Loci: Hyperbolas

MA 341 – Topics in Geometry
Lecture 26

Conic Sections

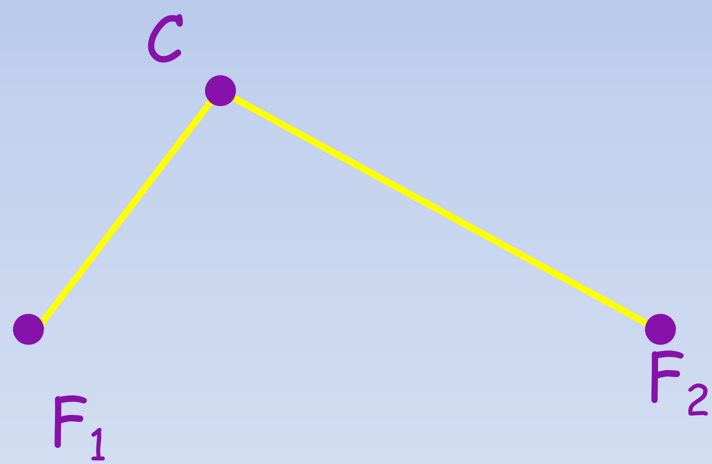
Believed first used by Apollonius of Perga
hyperbola comes from Greek word meaning "excessive"
ellipse comes from Greek word for "deficient"
parabola comes from Greek word for "comparable"
May refer to eccentricity of curves
- hyperbola - greater than one
- ellipse - less than one
- parabola - exactly one

Hyperbola

Given two points F_1 and F_2 and $s > 0$.

Find the locus of points, C, so that

$$|CF_1 - CF_2| = s$$



Hyperbola

Called a hyperbola with foci F_1 and F_2.

Let V be point on F_1F_2 so that

$$VF_2 = \tfrac{1}{2}(F_1F_2 - s)$$

Then

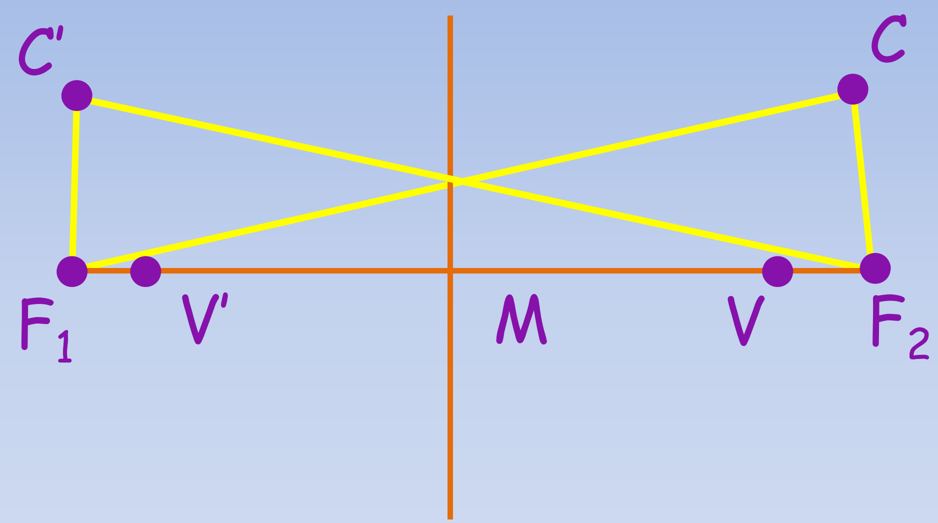
$$VF_1 - VF_2 = (F_1F_2 - VF_2) - VF_2 = F_1F_2 - 2VF_2 = s$$

If we reflect V across the midpoint we get a second point, V', on the hyperbola.

V, V' – vertices of hyperbola

Hyperbola

M = midpoint of F_1F_2



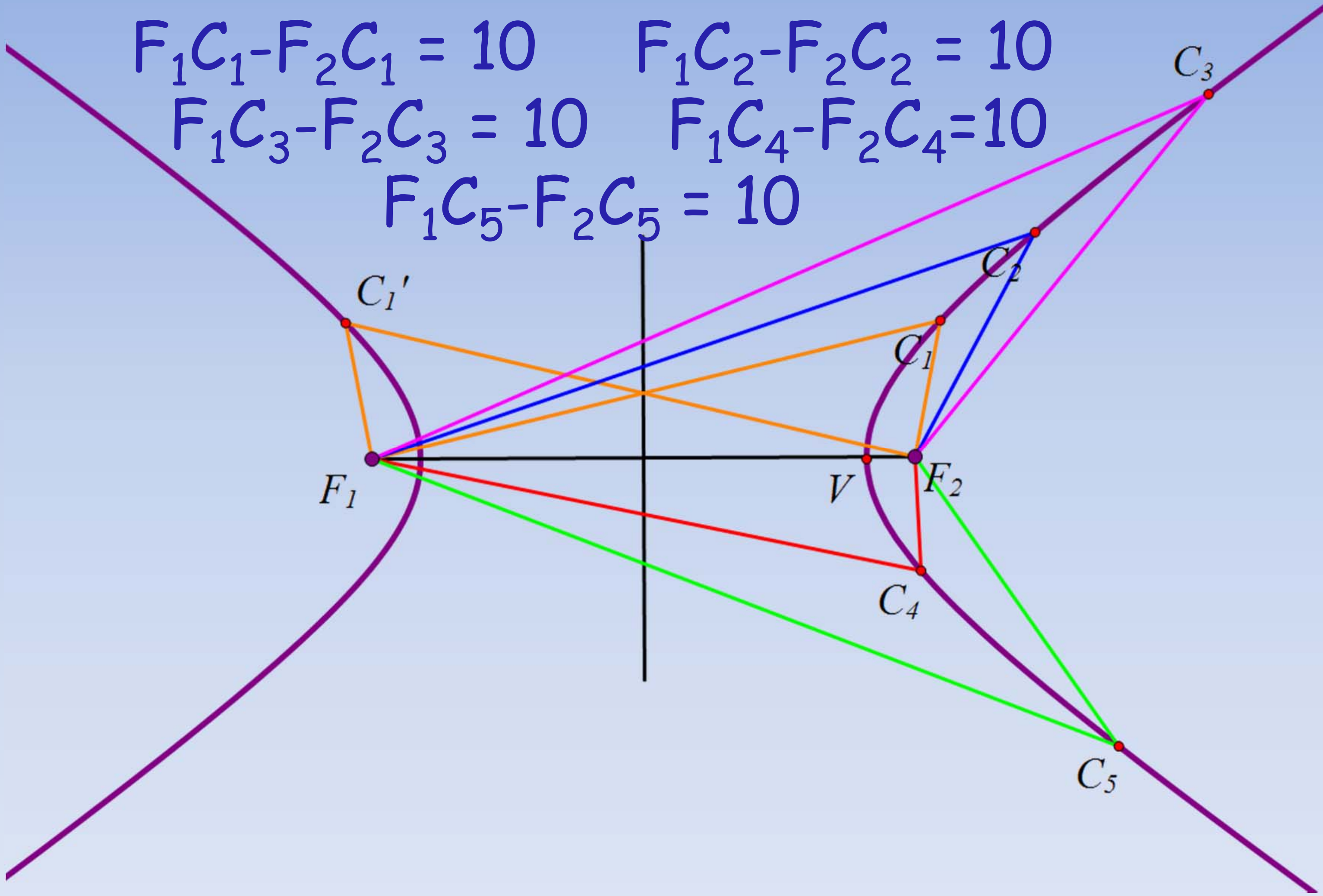
Hyperbola

The common difference is 10:

$F_1C_1 - F_2C_1 = 10$ $F_1C_2 - F_2C_2 = 10$

$F_1C_3 - F_2C_3 = 10$ $F_1C_4 - F_2C_4 = 10$

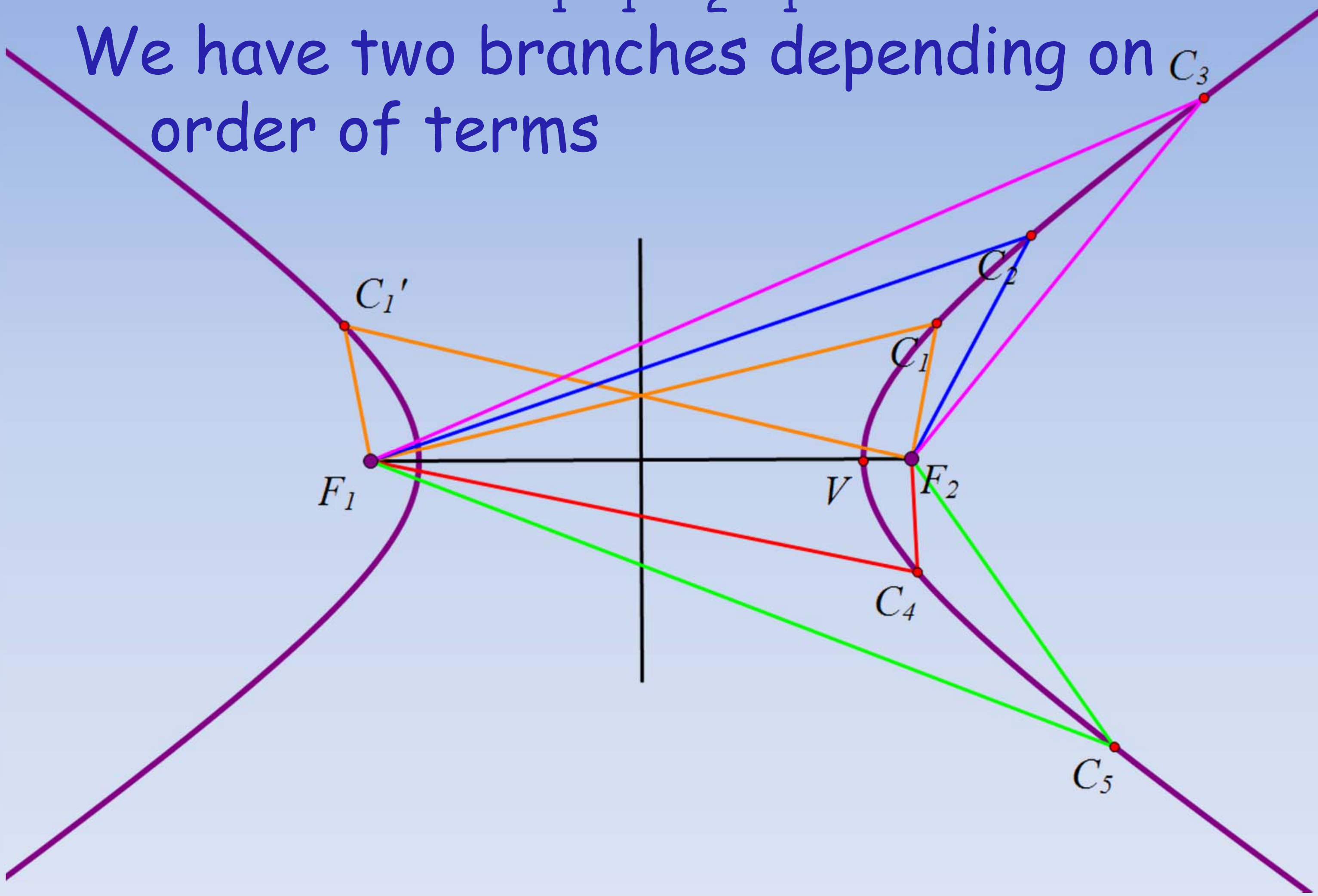
$F_1C_5 - F_2C_5 = 10$



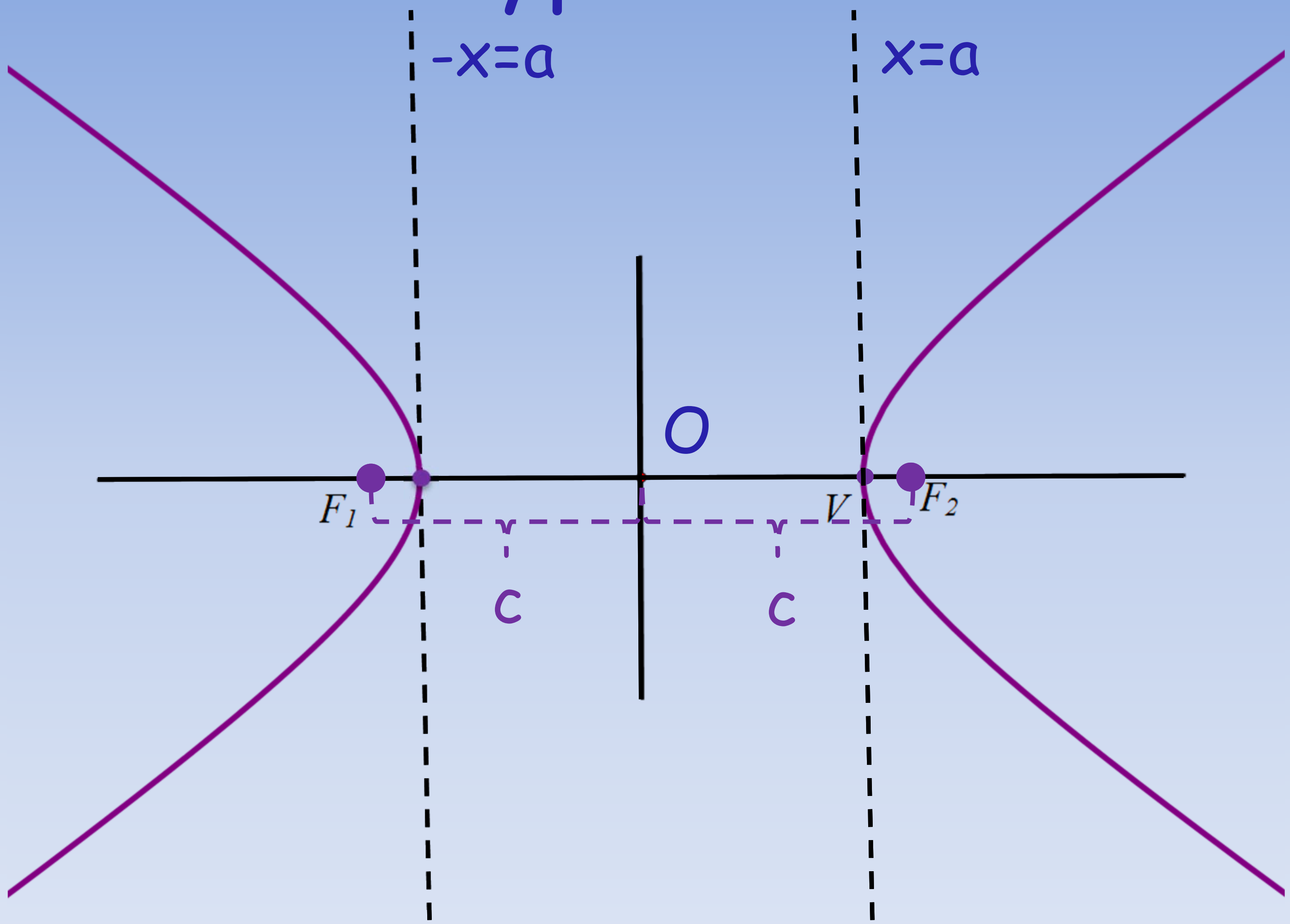
Hyperbola

$F_1C'_1 - F_2C'_1 = -10$

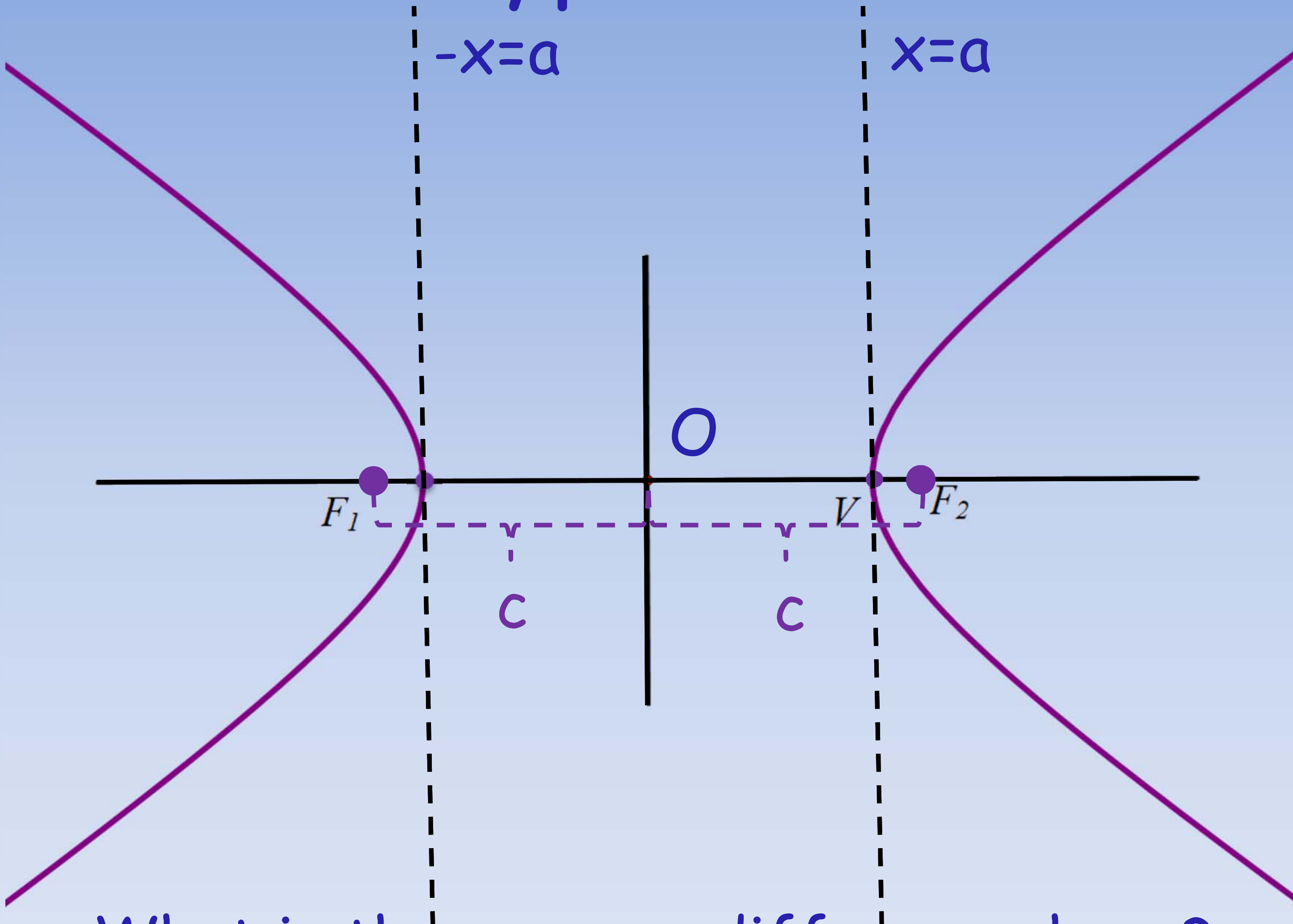
We have two branches depending on order of terms



Hyperbola

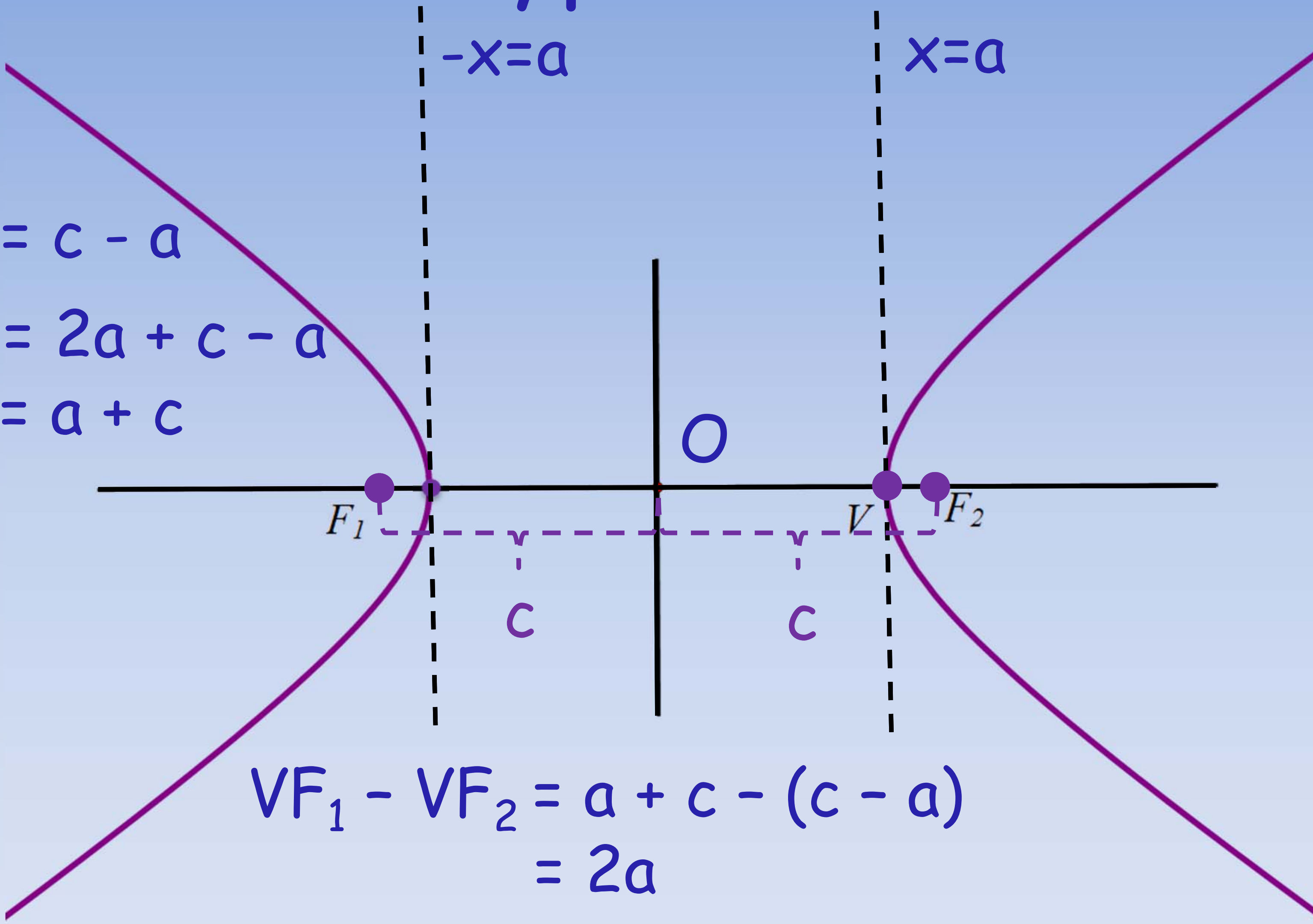


Hyperbola



What is the common difference here?

Hyperbola



$VF_2 = c - a$

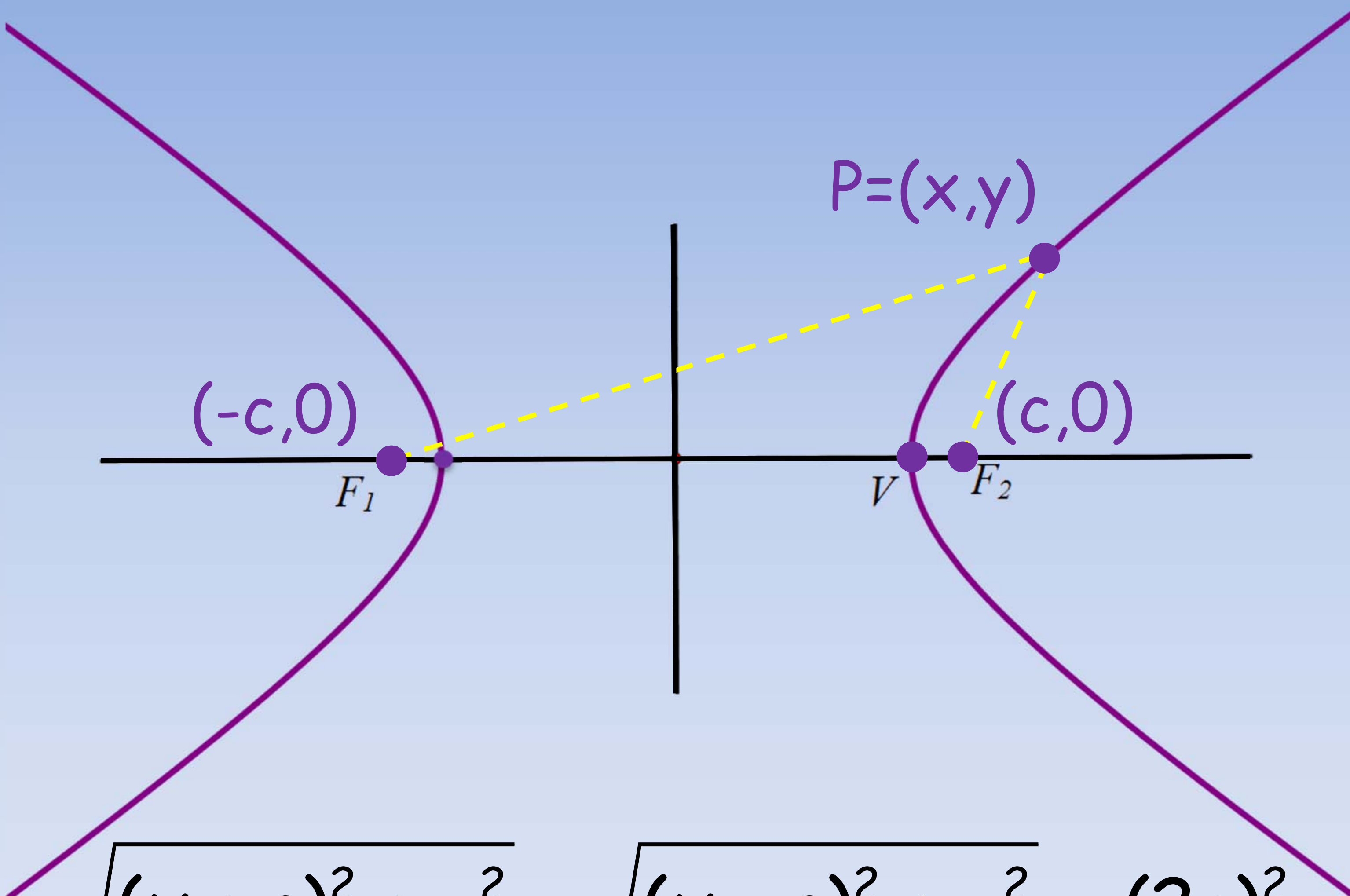
$VF_1 = 2a + c - a$

$= a + c$

$VF_1 - VF_2 = a + c - (c - a)$

$= 2a$

Hyperbola



$$\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}=(2a)^2$$

Hyperbola

$$\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}=(2a)^2$$

$$(x+c)^2+y^2-2\sqrt{(x+c)^2+y^2}\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2=4a^2$$

$$x^2+y^2+c^2-2a^2=\sqrt{(x+c)^2+y^2}\sqrt{(x-c)^2+y^2}$$

$$(x^2+y^2+c^2-2a^2)^2=((x+c)^2+y^2)((x-c)^2+y^2)$$

$$x^4+y^4+c^4+4a^4-4a^2c^2-4a^2x^2-4a^2y^2+2c^2x^2+2c^2y^2+2x^2y^2$$

$$=x^4+y^4+c^4-2c^2x^2+2c^2y^2+2x^2y^2$$

$$4a^4-4a^2c^2-4a^2x^2-4a^2y^2+4c^2x^2=0$$

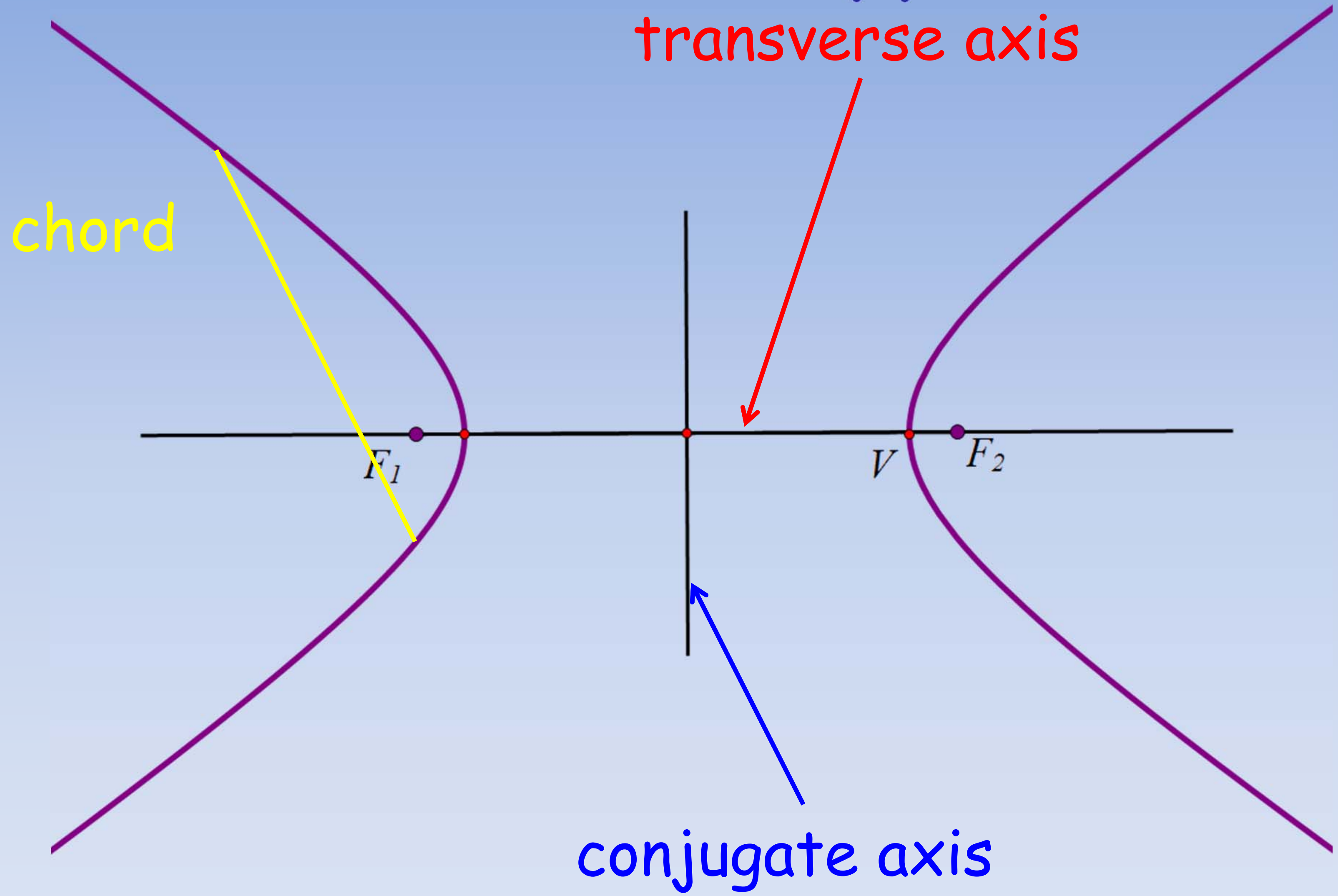
$$(c^2-a^2)x^2-a^2y^2=a^2(c^2-a^2)$$

Let $b^2=c^2-a^2$

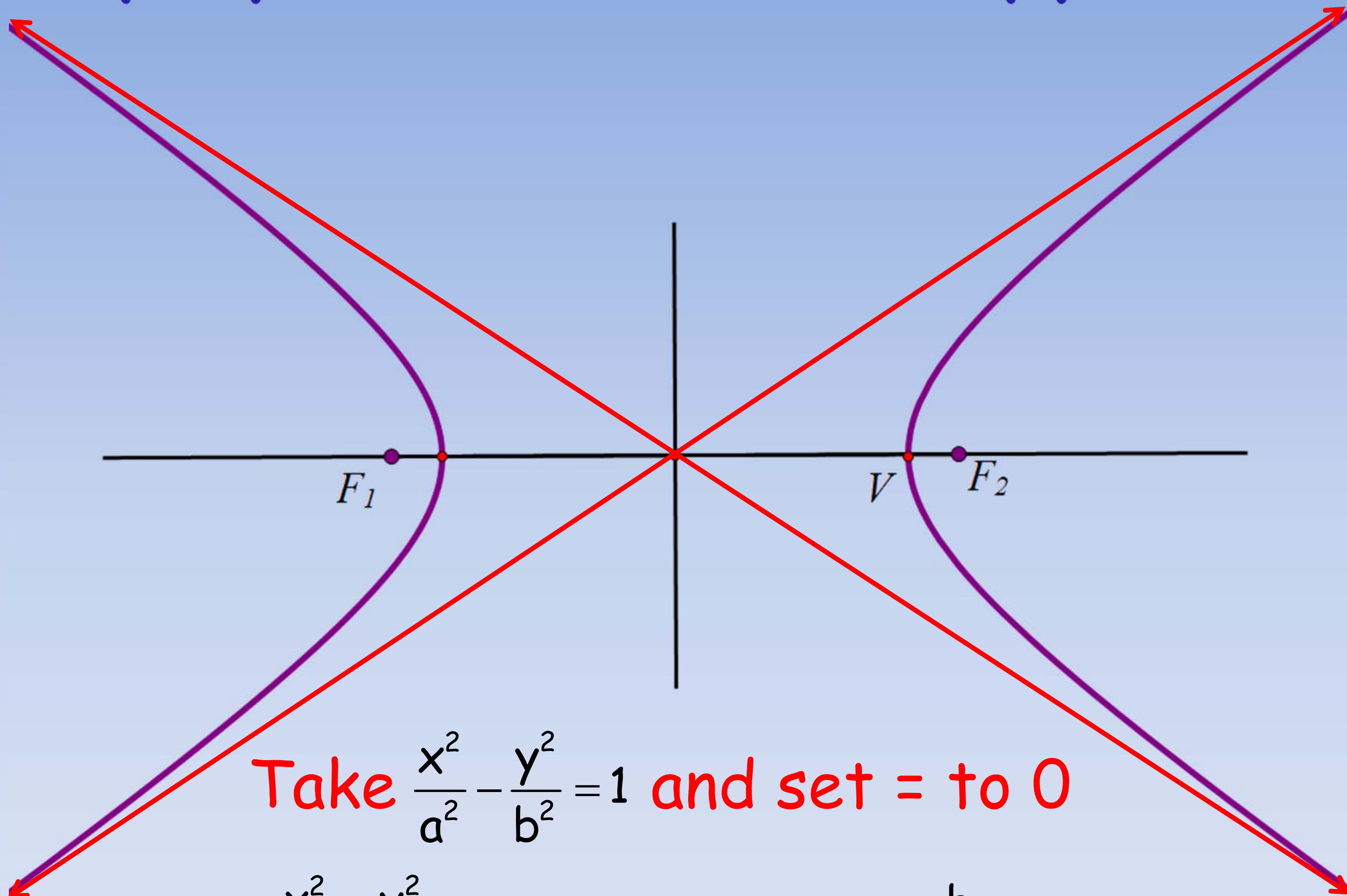
$$b^2x^2-a^2y^2=a^2b^2$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$ Standard Form

Parts of the Hyperbola



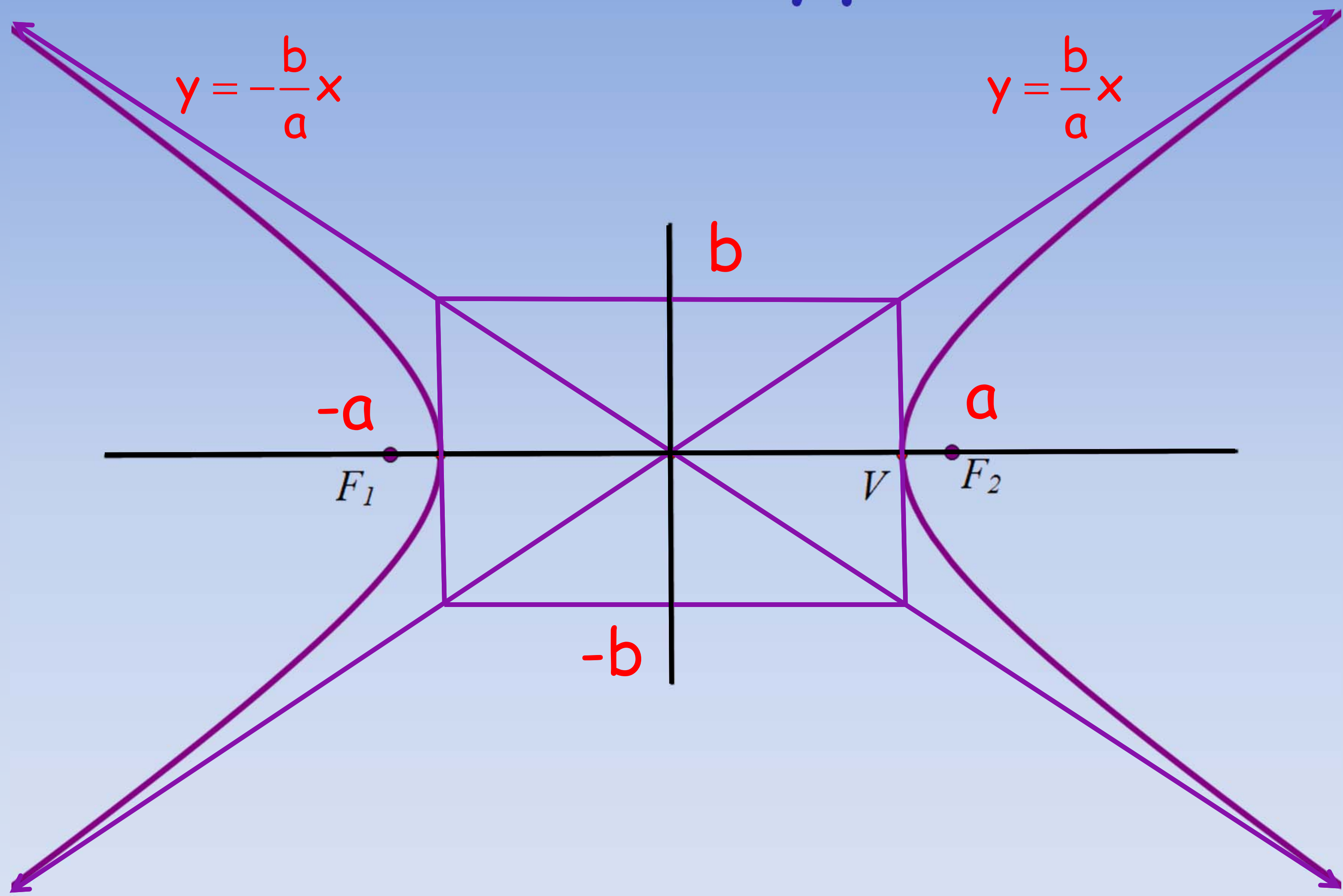
Asymptotes of the Hyperbola



Take $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and set = to 0

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \qquad y = \pm \frac{b}{a} x$$

To Draw a Hyperbola



Hyperbolas with Center (0,0)

Standard equation	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$
Focal axis	x-axis	y-axis
Foci	$(\pm c,0)$	$(0,\pm c)$
Vertices	$(\pm a,0)$	$(0,\pm a)$
Semitransverse axis	a	a
Semiconjugate axis	b	b
Pythagorean relation	$c^2=a^2+b^2$	$c^2=a^2+b^2$
Asymptotes	$y=\pm\frac{b}{a}x$	$y=\pm\frac{a}{b}x$

Hyperbolas with Center (h,k)

Standard equation	$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$	$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$
Focal axis	$y = k$	$x = h$
Foci	$(h \pm c, k)$	$(h, k \pm c)$
Vertices	$(h \pm a, k)$	$(h, k \pm a)$
Semitransverse axis	a	a
Semiconjugate axis	b	b
Pythagorean relation	$c^2 = a^2 + b^2$	$c^2 = a^2 + b^2$
Asymptotes	$y = \pm\frac{b}{a}(x-h)+k$	$y = \pm\frac{a}{b}(x-h)+k$

Eccentricity

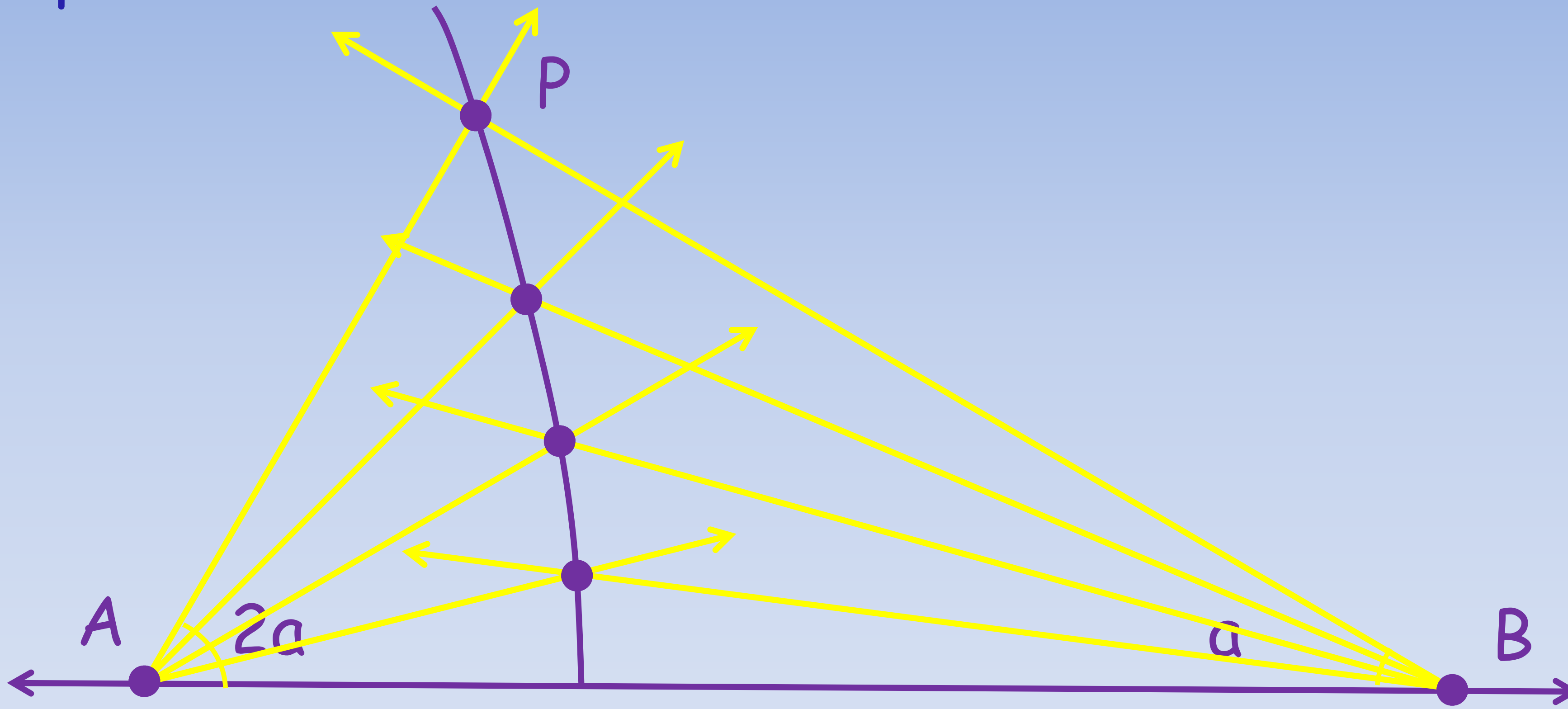
The eccentricity of a hyperbola is

$$e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a}$$

where a is semitransverse axis, b is semiconjugate axis, and c is distance from center to focus.

Example

Given two fixed points A and B, find the loci of points P so that $\angle PAB = 2\angle PBA$.



Example

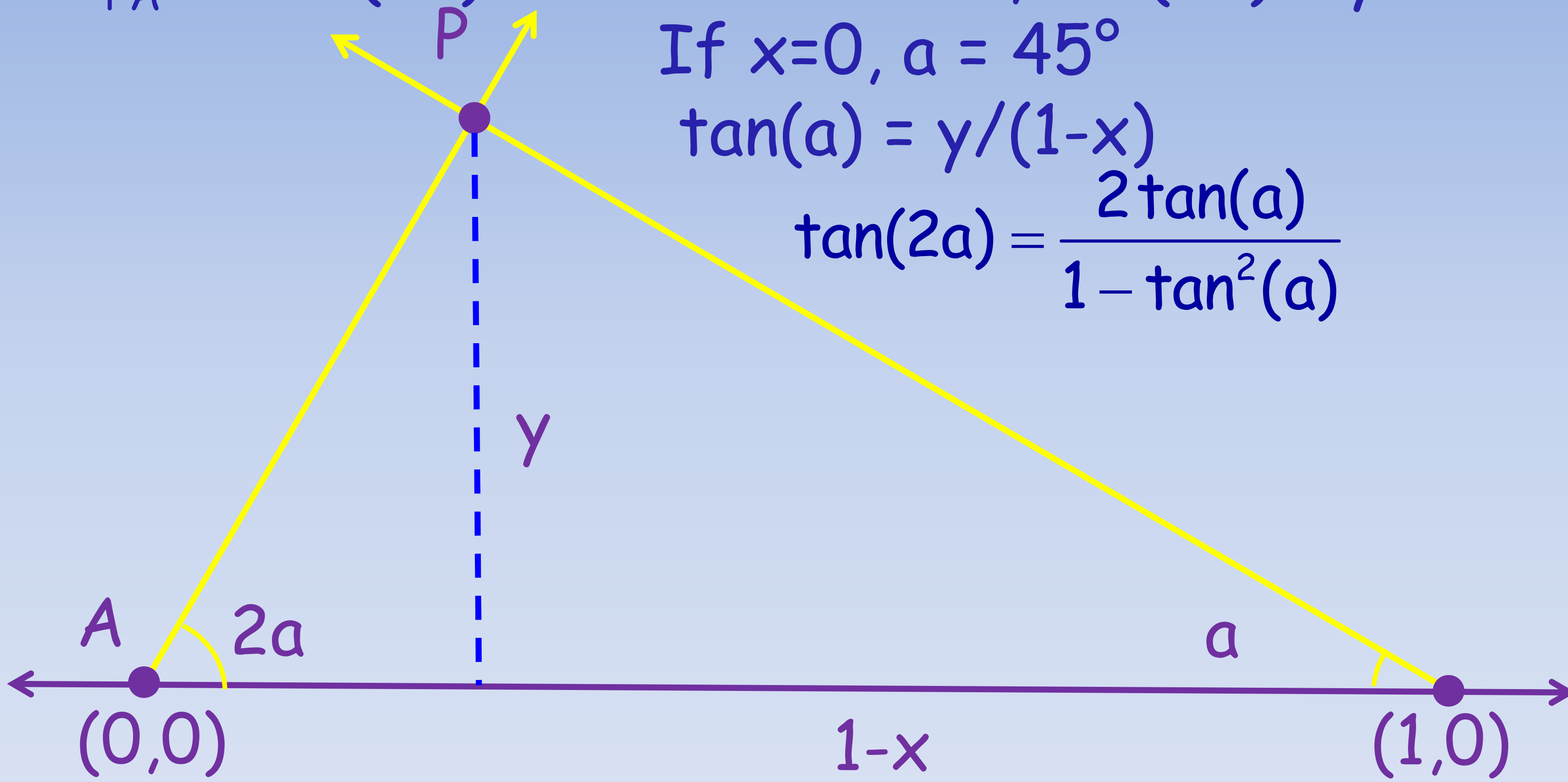
Put origin at A and B at (1,0).

m_{PA} = tan (2a) hence for $x \neq 0$, tan(2a) = y/x.

If x=0, a = 45°

tan(a) = y/(1-x)

$$\tan(2a) = \frac{2\tan(a)}{1-\tan^2(a)}$$



Example

$$\tan(2a) = \frac{2\tan(a)}{1-\tan^2(a)}$$

$$\frac{y}{x} = \tan(2a) = 2\frac{\frac{y}{1-x}}{1-\frac{y^2}{(1-x)^2}}$$

$$\frac{y}{x} = 2\frac{y(1-x)}{(1-x)^2-y^2}$$

$$y((1-x)^2-y^2) = 2xy(1-x)$$

Example

$$y((1-x)^2 - y^2) = 2xy(1-x)$$

$$1 - 4x + 3x^2 - y^2 = 0$$

$$\frac{(x - \frac{2}{3})^2}{(\frac{1}{3})^2} - \frac{y^2}{(\frac{1}{\sqrt{3}})^2} = 1$$

Hyperbola – center at (2/3,0)

a = 1/3 and b=√3

$a^2 + b^2 = 4/9$, so c = ±2/3 and foci are at (0,0) and (4/3,0)

Example II

Show that the graph of the curve $xy = 1$ is a hyperbola.

Center = (0,0)

Vertices = (1,1) and (-1,-1)

$a = \sqrt{2}$

Asymptotes are perpendicular

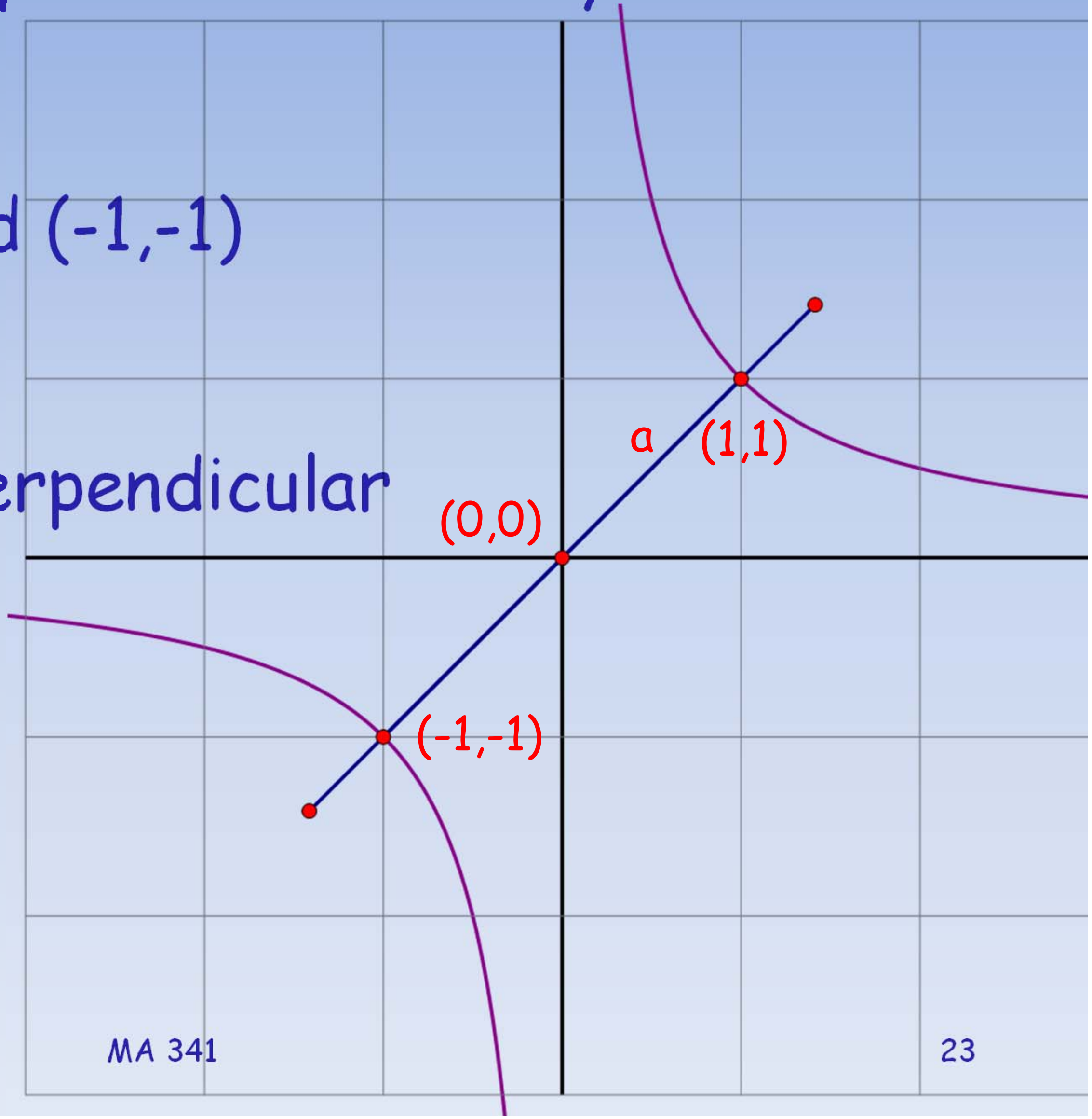
So, $b = a$

$c^2 = a^2 + b^2 = 2+2=4$

$c=2$

$F_1 = (-\sqrt{2},-\sqrt{2})$

$F_2 = (\sqrt{2}, \sqrt{2})$



Example II

To show a hyperbola we need to show

$$|PF_1 - PF_2| = 2\sqrt{2}$$

$P = (x,y)$, $x \neq 0$ and $y=1/x$.

$$\left|PF_1 - PF_2\right| = \sqrt{\left(-\sqrt{2}-x\right)^2 + \left(-\sqrt{2}-\frac{1}{x}\right)^2} - \sqrt{\left(\sqrt{2}-x\right)^2 + \left(\sqrt{2}-\frac{1}{x}\right)^2}$$

$$\left|PF_1 - PF_2\right| = \sqrt{4 + 2\sqrt{2}\left(x+\frac{1}{x}\right) + \left(x^2+\frac{1}{x^2}\right)} - \sqrt{4 - 2\sqrt{2}\left(x+\frac{1}{x}\right) + \left(x^2+\frac{1}{x^2}\right)}$$

Let $u = x + 1/x$

$$\left|PF_1 - PF_2\right| = \sqrt{2 + 2\sqrt{2}u + u^2} - \sqrt{2 - 2\sqrt{2}u + u^2}$$

$$\left|PF_1 - PF_2\right| = \left\|u + \sqrt{2}\right| - \left|u - \sqrt{2}\right\|$$

$u^2 = x + 2 + 1/x > 2$ so $|u| \geq \sqrt{2}$

Example II

If u ≤ -√2

$$|PF_1 - PF_2| = |-u - \sqrt{2} - (-u + \sqrt{2})| = |-2\sqrt{2}| = 2\sqrt{2}$$

If u ≥ √2

$$|PF_1 - PF_2| = |u + \sqrt{2} - (u - \sqrt{2})| = 2\sqrt{2}$$

Therefore, P lies on the hyperbola.