


## Loci: Hyperbolas

MA 341 - Topics in *Geometry*  
Lecture 26




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## Conic Sections

Believed first used by Apollonius of Perga  
 hyperbola comes from Greek word meaning "excessive"  
 ellipse comes from Greek word for "deficient"  
 parabola comes from Greek word for "comparable"  
 May refer to eccentricity of curves  
 hyperbola - greater than one  
 ellipse - less than one  
 parabola - exactly one

31-Oct-2011 MA 341 2

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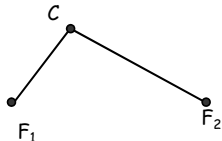
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## Hyperbola

Given two points  $F_1$  and  $F_2$  and  $s > 0$ .  
 Find the locus of points,  $C$ , so that  
 $|CF_1 - CF_2| = s$



31-Oct-2011 MA 341 3

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### Hyperbola

Called a hyperbola with foci  $F_1$  and  $F_2$ .

Let  $V$  be point on  $F_1F_2$  so that  

$$VF_2 = \frac{1}{2}(F_1F_2 - s)$$

Then  

$$VF_1 - VF_2 = (F_1F_2 - VF_2) - VF_2 = F_1F_2 - 2VF_2 = s$$

If we reflect  $V$  across the midpoint we get a second point,  $V'$ , on the hyperbola.

$V, V'$  - vertices of hyperbola

31-Oct-2011 MA 341 4

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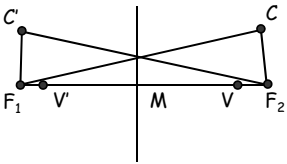
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### Hyperbola

$M$  = midpoint of  $F_1F_2$



31-Oct-2011 MA 341 5

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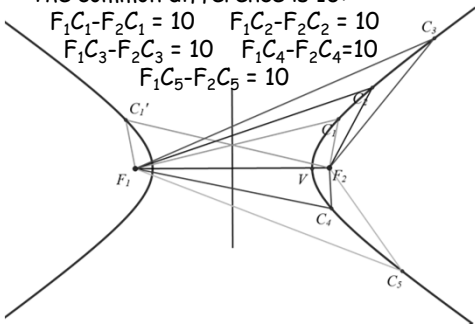
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### Hyperbola

The common difference is 10:

$$F_1C_1 - F_2C_1 = 10 \quad F_1C_2 - F_2C_2 = 10$$

$$F_1C_3 - F_2C_3 = 10 \quad F_1C_4 - F_2C_4 = 10$$

$$F_1C_5 - F_2C_5 = 10$$


31-Oct-2011 MA 341 6

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### Hyperbola

$F_1C'_1 - F_2C'_1 = -10$

We have two branches depending on order of terms

31-Oct-2011 MA 341 7

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### Hyperbola

31-Oct-2011 MA 341 8

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### Hyperbola

What is the common difference here?

31-Oct-2011 MA 341 9

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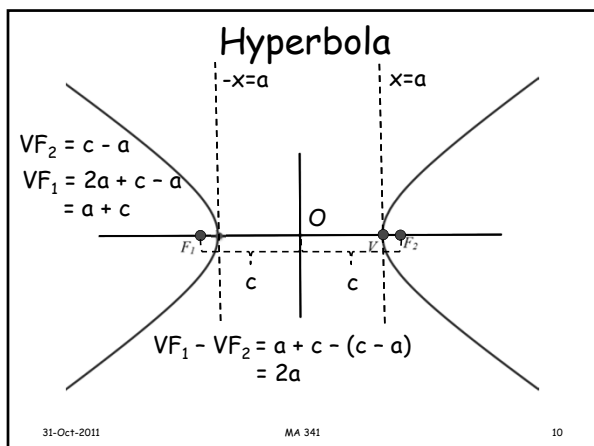
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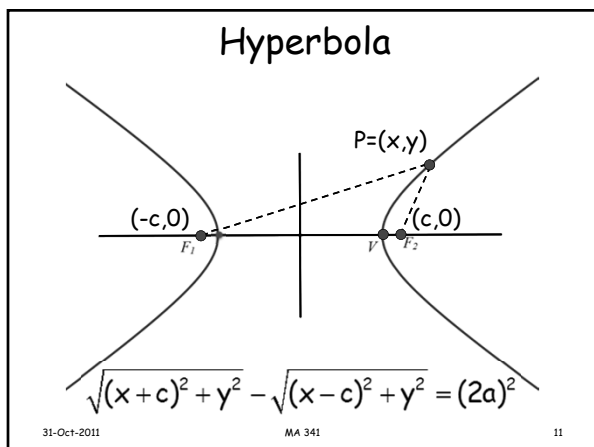
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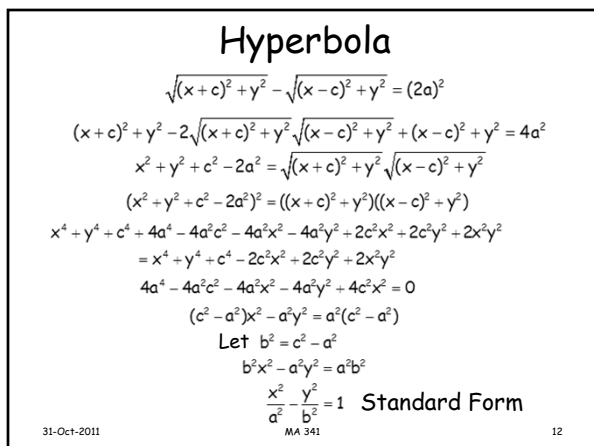
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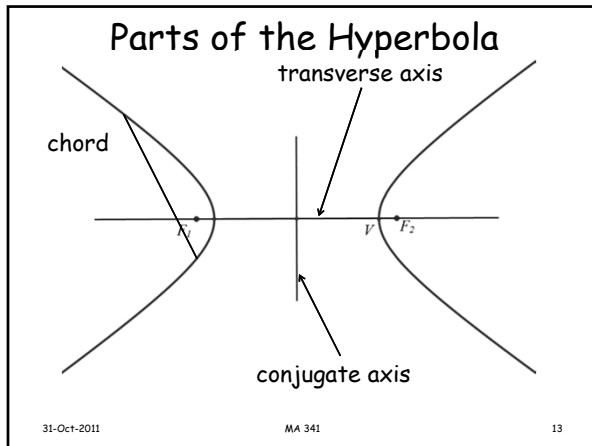
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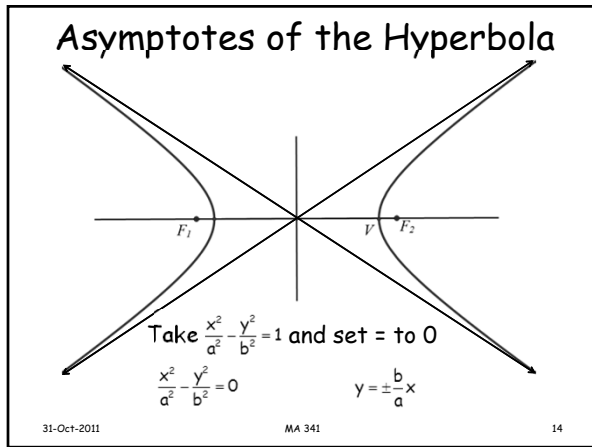
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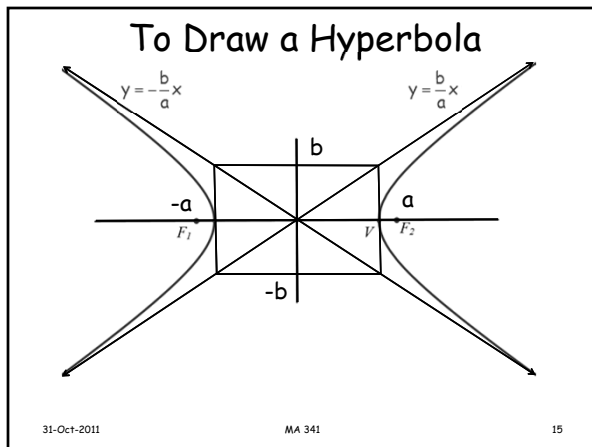
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### Hyperbolas with Center (0,0)

Standard equation	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
Focal axis	x-axis	y-axis
Foci	$(\pm c, 0)$	$(0, \pm c)$
Vertices	$(\pm a, 0)$	$(0, \pm a)$
Semitransverse axis	a	a
Semiconjugate axis	b	b
Pythagorean relation	$c^2 = a^2 + b^2$	$c^2 = a^2 + b^2$
Asymptotes	$y = \pm \frac{b}{a}x$	$y = \pm \frac{a}{b}x$

31-Oct-2011

MA 341

16

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### Hyperbolas with Center (h,k)

Standard equation	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
Focal axis	y = k	x = h
Foci	$(h \pm c, k)$	$(h, k \pm c)$
Vertices	$(h \pm a, k)$	$(h, k \pm a)$
Semitransverse axis	a	a
Semiconjugate axis	b	b
Pythagorean relation	$c^2 = a^2 + b^2$	$c^2 = a^2 + b^2$
Asymptotes	$y = \pm \frac{b}{a}(x-h) + k$	$y = \pm \frac{a}{b}(x-h) + k$

31-Oct-2011

MA 341

17

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### Eccentricity

The eccentricity of a hyperbola is

$$e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a}$$

where a is semitransverse axis, b is semiconjugate axis, and c is distance from center to focus.

31-Oct-2011

MA 341

18

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### Example

Given two fixed points A and B, find the loci of points P so that  $\angle PAB = 2\angle PBA$ .

31-Oct-2011 MA 341 19

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### Example

Put origin at A and B at (1,0).  
 $m_{PA} = \tan(2a)$  hence for  $x \neq 0$ ,  $\tan(2a) = y/x$ .  
 If  $x=0$ ,  $a = 45^\circ$   
 $\tan(a) = y/(1-x)$   
 $\tan(2a) = \frac{2\tan(a)}{1-\tan^2(a)}$

31-Oct-2011 MA 341 20

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### Example

$$\tan(2a) = \frac{2\tan(a)}{1-\tan^2(a)}$$

$$\frac{y}{x} = \tan(2a) = 2 \frac{\frac{y}{1-x}}{1 - \frac{y^2}{(1-x)^2}}$$

$$\frac{y}{x} = 2 \frac{y(1-x)}{(1-x)^2 - y^2}$$

$$y((1-x)^2 - y^2) = 2xy(1-x)$$

31-Oct-2011 MA 341 21

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### Example

$$y((1-x)^2 - y^2) = 2xy(1-x)$$

$$1 - 4x + 3x^2 - y^2 = 0$$

$$\frac{(x - \frac{2}{3})^2}{(\frac{1}{3})^2} - \frac{y^2}{(\frac{1}{\sqrt{3}})^2} = 1$$

Hyperbola - center at (2/3,0)  
 $a = 1/3$  and  $b = \sqrt{3}$   
 $a^2 + b^2 = 4/9$ , so  $c = \pm 2/3$  and foci are at (0,0) and (4/3,0)

31-Oct-2011 MA 341 22

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### Example II

Show that the graph of the curve  $xy = 1$  is a hyperbola.  
 Center = (0,0)  
 Vertices = (1,1) and (-1,-1)

$a = \sqrt{2}$   
 Asymptotes are perpendicular  
 So,  $b = a$   
 $c^2 = a^2 + b^2 = 2 + 2 = 4$   
 $c = 2$   
 $F_1 = (-\sqrt{2}, -\sqrt{2})$   
 $F_2 = (\sqrt{2}, \sqrt{2})$

31-Oct-2011 MA 341 23

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### Example II

To show a hyperbola we need to show  
 $|PF_1 - PF_2| = 2\sqrt{2}$   
 $P = (x, y), x \neq 0$  and  $y = 1/x$ .

$$|PF_1 - PF_2| = \sqrt{(-\sqrt{2} - x)^2 + (-\sqrt{2} - \frac{1}{x})^2} - \sqrt{(\sqrt{2} - x)^2 + (\sqrt{2} - \frac{1}{x})^2}$$

$$|PF_1 - PF_2| = \sqrt{4 + 2\sqrt{2}(x + \frac{1}{x}) + (x^2 + \frac{1}{x^2})} - \sqrt{4 - 2\sqrt{2}(x + \frac{1}{x}) + (x^2 + \frac{1}{x^2})}$$

Let  $u = x + 1/x$

$$|PF_1 - PF_2| = \sqrt{4 + 2\sqrt{2}u + u^2} - \sqrt{4 - 2\sqrt{2}u + u^2}$$

$$|PF_1 - PF_2| = |u + \sqrt{2}| - |u - \sqrt{2}|$$

$u^2 = x + 2 + 1/x > 2$  so  $|u| \geq \sqrt{2}$

31-Oct-2011 MA 341 24

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### Example II

If  $u \leq -\sqrt{2}$

$$|PF_1 - PF_2| = |-u - \sqrt{2} - (-u + \sqrt{2})| = |-2\sqrt{2}| = 2\sqrt{2}$$

If  $u \geq \sqrt{2}$

$$|PF_1 - PF_2| = |u + \sqrt{2} - (u - \sqrt{2})| = 2\sqrt{2}$$

Therefore, P lies on the hyperbola.

31-Oct-2011

MA 341

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