Hyperbolic Functions

MA 341 – Topics in Geometry
Lecture 27

UK

History of Hyperbolic Functions

Johann Heinrich Lambert
1728 - 1777

Vincenzo Riccati
1707-1775

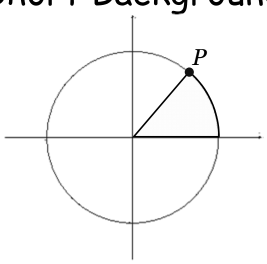
Riccati's Work

- Developed hyperbolic functions
- Proved consistency using only geometry of unit hyperbola $x^2 - y^2 = 1$
- Followed father's interests in DE's arising from geometrical problems.
- Developed properties of the hyperbolic functions from purely geometrical considerations.

Lambert's Work

- First to introduce hyperbolic functions in trigonometry
- Tied them to geometry
- Used them in solving certain differential equations

Short Background

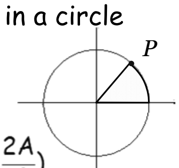


Unit circle
$x^2 + y^2 = 1$

Circular Relation

Arclength relation to area in a circle

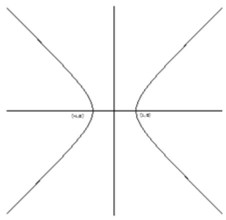
$$L = r\theta \text{ and } A = r^2\frac{\theta}{2} \Rightarrow L = \frac{2A}{r}$$

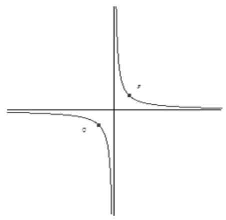


$$P = (r\cos\theta, r\sin\theta) = (r\cos\frac{2A}{r^2}, r\sin\frac{2A}{r^2})$$

Does this hold in hyperbolas?

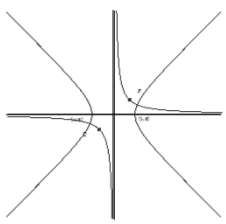
Unit Hyperbolas



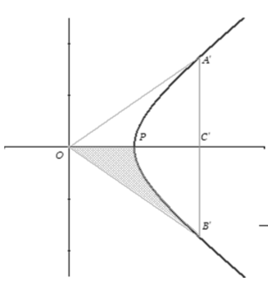


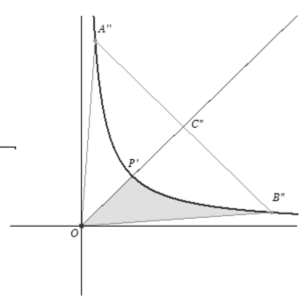
$x^2 - y^2 = 1$ $2xy = 1$

Derivation

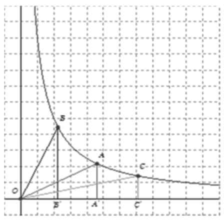


Derivation





Hyperbolic Areas



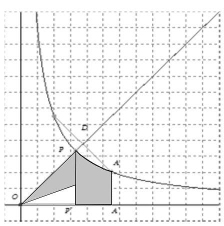
The graph is xy = k.
Area of $\Delta OAA'$

$$\text{Area} = \frac{xy}{2} = \frac{k}{2}$$

Note then that

Area($\Delta OAA'$) = Area($\Delta OBB'$) = Area($\Delta OCC'$)

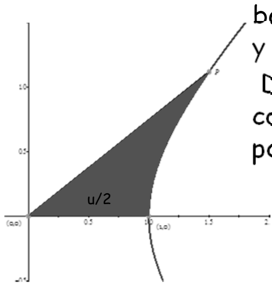
Area considerations for xy=1



Area($\Delta OAA'$)= Area($\Delta OPP'$)
Area(ΔOPQ) = Area($AA'PP'$)
(Subtract Area($\Delta OQP'$) from above)
Area($APP'A'$) = Area(OAP)
(Add Area(QAP) to above)

$$\text{Area}(AOP) = \int_1^a \frac{dx}{x} = \ln(a) = u$$

Hyperbolic Trigonometric



Let u/2 = area bounded by x-axis, y = x, and curve.
Define the coordinates of the point P by
x = ch(u)
y = sh(u)

Properties of the functions ch(u) and sh(u)

1. $ch(u)^2 - sh(u)^2 = 1$ – Obvious(?).
2. ch(u + v) = ch(u)ch(v) + sh(u)sh(v)
3. sh(u + v) = sh(u)ch(v) + ch(u)sh(v)

We will prove 2 & 3 shortly.

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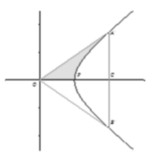
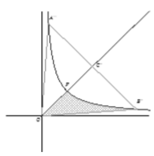
Second Fundamental Property

Area(PC'A')=Area(ΔOC'A') - u/2

Rotate counterclockwise through π/4. We do not change area!!

Rotation carries $x^2 - y^2 = 1$ to 2xy = 1

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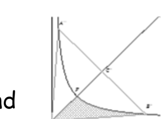
Second Fundamental Property

B'=(ch(u), – sh(u)) = (c_1,s_1)

B''= (x,1/2x), P' = (1/√2, 1/√2)

B''C'' = B'C' = s_1 and OC''=OC' = c_1

Area bounded by OP', y = 1/2x and OB':



$$K = \left[\frac{1}{2}\times\frac{1}{\sqrt{2}}\times\frac{1}{\sqrt{2}}\right] + \int_{1/\sqrt{2}}^{x_0}\frac{dx}{2x} - \left[\frac{1}{2}\times x_0\times\frac{1}{2x_0}\right] = \frac{1}{2}\ln x\Big|_{1/\sqrt{2}}^{x_0} = \frac{1}{2}\ln\left(\sqrt{2}x_0\right)$$

$$\frac{u}{2} = \frac{1}{2}\ln\left(\sqrt{2}x_0\right)$$

$$e^u = \sqrt{2}x_0, \quad x_0 = \frac{e^u}{\sqrt{2}} \text{ and } y_0 = \frac{1}{\sqrt{2}e^u}$$

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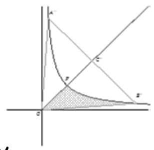
Second Fundamental Property

Now we need to find *C''*.

B''*C''* ⊥ *OC''* so it has slope -1 and equation

$$y - \frac{1}{2x_0} = -(x - x_0)$$



Since *C''* lies on the diagonal, x = y and

$$x = \frac{x_0}{2} + \frac{1}{4x_0} = y$$

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Second Fundamental Property

Thus, the distance B''*C''* is given by the *distance formula*

$$B''C'' = \left[\left(x_0 - \left[\frac{x_0}{2} + \frac{1}{4x_0}\right]\right)^2 + \left(\frac{1}{2x_0} - \left[\frac{x_0}{2} + \frac{1}{4x_0}\right]\right)^2\right]^{1/2}$$

$$= \sqrt{2}\left(\frac{x_0}{2} - \frac{1}{4x_0}\right)$$

This last term is positive if $x_0 > 1/\sqrt{2}$

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Second Fundamental Property

Recall that $x_0 = e^u/\sqrt{2}$ and B''*C''* = sh(u). Therefore

$$sh(u) = \sqrt{2}\left(\frac{e^u}{2\sqrt{2}} - \frac{\sqrt{2}}{4e^u}\right) = \frac{e^u - e^{-u}}{2}$$

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Second Fundamental Property

$$O''C'' = \left[\left(\left[\frac{x_0}{2} + \frac{1}{4x_0}\right]\right)^2 + \left(\left[\frac{x_0}{2} + \frac{1}{4x_0}\right]\right)^2\right]^{1/2}$$

$$= \sqrt{2}\left(\frac{x_0}{2} + \frac{1}{4x_0}\right)$$

$$ch(u) = \sqrt{2}\left(\frac{e^u}{2\sqrt{2}} + \frac{\sqrt{2}}{4e^u}\right) = \frac{e^u + e^{-u}}{2}$$

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Hyperbolic Trigonometric Functions

Traditionally, we have:

ch(u) = cosh(u)

sh(u) = sinh(u)

Define the remaining 4 hyperbolic trig functions as expected:

tanh(u), coth(u), sech(u), csch(u)

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Properties of the functions cosh(u) and sinh(u)

1. $\cosh(u)^2 - \sinh(u)^2 = 1$
2. cosh(u + v) = cosh(u)cosh(v) + sinh(u)sinh(v)
3. sinh(u + v) = sinh(u)cosh(v) + cosh(u)sinh(v)

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Evaluation of the functions cosh(u) and sinh(u)

With a little work, we can show the following two identities:

$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

Properties of cosh(u) and sinh(u)

$$\sinh u \cosh v + \cosh u \sinh v = \frac{e^{u} - e^{-u}}{2}\frac{e^{v} + e^{-v}}{2} + \frac{e^{u} + e^{-u}}{2}\frac{e^{v} - e^{-v}}{2}$$

$$= \frac{e^{u+v} - \cancel{e^{v-u}} + \cancel{e^{u-v}} - e^{-(u+v)}}{4} + \frac{e^{u+v} + \cancel{e^{v-u}} - \cancel{e^{u-v}} - e^{-(u+v)}}{4}$$

$$= \frac{2e^{u+v} - 2e^{-(u+v)}}{4} \quad = \frac{e^{u+v} - e^{-(u+v)}}{2}$$

$$= \sinh(u+v)$$

Properties of cosh(u) and sinh(u)

Note: $\sinh^{2}x = \left(\frac{e^{x} - e^{-x}}{2}\right)^{2} = \frac{e^{2x} - 2 + e^{-2x}}{4}$

$$\cosh^{2}x = \left(\frac{e^{x} + e^{-x}}{2}\right)^{2} = \frac{e^{2x} + 2 + e^{-2x}}{4}$$

$$\cosh^{2}x - \sinh^{2}x = \frac{\cancel{e^{2x}} + 2 + \cancel{e^{-2x}}}{4} - \frac{\cancel{e^{2x}} - 2 + \cancel{e^{-2x}}}{4}$$

$$= \frac{4}{4} = 1$$

More Hyperbolic Functions

There are four more hyperbolic trig functions:

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{e^x + e^{-x}}{e^x - e^{-x}}, x \neq 0$$

$$\text{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^x + e^{-x}}$$

$$\text{csch}(x) = \frac{1}{\sinh(x)} = \frac{2}{e^x - e^{-x}}, x \neq 0$$

Circular Functions | **Hyperbolic Functions**

$$\cos^2 x + \sin^2 x = 1 \qquad \cosh^2 x - \sinh^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x \qquad 1 - \tanh^2 x = \text{sech}^2 x$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 + \tan x \tan y}$$

$$\tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$

$$\sin^2 \frac{x}{2} = \frac{1 - \cos x}{2} \qquad \sinh^2 \frac{x}{2} = \frac{\cosh x - 1}{2}$$

$$\cos^2 \frac{x}{2} = \frac{1 + \cos x}{2} \qquad \cosh^2 \frac{x}{2} = \frac{\cosh x + 1}{2}$$

$$\tan^2 \frac{x}{2} = \frac{1 - \cos x}{1 + \cos x} \qquad \tanh^2 \frac{x}{2} = \frac{\cosh x - 1}{\cosh x + 1}$$

$$\tan \frac{x}{2} = \frac{\sin x}{1 + \cos x} \qquad \tanh \frac{x}{2} = \frac{\sinh x}{\cosh x + 1}$$

$$\tan \frac{x}{2} = \frac{1 - \cos x}{\sin x} \qquad \tanh \frac{x}{2} = \frac{\cosh x - 1}{\sinh x}$$

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} \qquad \sinh x = 2 \sinh \frac{x}{2} \cosh \frac{x}{2}$$

$$\sin x \pm \sin y = 2\sin\tfrac{1}{2}(x \pm y)\cos\tfrac{1}{2}(x \mp y)$$

$$\sinh x \pm \sinh y = 2\sinh\tfrac{1}{2}(x \pm y)\cosh\tfrac{1}{2}(x \mp y)$$

$$\cos x + \cos y = 2\cos\tfrac{1}{2}(x + y)\cos\tfrac{1}{2}(x - y)$$

$$\cosh x + \cosh y = 2\cosh\tfrac{1}{2}(x + y)\cosh\tfrac{1}{2}(x - y)$$

$$\cos x - \cos y = -2\sin\tfrac{1}{2}(x + y)\sin\tfrac{1}{2}(x - y)$$

$$\cosh x - \cosh y = 2\sinh\tfrac{1}{2}(x + y)\sinh\tfrac{1}{2}(x - y)$$

Hyperbolic Trig Functions

The hyperbolic trigonometric functions satisfy the following properties

$$\cosh(x)^2 - \sinh(x)^2 = 1$$

$$\sinh(x + y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$$

$$\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$$

$$\cosh(-x) = \cosh(x)$$

$$\sinh(-x) = -\sinh(x)$$

Hyperbolic Trig Functions Certain Values

$\sinh 0 = 0$

$\cosh 0 = 1$

$\tanh 0 = 0$

$\coth 0 = \text{undefined}$

$\operatorname{sech} 0 = 1$

$\operatorname{csch} 0 = \text{undefined}$

$$\sinh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} = \frac{2 - 1/2}{2} = \frac{3}{4}$$

$$\cosh(\ln 2) = \frac{e^{\ln 2} + e^{-\ln 2}}{2} = \frac{2 + 1/2}{2} = \frac{5}{4}$$

$$\tanh(\ln 2) = \frac{\sinh(\ln 2)}{\cosh(\ln 2)} = \frac{3}{5}$$

$$\coth(\ln 2) = \frac{5}{3}$$

$$\operatorname{sech}(\ln 2) = \frac{4}{5}$$

$$\operatorname{csch}(\ln 2) = \frac{4}{3}$$

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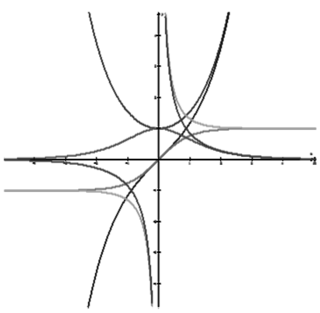
The Graphs

— cosh *x*
— sinh *x*
— tanh *x*
— coth *x*
— sech *x*
— csch *x*



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Hyperbolic Trig Functions Derivatives

$$\frac{d}{dx}\sinh x = \frac{d}{dx}\frac{e^{x} - e^{-x}}{2} = \frac{e^{x} + e^{-x}}{2} = \cosh x$$

$$\frac{d}{dx}\cosh x = \frac{d}{dx}\frac{e^{x} + e^{-x}}{2} = \frac{e^{x} - e^{-x}}{2} = \sinh x$$

$$\frac{d}{dx}\sinh x = \cosh x \qquad \frac{d}{dx}\cosh x = \sinh x$$

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Hyperbolic Trig Functions

From their definitions and the rules of derivatives we get

$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x \qquad \frac{d}{dx}\coth x = -\operatorname{csch}^2 x$$

$$\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x \qquad \frac{d}{dx}\operatorname{csch} x = -\operatorname{csch} x \coth x$$

Hyperbolic Trig Functions

Since the exponential function has a power series expansion

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

The hyperbolic trig functions have power series expansions

$$\cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \qquad \sinh x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

Hyperbolic Trig Functions

Recall that the Maclaurin series for the sine and cosine are:

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \qquad \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

VERY SIMILAR!!!!

Hyperbolic Trig Functions

Replace x by ix, where $i^2 = -1$:

$$\cos(ix) = \sum_{n=0}^{\infty}\frac{(-1)^n (ix)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)^n i^{2n} x^{2n}}{(2n)!} = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!} = \cosh(x)$$

$$\sin(ix) = \sum_{n=0}^{\infty}\frac{(-1)^n (ix)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n i^{2n+1} x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{ix^{2n+1}}{(2n+1)!} = i\sinh(x)$$

$$\cosh(x) = \cos(ix) = \cos\left(\frac{x}{i}\right)$$

$$\sinh(x) = -i\sin(ix) = i\sin\left(\frac{x}{i}\right)$$

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Inverse Hyperbolic Trig Functions

$$y = \text{arcsinh}\, x = \sinh^{-1}x \iff x = \sinh y$$

$$x = \frac{e^y - e^{-y}}{2} \qquad e^y - e^{-y} = 2x$$

$$\left(e^y\right)^2 - 2xe^y + 1 = 0 \qquad e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$$

$$y = \ln\left(x + \sqrt{x^2+1}\right)$$

$$\sinh^{-1}x = \text{arcsinh}\, x = \ln\left(x + \sqrt{x^2+1}\right)$$

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Inverse Hyperbolic Trig Functions

$$\sinh^{-1}x = \text{arcsinh}\, x = \ln\left(x + \sqrt{x^2+1}\right)$$

$$\cosh^{-1}x = \text{arccosh}\, x = \ln\left(x + \sqrt{x^2-1}\right)$$

$$\tanh^{-1}x = \text{arctanh}\, x = \ln\left(\frac{\sqrt{1+x}}{\sqrt{1-x}}\right)$$

$$\coth^{-1}x = \text{arccoth}\, x = \ln\left(\frac{\sqrt{x+1}}{\sqrt{x-1}}\right)$$

$$\text{sech}^{-1}x = \text{arcsech}\, x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$$

$$\text{csch}^{-1}x = \text{arccsch}\, x = \ln\left(\frac{1+\sqrt{x^2+1}}{x}\right)$$

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Why did you do that?!?

These inverse hyperbolic trigonometric functions often appear in antiderivative formulas instead of the logarithms

$$\int \frac{dx}{\sqrt{x^2-9}} = \ln\left|x+\sqrt{x^2-9}\right|+C \quad \text{using } x = 3 \sec u$$

$$= \cosh^{-1}\left(\frac{x}{3}\right)+C \quad \text{using } x = 3 \cosh u$$

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Catenary & Chains

What shape does a chain take when hanging freely between two pegs?

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Catenary & Chains

This question and curve studied in 1691 by

- Leibniz
- Huygens
- Johann Bernoulli

Named *catenaria* – Latin

AKA: funicular curve, velar curve

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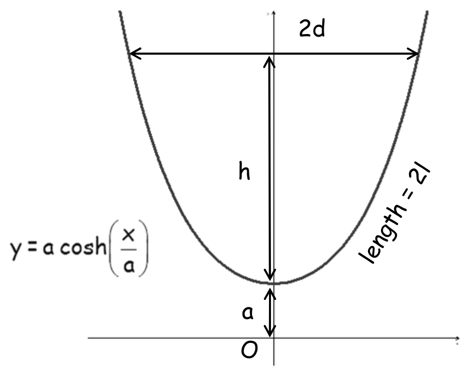
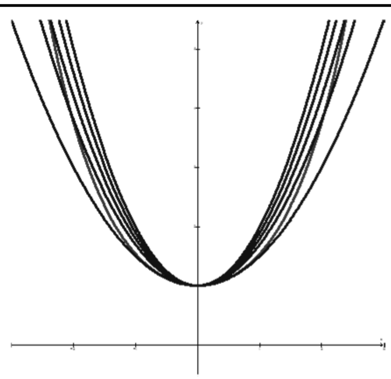


- The catenary is the form taken by a flexible, thin, homogeneous, inextensible wire suspended between two points, placed in a uniform gravitational field;
- Galileo thought it was an arc of a parabola,
- Leibniz, Jean Bernoulli, and Huygens showed in 1691 it was not



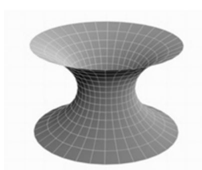
Parabolas are too “pointy”

Gateway Arch
St. Louis

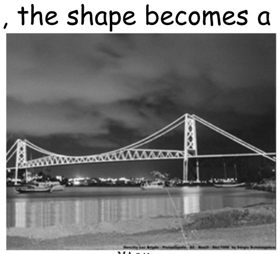
Parking
Structure
Lyon, France

Airship Hangar
Ecausseville, France

Euler proved in 1744 that the catenary is the curve which, when rotated about the x-axis, gives the surface of minimum surface area (the catenoid) for the given bounding circle.



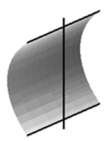
Cables on a suspension bridge are catenaries before the road bed is attached. Once the road bed is attached, the shape becomes a parabola.



A true hanging bridge though takes the shape of a catenary.

Velar Curve - Bernoulli

Profile of a rectangular sail attached to 2 horizontal bars, swollen by a wind blowing perpendicular to the bar



Backpacking – Catenary Tarp

"To help our members answer, a catenary cut tarp (or "cat" cut for short) is a tarp with the natural "sag" that gravity imposes in a line or chain suspended between two points, cut into the fabric along a seam. This results in a shape the opposite of an arch shape. It's done mainly to reduce flapping in wind, although the loss of that little bit of material also makes the tarp very slightly lighter in weight than a flat cut tarp."

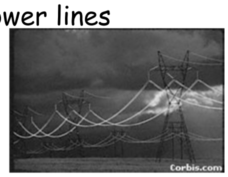
Other Uses

Anchor rope for anchoring marine vessels– shape is mostly a catenary

Sail design for racing sloops

Waves propagating through a narrow canal

Power lines



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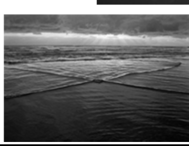
Other Uses

Waves propagating through a narrow canal – solitons – solitary wave



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Other Uses

Power lines

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Other Uses

??????????????

Are these supersized catenaries?

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