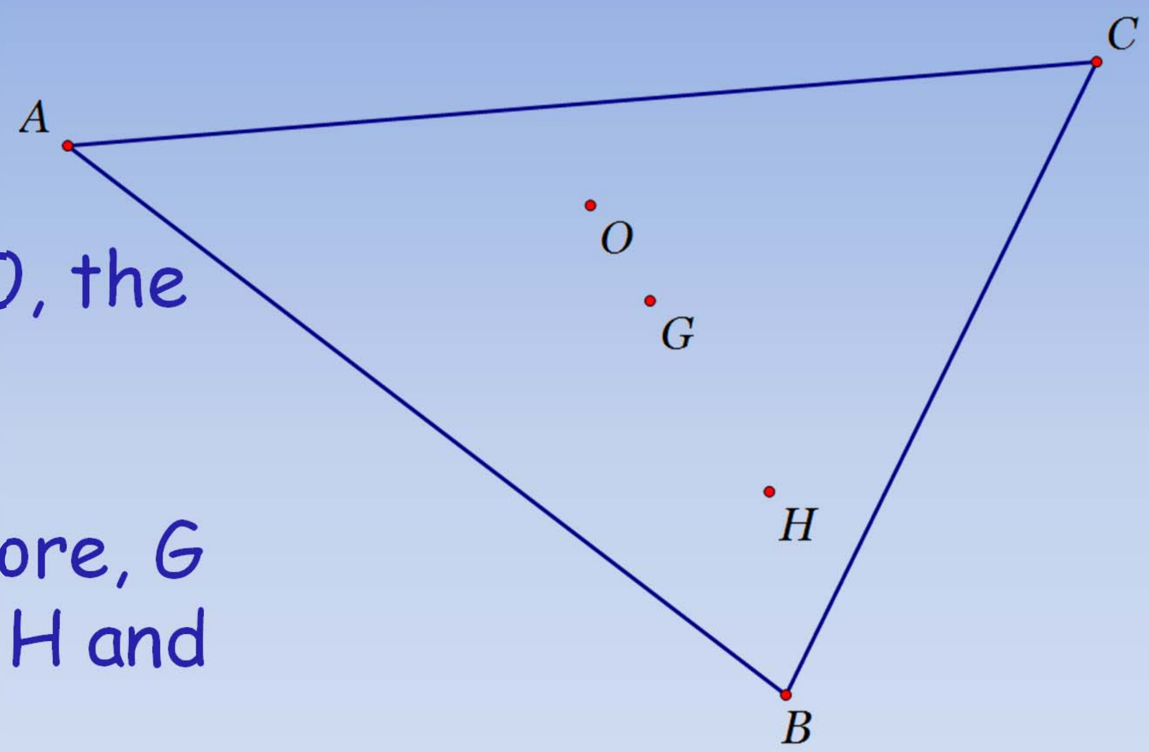
Suppose G=O, X = midpt of BC, Y=midpt of AC.
G=O $\Rightarrow$ AG=BG.
G=centroid $\Rightarrow$ 3/2 AG= 3/2 BG $\Rightarrow$ AX=BY $\Rightarrow$GX=GY. By SAS ΔAGY $\cong$ ΔBGX $\Rightarrow$ BX=AY $\Rightarrow$ BC=AC.



The Euler Segment

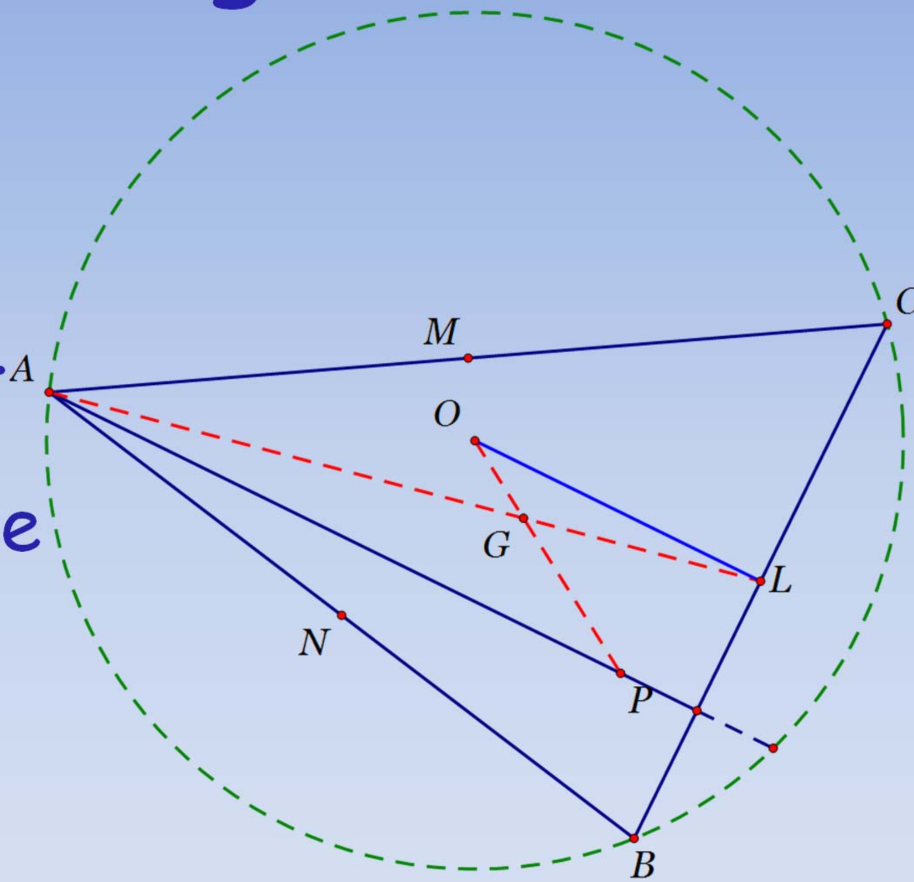
The circumcenter O, the centroid G, and the orthocenter H are collinear. Furthermore, G lies between O and H and

$$\frac{GH}{OG} = 2$$



The Euler Segment

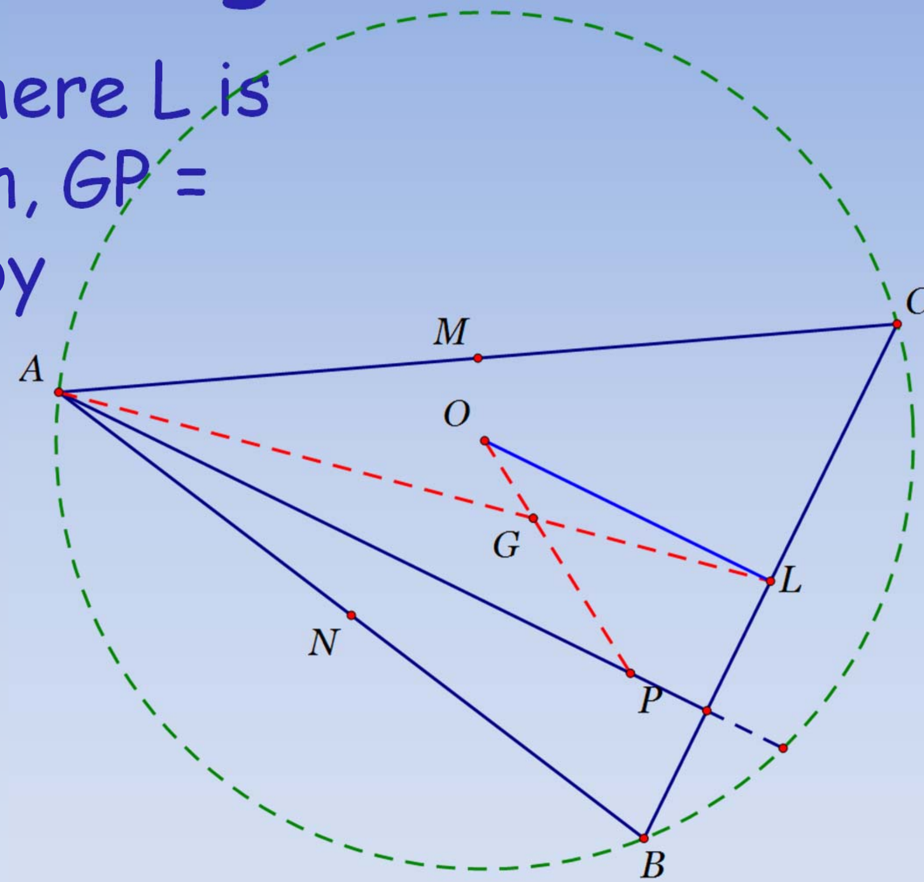
(Symmetric Triangles)
Extend OG twice its length to a point P, that is GP = 2OG. We need to show that P is the orthocenter.



The Euler Segment

Draw the median, AL, where L is the midpoint of BC. Then, GP = 2OG and AG = 2GL and by vertical angles we have that $\angle AGH \cong \angle LGO$

Then $\triangle AHG \sim \triangle LOG$ and OL is parallel to AP.



The Euler Segment

Since OL is perpendicular to BC, so it AP, making P lie on the altitude from A.

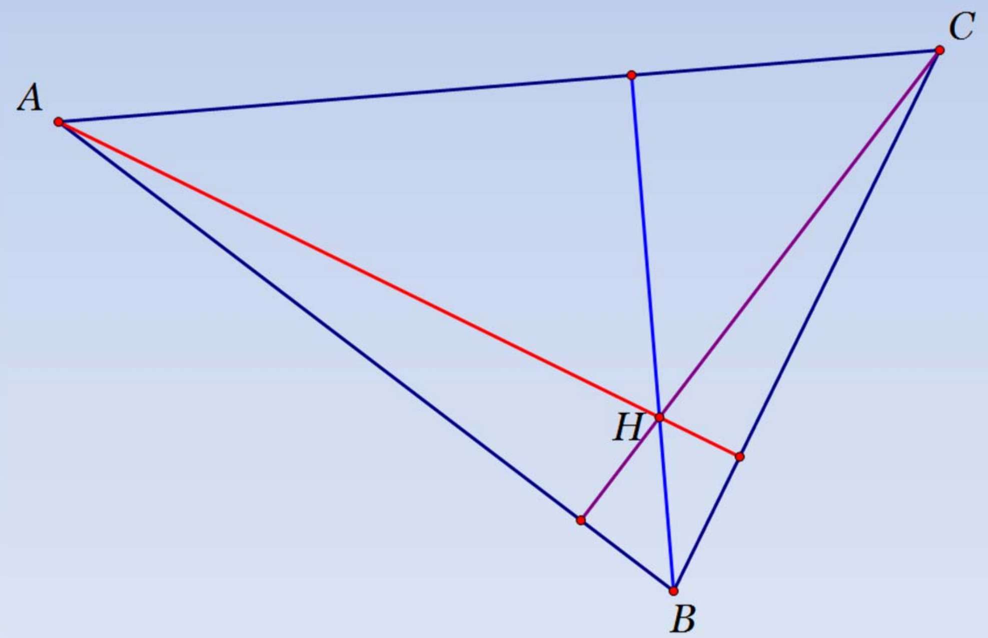
Repeating this for each of the other vertices gives us our result. By construction GP = 2OG.

This line segment is called the Euler Segment of the triangle.

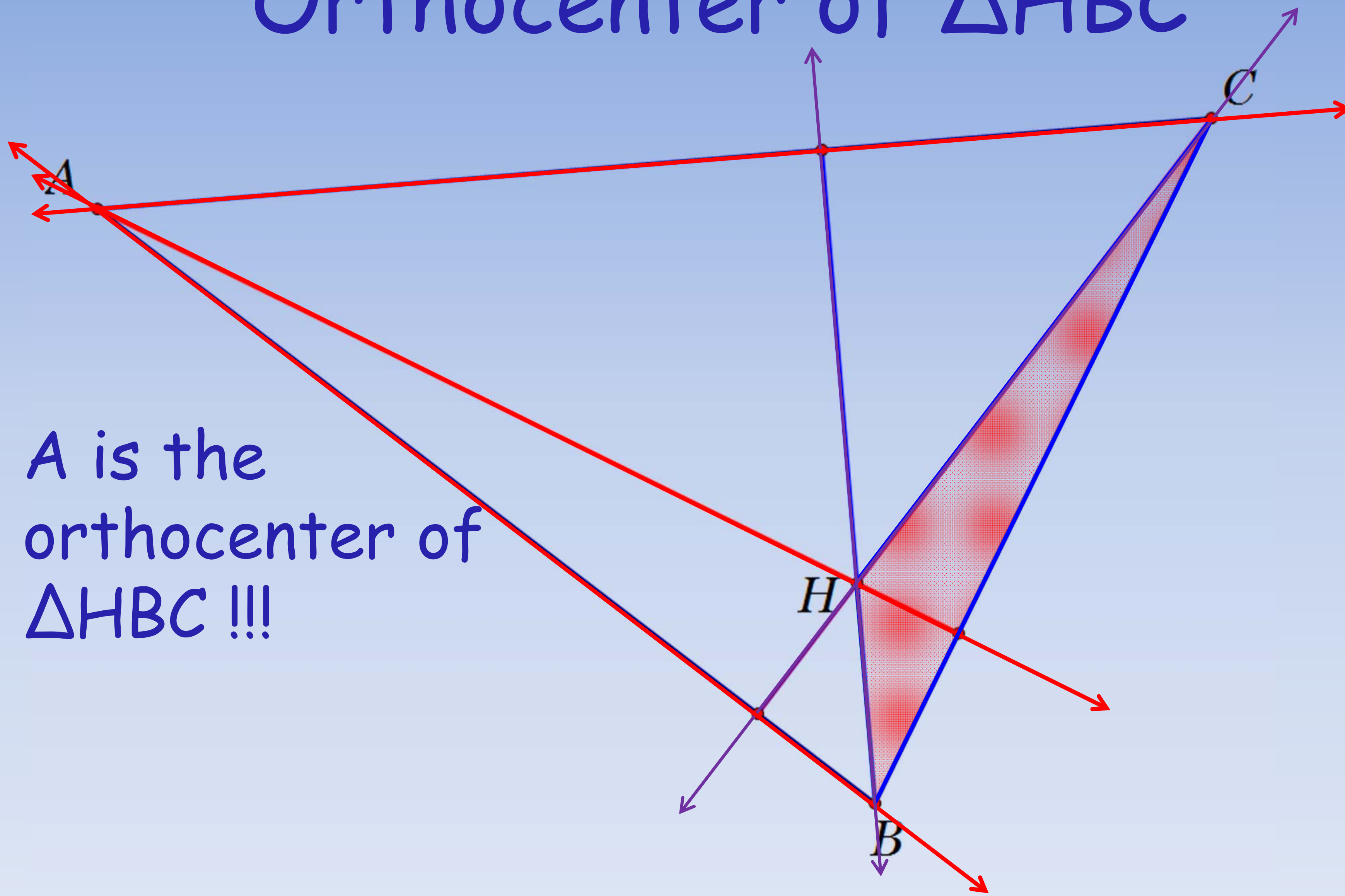
Orthic Quadruple

Let A, B, C, and H be four distinct points with A, B, and C noncollinear and H the orthocenter of ΔABC.

Line determined by 2 of these points is perpendicular to the line determined by other 2 points!!



Orthocenter of ΔHBC



A is the orthocenter of ΔHBC !!!

Orthic Quadruple

- H is the orthocenter of ΔABC
- A is the orthocenter of ΔHBC
- B is the orthocenter of ΔHAC
- C is the orthocenter of ΔHAB

{A, B, C, H} is called Orthic Quadruple

Orthic Quadruple

Given three points {A, B, C} there is always a fourth point, H, making an orthic quadruple UNLESS

1. A, B, C collinear
2. A, B, C form a right triangle

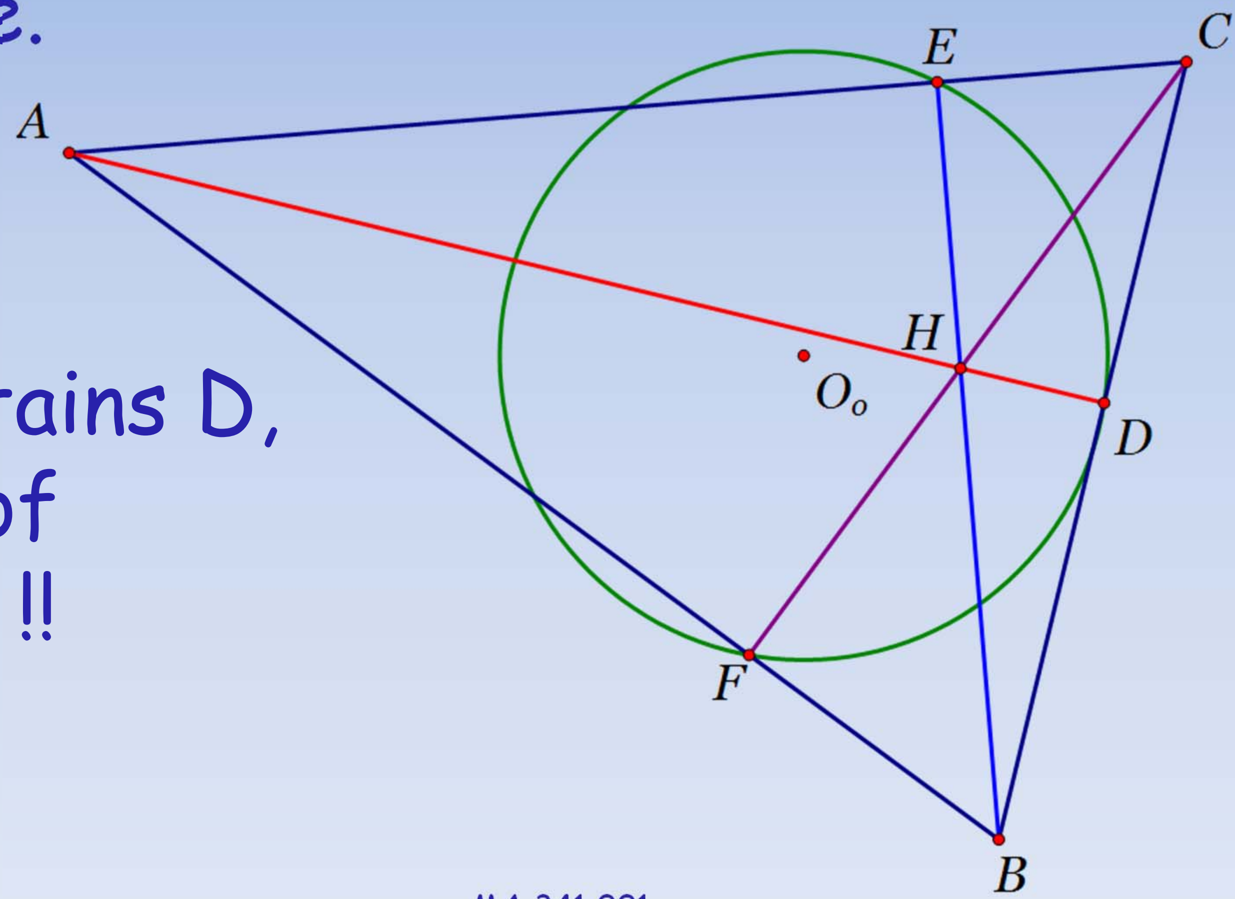
Orthic Triangle

Let A, B, C form a triangle and let D, E, F denote the intersections of the altitudes from A, B, and C with the lines $\overleftrightarrow{BC}$, $\overleftrightarrow{AC}$, and $\overleftrightarrow{AB}$ respectively. The triangle DEF is called the orthic triangle.

Theorem: The orthic triangles of each of the four triangles determined by an orthic quadruple are all the same.

Circumcircle of Orthic Triangle

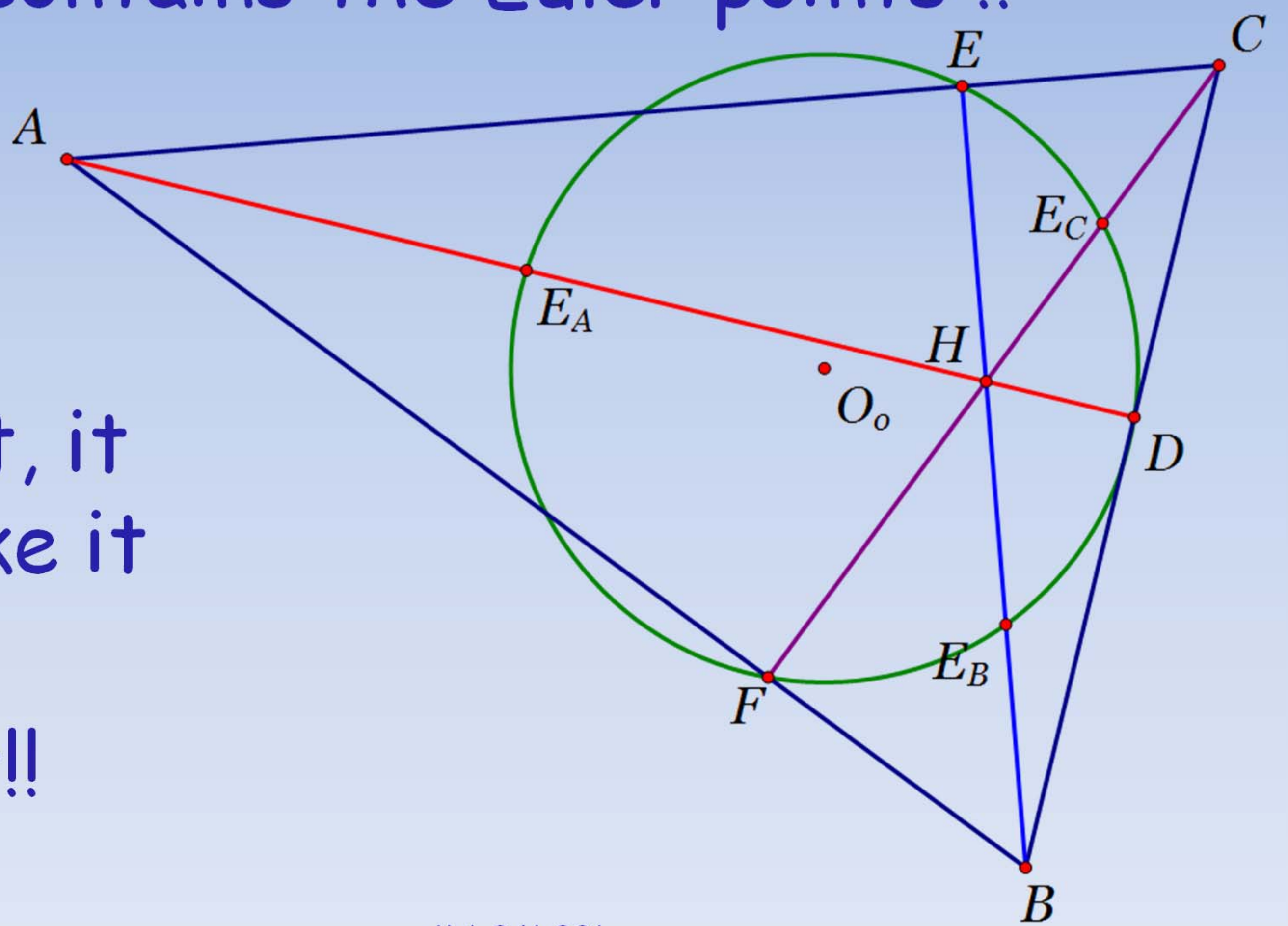
Consider the circumcircle of the orthic triangle.



It contains D, E, F - of course !!

Circumcircle of Orthic Triangle

It also contains the Euler points !!

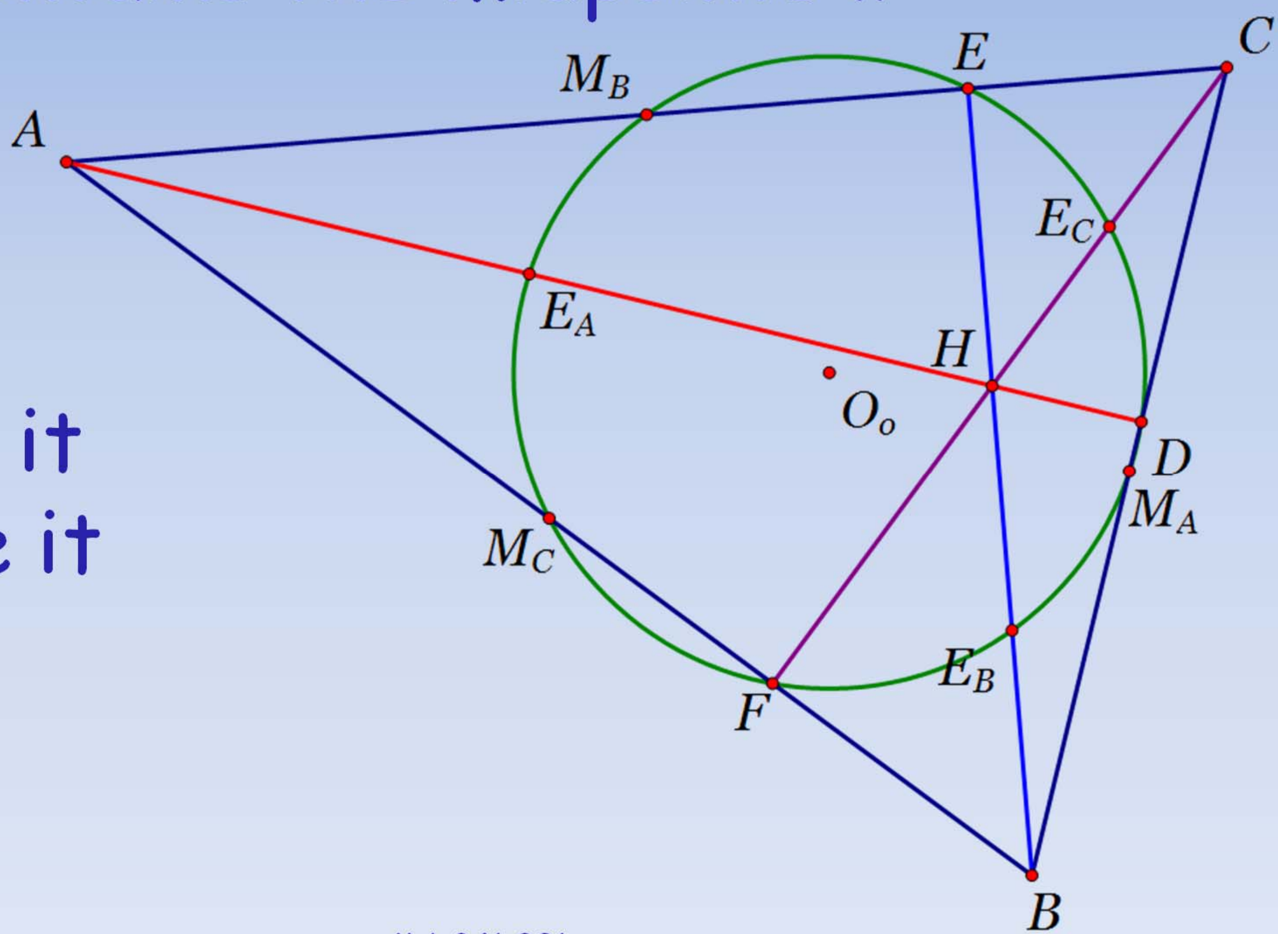


At least, it looks like it does.

Prove it!!

Circumcircle of Orthic Triangle

It also contains the midpoints !!



At least, it looks like it does.

Nine Point Circle Theorem

Theorem: For any triangle the following nine points all lie on the same circle: the three feet of the altitudes, the three Euler points, and the three midpoints of the sides. Furthermore, the line segments joining an Euler point to the midpoint of the opposite side is a diameter of this circle.

Sometimes called Feuerbach's Circle.