The Divine Proportion

MA 341 – Topics in Geometry
Lecture 20

What is the Divine Proportion?

In mathematics and the arts, two quantities are in the golden ratio if the ratio of the sum of the quantities to the larger quantity is equal to the ratio of the larger quantity to the smaller one.

$$\frac{a+b}{a} = \frac{a}{b} \quad \text{for } a > b$$

What is the Divine Proportion?

Other names frequently used for the golden ratio are the

golden section	golden mean
golden cut	golden number
golden proportion	extreme ratio
mean ratio	medial section
divine proportion	divine section
mean of Phidias	

Denoted by phi = φ

Value?

Does this ratio have a number associated with it, like π =
3.14159265358979323846264338327950288419716939937510582097494459230781640628620899862803482534211706798214808651328230664709384460955058223172535940812848111745028410270193852110555964462294895493038196442881097566593344612847564823378678316527120190914564856692346034861045432664821339360726024914127372458700660631558817488152092096282925409171536436789259036001133053054882046652138414695194151160943305727036575959195309218611738193261179310...

Value?

If $a/b = \varphi$, then

$$\frac{a}{b} = \frac{a+b}{a}$$

$$\frac{a}{b} = 1 + \frac{b}{a}$$

$$\varphi = 1 + \frac{1}{\varphi}$$

$$\varphi^2 = \varphi + 1$$

$$\varphi^2 - \varphi - 1 = 0$$

Value?

$$\varphi = \frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2}$$

$$\varphi = \frac{1 \pm \sqrt{5}}{2}$$

Which is it? Is it + or -?

What do we know about φ?

$a > b$ so $\varphi > 1$ and $\varphi = \frac{1 + \sqrt{5}}{2}$

Phi

$$\varphi = \frac{1+\sqrt{5}}{2} =$$

1.61803398874989484820458683436563811772030917980576286213544862270526046281890244970720720418939113748475408807538689175212663386222353693179318006076672635443338908659593958290563832266131992829026788067520876689250171169620703222104321626954862629631361443814975870122034080588795445474924618569536486444924104432077134494704956584678850987433944221254487706647809158846074998871240076521705751797883416...

Properties of φ

From earlier we have that

$$\varphi = 1 + \frac{1}{\varphi}$$

Therefore,

$$\frac{1}{\varphi} = \varphi - 1$$

Properties of φ

$$\frac{1}{\varphi} = \varphi - 1$$

$$\frac{1}{\varphi} = \frac{2}{1+\sqrt{5}} = \frac{2\left(1-\sqrt{5}\right)}{\left(1+\sqrt{5}\right)\left(1-\sqrt{5}\right)}$$

$$= \frac{2\left(1-\sqrt{5}\right)}{-4}$$

$$= \frac{-1+\sqrt{5}}{2}$$

Φ and φ

Sometimes authors use:

$\Phi = 1.6180\ldots$

and

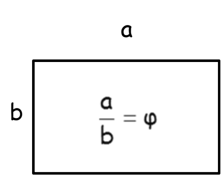
$\varphi = 0.6180\ldots = 1/\Phi$

History

Euclid's Elements provides first known written definition of golden mean:
"A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the less."
Euclid gives a construction for cutting a line "in extreme and mean ratio", i.e. the golden ratio.
Several propositions and their proofs employ the golden ratio.
Some of these propositions show that the golden ratio is an irrational number.

Golden Rectangle

- A rectangle is called a golden rectangle if its sides are in the ratio of the golden mean:

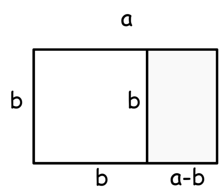


Golden Rectangle

Remove a square from the golden rectangle:

$$\frac{a}{b} = \varphi \Rightarrow a = b\varphi$$

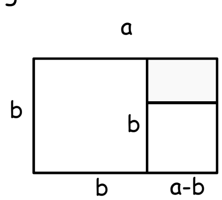
$$\frac{b}{a-b} = \frac{b}{b\varphi - b} = \frac{1}{\varphi - 1} = \frac{1}{1/\varphi} = \varphi$$



Golden Rectangle

The remaining rectangle is a golden rectangle!

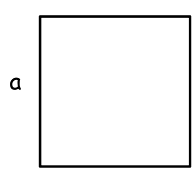
Do it again!



Golden Rectangle

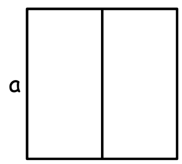
This actually gives us a construction for a golden rectangle using a compass and straightedge – thus GeoGebra or Sketchpad

Start with a square



Golden Rectangle

Find midpoint of the base and split square in two.



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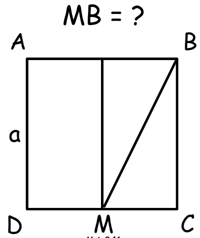
Golden Rectangle

Construct diagonal MB.

MC = a/2 and BC = a

MB = ?



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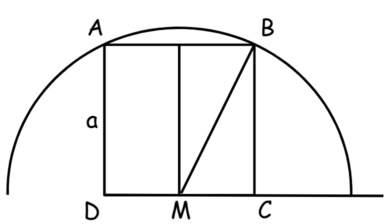
Golden Rectangle

Construct circle with radius MB centered at M.



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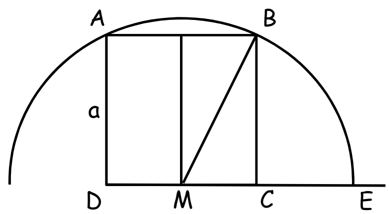
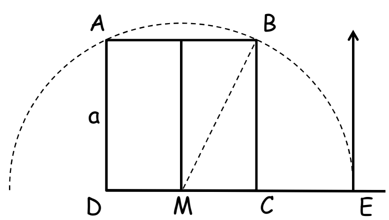
Golden Rectangle

Mark point of intersection E.



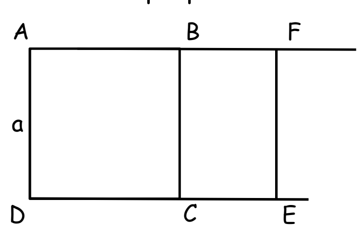
Golden Rectangle

Construct perpendicular at E.



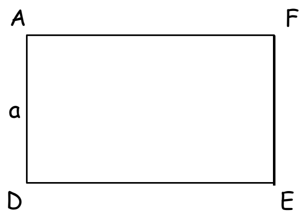
Golden Rectangle

Extend AB to meet this perpendicular.



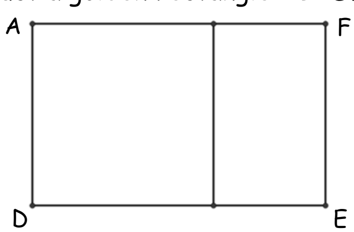
Golden Rectangle

AFED is a golden rectangle.



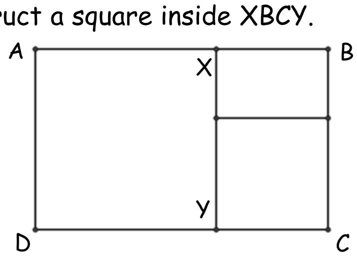
Golden Spiral

Construct a golden rectangle ABCD.



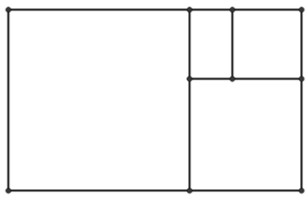
Golden Spiral

Construct a square inside XBCY.



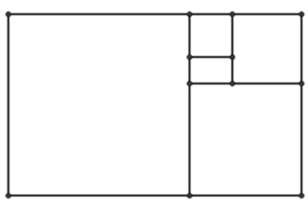
Golden Spiral

Construct another square inside the smaller golden rectangle..



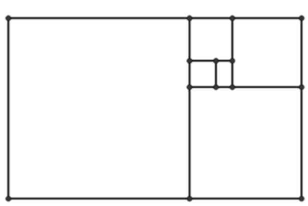
Golden Spiral

Again



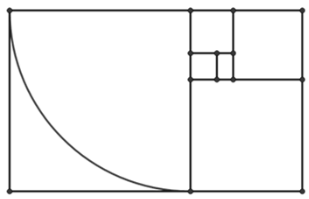
Golden Spiral

Again



Golden Spiral

In each square construct a quarter circle:



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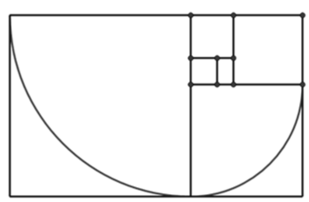
Golden Spiral

In each square construct a quarter circle:



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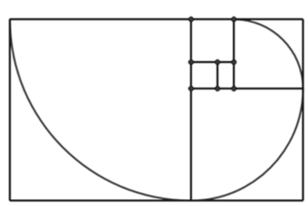
Golden Spiral

In each square construct a quarter circle:



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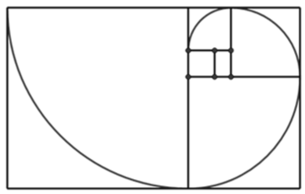
Golden Spiral

In each square construct a quarter circle:



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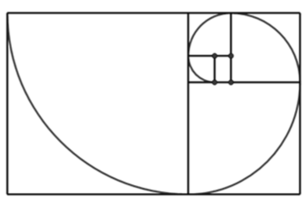
Golden Spiral

In each square construct a quarter circle:



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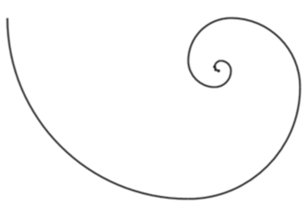
Golden Spiral

In each square construct a quarter circle:



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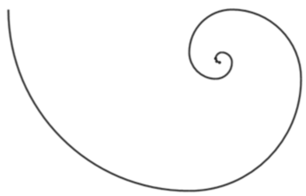
Golden Spiral

This does give a logarithmic spiral:

$\Theta = a \ln(b \cdot r)$, in polar coordinates



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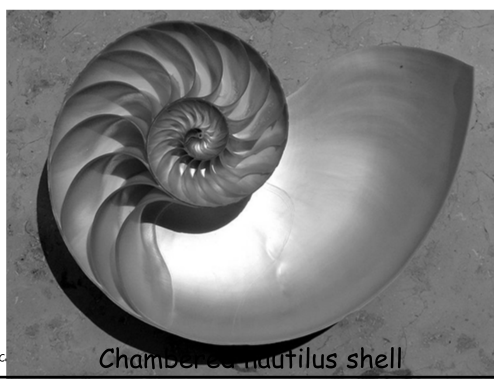
Golden Spirals?



Chambered nautilus shell

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Golden Spirals?

Spiral galaxies

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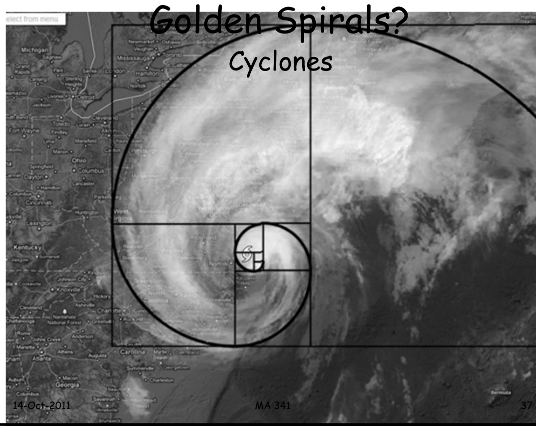
Golden Spirals?

Cyclones



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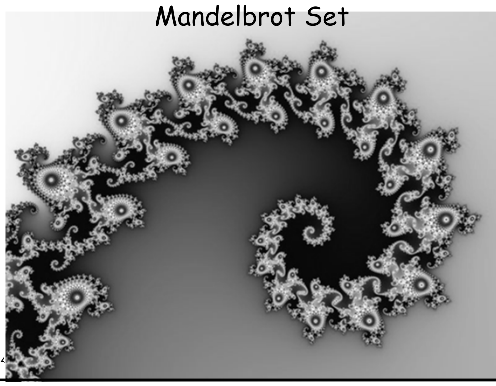
Golden Spirals?

Mandelbrot Set



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Golden Spirals?

Flower heads

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Golden Spirals?

Phyllotaxis
(http://www.math.smith.edu/phyllo//)




How?

Consider the following list of numbers:

1,1,2,3,5,8,13,21,34,55,89,144,…

This is the Fibonacci sequence $\{F_n\}$. We are interested in the quotients

$$F_{n+1}/F_n$$

Fibonacci Connection

F_0=1	F_9=55	F_5/F_4 = 1.6
F_1=1	F_{10}=89	F_6/F_5 = 1.625
F_2=2	F_{11}=144	F_7/F_6 = 1.6153.
F_3=3	F_{12}=233	F_8/F_7 = 1.6190
F_4=5	F_{13}=377	F_9/F_8 = 1.6176
F_5=8	F_1/F_0 = 1	F_{10}/F_9 = 1.618181
F_6=13	F_2/F_1 = 2	F_{11}/F_{10} = 1.617977
F_7=21	F_3/F_2 =1.5	F_{12}/F_{11} = 1.618055
F_8=34	F_4/F_3 = 1.6667	F_{13}/F_{12} = 1.618025

Fibonacci Connection

Is it true that:

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi \ ?$$

Note: Since $\varphi^2 = \varphi + 1$, multiplying by φ^{n-1} gives a Fibonacci type relationship:

$$\varphi^{n+1} = \varphi^n + \varphi^{n-1}$$

$$(F_{n+1} = F_n + F_{n-1})$$

And it so happens that

$$F_n = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}} = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}$$

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Other representations?

Consider the following sequence:

$$a_0 = 1, a_1 = \sqrt{1+\sqrt{a_0}}, \ldots, a_n = \sqrt{1+\sqrt{a_{n-1}}}$$

$$a_n = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}}$$

What is lim a_n ?

First, we need to know that $\{a_n\}$ has a limit. This can be shown with calculus.

Let L = lim a_n

Then,

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Other representations?

$$L = \lim_{n\to\infty} \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}}$$

$$L = \sqrt{1+\lim_{n\to\infty}\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}}}$$

$$L = \sqrt{1+L}$$

$$L^2 = L + 1$$

Hey!!! L = φ, so

$$\varphi = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}}$$

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Another representations

Consider the following sequence:

$$b_0 = 1, b_1 = 1 + \frac{1}{1+b_0}, \ldots, b_n = 1 + \frac{1}{1+b_{n-1}}$$

$$b_n = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}}$$

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Other representations?

What is lim b_n ?

First, we need to know that $\{b_n\}$ has a limit. This can be shown with calculus.

Let L = lim b_n

Then, $L = 1 + \frac{1}{L}$

Again, L = φ.

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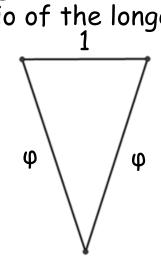
Golden Triangle

A golden triangle is an isosceles triangle where the ratio of the longer side to the base is φ.



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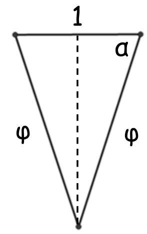
Golden Triangle

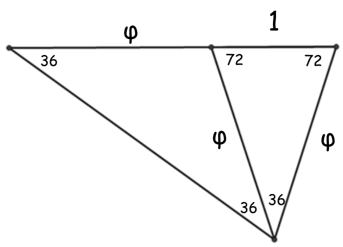
What are the angles?

$\cos(\alpha) = \frac{1}{2}/\varphi \Rightarrow \alpha = 72°$, making summit angle 36°



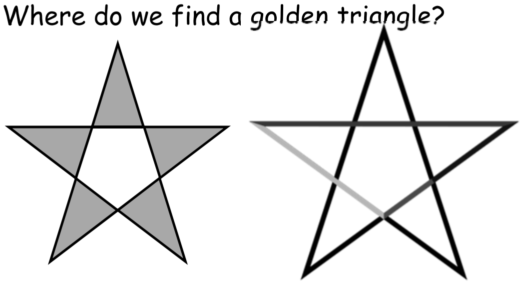
Golden Triangle

- $1 + \varphi = \varphi^2$, so the larger triangle is similar to the smaller with similarity φ.



Golden Triangle

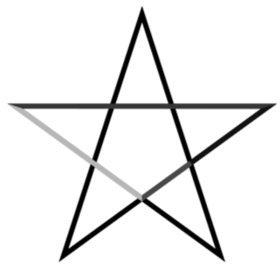
Where do we find a golden triangle?



Golden Triangle

In fact:

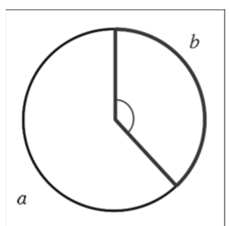
Red/green
= green/blue
= blue/pink
= φ



Golden Angle

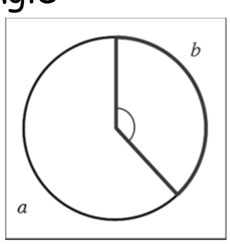
If ratio of arcs $a/b = \varphi$, then angle subtended by smaller arc is called golden angle.

It measures approximately 137.51°, or about 2.399963 radians.



Golden Angle

It is exactly



$$360\left(1-\frac{1}{\varphi}\right)=360(2-\varphi)=\frac{360}{\varphi^2}=180(3-\sqrt{5})$$

Golden Ratio and Art

- http://www.goldennumber.net/
- http://www.keplersdiscovery.com/DivineProportion.html
- http://spiralzoom.com/

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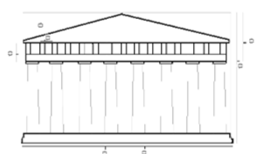
The Parthenon



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The Parthenon

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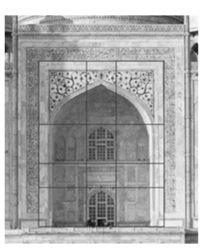
The Acropolis, Porch of the Maidens

Chartes Cathedral & UN Building

Taj Mahal

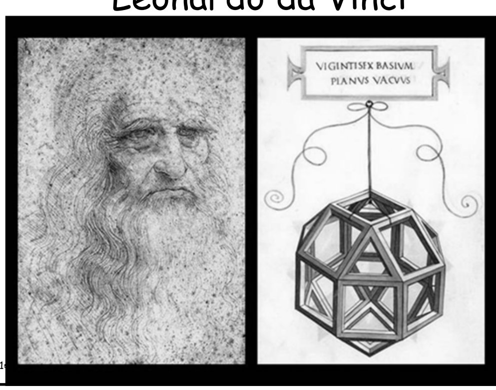


Fra Luca Pacioli

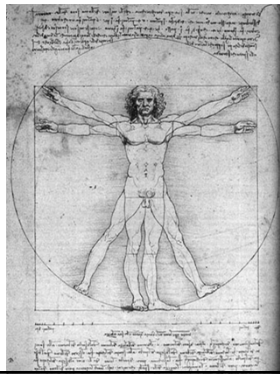
Pacioli's De divina proportione

- Written in Milan in 1496–98,
- Published in Venice in 1509
- The subject - mathematical and artistic proportion, especially mathematics of golden ratio and application in architecture.
- Leonardo da Vinci drew illustrations of regular solids in *De divina proportione* while living with and taking mathematics lessons from Pacioli.
- Discusses use of perspective by painters such as Piero della Francesca, Melozzo da Forlì, and Marco Palmezzano

Leonardo da Vinci



Leonardo da Vinci



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Facial Study

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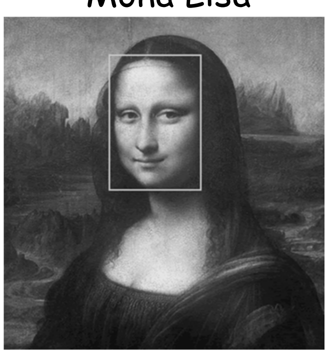
Mona Lisa



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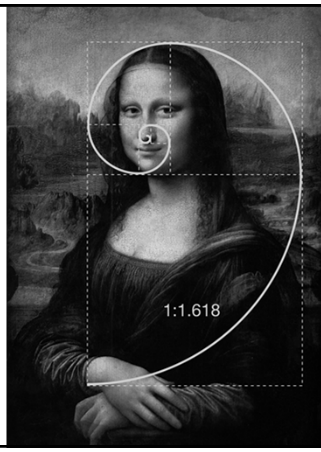
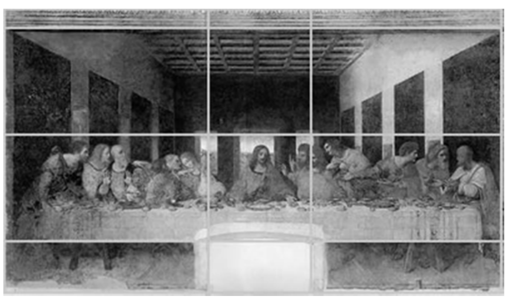
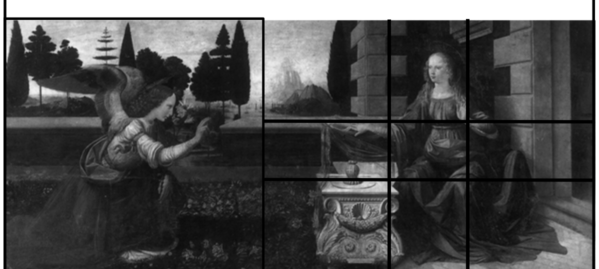


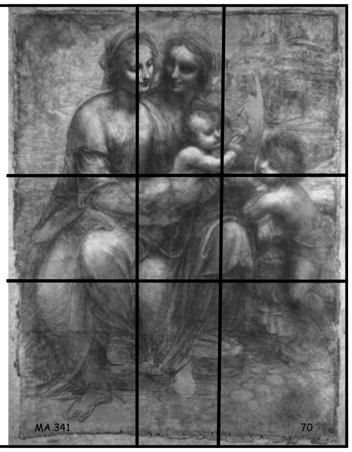
The Last Supper



The Annunciation



Madonna and Child with St. Anne and St. John



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Michaelangelo & Raphael

http://web.me.com/paulscott.info/place/pm10/pm10.html

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The Holy Family - Michelangelo

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The Crucifixion - Raphael

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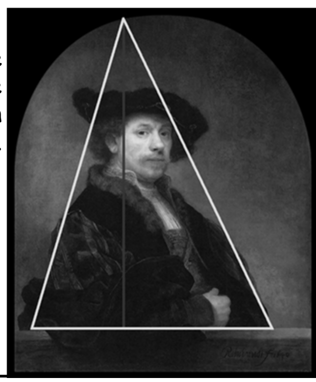
Self Portrait - Rembrandt

Red line divides base into golden mean.



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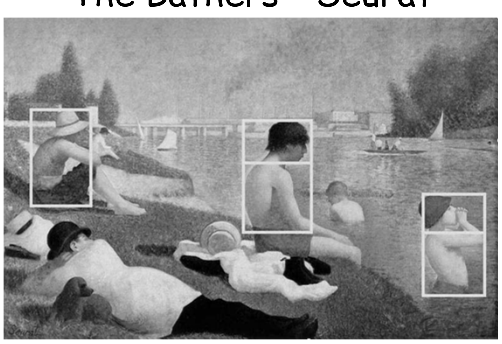
The Bathers - Seurat



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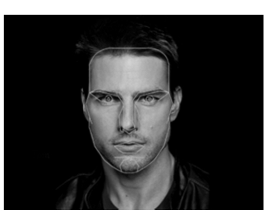
The Perfect Face

Dr. Stephen Marquardt
(http://www.beautyanalysis.com/index2_mba.htm)
Claims that this gives the most beautiful shape of human face
Used decagons and pentagons and embodies φ in all their dimensions.

The Perfect Face

This mask of the human face is based on the Golden Ratio. The proportions of the length of the nose, the position of the eyes and the length of the chin, all conform to some aspect of the Golden Ratio.



The Perfect Face?

The Perfect Face?

The Perfect Smile

- Front two teeth form a golden rectangle
- Also a golden ratio in height to width of center two teeth.
- Ratio of the width of the 2 center teeth to those next to them is φ.
- Ratio of width of smile to 3rd tooth from center is φ

See Donald Duck in Mathemagic Land by Disney Studios

http://www.youtube.com/watch?v=oT_Bxgah9zc

Without mathematics, there is no art.
- Fra. Luca Pacioli

14-Oct-2011 MA 341 82