Triangles IV

Euler Segment
Nine Point Circle Theorem

Triangle Information

1) Real numbers a_1, a_2, a_3 are sides of a triangle in the plane iff $a_i + a_j > a_k$ for any distinct choices of i,j,k.
2) Stewart's Theorem:

$$a^2n + b^2m = c(d^2 + mn)$$

3) Lengths of medians

$$2m_a^2 = b^2 + c^2 - \frac{1}{2}a^2$$

$$2m_b^2 = a^2 + c^2 - \frac{1}{2}b^2$$

$$2m_c^2 = a^2 + b^2 - \frac{1}{2}c^2$$

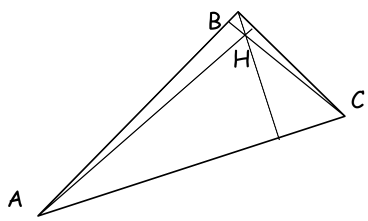
Triangle Information

4) $\frac{3}{4}(a + b + c) < m_a + m_b + m_c < a + b + c$

5) $m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$

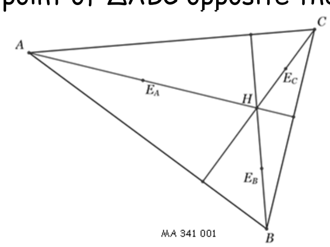
6) The sum of squares of lengths of segments joining centroid with vertices is one-third the sum of the squares of the lengths of sides.

Orthocenter



Euler Points

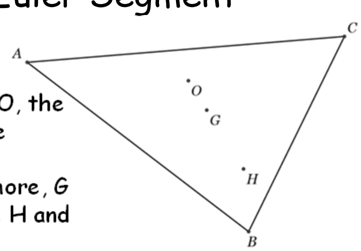
The midpoint of the segment from the vertex to the orthocenter is called the Euler point of ΔABC opposite the side.



The Euler Segment

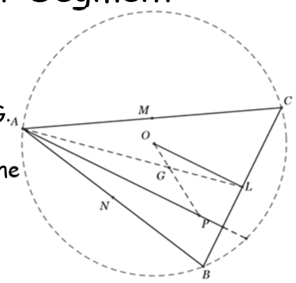
The circumcenter O, the centroid G, and the orthocenter H are collinear. Furthermore, G lies between O and H and

$$\frac{GH}{OG} = 2$$



The Euler Segment

(Symmetric Triangles)
Extend OG twice its length to a
point P, that is GP = 2OG. We
need to show that P is the orthocenter.

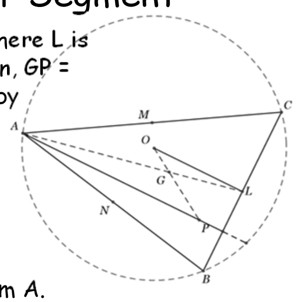


The Euler Segment

Draw the median, AL, where L is the midpoint of BC. Then, GP = 2OG and AG = 2GL and by vertical angles we have that $\angle AGH \cong \angle LGO$

Then $\triangle AHG \sim \triangle LOG$ and OL is parallel to AP.

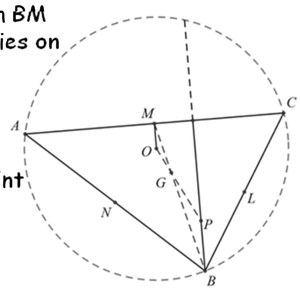
So P lies on altitude from A.



The Euler Segment

Do the same with median BM and we can show that P lies on the altitude through B.

Since it lies on two altitudes and the 3 altitudes meet in one point P must be orthocenter!



The Euler Segment

Note that by construction GP = 2OG.

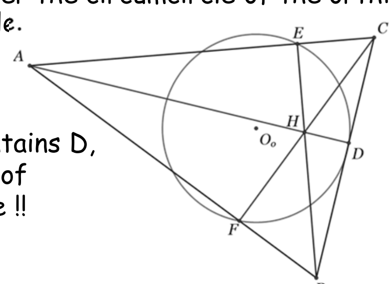
This line segment is called the Euler Segment of the triangle.

Orthic Triangle

Let A, B, C form a triangle and let H_A, H_B, H_C denote the intersections of the altitudes from A, B, and C with the lines $\overleftrightarrow{BC}$, $\overleftrightarrow{AC}$, and $\overleftrightarrow{AB}$ respectively. The triangle $\Delta H_AH_BH_C$ is called the orthic triangle.

Circumcircle of Orthic Triangle

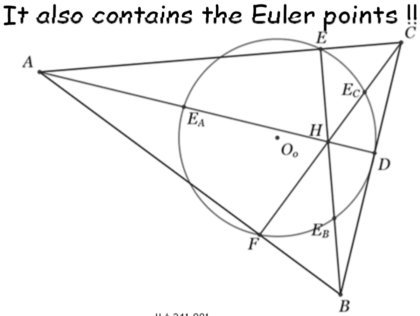
Consider the circumcircle of the orthic triangle.



It contains D, E, F – of course !!

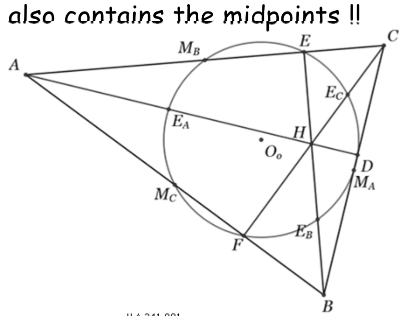
Circumcircle of Orthic Triangle

Claim: It *also contains the Euler points* !!



Circumcircle of Orthic Triangle

Claim: It *also contains the midpoints* !!



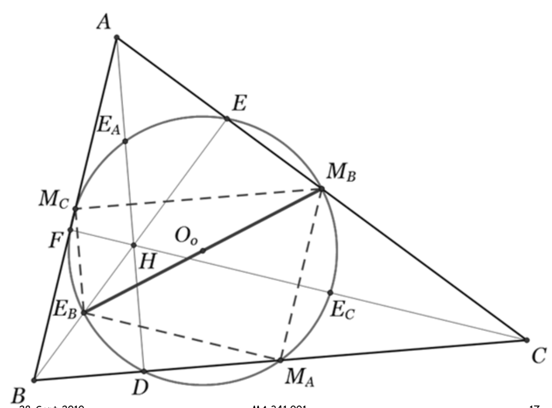
Nine Point Circle Theorem

Theorem: For any triangle the following nine points all lie on the same circle: the three feet of the altitudes, the three Euler points, and the three midpoints of the sides. Furthermore, the line segments joining an Euler point to the midpoint of the opposite side is a diameter of this circle.

Sometimes called Feuerbach's Circle.

Proof

Must show
1. H_A, H_B, H_C lie on circle
2. M_A, M_B, M_C lie on circle
3. E_A, E_B, E_C lie on circle
4. E_AM_A, E_BM_B, E_CM_C are diameters



Construct E_BM_B and consider unique circle with it as diameter.
$M_AM_B \parallel AB$ by midline theorem in ΔABC
In ΔBHC, $E_BM_A \parallel HC=CF$ – same theorem
$CF \perp AB \Rightarrow E_BM_A \perp M_AM_B \Rightarrow \angle E_BM_AM_B=90$.
Similarly, we show $\angle M_BM_CE_B=90$
Thus, M_A and M_C lie on this circle with diameter E_BM_B.

Circle contains M_A, M_B and M_C so it is the circumcircle of the medial triangle and has diameter E_BM_B.

Similarly, we can show that E_AM_A and E_CM_C are diameters of the circumcircle of the medial triangle.

Thus, this circle contains 6 of the 9 points.

Note that D is on the circle since $\angle E_ADM_A=90$. Similarly, E and F lie on circle.

Center & Radius of 9 Point Circle

What are the center and the radius of the Nine Point Circle? Are they related to any of the quantities we have already studied?

9 Point circle = circumcircle of medial Δ

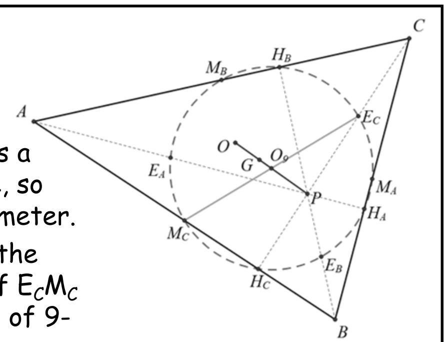
Side of medial triangle = ½ side of original triangle and all angles are equal.

Circumradius of triangle comes from Law of Sines

$$r = \tfrac{1}{2}R$$

$\angle E_CH_CM_C$ is a right angle, so E_CM_C a diameter.

Let O_9 be the midpoint of E_CM_C and center of 9-point circle.



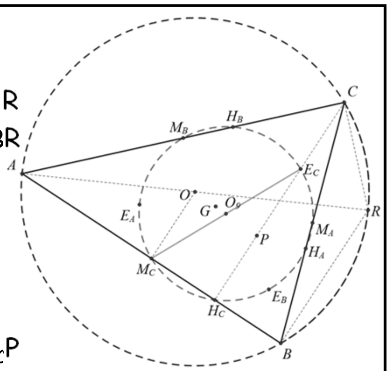
Construct AO intersecting circumcircle at R

Construct CR, BR

OM_C midline to ΔABR and $OM_C \parallel BR$

$\angle ABR=90°$ so $BR \parallel CP$

Thus, $OM_C \parallel E_C P$



Similarly, $BH_B \parallel CR$

$\Rightarrow$CRBP parallelogram

$\Rightarrow$ CP = RB

$\Rightarrow OM_C = \frac{1}{2}RB$

$= \frac{1}{2}CP$

$= PE_C$

$OE_C PM_C$ parallelogram

Diagonals bisect one another

$\Rightarrow O_9$ midpoint of OP, the Euler segment