

# Triangles IV

Euler Segment  
Nine Point Circle Theorem

# Triangle Information

1) Real numbers  $a_1, a_2, a_3$  are sides of a triangle in the plane iff  $a_i + a_j > a_k$  for any distinct choices of  $i, j, k$ .

2) Stewart's Theorem:

$$a^2n + b^2m = c(d^2 + mn)$$

3) Lengths of medians

$$2m_a^2 = b^2 + c^2 - \frac{1}{2}a^2$$

$$2m_b^2 = a^2 + c^2 - \frac{1}{2}b^2$$

$$2m_c^2 = a^2 + b^2 - \frac{1}{2}c^2$$

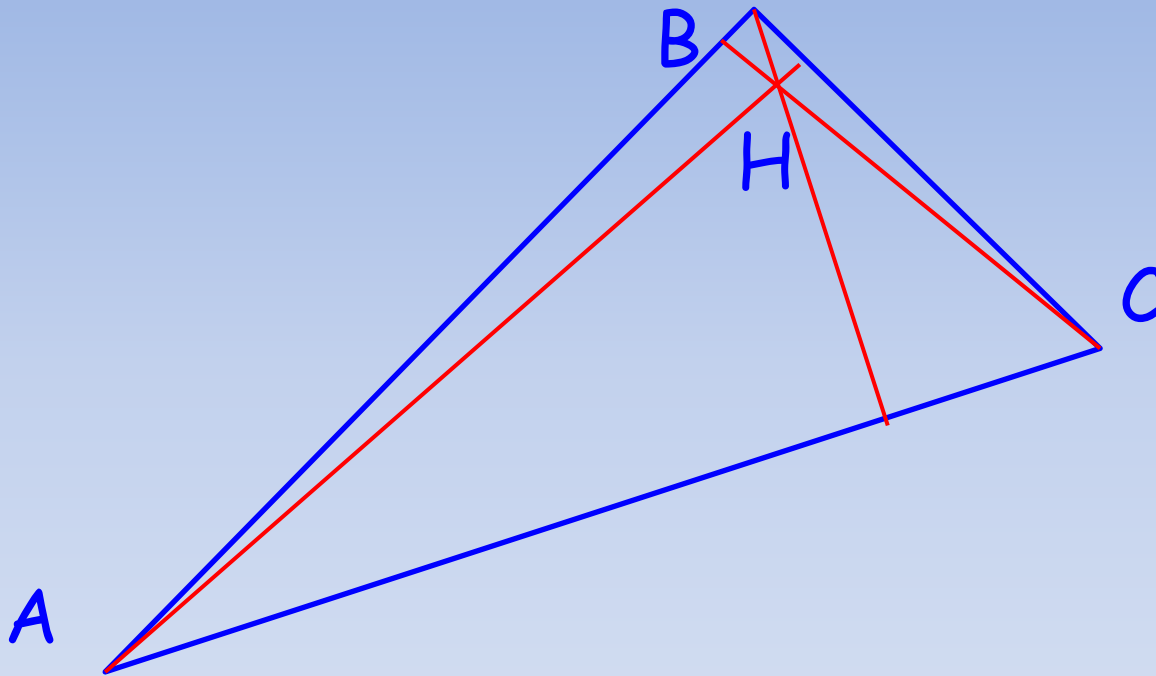
# Triangle Information

$$4) \quad \frac{3}{4}(a+b+c) < m_a + m_b + m_c < a+b+c$$

$$5) \quad m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$$

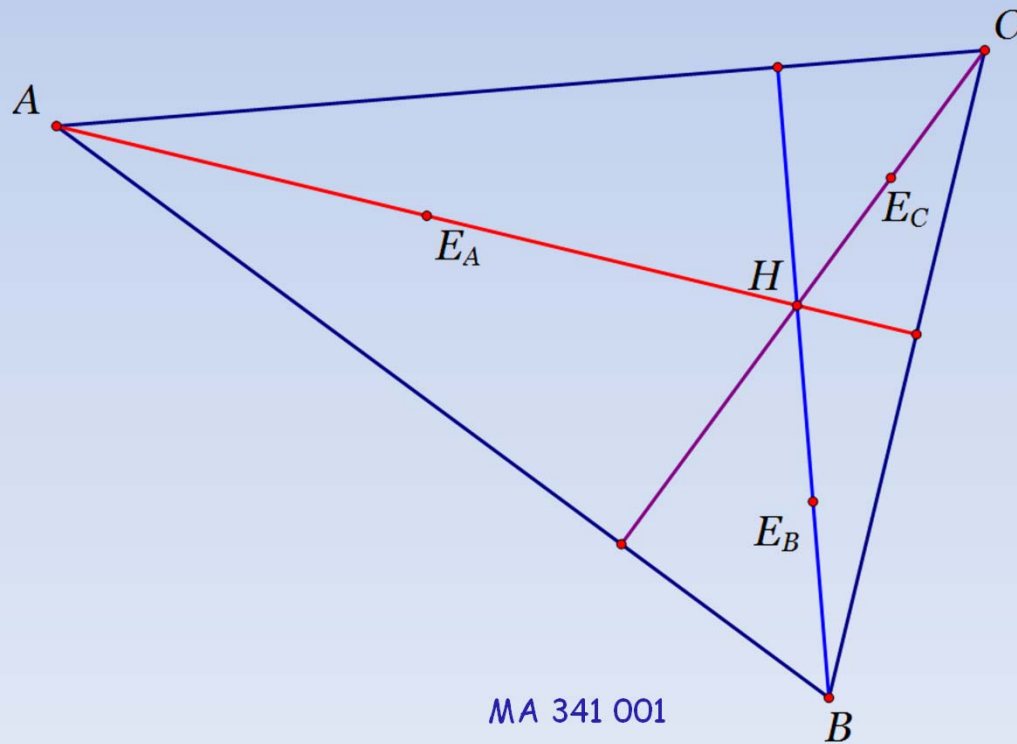
6) The sum of squares of lengths of segments joining centroid with vertices is one-third the sum of the squares of the lengths of sides.

# Orthocenter



# Euler Points

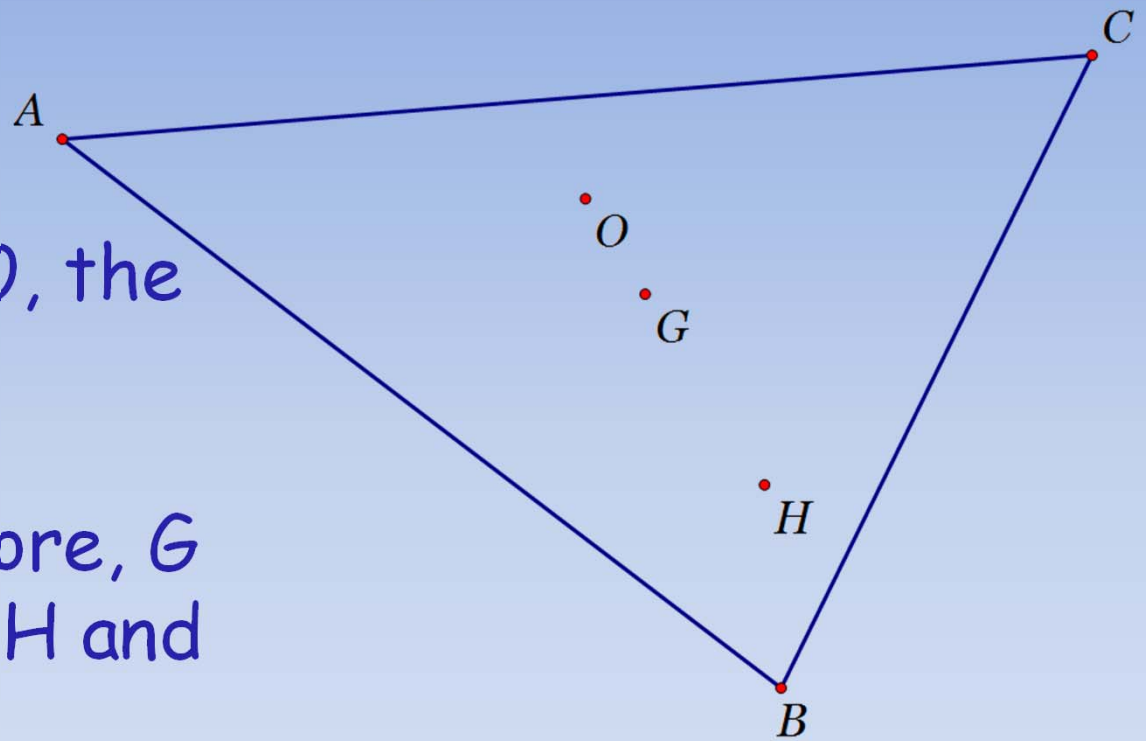
The midpoint of the segment from the vertex to the orthocenter is called the **Euler point** of  $\triangle ABC$  opposite the side.



# The Euler Segment

The circumcenter  $O$ , the centroid  $G$ , and the orthocenter  $H$  are collinear. Furthermore,  $G$  lies between  $O$  and  $H$  and

$$\frac{GH}{OG} = 2$$



# The Euler Segment

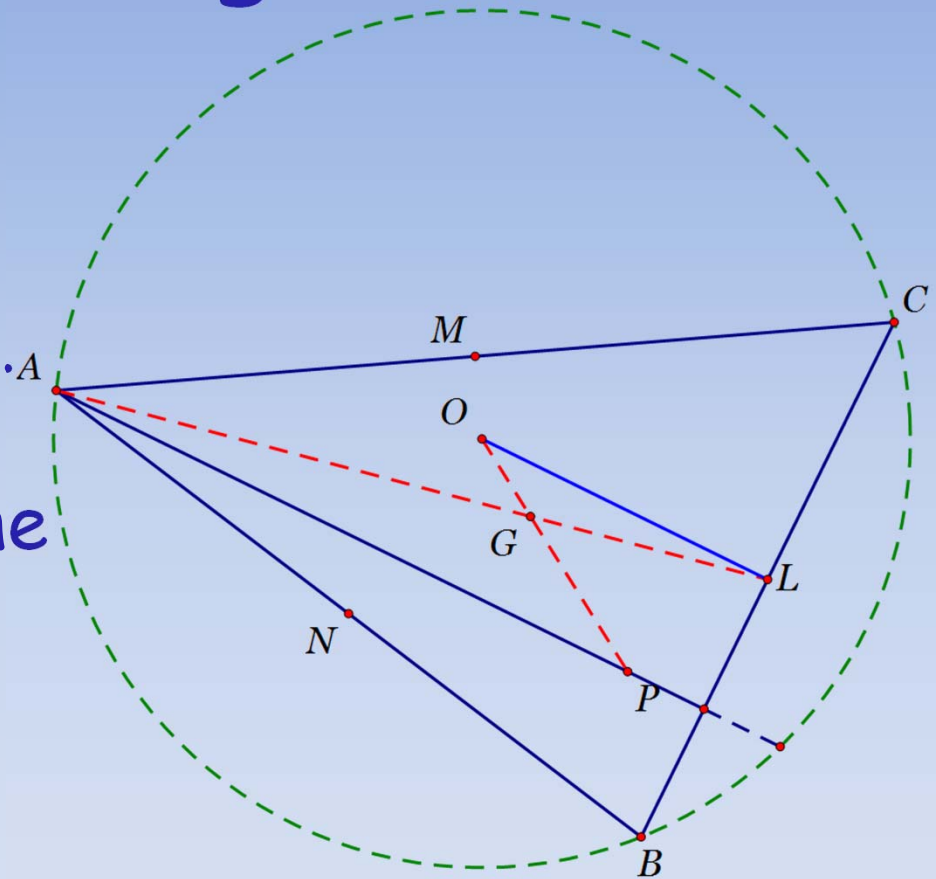
(Symmetric Triangles)

Extend  $OG$  twice its length to a

point  $P$ , that is  $GP = 2OG$ .

We

need to show that  $P$  is the orthocenter.

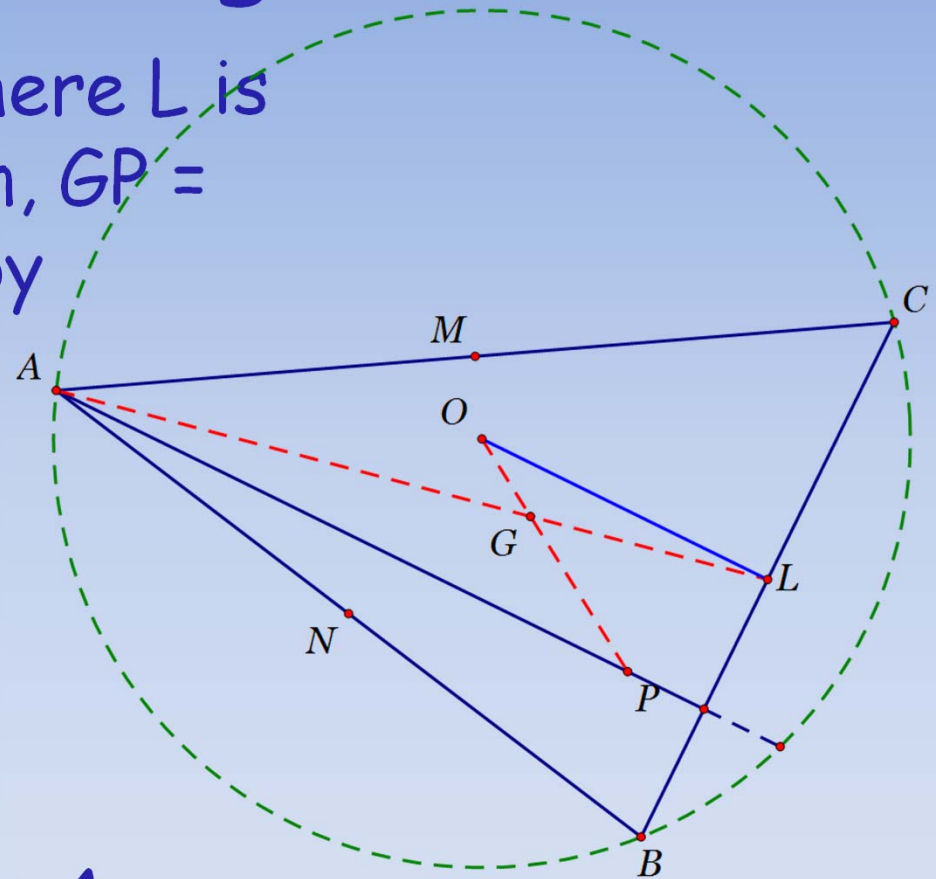


# The Euler Segment

Draw the median,  $AL$ , where  $L$  is the midpoint of  $BC$ . Then,  $GP = 2OG$  and  $AG = 2GL$  and by vertical angles we have that  $\angle AGH \cong \angle LGO$

Then  $\triangle AHG \sim \triangle LOG$  and  $OL$  is parallel to  $AP$ .

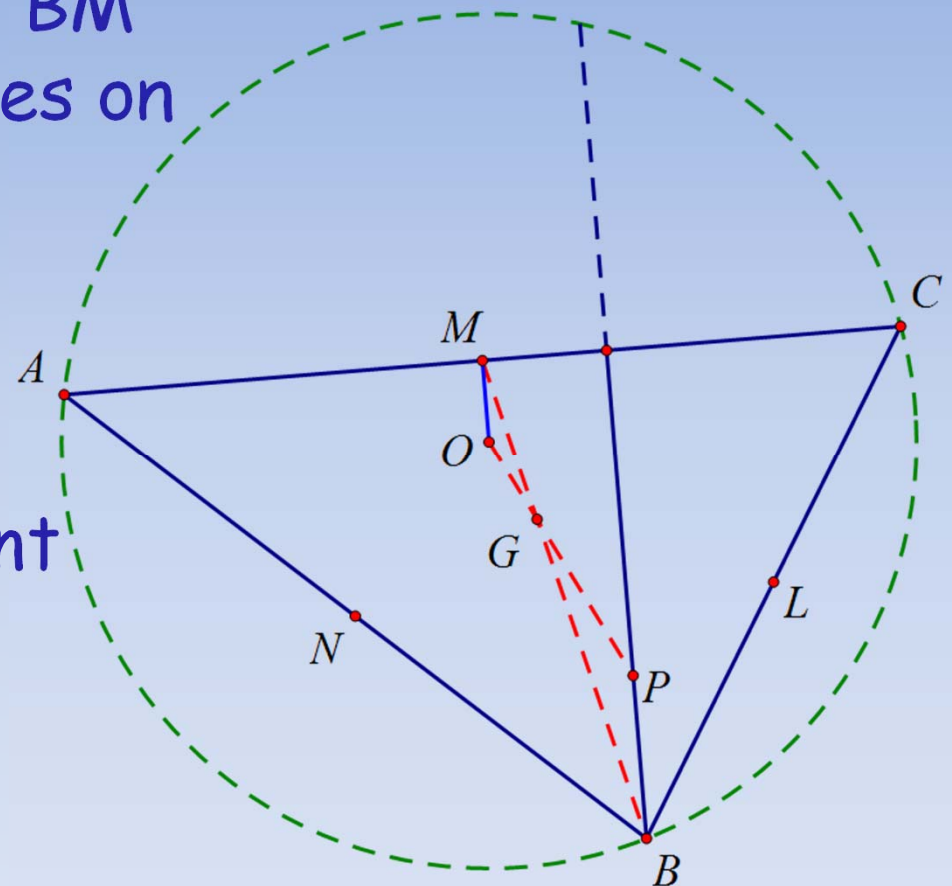
So  $P$  lies on altitude from  $A$ .



# The Euler Segment

Do the same with median  $BM$  and we can show that  $P$  lies on the altitude through  $B$ .

Since it lies on two altitudes and the 3 altitudes meet in one point  $P$  must be orthocenter!



# The Euler Segment

Note that by construction  $GP = 2OG$ .

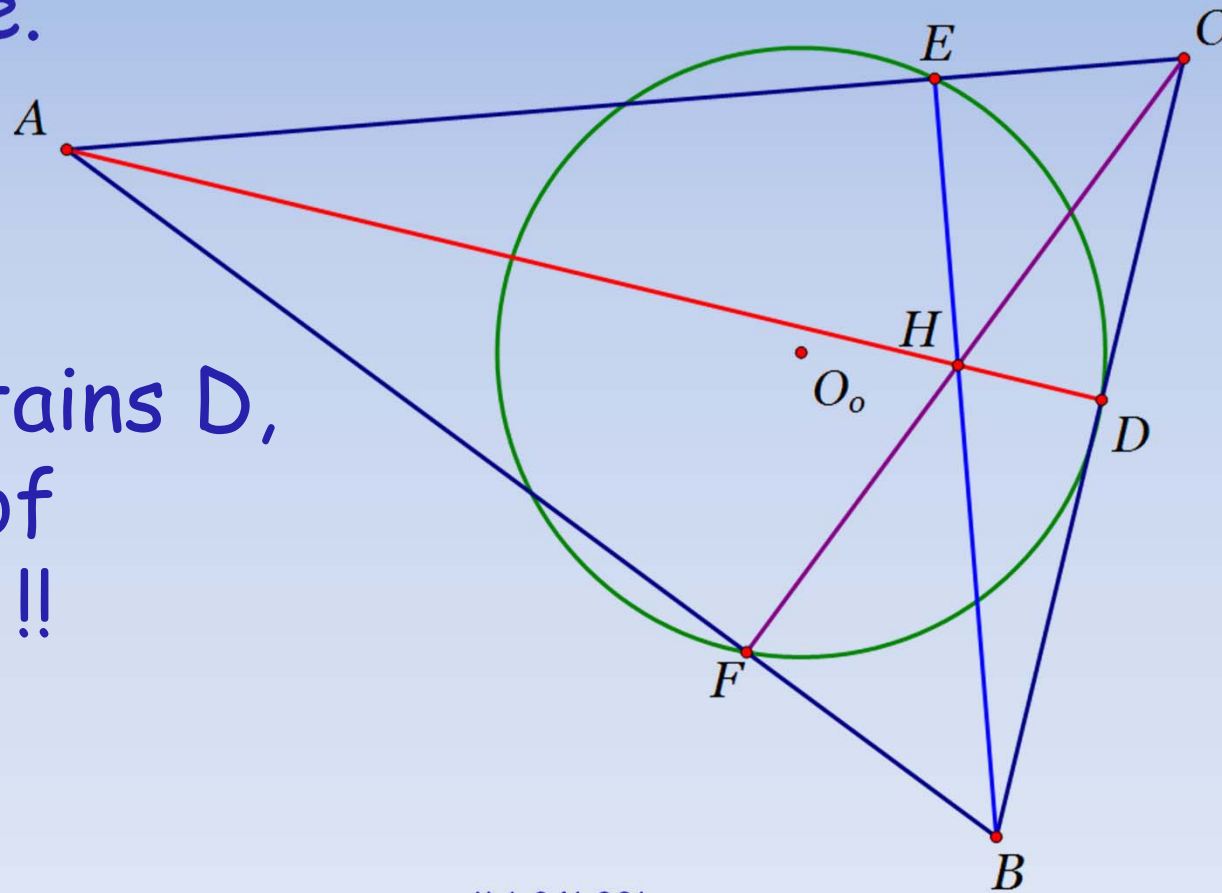
This line segment is called the Euler Segment of the triangle.

# Orthic Triangle

Let  $A, B, C$  form a triangle and let  $H_A, H_B, H_C$  denote the intersections of the altitudes from  $A, B,$  and  $C$  with the lines  $\overleftrightarrow{BC}$ ,  $\overleftrightarrow{AC}$ , and  $\overleftrightarrow{AB}$  respectively. The triangle  $\Delta H_A H_B H_C$  is called the orthic triangle.

# Circumcircle of Orthic Triangle

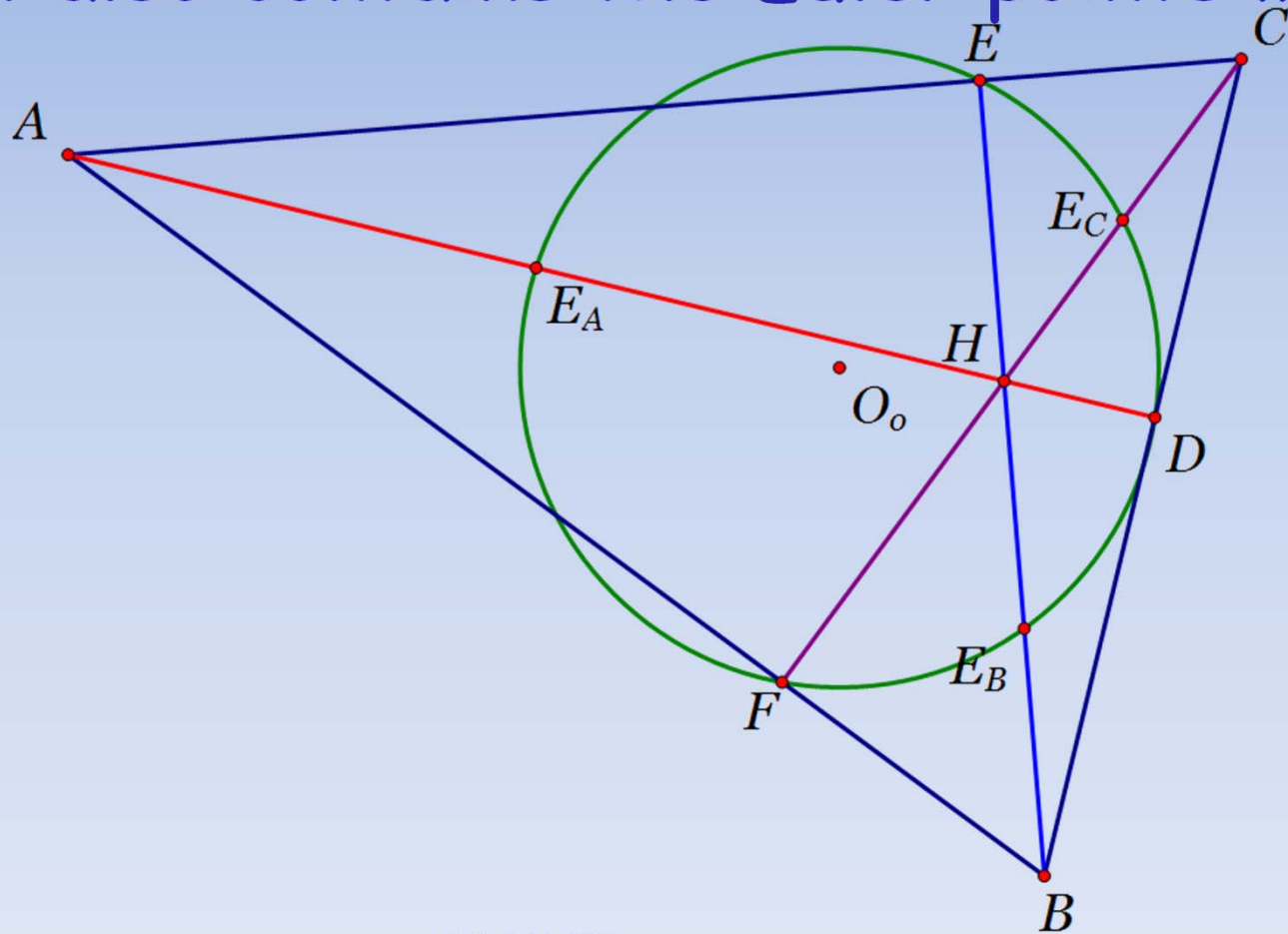
Consider the circumcircle of the orthic triangle.



It contains  $D$ ,  
 $E$ ,  $F$  - of  
course !!

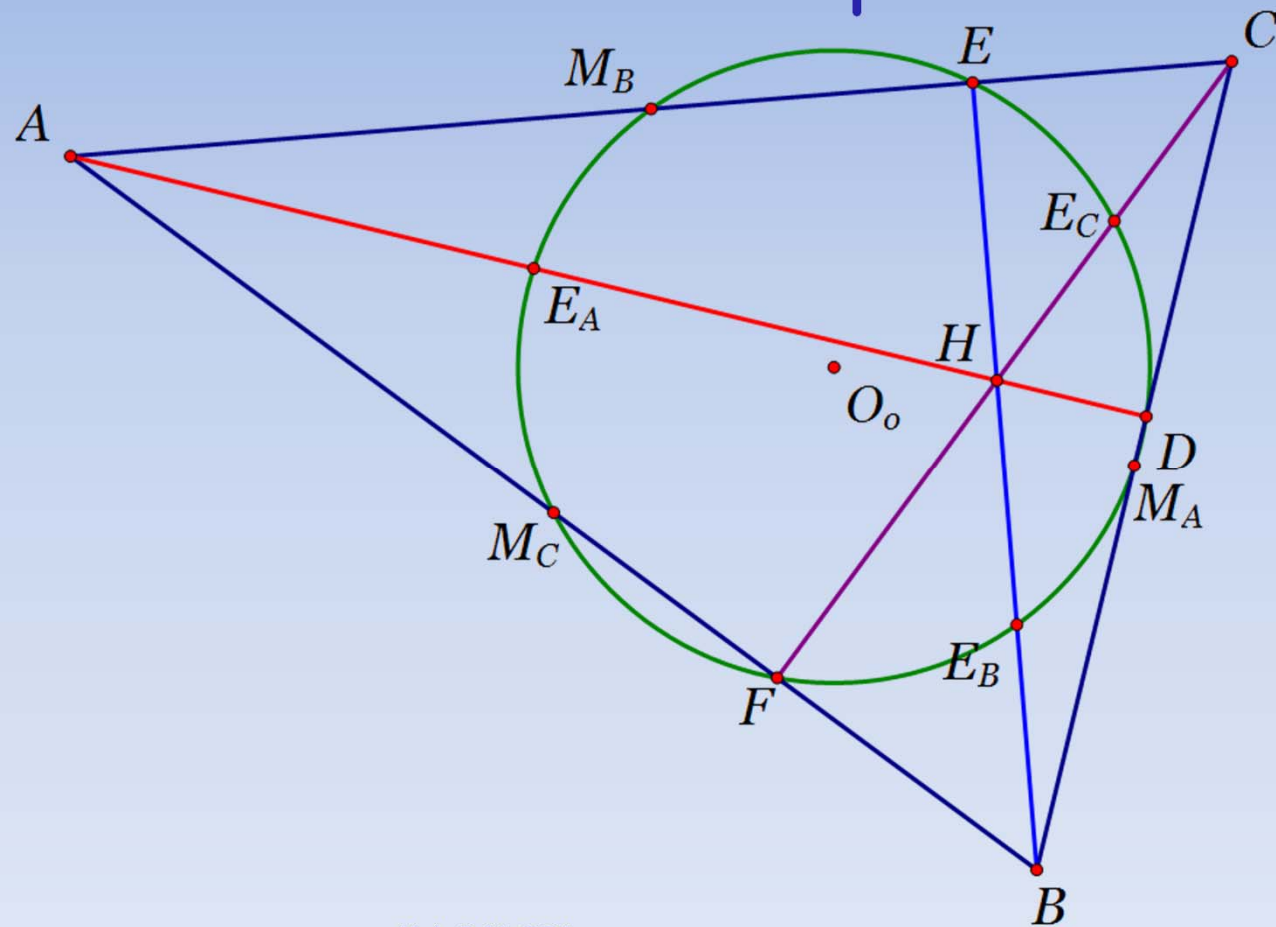
# Circumcircle of Orthic Triangle

Claim: It also contains the Euler points !!



# Circumcircle of Orthic Triangle

Claim: It also contains the midpoints !!



# Nine Point Circle Theorem

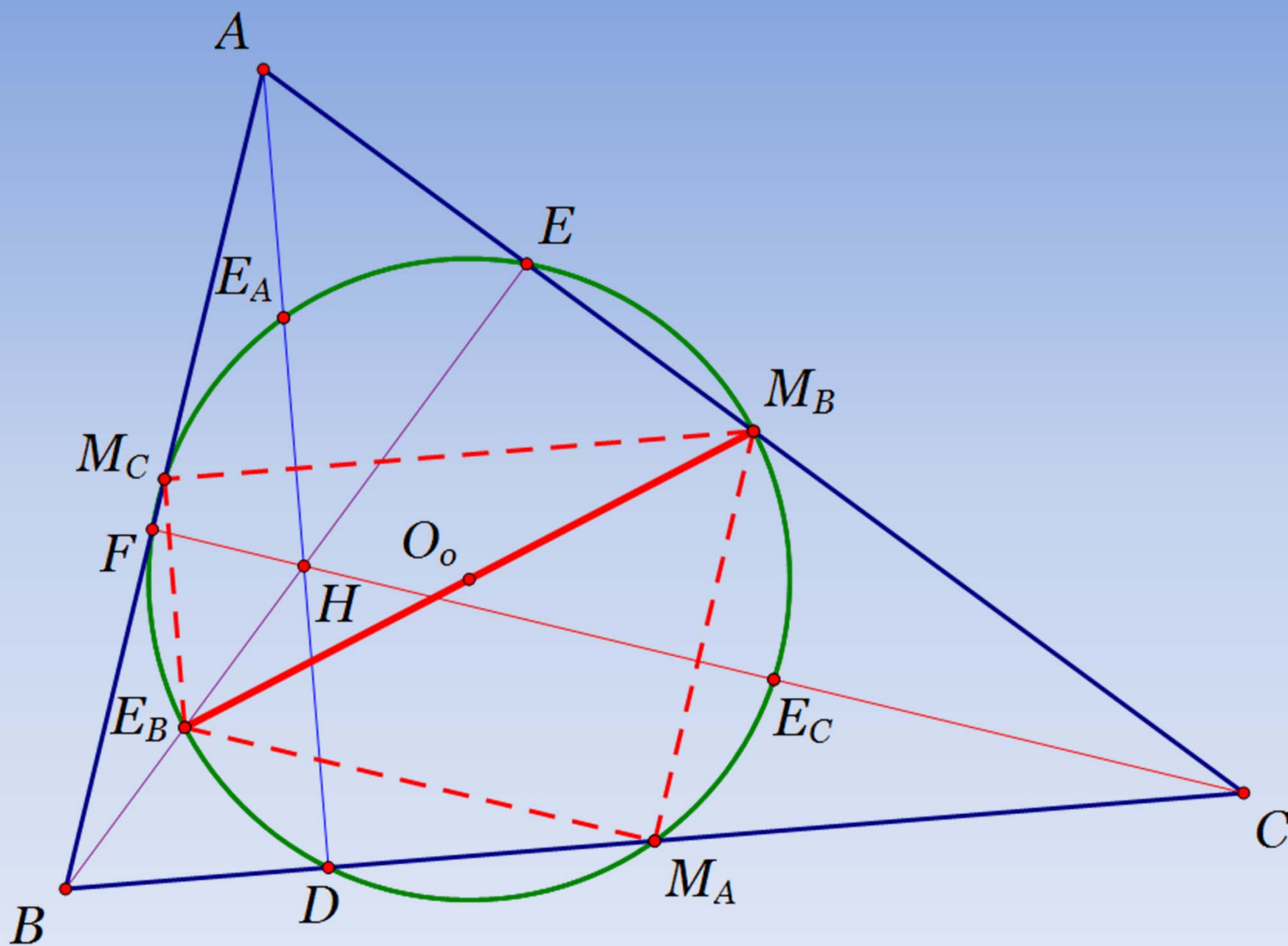
Theorem: For any triangle the following nine points all lie on the same circle: the three feet of the altitudes, the three Euler points, and the three midpoints of the sides. Furthermore, the line segments joining an Euler point to the midpoint of the opposite side is a diameter of this circle.

Sometimes called Feuerbach's Circle.

# Proof

Must show

1.  $H_A, H_B, H_C$  lie on circle
2.  $M_A, M_B, M_C$  lie on circle
3.  $E_A, E_B, E_C$  lie on circle
4.  $E_A M_A, E_B M_B, E_C M_C$  are diameters



Construct  $E_B M_B$  and consider unique circle with it as diameter.

$M_A M_B \parallel AB$  by midline theorem in  $\triangle ABC$

In  $\triangle BHC$ ,  $E_B M_A \parallel HC = CF$  - same theorem

$CF \perp AB \Rightarrow E_B M_A \perp M_A M_B \Rightarrow \angle E_B M_A M_B = 90^\circ$ .

Similarly, we show  $\angle M_B M_C E_B = 90^\circ$

Thus,  $M_A$  and  $M_C$  lie on this circle with diameter  $E_B M_B$ .

Circle contains  $M_A$ ,  $M_B$  and  $M_C$  so it is the circumcircle of the medial triangle and has diameter  $E_B M_B$ .

Similarly, we can show that  $E_A M_A$  and  $E_C M_C$  are diameters of the circumcircle of the medial triangle.

Thus, this circle contains 6 of the 9 points.

Note that  $D$  is on the circle since  $\angle E_A D M_A = 90$ . Similarly,  $E$  and  $F$  lie on circle.

# Center & Radius of 9 Point Circle

What are the center and the radius of the Nine Point Circle? Are they related to any of the quantities we have already studied?

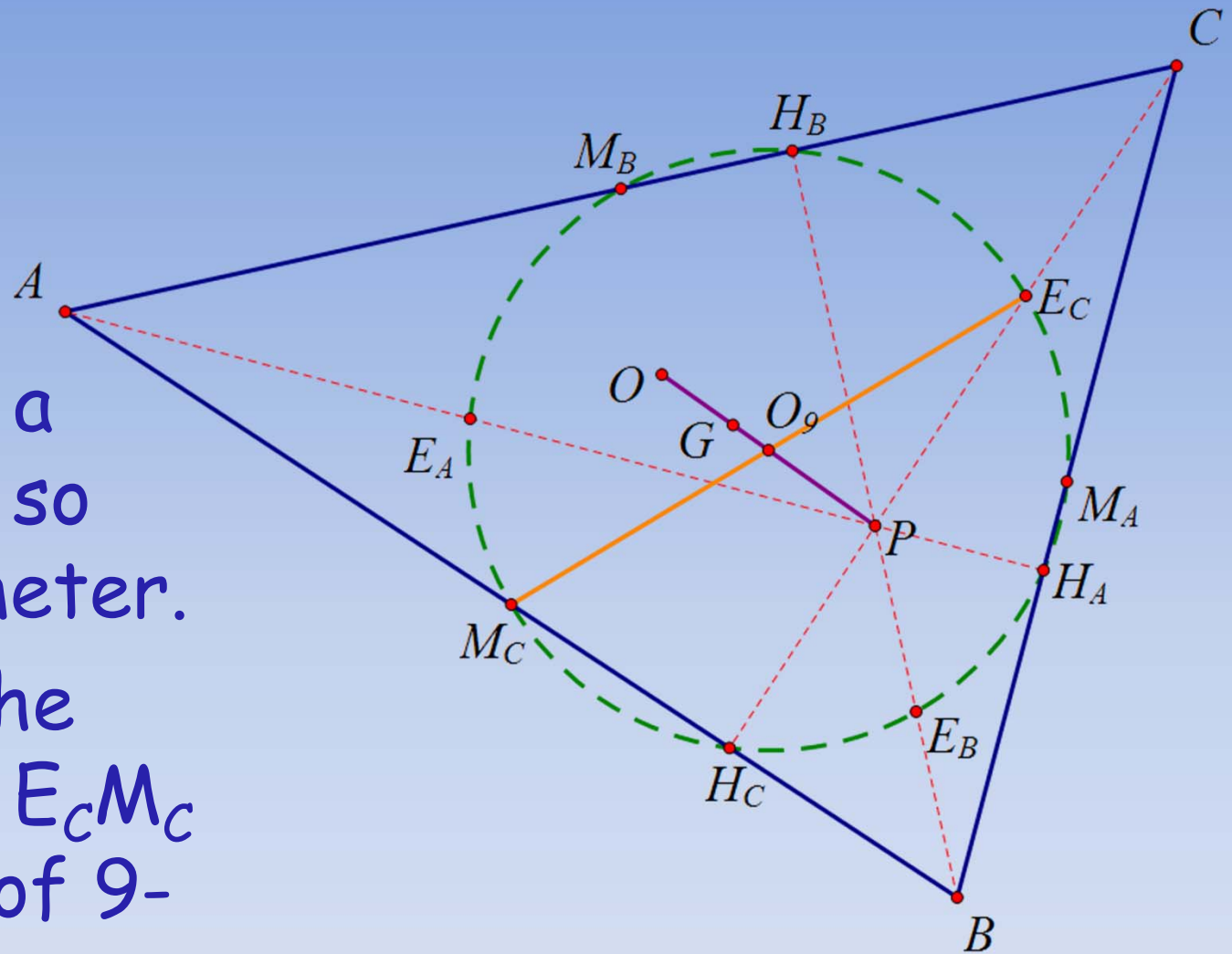
9 Point circle = circumcircle of medial  $\Delta$

Side of medial triangle =  $\frac{1}{2}$  side of original triangle and all angles are equal.

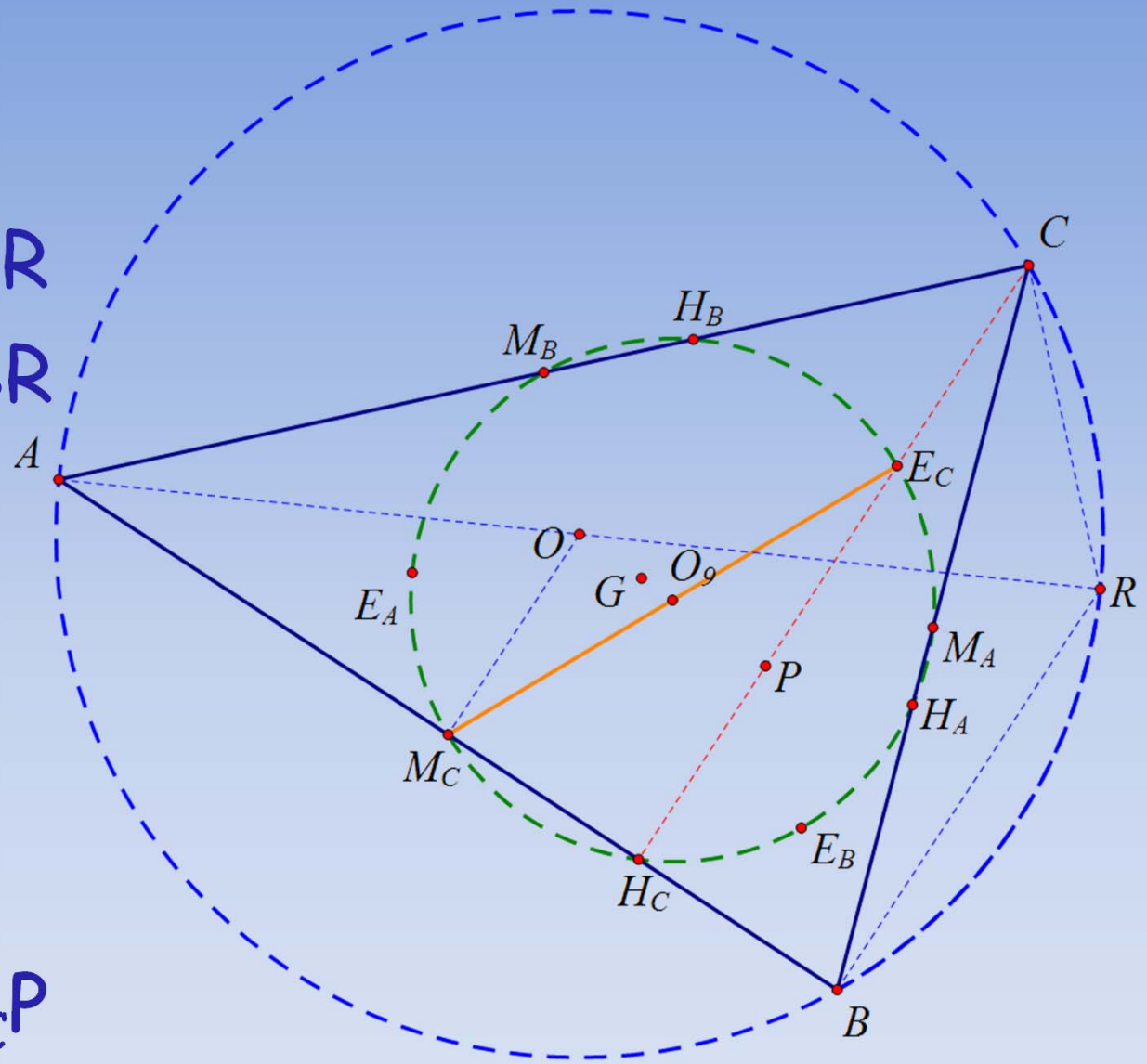
Circumradius of triangle comes from Law of Sines

$$r = \frac{1}{2}R$$

$\angle E_c H_c M_c$  is a  
 right angle, so  
 $E_c M_c$  a diameter.  
 Let  $O_9$  be the  
 midpoint of  $E_c M_c$   
 and center of 9-  
 point circle.



Construct  $AO$   
 intersecting  
 circumcircle at  $R$   
 Construct  $CR, BR$   
 $OM_C$  midline to  
 $\triangle ABR$  and  
 $OM_C \parallel BR$   
 $\angle ABR = 90^\circ$  so  
 $BR \parallel CP$   
 Thus,  $OM_C \parallel E_C P$



Similarly,  $BH_B \parallel CR$   
 $\Rightarrow CRBP$  parallelogram

$\Rightarrow CP = RB$

$\Rightarrow OM_C = \frac{1}{2}RB$   
 $= \frac{1}{2}CP$   
 $= PE_C$

$OE_CPM_C$  parallelogram

Diagonals bisect one another

$\Rightarrow O_9$  midpoint of  $OP$ , the Euler segment

