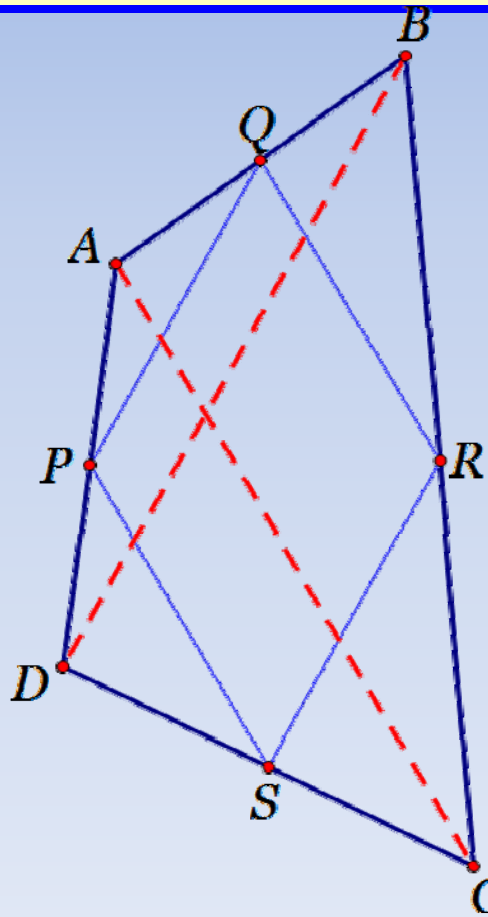
Quadrilateral Geometry

MA 341 – Topics in Geometry
Lecture 19

Varignon's Theorem I

The quadrilateral formed by joining the midpoints of consecutive sides of any quadrilateral is a parallelogram.

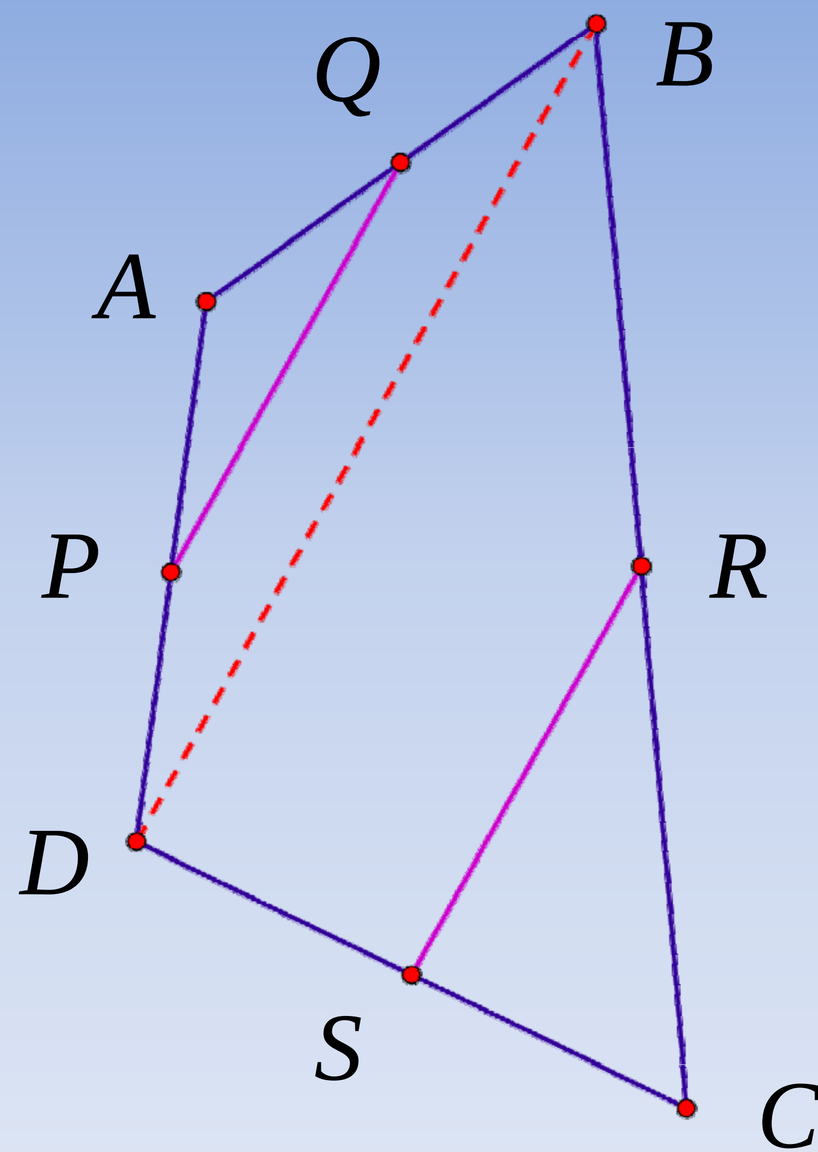
PQRS is a parallelogram.



Proof

PQ || BD
RS || BD

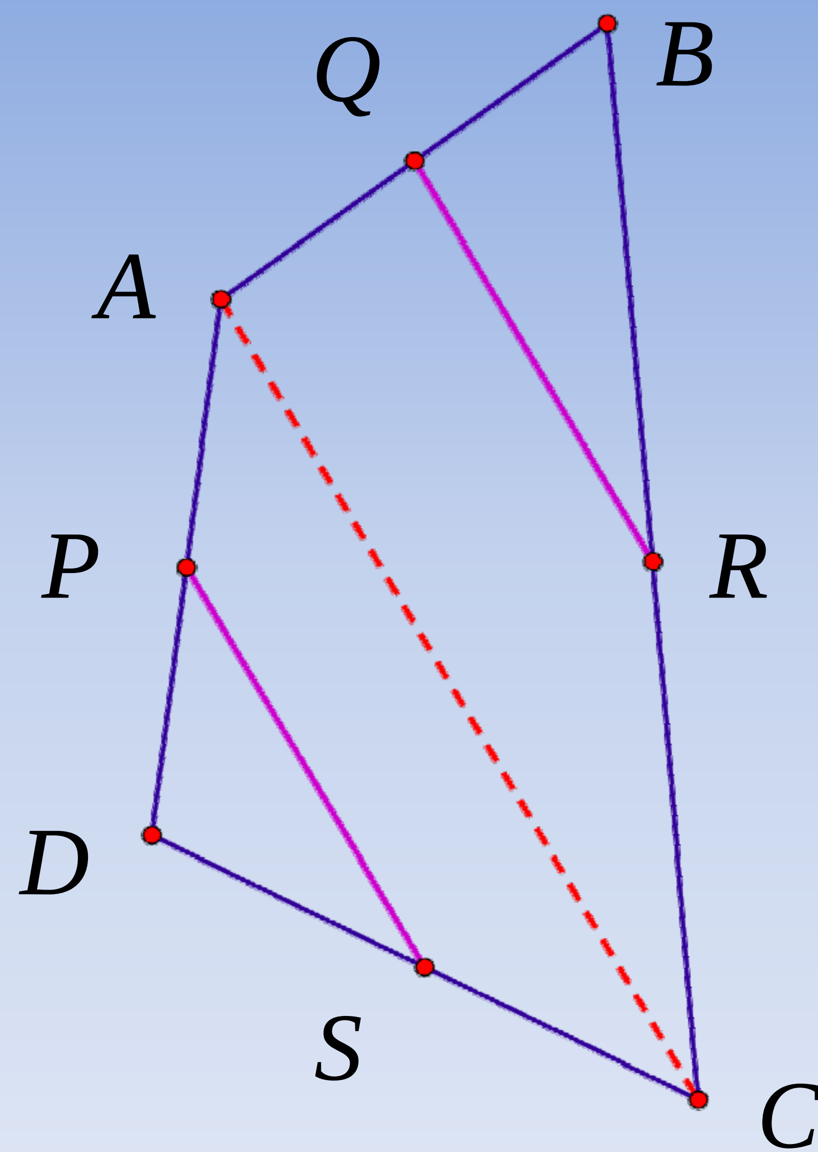
PQ || RS.



Proof

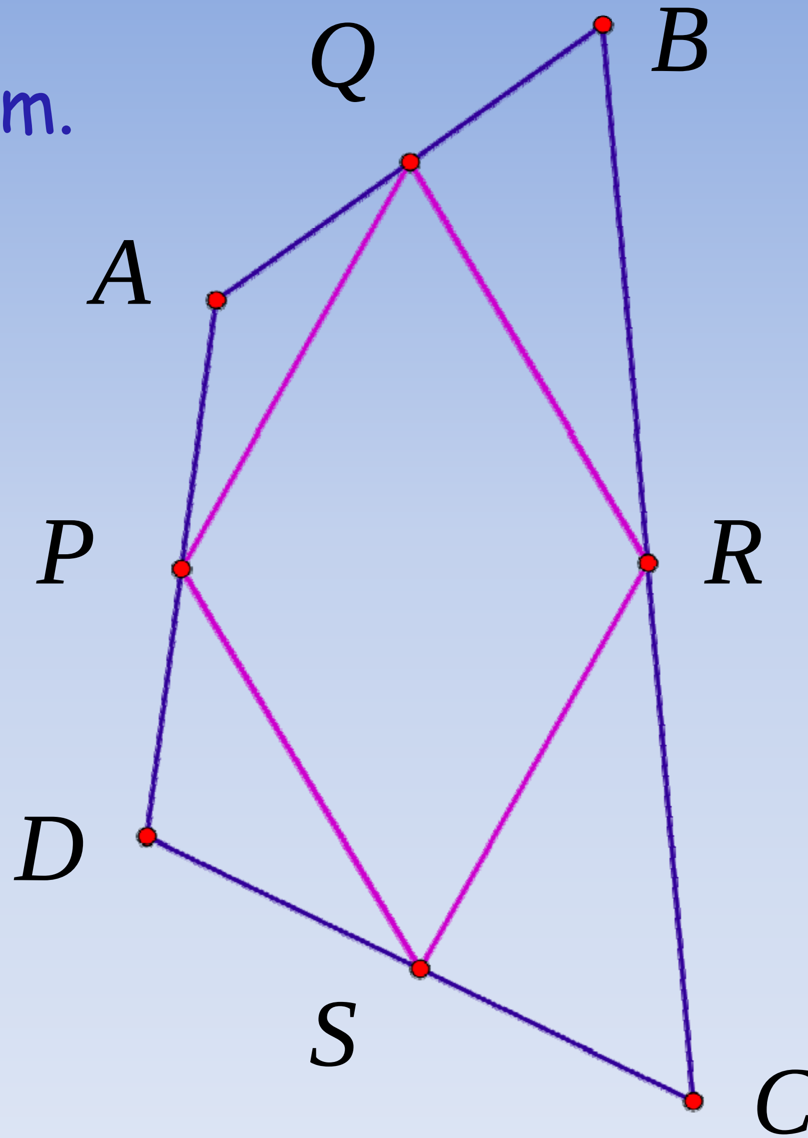
QR || AC
PS || AC

QR || PS.



Proof

PQRS is a parallelogram.



Starting with any quadrilateral gives us a parallelogram

What type of quadrilateral will give us

- a square?
- a rhombus?
- a rectangle?

Varignon's Corollary: Rectangle

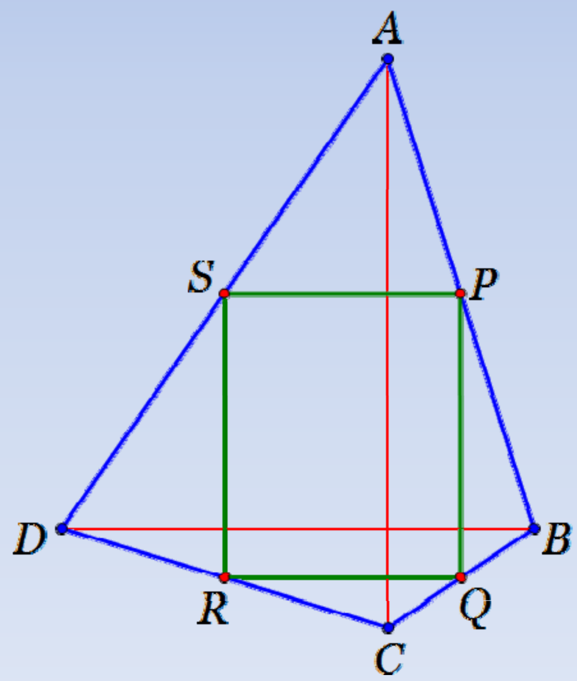
The quadrilateral formed by joining the midpoints of consecutive sides of a quadrilateral whose diagonals are perpendicular is a rectangle.

PQRS is a parallelogram

Each side is parallel to one of the diagonals

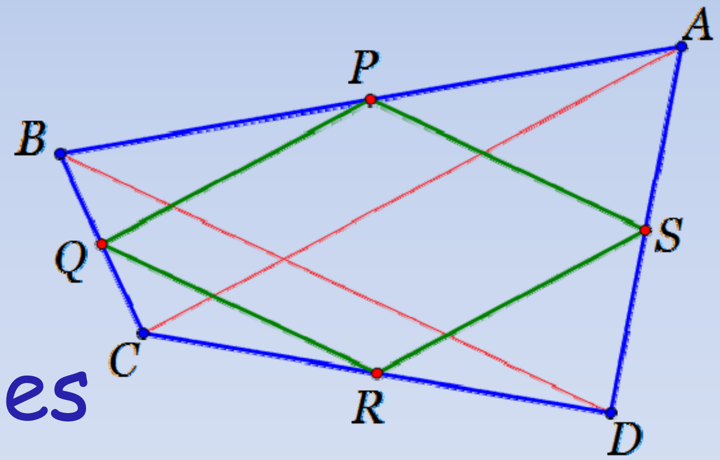
Diagonals perpendicular $\Rightarrow$ sides of parallelogram are perpendicular

$\Rightarrow$ parallelogram is a rectangle.



Varignon's Corollary: Rhombus

The quadrilateral formed by joining the midpoints of consecutive sides of a quadrilateral whose diagonals are congruent is a rhombus.



PQRS is a parallelogram

Each side is half of one of the diagonals

Diagonals congruent ⇒ sides of parallelogram are congruent

Varignon's Corollary: Square

The quadrilateral formed by joining the midpoints of consecutive sides of a quadrilateral whose diagonals are congruent and perpendicular is a square.

Quadrilateral Centers

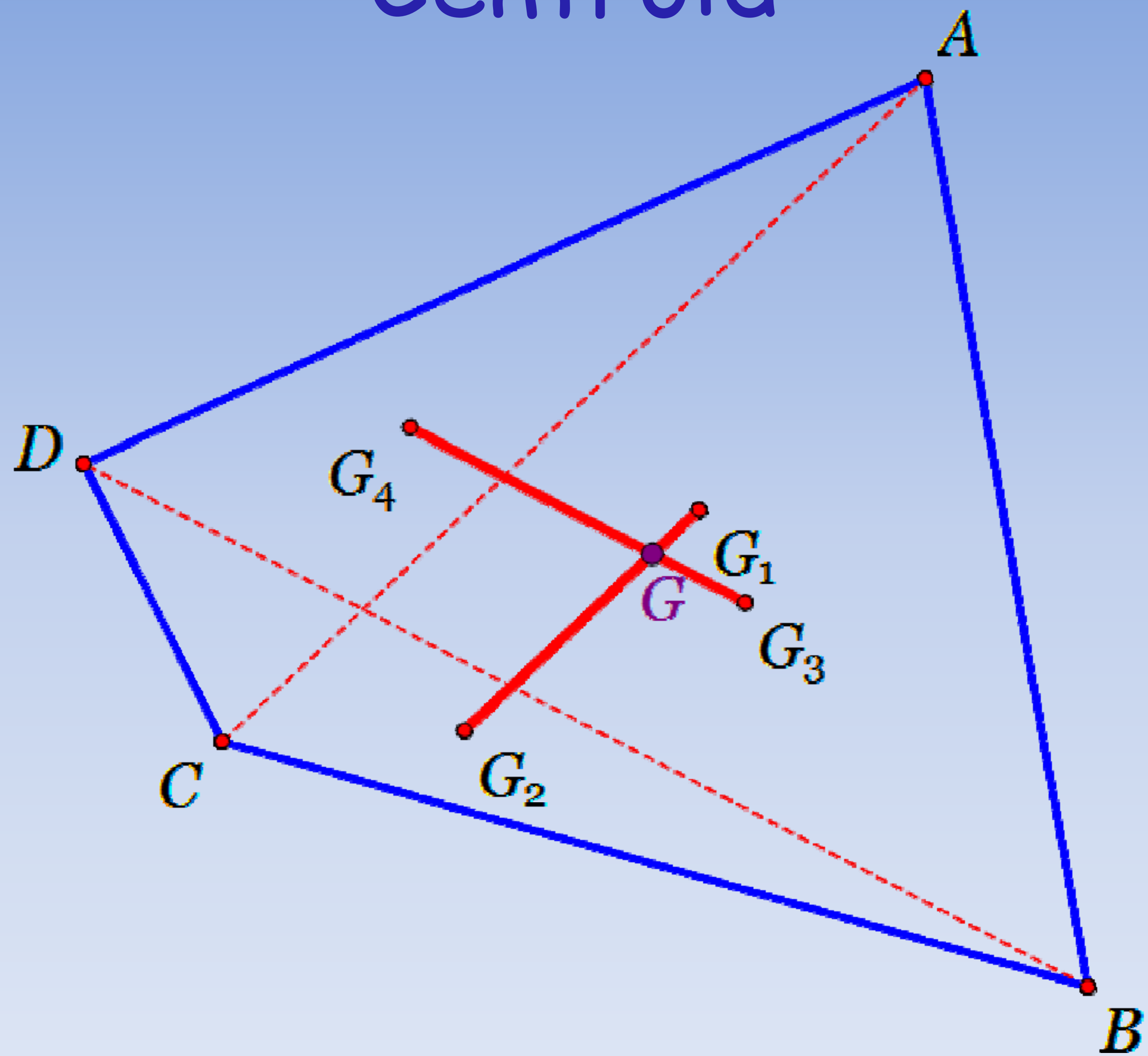
Each quadrilateral gives rise to 4 triangles using the diagonals.

P and Q = centroids of ΔABD and Δ CDB

R and S = centroids of ΔABC and Δ ADC

The point of intersection of the segments PQ and RS is the centroid of ABCD

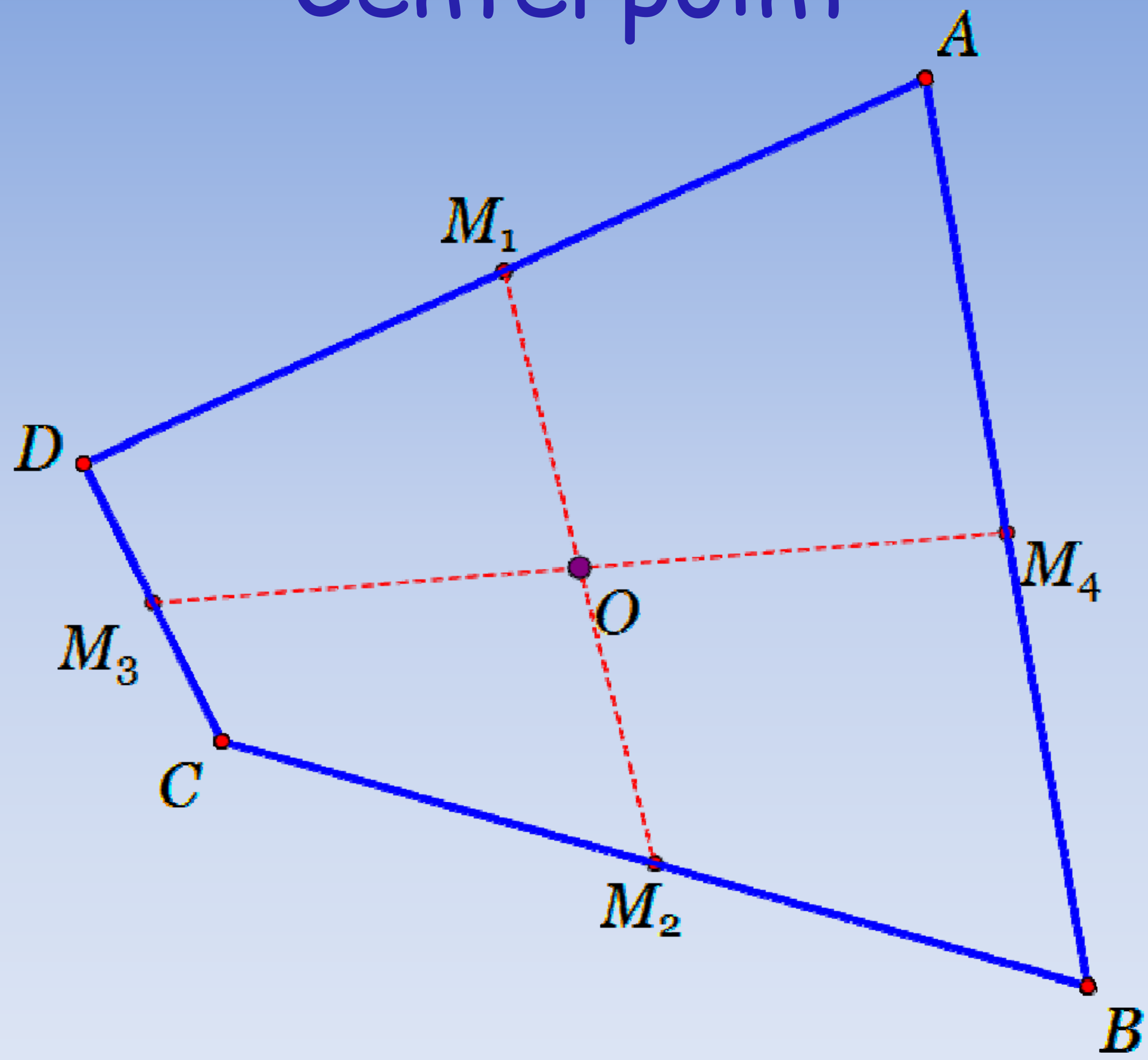
Centroid



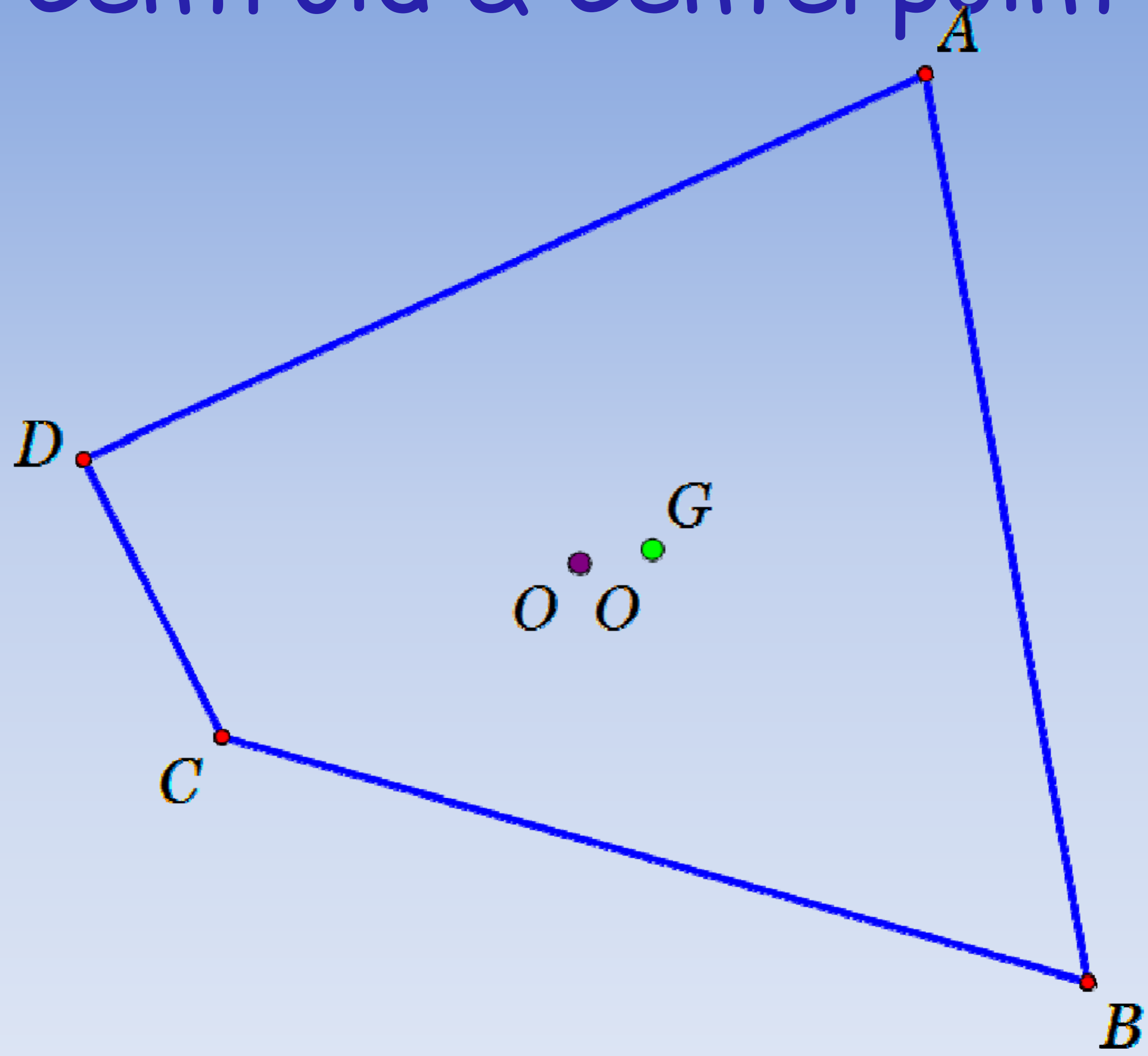
Quadrilateral Centers

The centerpoint of a quadrilateral is the point of intersection of the two segments joining the midpoints of opposite sides of the quadrilateral. Let us call this point O.

Centerpoint



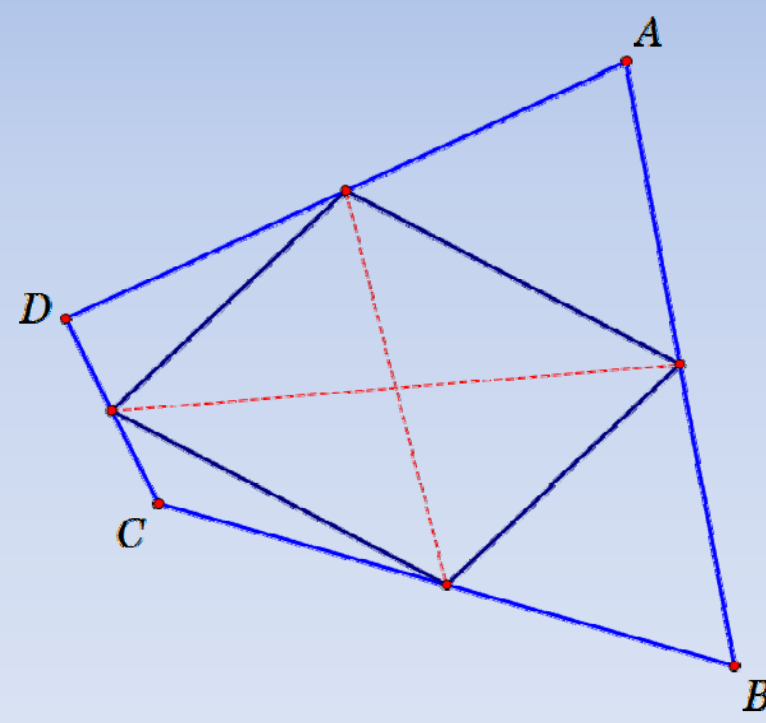
Centroid & Centerpoint



Theorem

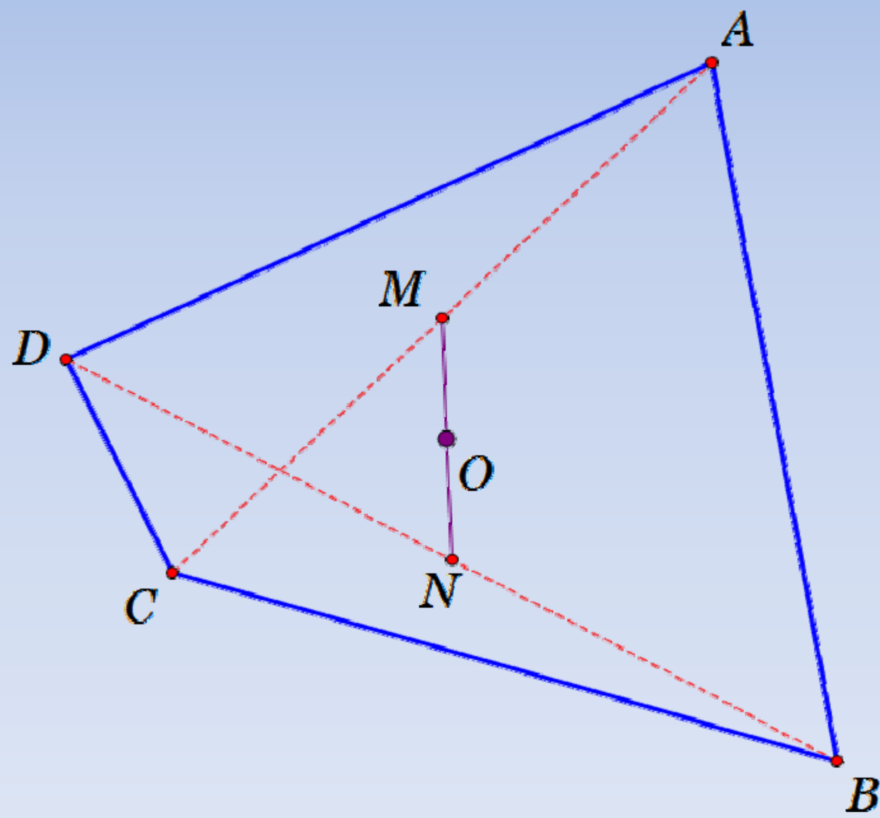
The segments joining the midpoints of the opposite sides of any quadrilateral bisect each other.

Proof:



Theorem

The segment joining the midpoints of the diagonals of a quadrilateral is bisected by the centerpoint.



Proof

Need to show that PMRN a parallelogram

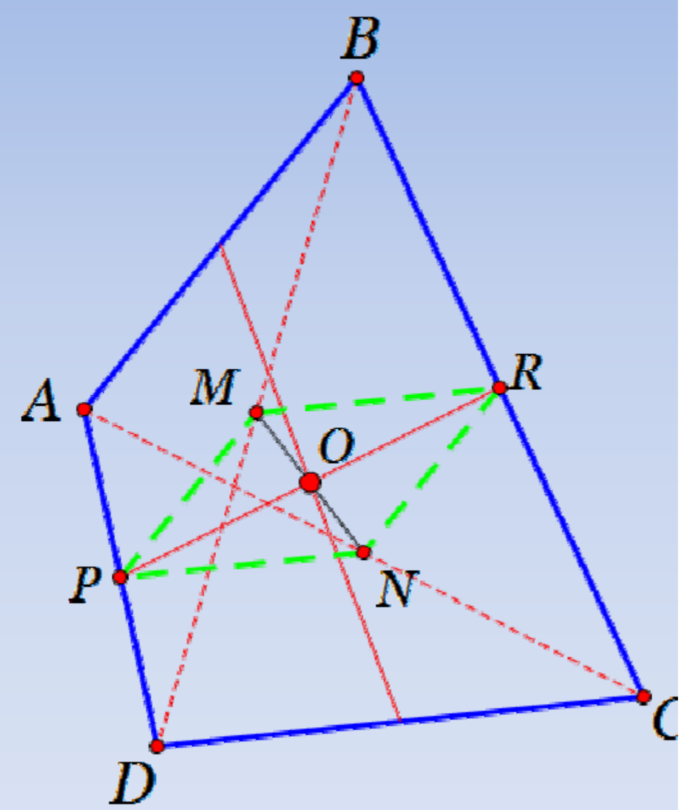
In ΔADC, PN a midline and $PN \| DC$ and $PN = \frac{1}{2}DC$

In ΔBDC, MR a midline and $MR \| DC$ and $MR = \frac{1}{2}DC$

$\Rightarrow MR \| PN$ and $MR = PN$

$\Rightarrow$PMRN a parallelogram

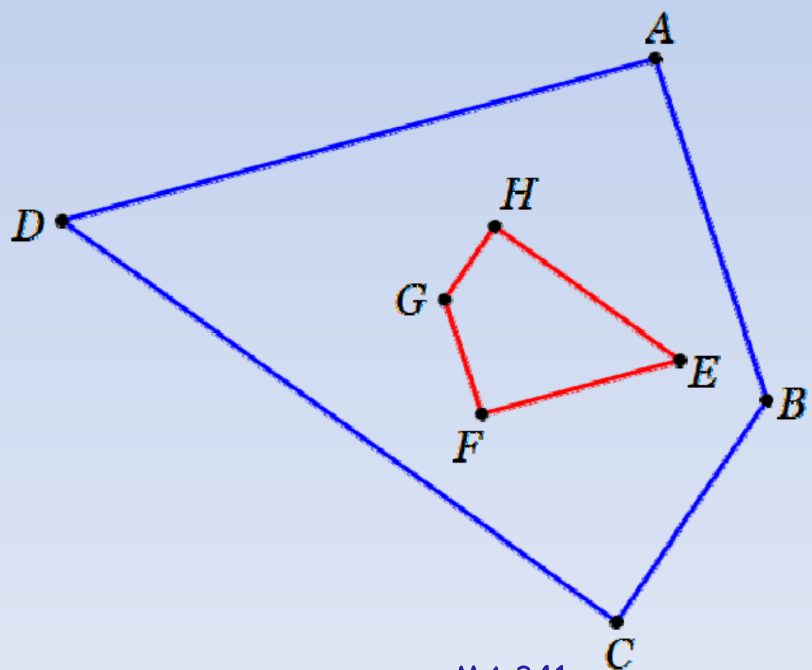
Diagonals bisect one another. Then MN intersects PR at its midpoint, which we know is O.



Theorem

Consider a quadrilateral ABCD and let E, F, G, H be the centroids of the triangles ΔABC, ΔBCD, ΔACD, and ΔABD.

1. $EF \| AD$, $FG \| AB$, $GH \| BC$, and $EH \| CD$;
2. $K_{ABCD} = 9\ K_{EFGH}$.



Proof

M_{BC} = midpoint of BC. Then AM_{BC} = median of ΔABC and E lies 2/3 of way between A and M_{BC}, $EM_{BC} = 1/3\ AM_{BC}$.

DM_{BC} = median of ΔDCB and $DF:FM_{BC}=2:1$

$\Rightarrow$ in ΔADM_{BC} we have EF || AD and EF= 1/3 AD.

Also

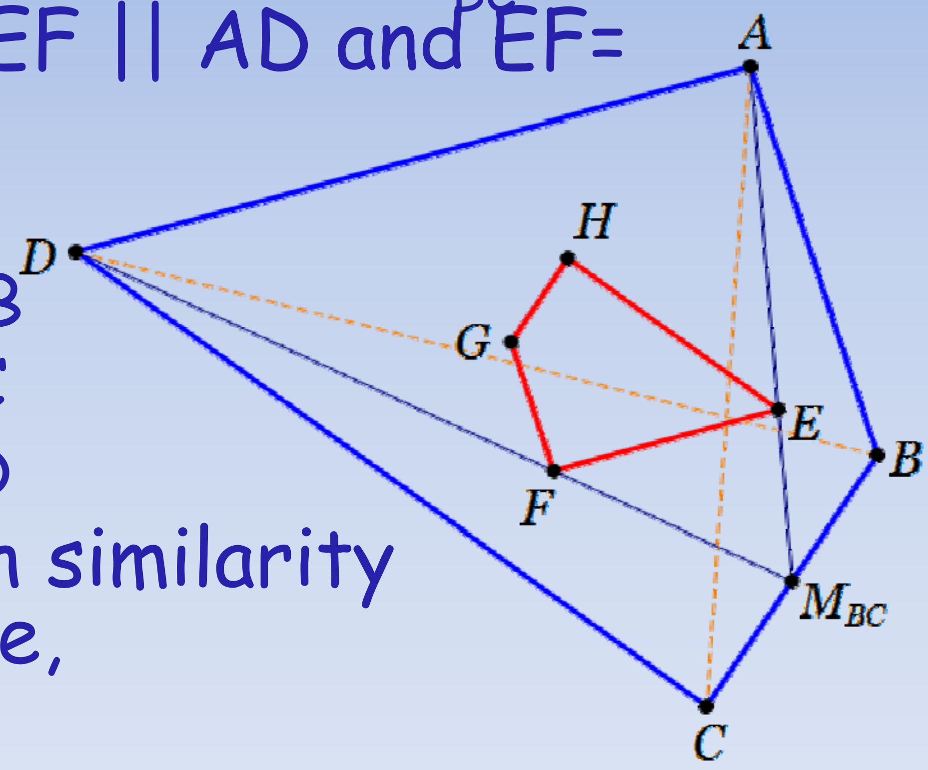
FG || AB and FG= 1/3 AB

GH || BC and GH=1/3 BC

EH || CD and EH=1/3 CD

Thus, EFGH~ADCB with similarity constant 1/3. Therefore,

$K_{ABCD} = 9K_{EFGH}$.

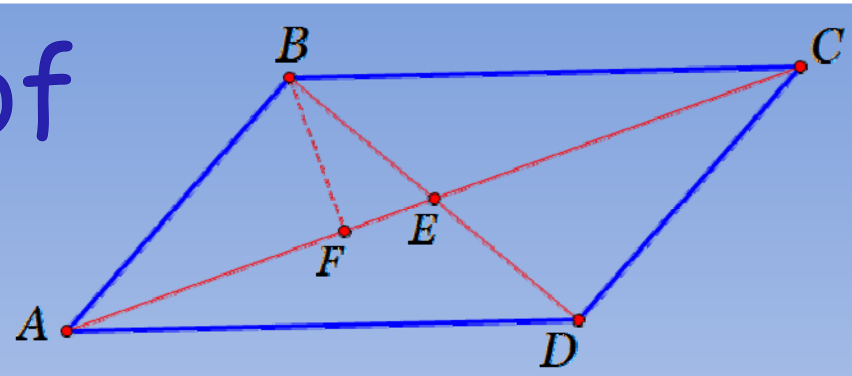


Theorem

The sum of the squares of the lengths of the sides of a parallelogram equals the sum of the squares of the lengths of the diagonals.

$$AB^2+BC^2+CD^2+AD^2=AC^2+BD^2$$

Proof



By Law of Cosines ΔABE

$$AB^2 = BE^2 + AE^2 - 2AE \cdot BE \cos(BEA)$$

Note that cos BEA=FE/BE, so

$$AB^2 = BE^2 + AE^2 - 2AE \cdot FE$$

Apply Stewart's Theorem to ΔEBC we have

$$BC^2 = BE^2 + EC^2 + 2EC \cdot FE$$

ABCD parallelogram $\Rightarrow$diagonals bisect each other

Thus AE=EC. Adding the first two equations we get

$$AB^2 + BC^2 = 2BE^2 + 2AE^2$$

Apply this same process to ΔCAD and we have

$$CD^2 + AD^2 = 2DE^2 + 2CE^2.$$

Proof

Now, add these equations and recall that AE=EC and BE=ED.

$AB^2+BC^2+CD^2+AD^2 = BE^2+2AE^2+2DE^2+2CE^2$

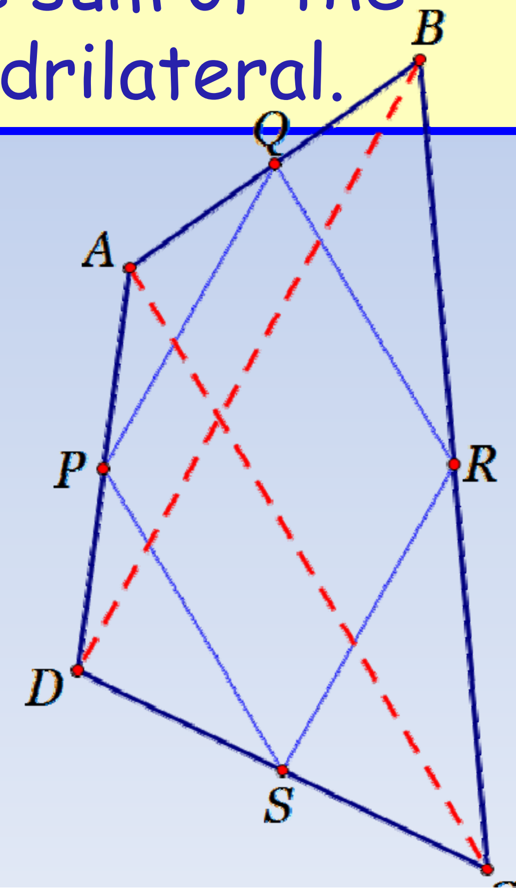
. $= 4AE^2 + 4BE^2$

. $= (2AE)^2 + (2BE)^2$

. $= AC^2 + BD^2$

Varignon's Theorem II

The area of the Varignon parallelogram is half that of the corresponding quadrilateral, and the perimeter of the parallelogram is equal to the sum of the diagonals of the original quadrilateral.



Proof

Recall SP = midline of ΔABD and $K_{ASP} = \frac{1}{4} K_{ABD}$

$K_{DSR} = \frac{1}{4} K_{DAC}$

$K_{CQR} = \frac{1}{4} K_{CBD}$

$K_{BPQ} = \frac{1}{4} K_{BAC}$

Therefore,

$$K_{ASP}+K_{DSR}+K_{CQR}+K_{BPQ} = \frac{1}{4}(K_{ABD}+K_{CBD})+\frac{1}{4}(K_{DAC}+K_{BAC})$$

$$= \frac{1}{4}K_{ABCD}+\frac{1}{4}K_{ABCD}$$

$$= \frac{1}{2} K_{ABCD}$$

Proof

Then,

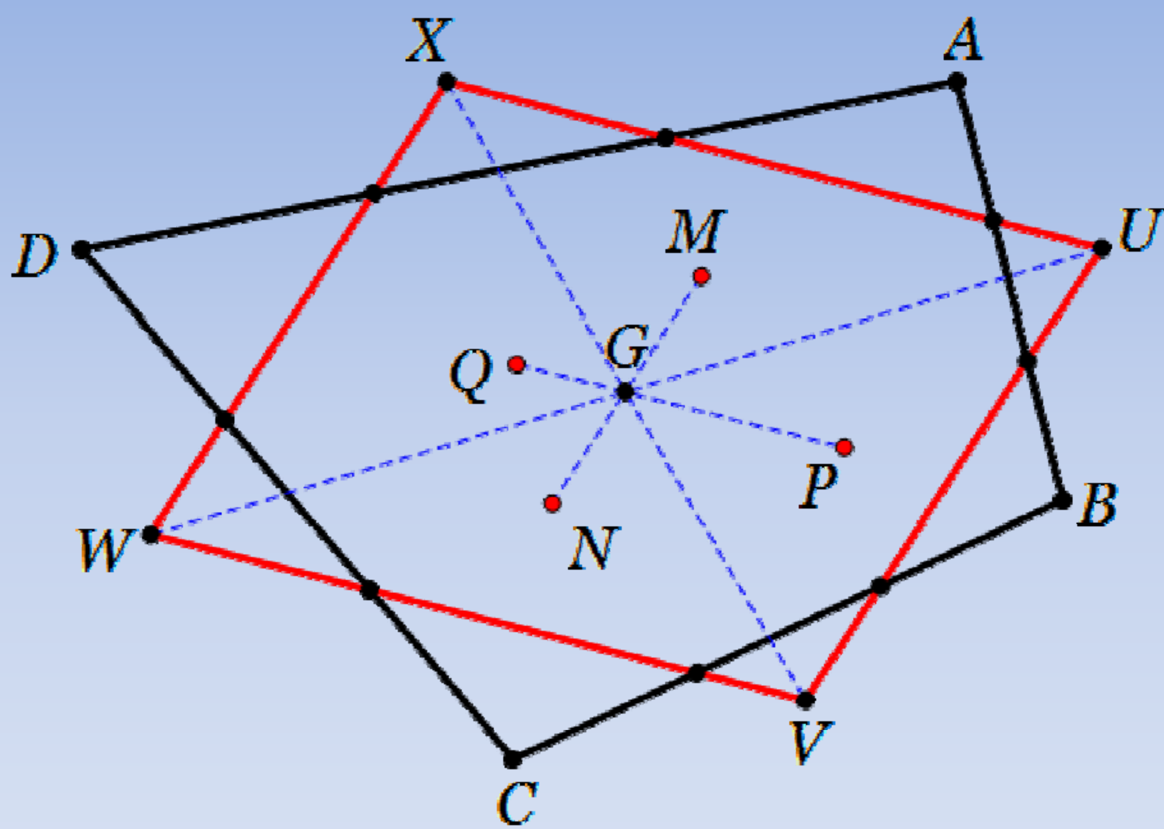
$$K_{PQRS} = K_{ABCD} - (K_{ASP}+K_{DSR}+K_{CQR}+K_{BPQ})$$
$$= K_{ABCD} - \frac{1}{2} K_{ABCD}$$
$$= \frac{1}{2} K_{ABCD}$$

Also $PQ = \frac{1}{2} AC = SR$ and $SP = \frac{1}{2} BD = QR$

Easy to see that the perimeter of the Varignon parallelogram is the sum of the diagonals.

Wittenbauer's Theorem

Given a quadrilateral ABCD a parallelogram is formed by dividing the sides of a quadrilateral into three equal parts, and connecting and extending adjacent points on either side of each vertex. Its area is 8/9 of the quadrilateral. The centroid of ABCD is the center of Wittenbauer's parallelogram (intersection of the diagonals).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