Quadrilaterals

MA 341 – Topics in Geometry
Lecture 22

Theorems

1. A convex quadrilateral is cyclic if and only if the four perpendicular bisectors of the sides are concurrent.
2. A convex quadrilateral is cyclic if and only if opposite angles are supplementary.

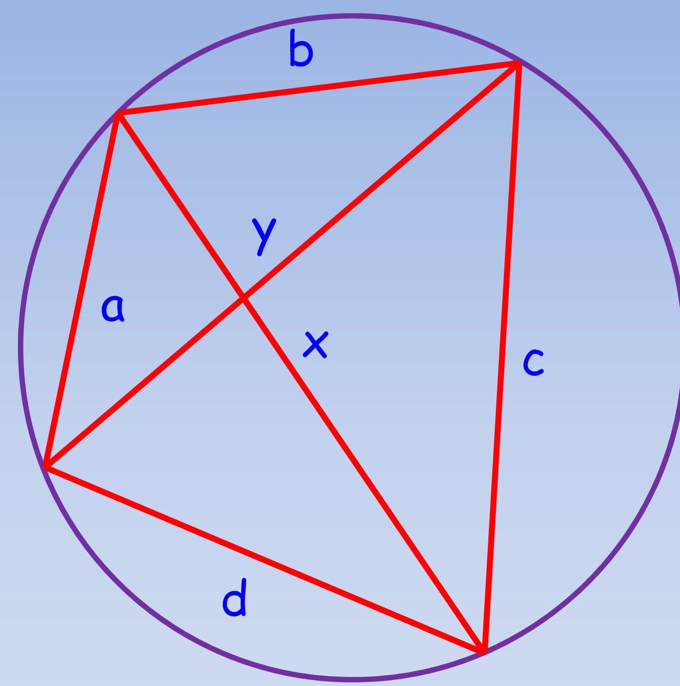
Ptolemy's Theorem

Let a, b, c, and d be the lengths of consecutive sides of a cyclic quadrilateral and let x and y be the lengths of the diagonals. Then

$$ac + bd = xy.$$

Also,

$$\frac{x}{y} = \frac{ad + bc}{ab + cd}$$



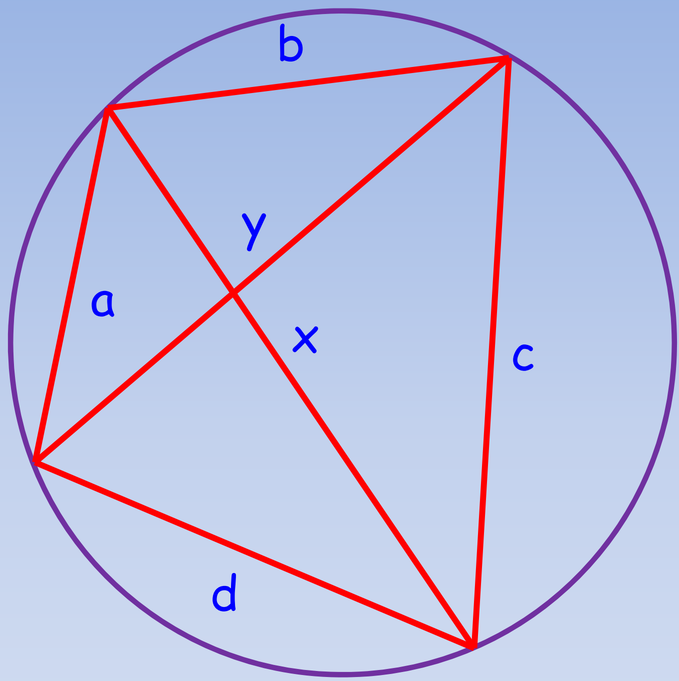
Ptolemy's Theorem

This leads to the following:

$$x = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}$$

and

$$y = \sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}}$$



Angles of a cyclic quadrilateral

If A is the angle between a and d

$$\cos(A) = \frac{a^2 + d^2 - b^2 - c^2}{2(ad + bc)}$$

$$\sin(A) = 2\frac{\sqrt{(s-a)(s-b)(s-c)(s-d)}}{ad + bc}$$

$$\tan\frac{A}{2} = \sqrt{\frac{(s-a)(s-d)}{(s-b)(s-c)}}$$

Angles of a cyclic quadrilateral

The angle, θ, between the diagonals

$$\tan\frac{\theta}{2} = \sqrt{\frac{(s-b)(s-d)}{(s-a)(s-c)}}$$

The circumradius is

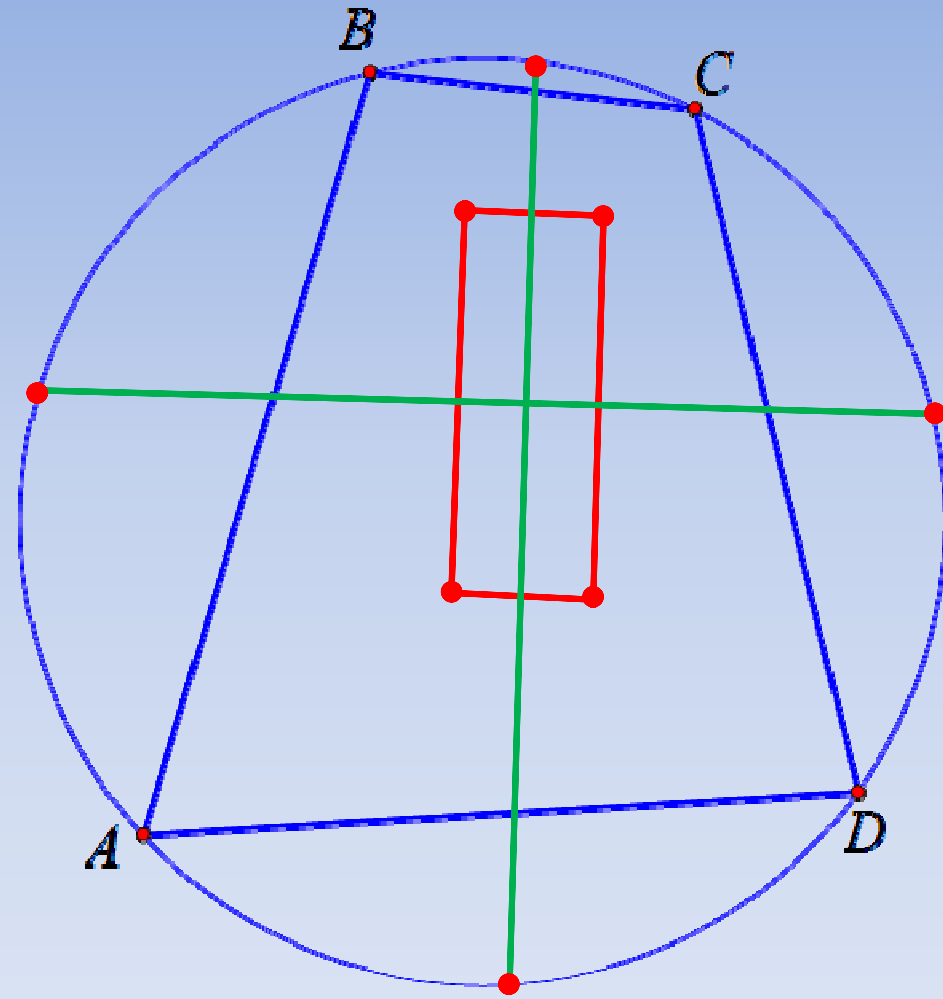
$$R = \frac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}}$$

(Parameshvara's formula)

Incenters of the triangles

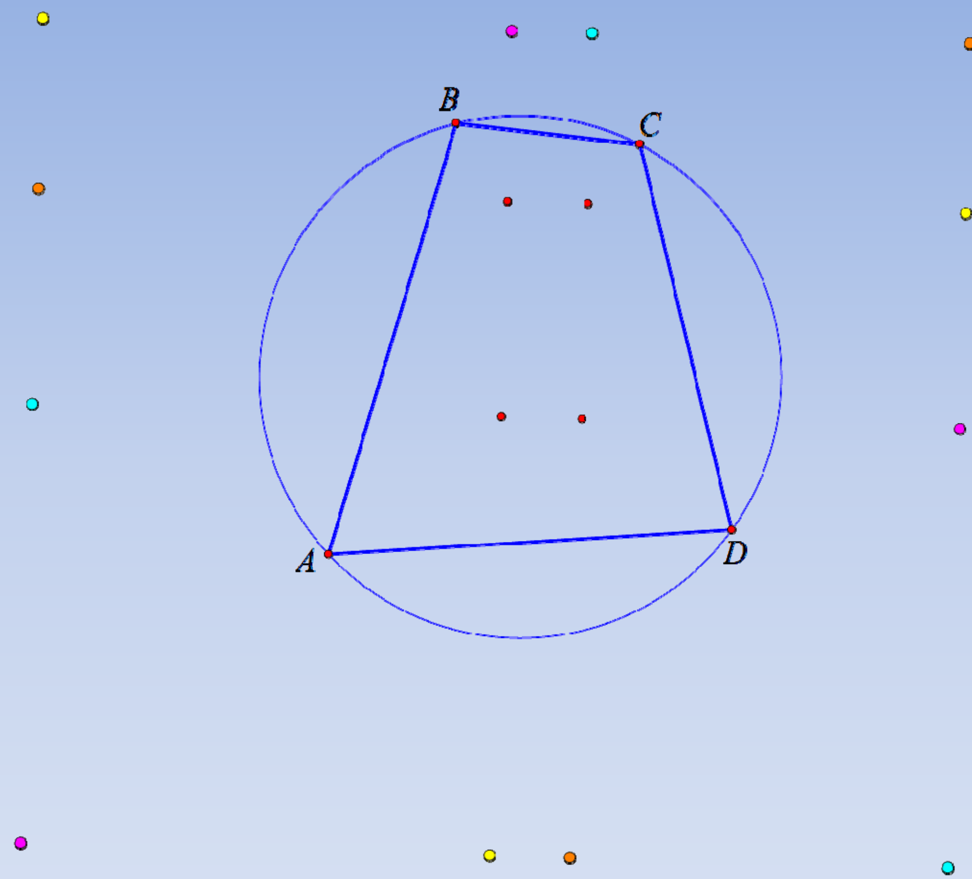
The incenters of 4 triangles form a rectangle

Sides parallel to lines through midarc points



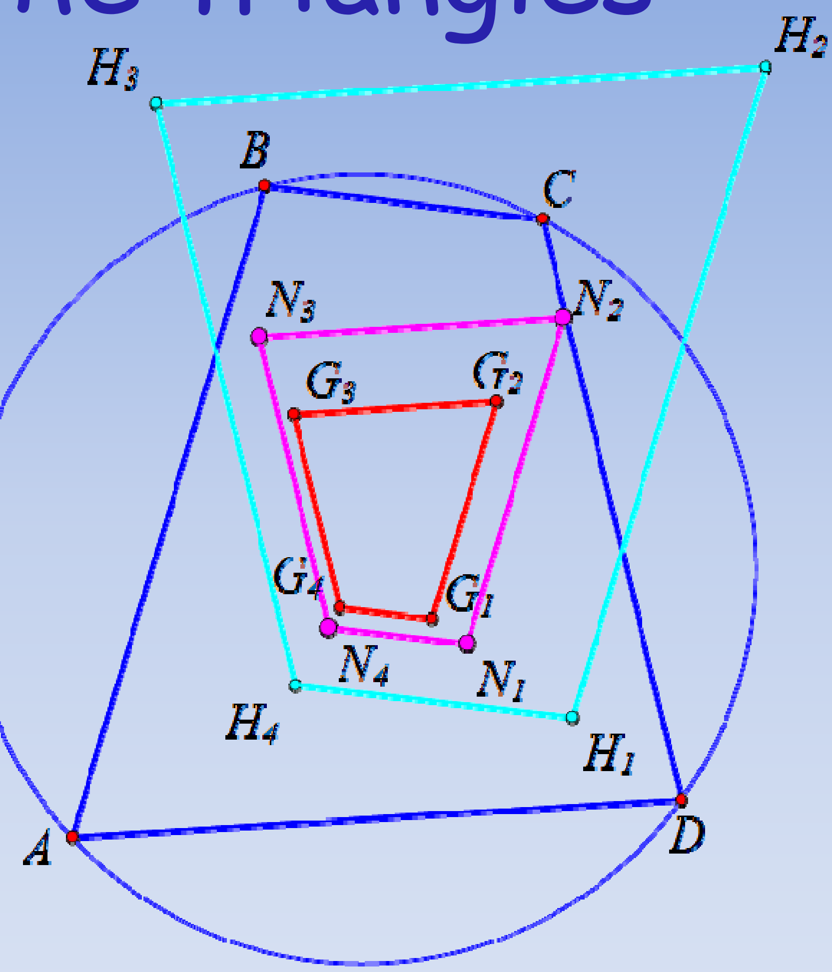
Incenters of the triangles

The incenters and excenters of 4 triangles form a 4 x 4 grid



Incenters of the triangles

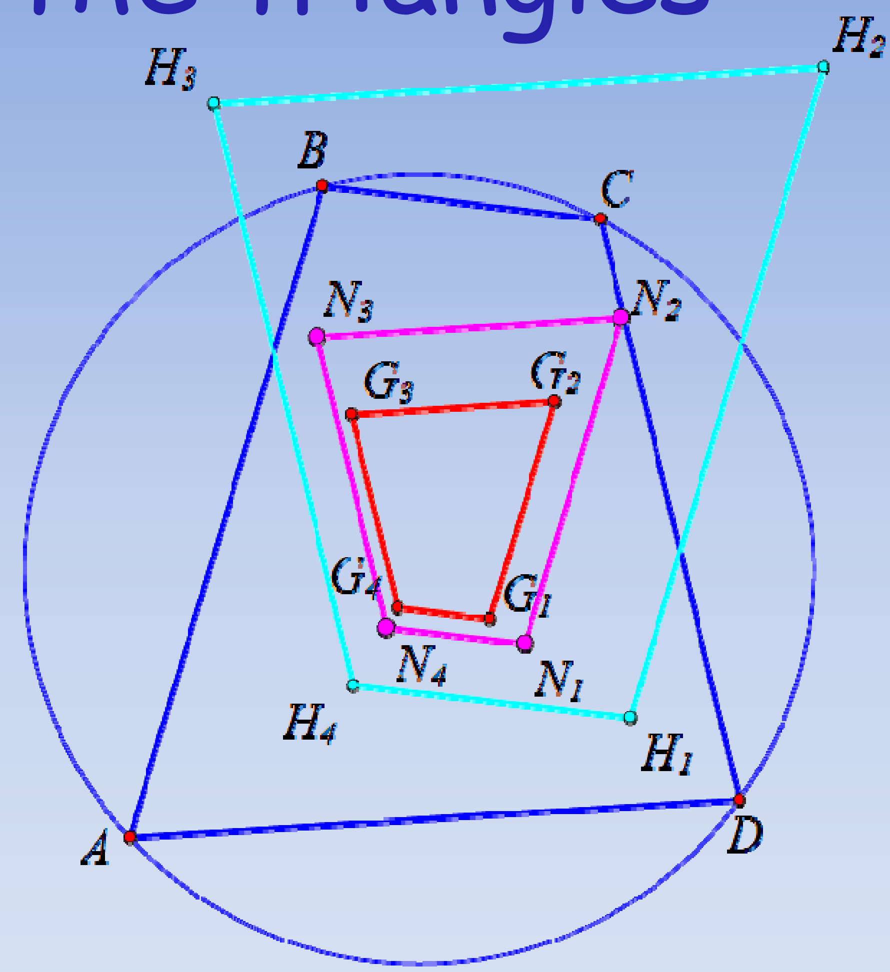
Consider quadrilaterals formed by centroids, 9-point circle centers and orthocenters of the 4 triangles



Incenters of the triangles

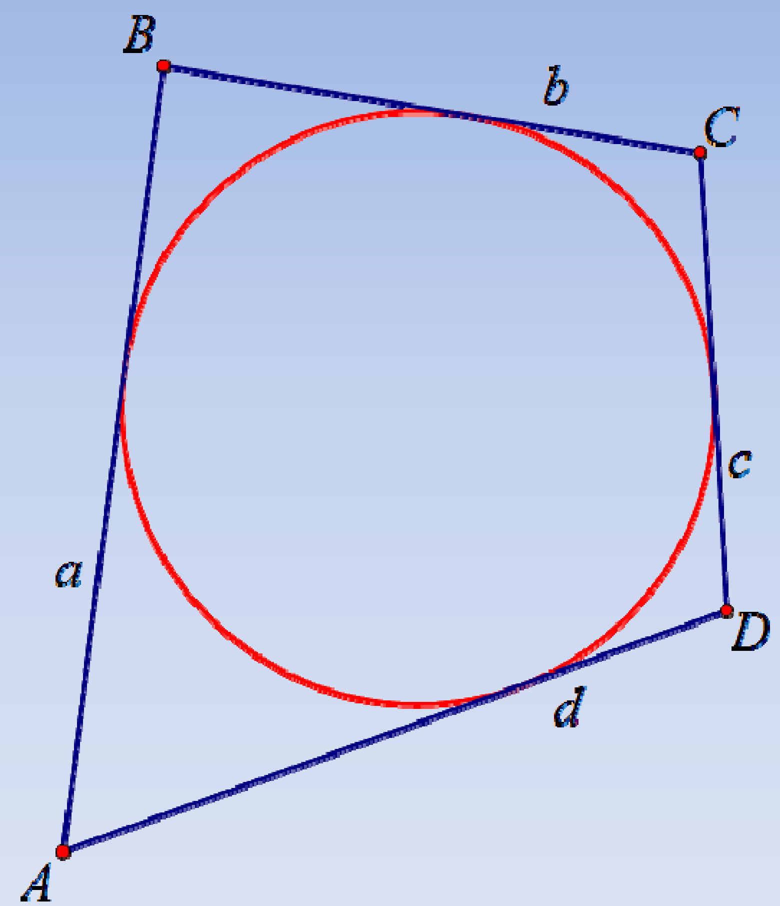
All are similar to ABCD.

$H_2H_1H_4H_3$ is congruent to ABCD.



Tangential Quadrilaterals

A convex quadrilateral whose sides all lie tangent to inscribed circle – the incircle



Tangential Quadrilaterals

Is a rectangle a tangential quadrilateral?

Is a square a tangential quadrilateral?

Is a parallelogram a tangential quadrilateral?

Tangential Quadrilaterals

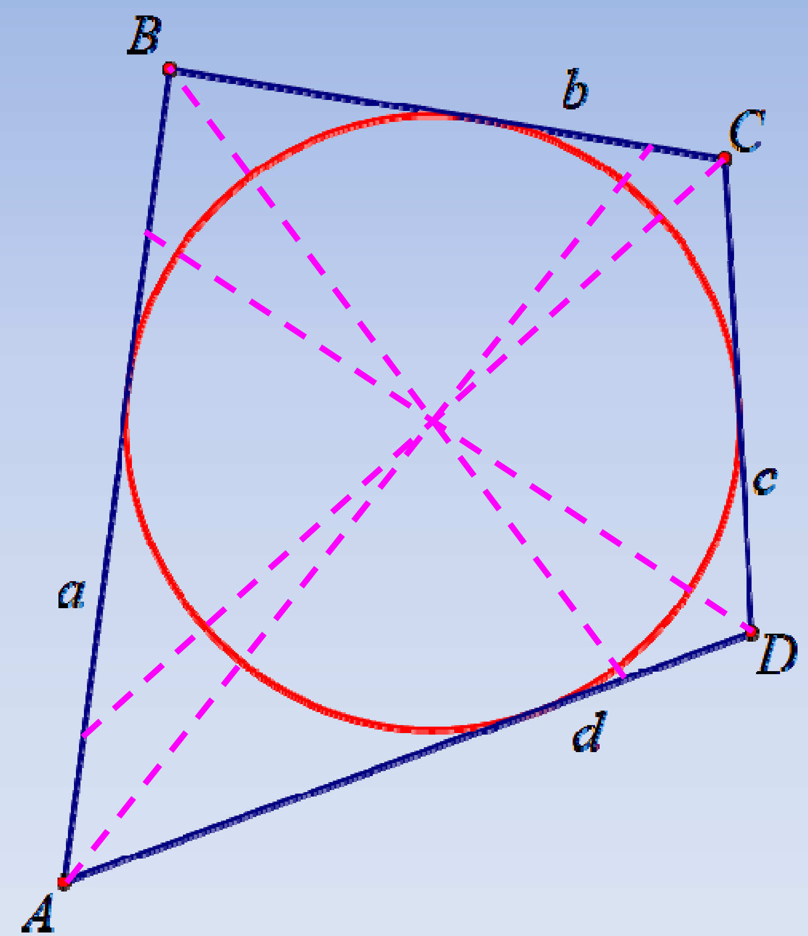
Is a rhombus a tangential quadrilateral?

Is a kite a tangential quadrilateral?

Tangential Quadrilaterals

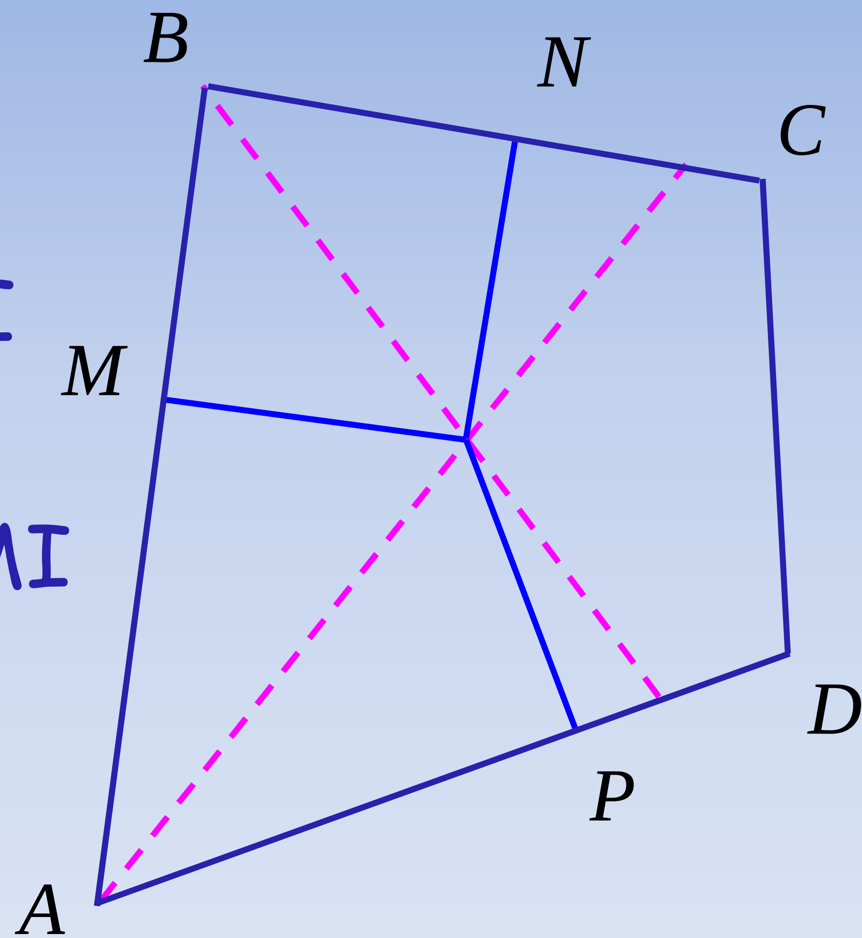
In triangle incenter = intersection of angle bisectors

What about quadrilaterals?



Tangential Quadrilaterals

I is equidistant from AD, AB and BC

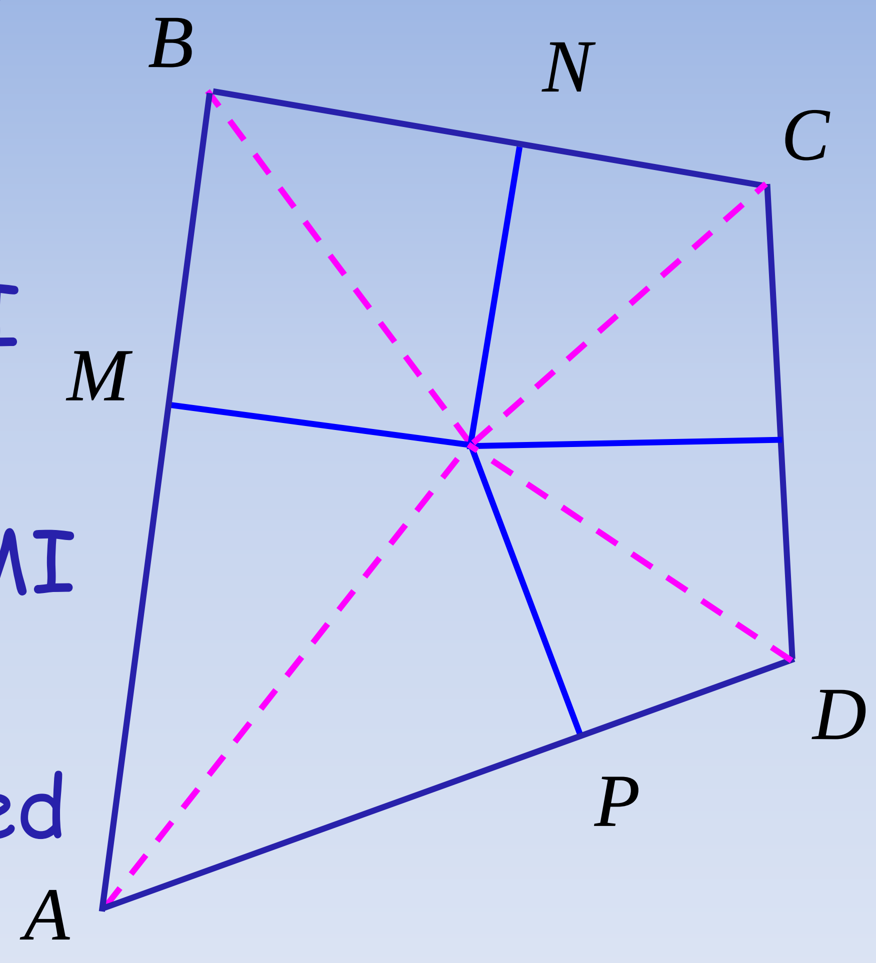


$\Delta API \cong \Delta AMI \Rightarrow PI \cong MI$

$\Delta MIB \cong \Delta NIB \Rightarrow NI \cong MI$

Tangential Quadrilaterals

I is equidistant from AD, AB, BC and CD



$\Delta API \cong \Delta AMI \Rightarrow PI \cong MI$

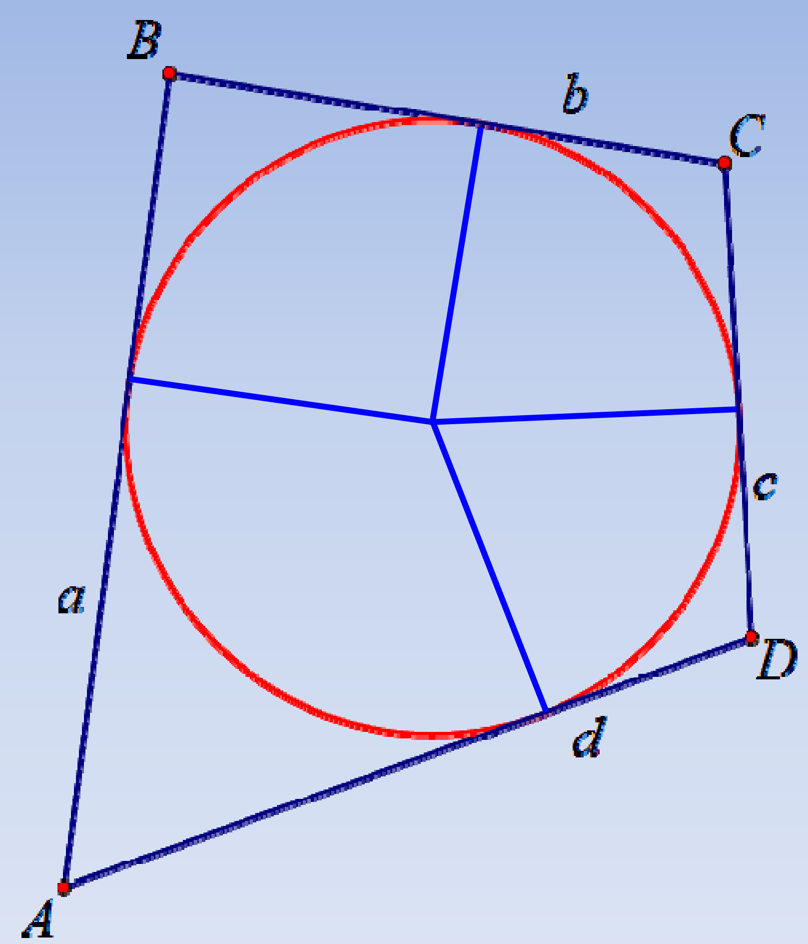
$\Delta MIB \cong \Delta NIB \Rightarrow NI \cong MI$

etc

Thus, there is an inscribed circle!

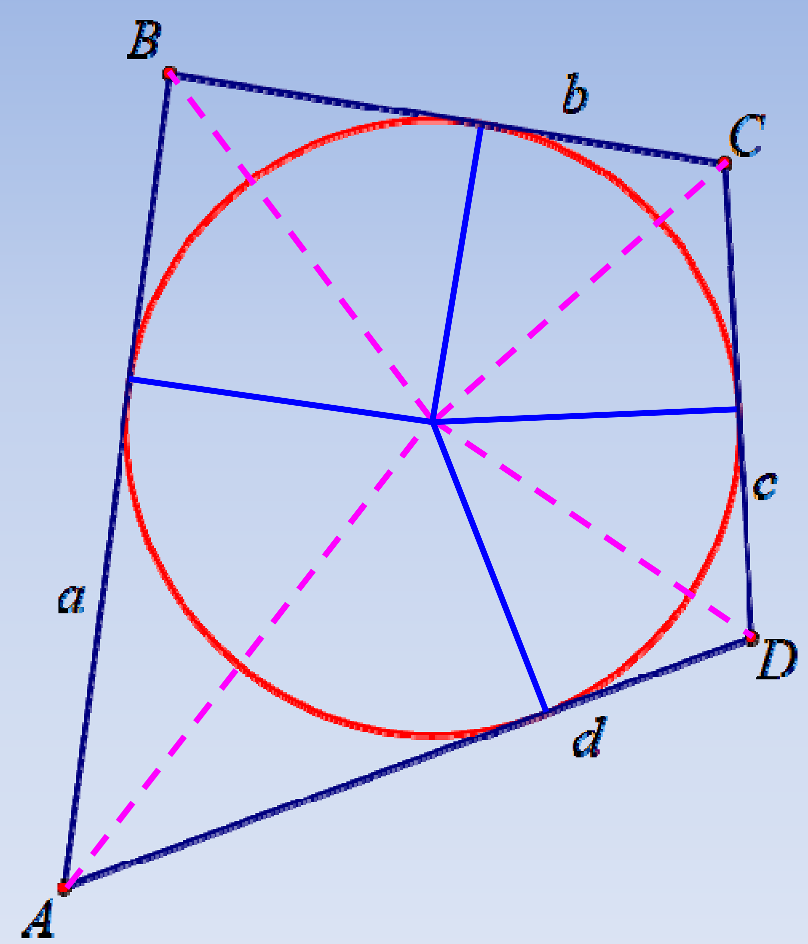
Tangential Quadrilaterals

If there is an inscribed circle, then center is equidistant from sides.



Tangential Quadrilaterals

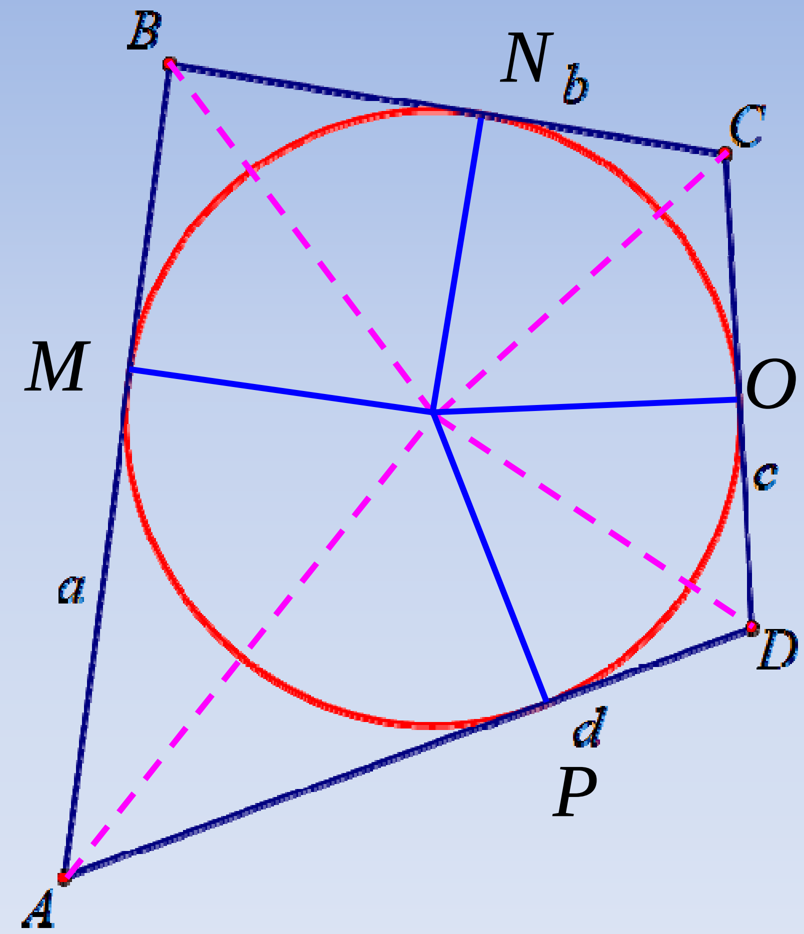
Then, BI, CI, DI, and AI are angle bisectors.



Theorem

A quadrilateral is tangential if and only if

$$a + c = b + d.$$



Proof

BM=BN

CN=CO

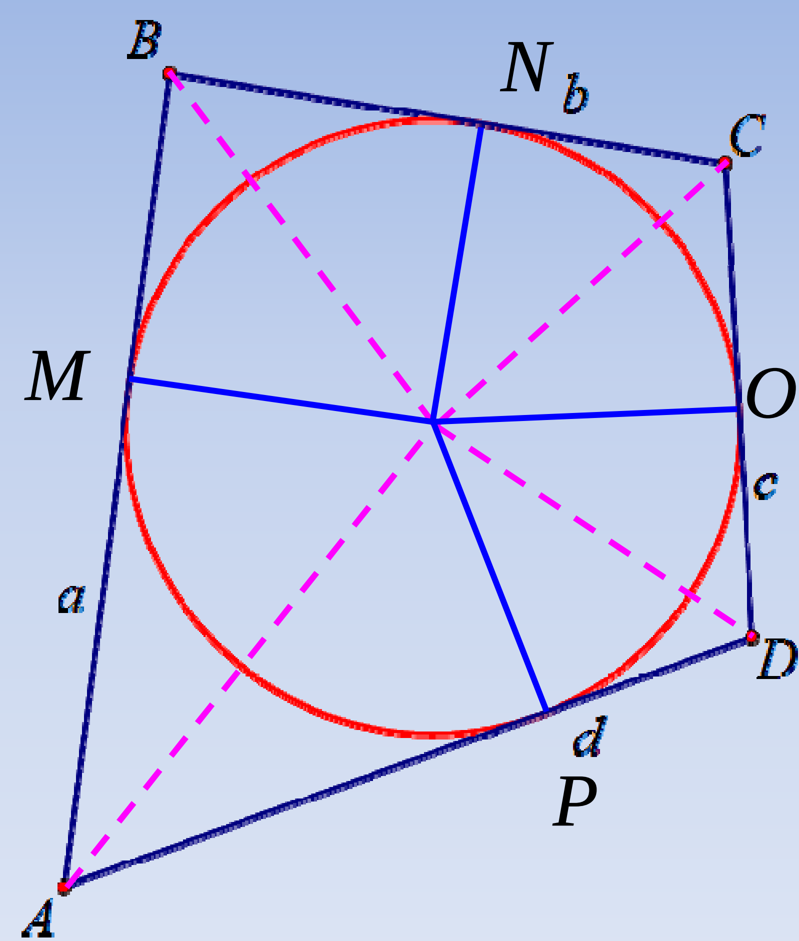
DO=DP

AP=AM

a + c = AM+MB+CO+OD

= AP+BN+CN+DP

= b + d



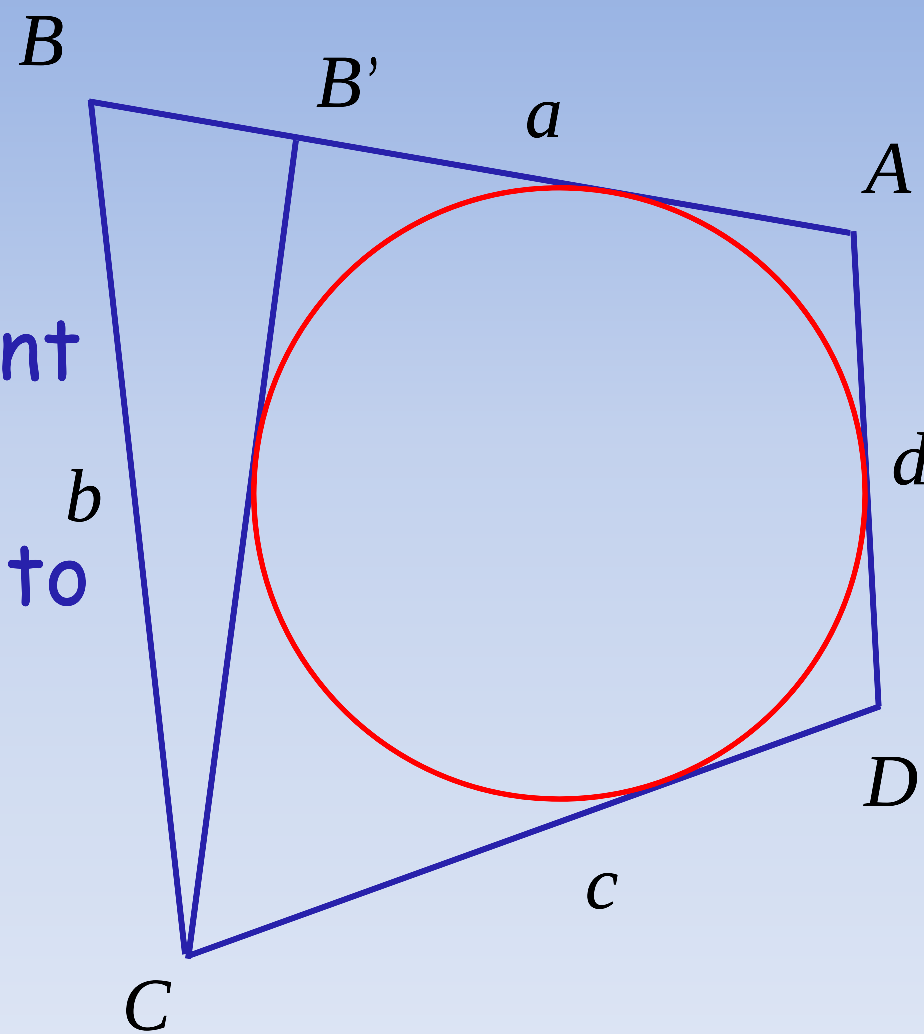
Proof

Assume a + c = b + d.

Construct circle tangent to AB, AD, CD.

Construct CB′ tangent to circle

WLOG AB′<AB



Proof

AB′CD tangential $\Rightarrow$

AB′ + c = d + B′C

a + c = d + b

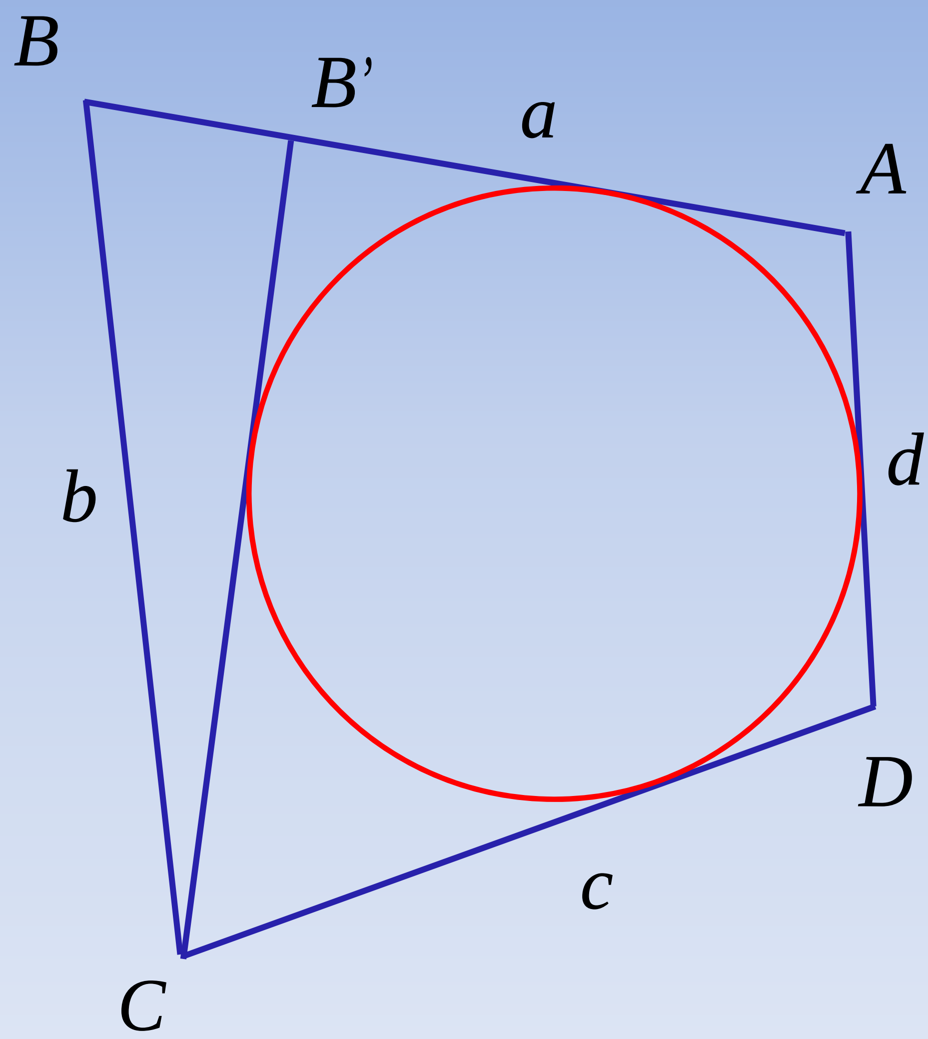
Subtraction

a - AB′ = b − B′C

BB′ + B′C = b

$\Rightarrow$ B = B′

$\Rightarrow$ ABCD tangential



Result

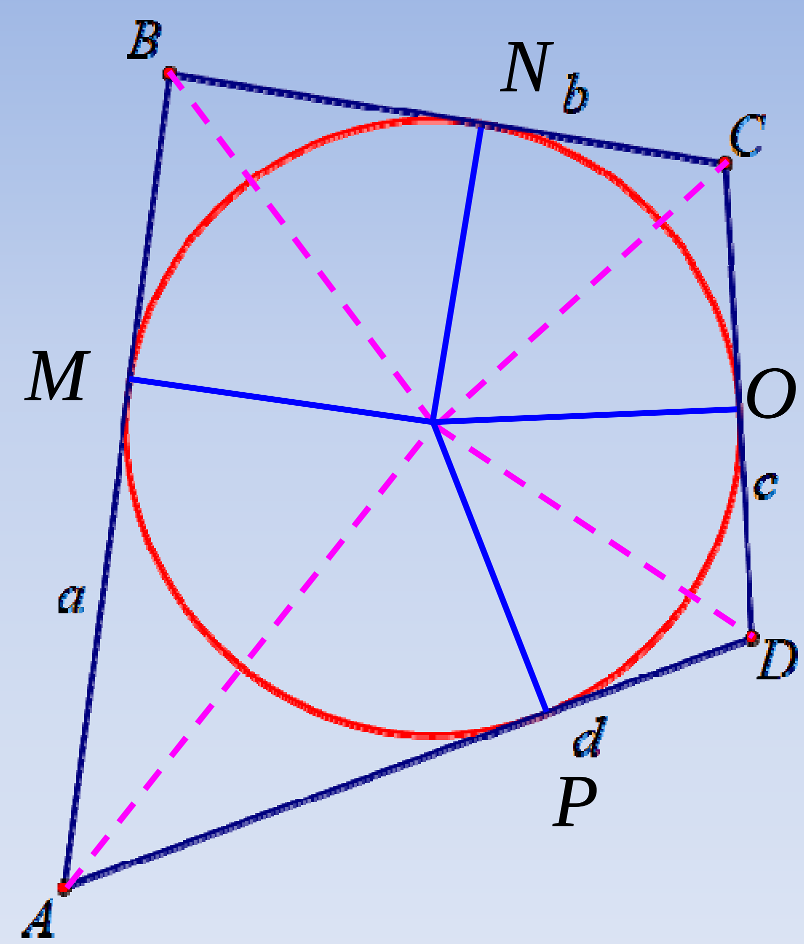
If ABCD tangential quadrilateral then

$$a+c=b+d=\frac{a+b+c+d}{2}=s$$

Area

For inradius r and semiperimeter s

K = rs



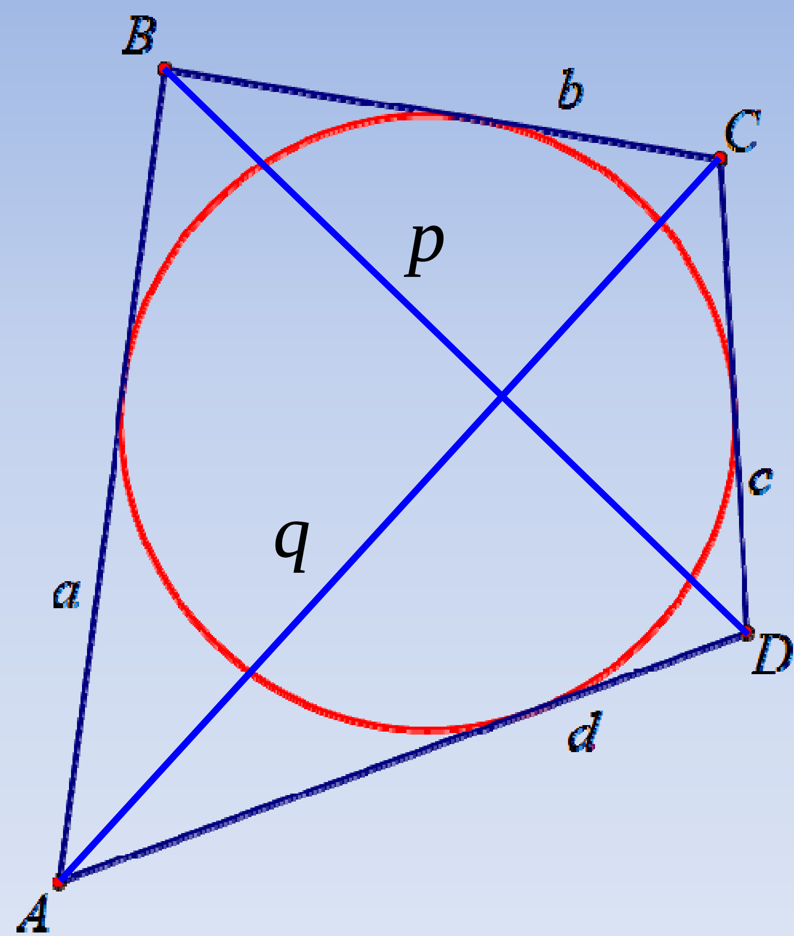
Area

Trigonometric formula

$$K = rs$$

$$K = \sqrt{abcd}\,\sin\left(\frac{B+D}{2}\right)$$

$$K = \frac{1}{2}\sqrt{p^2q^2-(ac-bd)^2}$$



Further Results

1. If a line cuts a tangential quadrilateral into two polygons with equal areas and equal perimeters, the line passes through the incenter
2. If M and N are midpoints of diagonals and incenter = I, then M, I, and N are collinear

Further Results

3. If incircle is tangent to AB, BC, CD, DA at X, Y, Z, W respectively, then lines XY, WZ and AC are concurrent
4. If I is incenter then

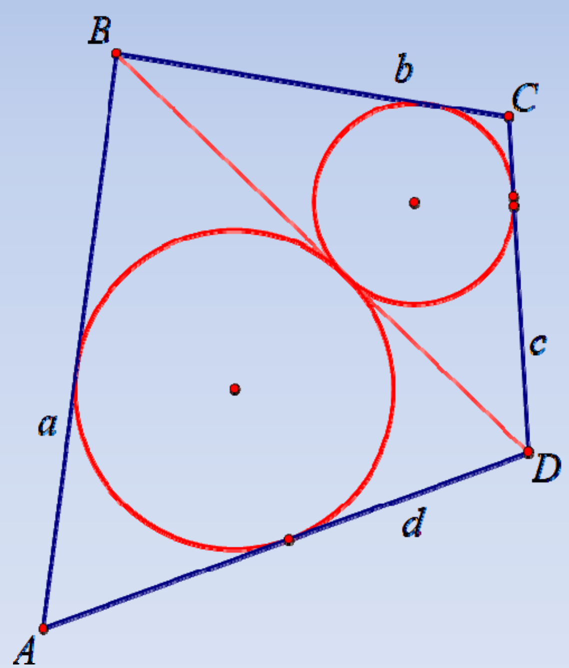
$$IA \bullet IC + IB \bullet ID = \sqrt{AB \bullet BC \bullet CD \bullet DA}$$

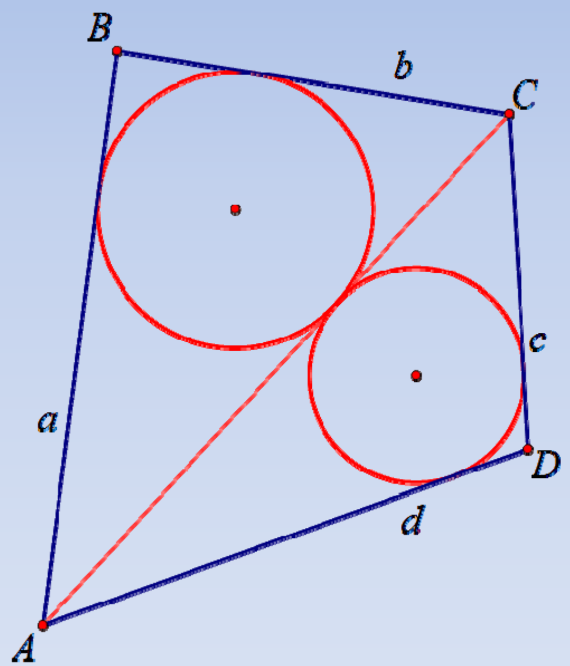
5. Incenter I of ABCD coincides with centroid of ABCD if and only if

$$IA \bullet IC = IB \bullet ID$$

Further Results

Look at the incircles of the triangles of a tangential quadrilateral.





Further Results